Ex 4.1 Alternative Solution

$$\int_{\mathbb{R}^d} |x - T(x)|^2 \, d\mu(x) + \int_{\mathbb{R}^d} |x + T(x)|^2 \, d\mu(x)$$

$$= \int \left(2|x|^2 + 2|T(x)|^2\right) d\mu(x) = 2\int |x|^2 \, d\mu(x) + 2\int |y|^2 \, d\nu(y) =: M.$$

So $$\max_{T : T_\# \mu = \nu} \int_{\mathbb{R}^d} |x - T(x)|^2 \, d\mu(x) = M - \min_{T : T_\# \mu = \nu} \int_{\mathbb{R}^d} |x + T(x)|^2 \, d\mu(x)$$

Let $S = -T$, Now

$$\begin{aligned} T_\# \mu = \nu &\iff \mu(T^{-1}(A)) = \nu(A) \quad \forall A \\ &\iff \mu(\{x : T(x) \in A\}) = \nu(A) \quad \forall A \\ &\iff \mu(\{x : -S(x) \in A\}) = \nu(A) \quad \forall A \\ &\iff \mu(\{x : S(x) \in A\}) = \nu(-A) \quad \forall A \\ &\iff S_\# \mu = \tilde{\nu} \quad \text{where } \tilde{\nu}(A) = \nu(-A). \end{aligned}$$

Hence $$\min_{T : T_\# \mu = \nu} \int_{\mathbb{R}^d} |x + T(x)|^2 \, d\mu(x) = \min_{S : S_\# \mu = \tilde{\nu}} \int_{\mathbb{R}^d} |x - S(x)|^2 \, d\mu(x)$$

By results from lectures $\exists \varphi$ s.t. $S = D\varphi$ where φ is convex (by Knott-Smith optimality). So $T = -D\varphi$ minimises $\int_{\mathbb{R}^d} |x + T(x)|^2 \, d\mu(x)$ over $T_\# \mu = \nu$ and so the same T maximises $\int_{\mathbb{R}^d} |x - T(x)|^2 \, d\mu(x)$ over all $T_\# \mu = \nu$.