Example Sheet 4

Introduction to Optimal Transport — Lent 2018-2019
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Exercise 4.1. Let $\mu \in \mathcal{P}_2(X)$, $\nu \in \mathcal{P}_2(X)$, where $X \subset \mathbb{R}^d$ is bounded, and assume that μ has a density with respect to the Lebesgue measure. Show that there exists a solution to

$$\max_{T\,:\,T_{\#}\mu=\nu} \int_{\mathbb{R}^d} |x - T(x)|^2 \,\mathrm{d}\mu(x)$$

and the optimal T is the gradient of a concave function.

Exercise 4.2. Let $\mu_i \in \mathcal{P}(\mathbb{R}^d)$ for $i = 1, \dots, n$ where each μ_i has compact support. Consider the multi-marginal problem

$$\min \left\{ \int_{\mathbb{R}^d \times \cdots \times \mathbb{R}^d} c(x_1, \dots, x_n) \,\mathrm{d}\pi(x_1, \dots, x_n) \;:\; \pi \in \mathcal{P}((\mathbb{R}^d)^n), (P_i)_{\#}\pi = \mu_i \forall i = 1, \dots, n \right\}$$

where $P_i(x_1, \dots, x_n) = x_i$. Suppose $\pi_0 \in \mathcal{P}((\mathbb{R}^d)^n)$ is admissible (i.e. $(P_i)_{\#}\pi = \mu_i$ for all i) and supported on the set

$$\left\{ (x_1, \dots, x_n) \in (\mathbb{R}^d)^n \;:\; x_1 + \cdots + x_n = \text{constant} \right\}.$$

(a) Prove that π_0 is optimal if the cost c is given by $c(x_1, \dots, x_n) = h(x_1 + \cdots + x_n)$ for some convex function $h : \mathbb{R}^d \to \mathbb{R}$.

(b) Prove π_0 is optimal for the cost $c(x_1, \dots, x_n) = -\sum_{i \neq j} |x_i - x_j|^2$.

Exercise 4.3. Prove that when $\nabla_x s(x,t)$ is invertible that

$$\frac{\partial}{\partial t} |\det(\nabla_x s(x,t))| = |\det(\nabla_x s(x,t))| \left[\operatorname{div} \left(\frac{\partial s}{\partial t}(s^{-1}(\cdot, t), t) \right) \right] \circ s(x,t).$$

Exercise 4.4. If $\phi : \mathbb{R}^d \to \mathbb{R}$ is C^2 then show that the following are equivalent:

1. (Hessian inequality) for all $x \in \mathbb{R}^d$,

$$D^2\phi(x) \geq \lambda \mathrm{Id}, \qquad \text{i.e. } \langle D^2\phi(x)\xi, \xi \rangle \geq \lambda |\xi|^2, \ \forall \xi \in \mathbb{R}^d;$$

2. (λ-monotonicity of $\nabla\phi$) for all $t \in [0,1]$, $x_0, x_1 \in \mathbb{R}^d$,

$$\langle \nabla\phi(x_0) - \nabla\phi(x_1), x_0 - x_1 \rangle \geq \lambda |x_0 - x_1|^2;$$

3. (λ-convexity inequality) for all $t \in [0,1]$, $x_0, x_1 \in \mathbb{R}^d$,

$$\phi(x_t) \leq (1-t)\phi(x_0) + t\phi(x_1) - \frac{\lambda}{2} t(1-t)|x_0 - x_1|^2$$

where $x_t = (1-t)x_0 + t x_1$;

4. (subgradient inequality) for all $x_0, x_1 \in \mathbb{R}^d$

$$\langle \nabla\phi(x_1), x_1 - x_0 \rangle - \frac{\lambda}{2}|x_1 - x_0|^2 \geq \phi(x_1) - \phi(x_0) \geq \langle \nabla\phi(x_0), x_1 - x_0 \rangle + \frac{\lambda}{2}|x_1 - x_0|^2.$$

Exercise 4.5. Show that ϕ is λ-convex if and only if $\tilde{\phi} = \phi - \frac{\lambda}{2}|\cdot|^2$ is convex.

Exercise 4.6. Let $\phi \in C^2(\mathbb{R}^d)$ be λ-convex and $u_0, v_0 \in \mathbb{R}^d$. Assume u and v satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = -\nabla\phi(u(t))\text{for } t \in (0,+\infty), \qquad u(0) = u_0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = -\nabla\phi(v(t))\text{for } t \in (0,+\infty) \qquad v(0) = v_0.$$

Show that $\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{2\lambda t}|u(t)-v(t)|^2\right) \leq 0$. Hence, show that the gradient flow starting from u_0 is unique.

Exercise 4.7. Show that if $\phi : \mathbb{R}^d \to \mathbb{R}$ is λ-convex then, for any $v \in \mathbb{R}^d$, $\Phi(u) = \alpha|u-v|^2 + \phi(u)$ is $(\alpha+\lambda)$-convex and admits a unique minimiser whenever $\alpha + \lambda > 0$.

Exercise 4.8. Let $\phi \in C^2(\mathbb{R}^d)$ be λ-convex, and assume that $t_\tau^{(n)} = n\tau$ and $U_\tau^{(n)}$ satisfies

$$\frac{U_\tau^{(n)} - U_\tau^{(n-1)}}{\tau} = -\nabla\phi(U_\tau^{(n)}) \quad n = 1, 2, \ldots$$

with $\lim_{\tau\to 0^+} U_\tau^{(0)} = u_0$. Show that U_τ converges uniformly on every compact set $[0,T]$ to u where u is the solution of

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t) = -\nabla\phi(u(t)), \quad t \in (0,+\infty) \qquad u(0) = u_0.$$
$$\sup_{t\in[0,T]} |u(t) - u_\tau(t)| \leq |u_0 - U_\tau^{(0)}| + C|\nabla\phi(u_0)|\tau.$$

Exercise 4.9. Let $(\mathcal{Z}, d)$ be a complete and separable metric space and $\phi : \mathcal{Z} \to \mathbb{R}$ a proper and lower semi-continuous function. Show (formally) that $u : (0,+\infty) \to \mathrm{Dom}(\phi) \subseteq \mathcal{Z}$ is an EVI_λ gradient flow if and only if for all $0 < s < t < +\infty$ and $v \in \mathrm{Dom}(\phi)$ we have

$$\frac{1}{2}e^{\lambda(t-s)}d^2(u(t), v) - \frac{1}{2}d^2(u(s), v) \leq E_\lambda(t-s)\left(\phi(v) - \phi(u(t))\right)$$

where $E_\lambda(t) = \frac{e^{\lambda t}-1}{\lambda}$.

Exercise 4.10. Let $(\mathcal{Z}, d)$ be a complete and separable metric space and $\phi : \mathcal{Z} \to \mathbb{R}$ a proper and lower semi-continuous function. Assume the EVI_λ gradient flow $u : (0,+\infty) \to \mathcal{Z}$ of ϕ exists for every initial condition. Show that ϕ is λ-convex along geodesics, i.e. if $\gamma : [0,1] \to \mathcal{Z}$ is a geodesic between $x_0 \in \mathcal{Z}$ and $x_1 \in \mathcal{Z}$ then

$$\phi(\gamma(t)) \leq (1-t)\phi(x_0) + t\phi(x_1) - \frac{\lambda}{2}t(1-t)d^2(x_0, x_1) \quad \forall t \in [0,1].$$

Exercise 4.11. Suppose for any $u_0 \in \mathbb{R}^d$ there exists a C^1 curve $u : (0,+\infty) \to \mathbb{R}^d$ that satisfies: $u(0) = u_0$, $\frac{\mathrm{d}u}{\mathrm{d}t}(t) = -\nabla\phi(u(t))$, and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u(t)-v|^2 + \frac{\lambda}{2}|u(t)-v|^2 \leq \phi(v) - \phi(u(t)) \quad \forall v \in \mathbb{R}^d, \forall t > 0.$$

Show that ϕ is λ-convex.

Exercise 4.12. Show that if $\mu_t \in \mathcal{P}(\mathbb{R}^d)$ has density ρ_t then

$$\frac{\partial}{\partial t}\mu_t + \nabla \cdot (v_t \mu_t) = 0$$

in the sense of distributions implies that

$$\frac{\partial \rho_t}{\partial t} + \nabla \cdot (v_t \rho) = 0$$

μ_t-almost everywhere.

Exercise 4.13. Let $\phi = \mathcal{U} + \mathcal{V} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ where

$$\mathcal{U}(\mu) = \begin{cases} \int_{\mathbb{R}^d} U(\rho(x))\,\mathrm{d}x & \text{if } \mu \ll \mathcal{L}^d \text{ and } \rho = \frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^d} \\ +\infty & \text{else} \end{cases}$$

$$\mathcal{V}(\mu) = \int_{\mathbb{R}^d} V(x)\,\mathrm{d}\mu(x)$$

and $U \in C^1(\mathbb{R})$. Show that $\frac{\partial \phi}{\partial \mu}(\mu) = U'(\rho) + V$ whenever $\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^d} = \rho$.

Exercise 4.14. Let $\zeta \in C_c^\infty(\mathbb{R}^d)$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and assume $\varepsilon > 0$ satisfies $\varepsilon \max_{\mathbb{R}^d} \|D^2\zeta\| < 1$. Define $T_\varepsilon = \mathrm{Id} + \varepsilon \nabla \zeta$. Show that $\pi_\varepsilon = (\mathrm{Id} \times T_\varepsilon)_{\#}\mu$ is an optimal transport plan for the Kantorovich optimal transport problem with quadratic cost between μ and $[T_\varepsilon]_{\#}\mu$.

Exercise 4.15. Given $D \in \mathbb{R}^{d\times d}$ let $F : [0,\infty) \to \mathbb{R}$ be defined by $F_D(\varepsilon) = \log|\det(\mathrm{Id} + \varepsilon D)|$. Show that, for $\varepsilon > 0$ sufficiently small, $\frac{\mathrm{d}F_D}{\mathrm{d}\varepsilon}(\varepsilon) = \mathrm{tr}((\mathrm{Id} + \varepsilon D)^{-1} D)$. [Hint: use Jacobi's theorem.]

Exercise 4.16. Let $\phi = \mathcal{V} : \mathcal{P}_2(\mathbb{R}^d) \to \overline{\mathbb{R}}$ where $\mathcal{V}(\mu) = \int_{\mathbb{R}^d} V(x)\,\mathrm{d}\mu(x)$. Using Theorem 8.29 show that the gradient flow μ_t in the Wasserstein space of ϕ satisfies the evolutionary PDE

$$\frac{\partial}{\partial t}\mu_t - \nabla \cdot (\mu_t \nabla V) = 0.$$

Exercise 4.17. Let $\phi = \mathcal{V} + \mathcal{U} : \mathcal{P}_2(\mathbb{R}^d) \to \overline{\mathbb{R}}$ where $\mathcal{V}(\mu) = \int_{\mathbb{R}^d} V(x)\,\mathrm{d}\mu(x)$ and

$$\mathcal{U}(\mu) = \begin{cases} \int_{\mathbb{R}^d} U(\rho(x))\,\mathrm{d}x & \text{if } \mu \ll \mathcal{L}^d \text{ and where } \rho = \frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^d} \\ +\infty & \text{else} \end{cases}$$

where $U(r) = \frac{r^m}{m-1}$ for $m > 1$. Using Theorem 8.29 show that the gradient flow μ_t in the Wasserstein space of ϕ satisfies the evolutionary PDE

$$\frac{\partial}{\partial t}\mu_t - \nabla \cdot (\nabla \rho_t^m + \rho_t \nabla V) = 0$$

where ρ_t is the density of μ_t.