Example Sheet 2

Introduction to Optimal Transport — Lent 2018-2019
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Exercise 2.1. Let $X = Y = \mathbb{R}^2$ and $c(x,y) = |x-y|^2$. Define

$$A = \left\{(x_1,x_2) \in \mathbb{R}^2 \,:\, x_1 = 0, x_2 \in [0,1]\right\}$$
$$B = \left\{(y_1,y_2) \in \mathbb{R}^2 \,:\, y_1 = 1, y_2 \in [0,1]\right\}$$
$$C = \left\{(y_1,y_2) \in \mathbb{R}^2 \,:\, y_1 = -1, y_2 \in [0,1]\right\}.$$

Define $\mu = \mathcal{H}^1\lfloor_A$ and $\nu = \frac{1}{2}\left(\mathcal{H}^1\lfloor_B + \mathcal{H}^1\lfloor_C\right)$ where $\mathcal{H}^1$ is the 1D Hausdorff measure. (a) Explain why the transport cost must be at least 1. (b) Divide A into $2n$ equal segments and B and C into n equal segments as follows

$$A_i = \left\{(x_1,x_2) \in \mathbb{R}^2 \,:\, x_1 = 0, x_2 \in \left[\frac{i-1}{2n}, \frac{i}{2n}\right]\right\} \qquad \text{for } i = 1,2\dots,2n$$
$$B_i = \left\{(y_1,y_2) \in \mathbb{R}^2 \,:\, y_1 = 1, y_2 \in \left[\frac{i-1}{n}, \frac{i}{n}\right]\right\} \qquad \text{for } i = 1,2\dots,n$$
$$C_i = \left\{(y_1,y_2) \in \mathbb{R}^2 \,:\, y_1 = -1, y_2 \in \left[\frac{i-1}{n}, \frac{i}{n}\right]\right\} \qquad \text{for } i = 1,2\dots,n.$$

Define the affine transportation map T_n that sends A_{2i-1} to B_i and A_{2i} to C_i, show that the cost of this transport map is less than $1 + \frac{1}{n}$. (c) Explain why there cannot exist an optimal transport map.

Exercise 2.2. Let $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$ where X and Y are Polish spaces. Assume that $\pi \in \Pi(\mu,\nu)$ is an optimal transport plan and the optimal transport cost is finite (for some measurable cost function $c : X \times Y \to [0,+\infty]$). Let $\tilde{\pi}$ be any non-negative measure on $X \times Y$ with $\tilde{\pi} \leq \pi$ and $\tilde{\pi}(X \times Y) > 0$. Define $\pi' = \frac{\tilde{\pi}}{\tilde{\pi}(X\times Y)}$, and $\mu' = P^X_{\#}\tilde{\pi}'$, $\nu' = P^Y_{\#}\tilde{\pi}'$ to be the marginals of π'. Show that π' is an optimal transport plan between μ' and ν'.

Exercise 2.3. Let $\mu \in \mathcal{P}(\mathbb{R})$ and $F(x) = \mu((-\infty,x])$ be its cumulative distribution function. Show that $F^{-1}(t) \leq x \Leftrightarrow F(x) \geq t$, where F^{-1} is the generalised inverse. If F is invertible then show $F^{-1}(F(x)) = x$ and $F(F^{-1}(t)) = t$.

Exercise 2.4. Using Corollary 3.2 from the notes find the optimal transport map between the measures $\mu \in \mathcal{P}(\mathbb{R})$ and $\nu \in \mathcal{P}(\mathbb{R})$ where μ is the uniform measure on $[0,1]$ and ν is the measure with density

$$g(y) = \begin{cases} 0 & \text{if } y \leq 0 \text{ or } y \geq 1 \\ 2y & \text{if } y \in (0,1). \end{cases}$$

Exercise 2.5. Show that if (φ,ψ) is an optimal pair for Kantorovich's dual problem then so is $(\varphi + c, \psi - c)$ for any $c \in \mathbb{R}$.

Exercise 2.6. Let X, Y be Polish and assume they are not compact (in particular assume that you can find sequences $x_n \in X$, $y_n \in Y$ such that $\|x_n\| \to +\infty$ and $\|y_n\| \to +\infty$). Recall that $C_0(Z)$ is the space of continuous functions on Z converging to 0 at infinity. Show that if $u \in C_0(X \times Y)$ then $\Xi(u) = +\infty$ unless $u = 0$, where Ξ is defined in the proof of Lemma 4.8. (This shows why we must first consider X, Y being compact.)

Exercise 2.7. Let $X, Y \subset \mathbb{R}^d$ and $\mu \in \mathcal{P}(X), \mu \in \mathcal{P}(Y)$. Suppose T is a continuous optimal transport map and (ϕ, ψ) are a differentiable optimal pair to the Kantorovich dual problem, i.e. $\mathbb{J}(\phi, \psi) = \mathbb{M}(T)$. Assume that $c : X \times Y \to [0, +\infty)$ is differentiable in it's first argument and continuous in it's second argument. Then show that

$$\phi(x_0) + \psi(T(x_0)) = c(x_0, T(x_0)), \qquad \nabla \phi(x_0) = \nabla_x c(x_0, T(x_0))$$

for all x_0 in the support of μ.

Exercise 2.8. Let $X = [0, 1], Y = [1, 2]$, μ be the uniform measure on X and ν be the uniform measure on Y. Show that $\varphi(x) = -x$ and $\psi(y) = y$ is an optimal pair for the Kantorovich dual problem when $c(x, y) = |x - y|$.

Exercise 2.9. Let $X = [0, 1], Y = [1, 2]$, μ be the uniform measure on X and ν be the uniform measure on Y. Derive an optimal pair for the Kantorovich dual problem when $c(x, y) = |x - y|^2$.

Exercise 2.10. Let $X = [0, 1]$, $Y = [1, 2]$, μ be the uniform measure on X and ν be the uniform measure on Y. Derive an optimal pair for the Kantorovich dual problem when $c(x, y) = \sqrt{|x - y|}$.

Exercise 2.11. Using the Fenchel-Rockafellar duality theorem derive the duality relationship for linear programming: for any $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$,

$$\sup_{Ax \leq b} c \cdot x = \min_{y \geq 0, A^\top y = c} b \cdot y.$$

[Hint: define the equivalence relation $\sim$ by $x_1 \sim x_2$ if and only if $Ax_1 \sim Ax_2$, then define the space $E = \mathbb{R}^n / \sim$ with the inner product $\langle x_1, x_2 \rangle_E = (Ax_1) \cdot (Ax_2)$.]

Exercise 2.12. Let X, Y be Euclidean spaces and $c : X \times Y \to [0, \infty)$ continuous and bounded. Let $\mu_k, \mu \in \mathcal{P}(X)$ and $\nu_k, \nu \in \mathcal{P}(Y)$ where μ_k and ν_k each have Lebesgue densities. Assume $\mu_k \stackrel{*}{\rightharpoonup} \mu$ and $\nu_k \stackrel{*}{\rightharpoonup} \nu$. Let π_k be any sequence of transport plans between μ_k and ν_k. (a) Show that π_k converges (up to a subsequence) to a transport plan $\pi \in \Pi(\mu, \nu)$. (b) [This part of the question is beyond the scope of the course - to answer it I used methods from the calculus of variations and the Wasserstein distance (see Chapter 7).] If π_k are optimal transport plans between μ_k and ν_k and $\liminf_{k \to \infty} \int_{X \times Y} c(x, y) \, \mathrm{d}\pi_k(x, y) < +\infty$ then show that π is an optimal transport plan between μ and ν.