# Exercise Sheet 4 Solutions

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## Intro to OT

**Q.1** Let $N = \underset{x,y \in X}{\sup}\ c(x,y)$ so $N - c(x,y) \geq 0 \quad \forall x,y \in X$.

and $c(x,y) = \frac{1}{2}|x-y|^2$

Then
$$\max_{\pi \in \Pi(\mu,\nu)} \mathbb{K}_c(\pi) = N - \min_{\pi \in \Pi(\mu,\nu)} \mathbb{K}_{N-c}(\pi)$$

$$= N - \max_{(\varphi,\psi) \in \Phi_{N-c}} \mathbb{J}(\varphi,\psi) \quad \text{by duality}$$

$$= - \max_{(\varphi,\psi) \in \Phi_{-c}} \mathbb{J}(\varphi,\psi)$$

$$= M - \max_{(\tilde\varphi,\tilde\psi) \in \tilde\Phi_{-c}} \mathbb{J}(\tilde\varphi,\tilde\psi)$$

where we identify $(\varphi,\psi) \in \Phi_c$ with $(\tilde\varphi,\tilde\psi) = (\varphi + \frac{1}{2}|\cdot|^2, \psi + \frac{1}{2}|\cdot|^2) \in \tilde\Phi_{-c}$

where $\mathbb{K}_c(\pi) = \int_{X\times X} c(x,y)\, d\pi(x,y)$.

$$M = \frac{1}{2}\int_X |x|^2\, d\mu(x) + \frac{1}{2}\int_X |y|^2\, d\nu(y).$$

$$\tilde\Phi_{-c} = \{(\tilde\varphi,\tilde\psi) \in C^1(\mu) \times C^1(\nu) : \tilde\varphi(x) + \tilde\psi(y) \leq x\cdot y\}$$

Since
$$\max_{\pi \in \Pi(\mu,\nu)} \mathbb{K}_c(\pi) = M - \min_{\pi \in \Pi(\mu,\nu)} \int_{X\times X} x\cdot y\, d\pi(x,y)$$

Then we have
$$\min_{\pi \in \Pi(\mu,\nu)} \int_{X\times X} x\cdot y\, d\pi(x,y) = \max_{(\tilde\varphi,\tilde\psi) \in \tilde\Phi_{-c}} \mathbb{J}(\tilde\varphi,\tilde\psi)$$

Let $(\varphi,\psi)\in\Phi_{-c}$ maximize $J$ over $\Phi_{-c}$. By Thm 4.10 we have that $\varphi=\varphi^{(-c)(-c)}$ and $\psi=\varphi^{-c}$. It follows that $(\hat\varphi,\hat\psi)=\left(\varphi+\tfrac12|\cdot|^2,\ \psi+\tfrac12|\cdot|^2\right)$ maximize $J$ over $\tilde\Phi_{-c}$.

Now,

$$\begin{aligned}
\hat\psi(y) &= \psi(y)+\tfrac12|y|^2\\
&= \inf_z\left(-\tfrac12|z-y|^2-\varphi(z)+\tfrac12|y|^2\right)\\
&= \inf_z\left(-\tfrac12|z|^2+z\cdot y-\varphi(z)\right)\\
&= \inf_z\left(z\cdot y-\hat\varphi(z)\right)\\
&= -\sup_z\left((-y)\cdot z-(-\hat\varphi(z))\right)\\
&= -(-\hat\varphi)^*(-y)
\end{aligned}$$

And,

$$\begin{aligned}
-\hat\varphi(x) &= -\varphi(x)-\tfrac12|x|^2\\
&= -\inf_y\left(-\tfrac12|x-y|^2-\varphi^{-c}(y)+\tfrac12|x|^2\right)\\
&= \sup_y\left(\tfrac12|x-y|^2+\varphi^{-c}(y)-\tfrac12|x|^2\right)\\
&= \sup_y\left(\tfrac12|y|^2-x\cdot y-(-\hat\varphi)^*(-y)-\tfrac12|y|^2\right)\\
&= \sup_y\left(x\cdot(-y)-(-\hat\varphi)^*(-y)\right)\\
&= (-\hat\varphi)^{**}(x)
\end{aligned}$$

using $\varphi^{-c}(y)=\hat\psi(y)-\tfrac12|y|^2 = -(-\hat\varphi)^*(-y)-\tfrac12|y|^2$

So $-\hat{\varphi}$ is convex. Hence $\hat{\varphi}$ is concave.

In particular we have shown that $\exists$ a concave maximiser of $\mathbb{J}$ over $\hat{\Phi}_{-c}$.

Let $\pi^{+} \in \Pi(\mu,\nu)$ minimise $\mathbb{K}$ over $\Pi(\mu,\nu)$ / maximise $\int_{X\times X} x\cdot y \, d\pi(x,y)$ over $\Pi(\mu,\nu)$

and $(\hat{\varphi}, -(-\hat{\varphi})^{*}(-\cdot))$ maximise $\mathbb{J}$ over $\hat{\Phi}_{-c}$.

By duality,

$$\int_{X\times X} x\cdot y \, d\pi^{+}(x,y) = \int_X \hat{\varphi}(x)\, d\mu(x) - \int (-\hat{\varphi})^{*}(-y)\, d\nu(y)$$

$$\Rightarrow \quad \int_{X\times X}\left(x\cdot y - \hat{\varphi}(x) + (-\hat{\varphi})^{*}(-y)\right) d\pi^{+}(x,y) = 0$$

Since $-x\cdot y \le (-\hat{\varphi})(x) + (-\hat{\varphi})^{*}(-y)$ then

$$x\cdot y - \hat{\varphi}(x) + (-\hat{\varphi})^{*}(-y) \geq 0$$

So $$x\cdot y - \hat{\varphi}(x) + (-\hat{\varphi})^{*}(-y) = 0 \quad \text{for } \pi^{+}\text{ a.e. } (x,y)$$

By Prop 6.5 this implies $-y \in \partial(-\hat{\varphi})(x)$

$\Rightarrow$ $-y = \nabla(-\hat{\varphi})(x)$ $\mathcal{L}$-a.e. and $\therefore$ $\mu$-a.e.

$\Rightarrow$ $y = \nabla\hat{\varphi}(x)$ a.e. If we let $T^{+}(x) = \nabla\hat{\varphi}(x)$ then we can see that $\mathbb{M}(T^{+}) = \mathbb{K}(\pi^{+})$ from which it follows that $T^{+}$ is optimal.

4.2 (a)

For any admissible $\pi$ we have

$$\int_{\mathbb{R}^d\times\mathbb{R}^d\times\dots\times\mathbb{R}^d} c(x_1, x_2, \dots, x_n)\, d\pi_q(x_1, x_2, \dots, x_n)$$

$$= \int_{\mathbb{R}^d\times\mathbb{R}^d\times\dots\times\mathbb{R}^d} h(x_1+x_2+\dots+x_n)\, d\pi_q(x_1,x_2,\dots,x_n)$$

$$\geq h\left(\int_{\mathbb{R}^d\times\mathbb{R}^d\times\dots\times\mathbb{R}^d}(x_1+x_2+\dots+x_n)\, d\pi_q(x_1,x_2,\dots,x_n)\right) \quad \text{by Jensen's inequality}$$

$$= h(c) \quad \text{where } c = \int_{\mathbb{R}^d} x_1\, d\mu_1(x_1) + \dots + \int_{\mathbb{R}^d} x_n\, d\mu_n(x_n)$$

On the other hand for ~~any admissible~~ $\pi_0$ we have

$$\int_{\mathbb{R}^d\times\mathbb{R}^d\times\dots\times\mathbb{R}^d} c(x_1, x_2, \dots, x_n)\, d\pi_0(x_1, x_2, \dots, x_n)$$

$$\cancel{\geq h\left(\int_{\mathbb{R}^{nd}}(x_1+x_2+\dots+x_n)\, d\pi_0(x_1,x_2,\dots,x_n)\right)} \quad \text{~~from ineq. as above~~}$$

$$= h(\alpha) \quad \text{where } x_1+\dots+x_n = \alpha \text{ on the support of } \pi_0$$

with $h$ where $\alpha = c$.

This follows easily as

$$\alpha = \int_{\mathbb{R}^d\times\dots\times\mathbb{R}^d}(x_1+x_2+\dots+x_n)\, d\pi_0(x_1,\dots,x_n) = \int_{\mathbb{R}^d} x_1\, d\mu_1(x_1) + \dots + \int_{\mathbb{R}^d} x_n\, d\mu_n(x_n) = c.$$

(b) Note,

$$\int_{\mathbb{R}^d\times\cdots\times\mathbb{R}^d} c(x_1,\ldots,x_n)\, d\pi(x_1,\ldots,x_n)$$

$$= -\sum_{i=1}^n \sum_{j=1}^n \int_{\mathbb{R}^d\times\cdots\times\mathbb{R}^d} |x_i-x_j|^2 \, d\pi(x_1,\ldots,x_n)$$

$$= -\sum_{i=1}^n \sum_{j=1}^n \int_{\mathbb{R}^d\times\cdots\times\mathbb{R}^d}\left(|x_i|^2+|x_j|^2-2x_i\cdot x_j\right) d\pi(x_1,\ldots,x_n)$$

$$= -\int_{\mathbb{R}^d\times\cdots\times\mathbb{R}^d}\left(2n\sum_{i=1}^n|x_i|^2 - 2\sum_{i,j=1}^n x_i\cdot x_j\right) d\pi(x_1,\ldots,x_n) \quad =(*)$$

Now $\quad |x_1+\cdots+x_n|^2 = \cancel{\sum_{i=1}^n|x_i|^2} + 2\sum_{i,j=1}^n x_i\cdot x_j$

So

$$(*) = -\int_{\mathbb{R}^d\times\cdots\times\mathbb{R}^d}\left(2n\sum_{i=1}^n|x_i|^2 - 2|x_1+\cdots+x_n|^2\right) d\pi(x_1,\ldots,x_n)$$

$$= \underbrace{-2n\sum_{i=1}^n\int_{\mathbb{R}^d}|x_i|^2\, d\mu_i(x_i)}_{=:M} + 2\int_{\mathbb{R}^d\times\cdots\times\mathbb{R}^d}|x_1+\cdots+x_n|^2\, d\pi(x_1,\ldots,x_n)$$

$$= M + \int_{\mathbb{R}^d\times\cdots\times\mathbb{R}^d} h(x_1+\cdots+x_n)\, d\pi(x_1,\ldots,x_n)$$

where $h = 2|x_1+\cdots+x_n|^2$, since $h$ is convex then by part (c)

$\pi_0$ is optimal.

Q.3 Assume $\det(\nabla_x s(x,t)) > 0$

Then $\frac{\partial}{\partial t} \left| \det(\nabla_x s(x,t)) \right|$

$$= \frac{\partial}{\partial t} \det(\nabla_x s(x,t))$$

$$= \det(\nabla_x s(x,t)) \operatorname{tr}\left( [\nabla_x s(x,t)]^{-1} \frac{\partial}{\partial t} \nabla_x s(x,t) \right)$$ by Jacobi's formula.

Now we notice:

$$\nabla_y \left( \frac{\partial s}{\partial t}(s^{-1}(y,t),t) \right) = \frac{\partial}{\partial t}(\nabla_x s)(s^{-1}(y,t),t)\,(\nabla_y s^{-1})(y,t)$$

So

$$\nabla_y \left( \frac{\partial s}{\partial t}(s^{-1}(y,t),t) \right) \Big|_{y=s(x,t)} = \frac{\partial}{\partial t}(\nabla_x s)(x,t)\,(\nabla_x s^{-1})(s(x,t),t)$$

$$= \frac{\partial}{\partial t}(\nabla_x s)(x,t)\,[(\nabla_x s)(x,t)]^{-1}$$

Hence,

$$\frac{\partial}{\partial t} \left| \det(\nabla_x s(x,t)) \right| = \det(\nabla_x s(x,t)) \operatorname{tr}\left( [\nabla_x s(x,t)]^{-1} \nabla_y \left( \frac{\partial s}{\partial t}(s^{-1}(y,t),t) \right)\Big|_{y=s(x,t)} (\nabla_x s)(x,t) \right)$$

$$= \det(\nabla_x s(x,t)) \operatorname{tr} \nabla_y \left( \frac{\partial s}{\partial t}(s^{-1}(y,t),t) \right)\Big|_{y=s(x,t)}$$ using cyclic permutation property of trace

$$= \left| \det(\nabla_x s(x,t)) \right| \left( \operatorname{div} \frac{\partial s}{\partial t}(s^{-1}(y,t),t) \right)\Big|_{y=s(x,t)}$$

9.4

Assum (1) wts (2)

$\langle \nabla\phi(x_0) - \nabla\phi(x_1), x_0 - x_1 \rangle$

$= \langle D^2\phi(x)(x_0 - x_1), x_0 - x_1 \rangle$ by Taylors theorem (MVT) for some $x$

$\geq \lambda |x_1 - x_0|^2$ by 1.

Assum 2. wts 1

$\cancel{\phi(x_t) = \phi((1-t)x_0 + tx_1)}$

$\cancel{= \phi(x_0}$

Using the definition of derivative ( $f(x) = f(y) + Df(y)(x-y) + o(\|x-y\|)$ )

we have

$\langle D^2\phi(x)\xi, \xi \rangle = \langle \nabla\phi(x+\xi) - \nabla\phi(x), \xi \rangle + \langle \gamma(\xi), \xi \rangle$ where $\frac{\|\gamma(\xi)\|}{\|\xi\|} \to 0$

$\geq \lambda |\xi|^2 + \langle \gamma(\xi), \xi \rangle$ by (2)

Let $\xi = \alpha\hat{\xi}$ where $\alpha = |\xi|$ and $|\hat{\xi}| = 1$, then

$\langle D^2\phi(x)\hat{\xi}, \hat{\xi} \rangle \geq \lambda - \frac{|\gamma(\hat{\xi}\alpha)|}{\alpha}$ $\forall \alpha$, let $\alpha \to 0$

$\Rightarrow \langle D^2\phi(x)\hat{\xi}, \hat{\xi} \rangle \geq \lambda$ $\forall |\hat{\xi}| = 1$. Sub back $\hat{\xi} = \frac{\xi}{|\xi|}$ then

$\langle D^2\phi(x)\xi, \xi \rangle \geq |\xi|^2\lambda$

So (1) $\Leftrightarrow$ (2)

6

Assume (1) WTS (3)

$$\phi(x_0) = \phi(x_t) + \nabla\phi(x_t)\cdot(x_0 - x_t) + \tfrac{1}{2}\langle D^2\phi(\hat{x})(x_t - x_0), x_t - x_0\rangle$$

for some $\hat{x}$ by Taylor's theorem

$$\geq \phi(x_t) - \nabla\phi(x_t)\cdot(x_0 - x_t) + \frac{\lambda}{2}|x_t - x_0|^2 \quad \text{by (1)}$$

Similarly

$$\phi(x_1) \geq \phi(x_t) - \nabla\phi(x_t)\cdot(x_1 - x_t) + \frac{\lambda}{2}|x_t - x_1|^2$$

so

$$(1-t)\phi(x_0) + t\phi(x_1) \geq \phi(x_t) - \nabla\phi(x_t)\cdot\big((1-t)(x_0 - x_t) + t(x_1 - x_t)\big)$$

$$+\frac{\lambda}{2}(1-t)|x_t - x_0|^2 + \frac{\lambda}{2}t|x_t - x_1|^2$$

$$= \phi(x_t) - \nabla\phi(x_t)\cdot\big(\underbrace{(1-t)x_0 + tx_1}_{x_t} - x_t\big)$$

$$+\frac{\lambda}{2}|t(x_1 - x_0)|^2(1-t) + \frac{\lambda}{2}t|(1-t)(x_0 - x_1)|^2$$

$$= \phi(x_t) + \frac{\lambda}{2}t^2(1-t)|x_0 - x_1|^2 + \frac{\lambda}{2}t(1-t)^2|x_0 - x_1|^2$$

$$= \phi(x_t) + \frac{\lambda}{2}t(1-t)|x_0 - x_1|^2$$

Assume (3) WTS (1)

Again by definition of derivatives

$$D^2\phi(x)\xi = \nabla\phi(x+\xi) - \nabla\phi(x) + \gamma_{2,x}(\xi)$$

$$\langle \nabla\phi(x), \xi\rangle = \phi(x+\xi) - \phi(x) + \gamma_{1,x}(\xi)$$

$$\langle \nabla\phi(x+\xi), \xi\rangle = \phi(x+2\xi) - \phi(x+\xi) + \gamma_{1,x+\xi}(\xi)$$

where $\frac{|\gamma_{2,x}(\xi)|}{|\xi|} \to 0$

$$\frac{|\gamma_{1,x}(\xi)|}{|\xi|} \to 0$$

and $\frac{|\gamma_{1,x+\xi}(\xi)|}{|\xi|} \to 0$ as $|\xi| \to 0$

Let $\bar\gamma(\xi) = \gamma_{2,x}(\xi) \pm \gamma_{1,x}(\xi) + \gamma_{1,x+\xi}(\xi)$

Then

$$\langle D^2\phi(x)\xi, \xi\rangle = \phi(x+2\xi) - 2\phi(x+\xi) \cancel{\phantom{xxxx}} + \phi(x) + \bar\gamma(\xi)$$

Let $x_0 = x$, $x_1 = x+2\xi$, $t = 1/2$ and $x_t = x+\xi$, then

$$\langle D^2\phi(x)\xi, \xi\rangle = \phi(x_1) - 2\phi(x_t) + \phi(x_0) + \bar\gamma(\xi)$$

$$\geq \phi(x_1) - \phi(x_0) - \phi(x_1) + \lambda t(1-t)|x_0 - x_1|^2 + \phi(x_1) + \bar\gamma(\xi) \text{ by (3)}$$

$$= \lambda|\xi|^2 + \bar\gamma(\xi)$$

so $\langle D^2\phi(x)\hat\xi, \hat\xi\rangle \geq \lambda + \frac{\bar\gamma(\xi)}{|\xi|}$, let $|\xi| \to 0 \Rightarrow \langle D^2\phi(x)\hat\xi, \hat\xi\rangle \geq \lambda$

let $\hat\xi = \frac{|\xi|}{|\xi|} \Rightarrow \langle D^2\phi(x)\xi, \xi\rangle \geq \lambda|\xi|^2$.

8

Assume (1) WTS (4)

$$\phi(x_1) = \phi(x_0) + \nabla\phi(x_0)\cdot(x_1 - x_0) + \frac{1}{2}\langle D^2\phi(\tilde{x})(x_1 - x_0), x_1 - x_0\rangle$$

by Taylor Thm

$$\Rightarrow \quad \phi(x_1) - \phi(x_0) \geq \nabla\phi(x_0)\cdot(x_1 - x_0) + \frac{\lambda}{2}|x_1 - x_0|^2$$

~~not~~

Switching $x_1 \leftrightarrow x_2$ implies

$$\phi(x_0) - \phi(x_1) \geq \nabla\phi(x_1)\cdot(x_0 - x_1) + \frac{\lambda}{2}|x_0 - x_1|^2$$

So (1) $\Rightarrow$ (4)

Now assume (4) WTS (2)

We have

$$\langle\nabla\phi(x_1), x_1 - x_0\rangle - \frac{\lambda}{2}|x_1 - x_0|^2 \geq \langle\nabla\phi(x_0), x_1 - x_0\rangle + \frac{\lambda}{2}|x_1 - x_0|^2$$

$$\Rightarrow \quad \langle\nabla\phi(x_1) - \nabla\phi(x_0), x_1 - x_0\rangle \geq \lambda|x_1 - x_0|^2$$

So (4) $\Rightarrow$ (2).

~~clearly (2) $\Leftrightarrow$ (4) so we are done.~~

## 4.5

One can easily check that

$$|x_t|^2 - (1-t)|x_0|^2 - t|x_1|^2 = -(1-t)t\,|x_0 - x_1|^2$$

Hence $\quad \hat{\phi}(x_t) \leq (1-t)\,\hat{\phi}(x_0) + t\,\hat{\phi}(x_1)$

$$\Leftrightarrow \quad \phi(x_t) - \frac{\lambda}{2}|x_t|^2 \leq (1-t)\,\phi(x_0) + t\,\phi(x_1) - \frac{\lambda}{2}\left((1-t)|x_0|^2 + t|x_1|^2\right)$$

$$\Leftrightarrow \quad \phi(x_t) \leq (1-t)\,\phi(x_0) + t\,\phi(x_1) - \frac{\lambda}{2}(1-t)t\,|x_0 - x_1|^2$$

## 4.6

$$\frac{d}{dt}\left(e^{2\lambda t}|u(t) - v(t)|^2\right) = 2\lambda e^{\lambda t}|u(t) - v(t)|^2 + 2e^{\lambda t}(u(t) - v(t))\cdot\left(\frac{du}{dt}(t) - \frac{dv}{dt}(t)\right)$$

$$= 2\lambda e^{\lambda t}|u(t) - v(t)|^2 - 2e^{\lambda t}(u(t) - v(t))\cdot(\nabla\phi(u(t)) - \nabla\phi(v(t)))$$

$$\leq 2\lambda e^{\lambda t}|u(t) - v(t)|^2 - 2\lambda e^{\lambda t}|u(t) - v(t)|^2 \quad \text{by Ex 4.4 (2)}$$

$$= 0.$$

## 4.7

Fix $u_0, u_1$ and $t \in [0,1]$, define $u_t = (1-t)u_0 + t u_1$.

$$\Phi(u_t) = \alpha |u_t - v|^2 + \phi(u_t)$$

$$= \alpha \left| (1-t)(u_0 - v) + t(u_1 - v) \right|^2 + \phi(u_t)$$

$$= \alpha \Big( (1-t)^2 |u_0 - v|^2 + t^2 |u_1 - v|^2 + 2t(1-t)(u_0 - v)\cdot(u_1 - v) \Big) + \phi(u_t)$$

$$= \alpha(1-t)|u_0 - v|^2 + \alpha\big((1-t)^2 - (1-t)\big)|u_0 - v|^2$$
$$+ \alpha t |u_1 - v|^2 + \alpha(t^2 - t)|u_1 - v|^2$$
$$+ \alpha t(1-t)(u_0 - v)\cdot(u_1 - v) + \phi(u_t)$$

$$= \alpha(1-t)|u_0 - v|^2 - \alpha(1-t)t|u_0 - v|^2$$
$$+ \alpha t |u_1 - v|^2 - \alpha(1-t)t|u_1 - v|^2$$
$$+ \alpha t(1-t)(u_0 - v)\cdot(u_1 - v) + \phi(u_t)$$

$$= \alpha(1-t)|u_0 - v|^2 + \alpha t|u_1 - v|^2 + \alpha t(1-t)\Big(\underbrace{(u_0 - v)\cdot(u_1 - v) - |u_0 - v|^2 - |u_1 - v|^2}_{= u_0 \cdot u_1 + u_1 \cdot v + u_0 \cdot v - |u_0|^2 - |v|^2 - |u_1|^2}\Big)$$
$$+ \phi(u_t)$$

$$\leq \alpha(1-t)|u_0 - v|^2 + \alpha t|u_1 - v|^2 + \alpha t(1-t)\Big(\overset{u_0 \cdot u_1 - \frac{1}{2}|u_0|^2 - \frac{1}{2}|u_1|^2}{\cancel{-\tfrac{1}{2}|u_0 - v|^2 - \tfrac{1}{2}|u_1 - v|^2}}\Big) + \phi(u_t)$$

by Young's inequality

$$\leq (1-t)\big(\alpha|u_0 - v|^2 + \phi(u_0)\big) + t\big(\alpha|u_1 - v|^2 + \phi(u_1)\big)$$
$$- \frac{\lambda}{2} t(1-t)|u_0 - u_1|^2 - \frac{\alpha}{2} t(1-t)|u_0 - u_1|^2$$

$$= (1-t)\,\Phi(u_0) + t\,\Phi(u_1) - \frac{(\lambda+\alpha)}{2}\, t\,(1-t)\,|u_0-u_1|^2$$

as required.

Since $\phi$ is $\lambda$-convex it is convex and i.e. ~~lsc~~ ~~[illegible]~~

By ex 4.5, and since convex functions can be bounded below by an affine function we have that $\exists\ a \in \mathbb{R}^d,\ b \in \mathbb{R}$ s.t. (by Ex 4.5)

$$\Phi(x) \geq a\cdot x + b + \frac{\hat\lambda}{2}|x|^2 \qquad , \quad \hat\lambda = \alpha+\lambda$$

$$\geq -\frac{|a|^2}{2\varepsilon} - \frac{\varepsilon}{2}|x|^2 + b + \frac{\hat\lambda}{2}|x|^2 \qquad \text{by Young's inequality } \forall \varepsilon>0$$

choose $\varepsilon = \frac{\hat\lambda}{2}$ then

$$\Phi(x) \geq -\frac{|a|^2}{\hat\lambda} + b + \frac{\hat\lambda}{4}|x|^2$$

$$\Rightarrow \quad \Phi(x) \geq \frac{\hat\lambda}{4}|x|^2 - C \quad \text{for some } C$$

choose $R = \dfrac{4(\Phi(0)+C)}{\hat\lambda}$, then if $|x| \geq R$ this implies

$$\Phi(u) \geq \Phi(0). \quad \text{In particular}$$

$$\inf_{u\in\mathbb{R}^d} \Phi(u) = \inf_{|u|\leq R} \Phi(u).$$

Since $\Phi$ is lsc it achieves its minimum on $B(0,R)$

· For uniqueness, suppose $u_1$ min $\Phi = \Phi(u_1) = \Phi(u_2)$, $u_1 \neq u_2$

Let $\bar{u} = \frac{1}{2}u_1 + \frac{1}{2}u_2$

Then $$\Phi(\bar{u}) \leq \frac{1}{2}\Phi(u_1) + \frac{1}{2}\Phi(u_2) - \frac{\hat{\lambda}}{8}|u_1 - u_2|^2$$

$$= \min \Phi - \frac{\hat{\lambda}}{8}|u_1-u_2|^2$$

$$< \min \Phi$$

a contradiction. so the minimum is unique

4.8

Since $\dfrac{u_\tau^{(n)} - u_\tau^{(n-1)}}{\tau} = -\nabla\phi(u_\tau^{(n)})$ then

$$\frac{\|u_\tau^{(n)} - u_\tau^{(n-1)}\|^2}{\tau} = -(\nabla\phi(u_\tau^{(n)}), u_\tau^{(n)} - u_\tau^{(n-1)})$$

$$\leq -(\nabla\phi(u_\tau^{(n-1)}), u_\tau^{(n)} - u_\tau^{(n-1)})$$

$$\leq \|\nabla\phi(u_\tau^{(n-1)})\| \, \|u_\tau^{(n)} - u_\tau^{(n-1)}\|$$

so $\dfrac{\|u_\tau^{(n)} - u_\tau^{(n-1)}\|^2}{\tau} \leq \|\nabla\phi(u_\tau^{(n-1)})\|$

$\|\nabla\phi(u_\tau^{(n)})\|$. By induction $\|\nabla\phi(u_\tau^{(n)})\| \leq \|\nabla\phi(u_\tau^{(0)})\|$

Hence $\|u_\tau'(t)\| \leq \sup_{n \in \mathbb{N}} \frac{1}{\tau}\|u_\tau^{(n)} - u_\tau^{(n-1)}\| \leq \sup_{n\in\mathbb{N}} \|\nabla\phi(u_\tau^{(n)})\| \leq \|\nabla\phi(u_\tau^{(0)})\|$

$\forall t \neq t_\tau^{(n)}$

It follows that $u_\tau$ is uniformly Lipschitz and so by the Arzelà-Ascoli Theorem, for any $T>0$ $\exists$ a uniformly convergent subsequence on $[0,T]$ (since $\|\nabla\phi(u_\tau^{(n)})\| \to \|\nabla\phi(u_0)\|$ we have that $\sup_{\tau_0 \geq \tau > 0} \sup_{t \neq t_\tau^{(n)}} \|u_\tau'(t)\| < +\infty$ and $u_\tau$ is bounded on $[0,T]$).

Say $u_\tau \to u$ in $C^0([0,T])$ along a subsequence.

Define $\bar{u}_\tau(t) := u_\tau^{(n)}$ if $t \in (t_\tau^{(n-1)}, t_\tau^{(n)}]$ to be the piecewise constant interpolation.

We have

(*) $u_\tau'(t) = -\nabla\phi(\bar{u}_\tau(t)) \quad \forall\, t \neq t_\tau^{(n)}$.

Since $|\bar{u}_\tau(t) - u_\tau(t)| \leq \tau \|\nabla\phi(u_\tau^{(0)})\|$ then $\bar{u}_\tau - u_\tau$ converges in $L^\infty$.

Formally passing to the limit $\tau \to \infty$ implies

$$u'(t) = -\nabla\phi(u(t)).$$

This can be made rigorous by considering the integral form of (*).

## 4.9

Assume $u$ is a $EVI_\lambda$ G.F.

i.e. $\frac{1}{2}\frac{d}{dt} d^2(u(t),v) + \frac{\lambda}{2} d^2(u(t),v) \leq \phi(v) - \phi(u(t)) \quad \forall t \in (0,\infty)$ and $v \in D(\phi)$

Then this is equivalent to

$$\frac{1}{2}\frac{d}{dt}\left(e^{\lambda(t-s)} d^2(u(t),v)\right) \leq e^{\lambda(t-s)}(\phi(v) - \phi(u(t)))$$

Integrating we obtain (between $t$ and $r \in [s,r]$)

$$\frac{1}{2} e^{\lambda(t-s)} d^2(u(t),v) - \frac{1}{2} d^2(u(s),v) \leq \int_s^r e^{\lambda(t-s)}(\phi(v) - \phi(u(t)))\,dt$$

$$\leq \int_s^r e^{\lambda(t-s)}(\phi(v) - \phi(u(r)))\,dt$$

$$= E_\lambda(t-s)\,(\phi(v) - \phi(u(r)))$$

where we use that $\phi(u(t)) \leq \phi(u(r)) \quad \forall\, t \leq r.$

---

We have $\phi(u(t))$ is decreasing by the following argument:
if we let $v = u_0$, then $d^2(u(t),u_0)$ must be non-decreasing, hence
$\frac{d}{dt} d^2(u(t),u_0) \geq 0$, so by $EVI_\lambda \Rightarrow 0 \leq \phi(u_0) - \phi(u(t))$ for $t$ small enough.

Now come the u solutions

$$\frac{1}{2} e^{\lambda t} d^2(u(t),v) - \frac{1}{2} e^{\lambda s} d^2(u(s),v) \le \frac{e^{\lambda t}-e^{\lambda s}}{\lambda}\left(\phi(v)-\phi(u(t))\right)$$

$$\frac{1}{2}\int_s^t \frac{d}{dr}\left(e^{\lambda r} d^2(u(r),v)\right) dr \qquad = \qquad \int_s^t e^{\lambda r}\, dr \left(\phi(v)-\phi(u(t))\right)$$

$$\Rightarrow \quad \frac{1}{2\delta}\int_{s-\delta}^{s} \frac{d}{dr}\left(e^{\lambda r} d^2(u(r),v)\right) dr \le \frac{1}{\delta}\int_{s-\delta}^{s} e^{\lambda r}\, dr \left(\phi(v)-\phi(u(s))\right)$$

$$\delta \to 0^+ \quad \Rightarrow \quad \frac{1}{2}\frac{d}{ds}\left(e^{\lambda s} d^2(u(s),v)\right) \le e^{\lambda s}\left(\phi(v)-\phi(u(s))\right)$$

$$\Rightarrow \quad \frac{1}{2}\lambda e^{\lambda s} d^2(u(s),v) + \frac{1}{2} e^{\lambda s}\frac{d}{ds} d^2(u(s),v) \le e^{\lambda s}\left(\phi(v)-\phi(u(s))\right)$$

$$\Rightarrow \quad \frac{\lambda}{2} d^2(u(s),v) + \frac{1}{2}\frac{d}{ds} d^2(u(s),v) \le \phi(v)-\phi(u(s))$$ as required.

## 4.10

Let $[0,1] \ni s \mapsto \gamma^{(s)}$ be a geodesic between $x_0$ and $x_1$,

i.e. $\gamma^{(0)} = x_0$ and $\gamma^{(1)} = x_1$. Let $\gamma_t^{(s)}$ be the EVI$_\lambda$-GF

with initial condition $\gamma^{(s)}$. Then

$$
\begin{cases}
(1) \quad \frac{1}{2} e^{\lambda t} d^2(\gamma_t^{(s)}, \gamma^{(0)}) - \frac{1}{2} d^2(\gamma^{(s)}, \gamma^{(0)}) \le E_\lambda(t) \left( \phi(\gamma^{(0)}) - \phi(\gamma_t^{(s)}) \right) \\
(2) \quad \frac{1}{2} e^{\lambda t} d^2(\gamma_t^{(s)}, \gamma^{(1)}) - \frac{1}{2} d^2(\gamma^{(s)}, \gamma^{(1)}) \le E_\lambda(t) \left( \phi(\gamma^{(1)}) - \phi(\gamma_t^{(s)}) \right)
\end{cases}
$$

$\forall s \in (0,1)$ and $t > 0$ by Ex 4.9.

Hence $(1-s)$ (eq. 1) $+ s$ (eq. 2) $\Rightarrow$

then

$$
\begin{aligned}
&\frac{1}{2} e^{\lambda t} \left( (1-s) d^2(\gamma_t^{(s)}, \gamma^{(0)}) + s d^2(\gamma_t^{(s)}, \gamma^{(1)}) \right) \\
&\quad - \frac{1}{2} \left( (1-s) d^2(\gamma^{(s)}, \gamma^{(0)}) + s d^2(\gamma^{(s)}, \gamma^{(1)}) \right) \\
&\qquad \le E_\lambda(t) \left( (1-s) \phi(\gamma^{(0)}) + s \phi(\gamma^{(1)}) - \phi(\gamma_t^{(s)}) \right)
\end{aligned}
$$

Using $(1-s)a^2 + sb^2 \ge s(1-s)(a+b)^2 \quad \forall a, b \in \mathbb{R}, \ s \in [0,1]$

Then
$$
\begin{aligned}
(1-s) d^2(\gamma_t^{(s)}, \gamma^{(0)}) + s d^2(\gamma_t^{(s)}, \gamma^{(1)}) &\ge s(1-s) \left( d(\gamma_t^{(s)}, \gamma^{(0)}) + d(\gamma_t^{(s)}, \gamma^{(1)}) \right)^2 \\
&\ge s(1-s) d^2(\gamma^{(0)}, \gamma^{(1)}) \quad \Delta\text{-inequality}
\end{aligned}
$$

And since $\gamma^{(s)}$ is a geodesic

$$
(1-s) d^2(\gamma^{(s)}, \gamma^{(0)}) + s d^2(\gamma^{(s)}, \gamma^{(1)}) = (1-s) s^2 d^2(\gamma^{(0)}, \gamma^{(1)}) + s(1-s)^2 d^2(\gamma^{(0)}, \gamma^{(1)})
$$

$$= s(1-s)\, d^2(\gamma^{(0)}, \gamma^{(1)})\,\big( s + 1-s \big)$$

$$= s(1-s)\, d^2(\gamma^{(0)}, \gamma^{(1)}).$$

Hence,

$$\frac{1}{2} e^{\lambda t}\, s(1-s)\, d^2(\gamma^{(0)}, \gamma^{(1)}) - \frac{1}{2}\, s(1-s)\, d^2(\gamma^{(0)}, \gamma^{(1)})$$
$$\le E_\lambda(t) \Big( (1-s)\, \phi(\gamma^{(0)}) + s\, \phi(\gamma^{(1)}) - \phi(\gamma_t^{(s)}) \Big)$$

$$\Rightarrow \quad \underbrace{\frac{e^{\lambda t} - 1}{2}}_{= \frac{\lambda E_\lambda(t)}{2}}\, s(1-s)\, d^2(\gamma^{(0)}, \gamma^{(1)}) \le E_\lambda(t) \Big( (1-s)\, \phi(\gamma^{(0)}) + s\, \phi(\gamma^{(1)}) - \phi(\gamma_t^{(s)}) \Big)$$

$$\Rightarrow \quad \phi(\gamma_t^{(s)}) \le (1-s)\, \phi(\underset{= x_0}{\gamma^{(0)}}) + s\, \phi(\underset{= x_1}{\gamma^{(1)}}) - \frac{\lambda}{2}\, s(1-s)\, d^2(\underset{= x_0}{\gamma^{(0)}}, \underset{= x_1}{\gamma^{(1)}})$$

Take $t \to 0$ implies the result.

## 4.11

~~Let $u(t)$ satisfy the condition in the question.~~ Let $u_0^{(0)}, u_0^{(1)} \in \mathbb{R}^n$

and ~~define $u^{(s)} = (1-s)\, u_0^{(0)} + s\, u^{(1)}$ for some $s \in (0,1)$.~~

Let $u^{(0)}(t)$, $u^{(1)}(t)$ be a $C^1$ curves that satisfies the condition in the question with

$u^{(0)}(0) = u^{(0)}$, $u^{(1)}(0) = u^{(1)}$.

Now $\langle \nabla \phi(u^{(0)}), u^{(1)} - u^{(0)} \rangle$ ~~$= \langle \frac{d}{dt} u^{(0)} \ldots$~~

$$= -\left\langle \frac{du^{(0)}}{dt}(0), u^{(1)} - u^{(0)} \right\rangle$$

$$\cancel{= \frac{1}{2}\frac{d}{dt} |u^{(0)}(t)|^2 \Big|_{t=0}}$$

$$= \frac{1}{2} \frac{d}{dt} \left| u^{(0)}(t) - u^{(1)} \right|^2 \Big|_{t=0}$$

$$\leq \phi(u^{(1)}) - \phi(u^{(0)}(0)) - \frac{\lambda}{2} |u^{(0)}(0) - u^{(1)}|^2$$

$$\Rightarrow \langle \nabla \phi(u^{(0)}), u^{(1)} - u^{(0)} \rangle + \frac{\lambda}{2} |u^{(1)} - u^{(0)}|^2 \leq \phi(u^{(1)}) - \phi(u^{(0)})$$

By symmetry

$$\langle \nabla \phi(u^{(1)}), u^{(0)} - u^{(1)} \rangle + \frac{\lambda}{2} |u^{(1)} - u^{(0)}|^2 \leq \phi(u^{(0)}) - \phi(u^{(1)})$$

Hence property (4) in Ex 4.4 holds. Hence $\phi$ is $\lambda$-convex.

9.12

$$\frac{\partial}{\partial t}\mu_t + \nabla\cdot(v_t\mu_t) = 0$$ in the sense of distributions

$$\Rightarrow \int_{\mathbb{R}}\int_{\mathbb{R}^d}\frac{\partial f_t}{\partial t}(x)\,d\mu_t(x)\,dt + \iint_{\mathbb{R}\ \mathbb{R}^d} v_t(x)\cdot\nabla f_t(x)\,d\mu_t(x)\,dt = 0$$ $\forall$ test function $f$.

$$\Rightarrow \int_{\mathbb{R}}\int_{\mathbb{R}^d}\frac{\partial f_t}{\partial t}(x)\,\rho_t(x)\,dx\,dt + \int_{\mathbb{R}}\int_{\mathbb{R}^d} v_t\cdot\nabla f_t(x)\,\rho_t(x)\,dx\,dt = 0$$

$$\Rightarrow \int_{\mathbb{R}}\int_{\mathbb{R}^d} f_t(x)\,\frac{\partial \rho_t}{\partial t}(x)\,dx\,dt + \int_{\mathbb{R}}\int_{\mathbb{R}^d} f_t(x)\,\nabla\cdot(v_t(x)\rho_t(x))\,dx\,dt = 0$$

$$\Rightarrow \frac{\partial \rho_t}{\partial t} + \nabla\cdot(v_t\rho_t) = 0$$ weakly.

9.13

$$\left\langle \frac{\partial \mathcal{U}}{\partial \mu}(\mu), \nu \right\rangle = \lim_{t \to 0^+} \frac{1}{t}\left( \mathcal{U}(\mu + t\nu) - \mathcal{U}(\mu) \right)$$

$$= \lim_{t \to 0^+} \frac{1}{t} \int_{\mathbb{R}^d} \left( U(\rho(x) + t\sigma(x)) - U(\rho(x)) \right) dx \qquad \text{where } \sigma = \frac{d\nu}{dx}$$

$$= \int_{\mathbb{R}^d} U'(\rho(x)) \cdot \sigma(x)\, dx$$

$$= \int_{\mathbb{R}^d} U'(\rho(x))\, d\nu(x)$$

Hence $\frac{\partial \mathcal{U}}{\partial \mu}(\mu) = U'(\rho(x))$.

$$\langle \mathcal{V}(\mu), \nu \rangle = \lim_{t \to 0^+} \frac{1}{t} \left( \mathcal{V}(\mu + t\nu) - \mathcal{V}(\mu) \right)$$

$$= \lim_{t \to 0^+} \frac{1}{t} \int_{\mathbb{R}^d} V(x)\, d(t\nu)(x)$$

$$= \int_{\mathbb{R}^d} V(x)\, d\nu(x)$$

so $\frac{\partial \mathcal{V}}{\partial \mu}(\mu) = V$

4.19

Let $\nu_\varepsilon = (T_\varepsilon)_\# \mu$

Clearly $T_\varepsilon$ is admissible between $\mu$ and $\nu$ and $\pi_\varepsilon \in \Pi(\mu, \nu_\varepsilon)$

It is enough to show that $\exists$ a convex function $\varphi$ s.t. $D\varphi = T_\varepsilon$.

Let $\varphi(x) = \frac{1}{2}|x|^2 + \varepsilon \zeta(x)$, then $D\varphi(x) = x + \varepsilon D\zeta(x) = T_\varepsilon(x)$ so

we are left to show $\varphi$ is convex. We need

$$D^2\varphi(x) \geq 0 \quad \forall x$$

Since $D^2\varphi(x) = I + \varepsilon D^2\zeta(x)$ then the required $\varepsilon \max_{\mathbb{R}^d} \|D^2\zeta\| < 1$

implies
$$\begin{aligned}
\langle D^2\varphi(x)\, \gamma, \gamma \rangle &= \langle \gamma, \gamma \rangle + \varepsilon \langle D^2\zeta(x)\gamma, \gamma \rangle \\
&\geq \|\gamma\|^2 - \varepsilon \max_x \|D^2\zeta(x)\| \, \|\gamma\|^2 \\
&= \|\gamma\|^2 \left(1 - \varepsilon \max_x \|D^2\zeta(x)\|\right) \\
&\geq 0 \quad \text{as required.}
\end{aligned}$$

By Theorem 6.2 (Brenier theorem) we have that $T_\varepsilon$ is the unique minimiser of MOT problem.

## 9.15

If $\varepsilon > 0$ is suff small then

$$F_D(\varepsilon) = \log\left(\det(\Sigma_A + \varepsilon D)\right)$$

Hence by Jacobi's formula:

$$\frac{\partial F_D}{\partial \varepsilon}(\varepsilon) = \frac{1}{\det(\Sigma_A + \varepsilon D)} \frac{\partial}{\partial \varepsilon} \det(\Sigma_A + \varepsilon D)$$

$$= \operatorname{tr}\left( (\Sigma_A + \varepsilon D)^{-1} \frac{\partial}{\partial \varepsilon}(\Sigma_A + \varepsilon D) \right)$$

$$= \operatorname{tr}\left( (\Sigma_A + \varepsilon D)^{-1} D \right)$$

**4.16** Letting $T_\varepsilon = \mathrm{Id} + \varepsilon \nabla \zeta$ for $\zeta \in C_c^\infty(\mathbb{R}^d)$ we have that, as in lecture notes, that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \left( \langle v_t(x), x-y \rangle + \frac{\lambda}{2}|y-x|^2 \right) d\pi_\varepsilon(x,y) \geq -\varepsilon \int_{\mathbb{R}^d} \langle v_t(x), \nabla \zeta(x) \rangle \, d\mu_t(x)$$

where $\pi_\varepsilon \in \Pi_{opt}(\mu_t, \mu_t^{(\varepsilon)})$, $\mu_t^{(\varepsilon)} = (T_\varepsilon)_\# \mu_t$.

Now

$$\frac{1}{\varepsilon}\left( \phi(\mu_t^{(\varepsilon)}) - \phi(\mu_t) \right) = \frac{1}{\varepsilon} \int_{\mathbb{R}^d} V(x) \, d\mu_t^{(\varepsilon)}(x) - \int_{\mathbb{R}^d} V(x) \, d\mu_t(x)$$

$$= \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \left( V(T_\varepsilon(x)) - V(x) \right) d\mu_t(x)$$

$$= \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \nabla V(x) \cdot (T_\varepsilon(x) - x) + \text{h.o.t.} \; d\mu_t(x)$$

$$= \int_{\mathbb{R}^d} \nabla V(x) \cdot \nabla \zeta(x) + \text{h.o.t.} \; d\mu_t(x).$$

Hence,

$$-\int_{\mathbb{R}^d} \langle v_t(x), \nabla \zeta(x) \rangle \, d\mu_t(x) \leq \int_{\mathbb{R}^d} \nabla V(x) \cdot \nabla \zeta(x) + O(\varepsilon) \; d\mu_t(x)$$

Letting $\varepsilon \to 0$ and $\zeta \mapsto -\zeta$ implies

$$-\int_{\mathbb{R}^d} \langle v_t(x), \nabla \zeta(x) \rangle \, d\mu_t(x) = \int_{\mathbb{R}^d} \nabla V(x) \cdot \nabla \zeta(x) \, d\mu_t \qquad \forall \zeta \in C_c^\infty(\mathbb{R}^d)$$

In weak form we can write

$-v_t \mu_t = \mu_t \nabla V$. Hence $0 = \frac{\partial \mu_t}{\partial t} + \nabla \cdot (\mu_t v_t) = \frac{\partial \mu_t}{\partial t} - \nabla \cdot (\mu_t \nabla V)$

9.17

Considering $\frac{1}{\varepsilon}\left(\phi(\mu_t^{(\varepsilon)}) - \phi(\mu_t)\right)$ where $\mu_t^{(\varepsilon)}$ is as in Ex 9.16 and the notes, we have:

$$\frac{1}{\varepsilon}\left(\phi(\mu_t^{(\varepsilon)}) - \phi(\mu_t)\right) = \frac{1}{m-1}\int_{\mathbb{R}^d}\left(\left(\rho_t^{(\varepsilon)}(x)\right)^m - \left(\rho_t(x)\right)^m\right) dx$$

$$+ \int_{\mathbb{R}^d} V(x)\, d\mu_t^{(\varepsilon)}(x) - \int_{\mathbb{R}^d} V(x)\, d\mu_t(x)$$

$$= \frac{1}{(m-1)}\int_{\mathbb{R}^d}\left(\left(\rho_t^{(\varepsilon)}(T_\varepsilon(x))\right)^{m-1} - \left(\rho_t(x)\right)^{m-1}\right) d\mu_t(x)$$

$$+ \varepsilon\int_{\mathbb{R}^d}\nabla V(x)\cdot\nabla\zeta(x) \text{ h.o.t.s } d\mu_t(x)$$

$$= \cancel{\frac{1}{(m-1)}}\int_{\mathbb{R}^d}\left(\left(\rho_t^{(\varepsilon)}(T_\varepsilon(x)) - \rho_t(x)\right)\left(\rho_t(x)\right)^{m-2} \text{ h.o.t.s}\right) d\mu_t(x)$$

$$+ \varepsilon\int_{\mathbb{R}^d}\nabla V(x)\cdot\nabla\zeta(x) \text{ h.o.t.s } d\mu_t(x)$$

$$= \int_{\mathbb{R}^d}\left(\frac{1}{|\det(\mathrm{Id}+\varepsilon D^2\zeta(x))|} - 1\right)\rho_t^m(x)\, dx$$

$$+ \varepsilon\int_{\mathbb{R}^d}\nabla V(x)\cdot\nabla\zeta(x)\, \rho_t(x)\, dx \quad \text{h.o.t.s.}$$

Let $F_D(\varepsilon) = \dfrac{1}{|\det(\mathrm{Id} + \varepsilon D)|}$

Then $$\frac{dF_D}{d\varepsilon} = -\frac{1}{|\det(\mathrm{Id}+\varepsilon D)|^2} \frac{d}{d\varepsilon} |\det(\mathrm{Id}+\varepsilon D)|$$

$$= -\frac{1}{|\det(\mathrm{Id}+\varepsilon D)|} \mathrm{tr}\left((\mathrm{Id}+\varepsilon D)^{-1} D\right) \quad \text{by Jacobi formula}$$

Hence $$\frac{1}{\varepsilon}\left(F_{D^2\xi(x)}(\varepsilon) - 1\right) = \frac{1}{\varepsilon}\left(F_{D^2\xi(x)}(\varepsilon) - F_{D^2\xi(x)}(0)\right)$$

$$= \frac{dF_{D^2\xi(x)}}{d\varepsilon}(0) + \text{h.o.t.s}$$

$$= -\mathrm{tr}\left(D^2\xi(x)\right) + \text{h.o.t.s}$$

$$= -\Delta\xi(x) + \text{h.o.t.s}$$

In particular,

$$\frac{1}{\varepsilon}\left(\phi(\mu_t^{(\varepsilon)}) - \phi(\mu_t)\right) = -\int_{\mathbb{R}^d} \Delta\xi(x)\rho_t^m(x)\,dx + \int_{\mathbb{R}^d} \nabla V(x)\cdot\nabla\xi(x)\rho_t(x)\,dx + \text{h.o.t.s}$$

Hence, as in Ex 1.16,

$$-\int_{\mathbb{R}^d} v_t(x)\cdot\nabla\xi(x)\rho_t(x)\,dx \le \int_{\mathbb{R}^d} -\Delta\xi(x)\rho_t^m(x) + \nabla V(x)\cdot\nabla\xi(x)\rho_t(x)\,dx + \text{h.o.t.s}$$

$$= \int_{\mathbb{R}^d} \nabla\xi(x)\cdot\nabla(\rho_t^m(x)) + \nabla V(x)\cdot\nabla\xi(x)\rho_t(x)\,dx + \text{h.o.t.s}$$

Letting $\epsilon \to 0$ and $Z_t \mapsto -Z_t$ we have

$$\cancel{V_t(x)\ \nabla Z}$$

$$-v_t(x)\,\rho_t(x) = \nabla\left(\rho_t^m(x)\right) + \nabla V(x)\,\rho_t(x)$$

which in turn implies,

$$0 = \frac{\partial \rho_t}{\partial t} + \nabla\cdot\left(\rho_t v_t\right) = \frac{\partial \rho_t}{\partial t} - \nabla\cdot\left(\nabla\left(\rho_t^m(x)\right) + \nabla V(x)\,\rho_t(x)\right)$$

as required.