Exercise Sheet 1

Introduction to Optimal Transport — Lent 2018-2019
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Exercise 1.1. Let $T_1 : \mathbb{R} \to \mathbb{R}$ be the translation $T_1(x) = x + 1$. Let μ be the uniform measure on $[0, 1]$ and ν be the uniform measure on $[1, 2]$. Show that $[T_1]_\# \mu = \nu$. Define $S : \mathbb{R} \to \mathbb{R}$ by $S(x) = 2x$, show that $S_\# \mu \neq \nu$.

Exercise 1.2. Assume $\pi = (\mathrm{Id} \times f)_\# \mu$ for some measurable function $f : X \to Y$. Show that disintegration of measures implies $\pi(\cdot|x) = \delta_{f(x)}$.

Exercise 1.3. Let $X, Y \subset \mathbb{R}^d$, $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$ and $T : X \to Y$ be measurable. Show that $T_\# \mu = \nu$ if and only if

$$\int_X \varphi(T(x)) \,\mathrm{d}\mu(x) = \int_Y \varphi(y) \,\mathrm{d}\nu(y)$$

for all bounded functions $\varphi : Y \to \mathbb{R}$.

Exercise 1.4. Let $X, Y \subset \mathbb{R}^d$ and $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$ have densities f and g with respect to the Lebesgue measure respectively. Assume $T : X \to Y$ is continuously differentiable and bijective. Show that $\nu = T_\# \mu$ if and only if $f(x) = g(T(x))|\det(\nabla T(x))|$.

Exercise 1.5. Define $T_2 : \mathbb{R} \to \mathbb{R}$ by $T_2(x) = 2 - x$ and $T_3 : \mathbb{R} \to \mathbb{R}$ by

$$T_3(x) = \begin{cases} x + \frac{3}{2} & \text{if } x \leq \frac{1}{2} \\ 2 - x & \text{if } x > \frac{1}{2}. \end{cases}$$

Let μ be the uniform measure on $[0, 1]$ and ν be the uniform measure on $[1, 2]$. Using exercise 1.3 show that $[T_2]\#\mu = \nu$ and $[T_3]\#\mu = \nu$.

Exercise 1.6. Let $\mu \in \mathcal{P}(\mathbb{R})$ be a symmetric probability measure. Show that $S_\# \mu = \mu$ where $S(x) = -x$.

Exercise 1.7 Let $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$. Assume $\pi = (\mathrm{Id} \times T)_\# \mu$ where $T : X \to Y$. Show $\pi \in \Pi(\mu, \nu)$ if and only if $\nu = T_\# \mu$.

Exercise 1.8. Let $X = [0, 1]$, $Y = [1, 2]$, μ be the uniform measure on X and ν be the uniform measure on Y. Let T_1, T_2, T_3 be defined as in exercises 1.1 and 1.5. Compute the Monge cost (i.e. $\mathbb{M}(T_i)$) for T_1, T_2, T_3 where (a) $c(x, y) = |x - y|^2$, (b) $c(x, y) = |x - y|$, and (c) $c(x, y) = \sqrt{|x - y|}$.

Exercise 1.9. Let $X = [0, 1]$, $Y = [1, 2]$, μ be the uniform measure on X and ν be the uniform measure on Y. Assume $c(x, y) = h(|x-y|)$ where $h : \mathbb{R}_+ \to \mathbb{R}$ is convex. Prove that T_1 as defined in exercise 1.1 is an optimal transport map. *Hint: use Jensen's inequality to show that* $\mathbb{M}(T) \geq h(1)$ *for all admissible transport maps.*

Exercise 1.10. Let $X = [0, 1]$, $Y = [1, 2]$, μ be the uniform measure on X and ν be the uniform measure on Y. Define $h : \mathbb{R}_+ \to \mathbb{R}$ by $h(s) = (s+1)\log(s+1)$ and assume $c(x, y) = h(|x - y|)$. Find an optimal transport map.

Exercise 1.11. Let $X, Y \subset \mathbb{R}$ be bounded, $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$, and $c(x,y) = h(y-x)$ where $h : \mathbb{R} \to \mathbb{R}$ is linear. Show that every admissible transport map is optimal.

Exercise 1.12. Let $X = [0,2]$, $Y = [1,3]$, μ be the uniform measure on X and ν be the uniform measure on Y. Assume $c(x,y) = |x-y|$. Let $T_1 : \mathbb{R} \to \mathbb{R}$ be as defined in exercise 1.1 and $T_4 : \mathbb{R} \to \mathbb{R}$ be defined by

$$T_4(x) = \begin{cases} x+2 & \text{if } x \leq 1 \\ x & \text{if } x > 1. \end{cases}$$

Show that T_1 and T_4 are both optimal.

Exercise 1.13. Let $X, Y \subseteq \mathbb{R}$, $c(x,y) = |x-y|^2$. Let $\mathcal{T}(\mu,\nu)$ be the Monge optimal transport cost between μ and ν for the cost c, i.e.

$$\mathcal{T}(\mu,\nu) = \inf_{T : T_{\#}\mu = \nu} \mathbb{M}(T).$$

For $a \in \mathbb{R}$ define the translation $\tau_a(x) = x - a$. Define $M_\mu = \int_{\mathbb{R}} x \, \mathrm{d}\mu(x)$ and $M_\nu = \int_{\mathbb{R}} y \, \mathrm{d}\nu(y)$. Assume $T_{\#}\mu = \nu$ for some map $T : \mathbb{R} \to \mathbb{R}$ and define $S = \tau_b \circ (T \circ \tau_{-a})$, i.e. $S(x) = T(x+a) - b$. Show (a) $S_{\#}([\tau_a]_{\#}\mu) = [\tau_b]_{\#}\nu$, and (b)

$$\mathcal{T}([\tau_a]_{\#}\mu, [\tau_b]_{\#}\nu) = \mathcal{T}(\mu,\nu) + (b-a)^2 + 2(a-b)(M_\nu - M_\mu).$$

Hint: Show that the LHS is less than or equal to the RHS, and that the RHS is less than or equal to the RHS by choosing T appropriately.