Asymptotic series expansions

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Algebraic equations

$$x^2 + \epsilon x - 1 = 0, \quad \epsilon \ll 1$$

$$x_\pm = -\frac{1}{2}\epsilon \pm \sqrt{1 + \frac{1}{4}\epsilon^2} \qquad = -\frac{1}{2}\varepsilon \pm \left(1 + \frac{1}{8}\varepsilon^2 + O(\varepsilon^4)\right)$$

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots$$

$O(1)$: $\quad x_0^2 - 1 = 0 \quad \Rightarrow \quad x_0 = \pm 1$ ✓

$O(\varepsilon)$: $\quad 2x_0 x_1 + x_0 = 0 \quad \Rightarrow \quad x_1 = -\frac{1}{2}$ ✓

$O(\varepsilon^2)$: $\quad 2x_0 x_2 + x_1^2 + x_1 = 0 \quad \Rightarrow \quad x_2 = \frac{-(x_1^2 + x_1)}{2x_0} = \frac{1}{8}\,\mathrm{sgn}(x_0)$ ✓

Singular? perturbations

$$\epsilon x^3 - x + 1 = 0, \quad \epsilon \ll 1$$

$\varepsilon = 0$, one root

$0 < \varepsilon \ll 1$, three roots

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots$$

$O(1)$: $-x_0 + 1 = 0 \Rightarrow x_0 = 1$

$$x = \delta(\epsilon) X, \quad X = \mathrm{ord}(1)$$

$$\epsilon\delta^3 X^3 - \delta X + 1 = 0$$

Dominant balance

$\delta X \sim 1 \Rightarrow \delta = 1 \qquad \delta X, 1 \gg \varepsilon \delta^3 X^3$

$\varepsilon \delta^3 X^3 \sim \delta X \Rightarrow \delta = \varepsilon^{-1/2} \qquad \varepsilon\delta^3 X^3, \delta X \gg 1$

$\varepsilon \delta^3 X^3 \sim 1 \Rightarrow \delta = \varepsilon^{-1/3} \qquad \varepsilon \delta^3 X^3, 1 \ll \delta X$ ✗

Non-integral powers

$$\epsilon x^3 - x + 1 = 0, \quad \epsilon \ll 1$$

$$x = \epsilon^{-1/2} X, \quad X = \mathrm{ord}(1) \qquad \Rightarrow X^3 - X + \epsilon^{1/2} = 0, \quad \epsilon \ll 1$$

$$X = X_0 + \varepsilon^{1/2} X_1 + \varepsilon X_2 + \cdots$$

$$O(1): \quad X_0^3 - X_0 = 0 \quad \Rightarrow \quad X_0 = \pm 1 \quad \text{as} \quad X = \mathrm{ord}(1)$$

In general, $X = X_0 + \delta_1(\epsilon) X_1 + \delta_2(\epsilon) X_2 + \ldots$

Hinch (1991) *Perturbation Methods* (Cambridge Texts in Applied Mathematics). Cambridge: Cambridge University Press.

Final remarks

- Remember to attempt the exercises for this topic before the live session on

2pm Thursday, 8 October

- If you find any typos/mistakes in the PDF notes, please email me at mt599@cam.ac.uk

Thank you for watching!