Example Sheet 3

Introduction to Optimal Transport — Lent 2018-2019
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Exercise 3.1. Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be defined by $T(x) = x + x_0$ for some fixed $x_0 \in \mathbb{R}^d$. For any probability measure $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ show that T is an optimal map from μ to $T_\# \mu$ with respect to the cost $c(x, y) = |x - y|^2$. [Hint: consider the potential $\varphi(x) = -2x \cdot x_0$.]

Exercise 3.2. Compute the convex conjugate of (a) $\varphi_1(x) = x \cdot x$, (b) $\varphi_2(x) = x \cdot x_0$ where $x_0 \in \mathbb{R}^d$ is fixed, (c) $\varphi_3(x) = 0$ if $x = x_0$ and $\varphi_3(x) = +\infty$ if $x \neq x_0$, and (d) $\varphi_4(x) = \frac{1}{p}|x|^p$ for $p \in (1, \infty)$.

Exercise 3.3. Suppose $\varphi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semi-continuous. Show that $y \in \partial\varphi(x) \Leftrightarrow x \in \partial\varphi^*(y)$.

Exercise 3.4. Assume $\varphi : \mathbb{R}^d \to \mathbb{R}$ is strictly convex, C^1 and satisfies

$$\lim_{|x| \to \infty} \frac{\varphi(x)}{|x|} = +\infty.$$

Show that $\nabla\varphi : \mathbb{R}^d \to \mathbb{R}^d$ is a bijection.

Exercise 3.5. Let $\Gamma \subset \mathbb{R}^d \times \mathbb{R}^d$. Show that (i) implies (ii) where

(i) there exists a lower semi-continuous convex function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that Γ is contained in the graph of $\partial\varphi$, i.e. if $(x, y) \in \Gamma$ then $y \in \partial\varphi(x)$;

(ii) for any choice of n and $(x_i, y_i) \in \Gamma$, $i = 1, \dots, n$, it holds

$$\sum_{i=1}^n x_i \cdot y_i \geq \sum_{i=1}^n x_{i+1} \cdot y_i$$

with the convention that $x_{n+1} = x_1$.

[In fact (i) and (ii) can be shown to be equivalent. Any Γ satisfying property (ii) is called cyclically monotone, note that the Knott-Smith optimality criterion implies that the support of any optimal transport plan is cyclically monotone - compare this to the definition in Proposition 3.3.]

Exercise 3.6. Let $c(x, y) = |x - y|^2$ and assume $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $\nu \in \mathcal{P}(\mathbb{R}^d)$ have finite second moments. Recall the definitions of $\mathbb{J}$, Φ_c (see Theorem 4.1) and $\tilde{\Phi}$ (see Theorem 6.1). Show that $(\tilde{\varphi}, \tilde{\psi}) \in \tilde{\Phi}$ minimise $\mathbb{J}$ over $\tilde{\Phi}$ if and only if $(\varphi, \psi) = (|\cdot|^2 - 2\tilde{\varphi}, |\cdot|^2 - 2\tilde{\psi})$ maximise $\mathbb{J}$ in Φ_c. Hence show that optimal maps T and optimal pairs $(\varphi, \psi) \in \Phi_c$ to the dual problem are related through

$$\nabla\varphi(x) = 2x - 2T(x)$$

whenever φ is differentiable.

Exercise 3.7. Use the above exercise and the hint in exercise 3.1 to show that $T(x) = x + x_0$ is the optimal transport map (with quadratic cost) between μ and $T_\#\mu$ for any $\mu \in \mathcal{P}(\mathbb{R}^d)$.

Exercise 3.8. Let μ and ν be the probability measures on $\mathbb{R}^2$ with densities $f(x) = \frac{1}{\pi}\chi_{B(0,1)}(x)$ and $g(y) = \frac{3|y|}{2\pi}\chi_{B(0,1)}(y)$ respectively. Find the optimal transport plan between μ and ν for the cost function $c(x, y) = |x - y|^2$. [Hint: use the ansatz $T(x) = \frac{x}{|x|^q}$.]

Exercise 3.9. Consider two sets of n points, $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^n$. Suppose all points are distinct and there are no three collinear points. Show that there exists a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that the line from x_i to $y_{\sigma(i)}$ does not intersect the line from x_j to $y_{\sigma(j)}$ for any $i \neq j$.