Cochain Approximations of Differential Forms

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PLAN OF THE TALK

- Maxwell Equations
- Functional spaces and their approximations
- Variational formulations
- Scalar products and their invariant part
- The Guardian Angel
- Conclusions

Recalling the Maxwell Equations

In order to present some examples of PDE problems that can be treated by suitable Finite Element spaces, and wanting to avoid too many changes in the field of application, I will concentrate in PDE problems coming from electromagnetism.

Electromagnetic phenomena are governed by Maxwell Equations which involve four fields: the electric and magnetic fields $\mathbf{E}$, $\mathbf{H}$, and the electric and magnetic inductions $\mathbf{D}$, $\mathbf{B}$, respectively.

The Maxwell equations

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \tag{1}$$

$$\frac{\partial \mathbf{D}}{\partial t} - \nabla \times \mathbf{H} = -\mathbf{J}, \tag{2}$$

$$\nabla \cdot \mathbf{D} = \rho, \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0; \tag{4}$$

ρ denotes the charge density and $\mathbf{J}$ the current density. It is easy to see that taking the divergence of (2) and comparing with the time derivative of (3) one gets the charge conservation equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

as a *necessary condition* for the existence of a solution.

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \qquad \frac{\partial \mathbf{D}}{\partial t} - \nabla \times \mathbf{H} = -\mathbf{J},$$

$$\nabla \cdot \mathbf{D} = \rho, \qquad \nabla \cdot \mathbf{B} = 0.$$

Fields and inductions are related to each other by **constitutive laws**:

$$\mathbf{E} = \boldsymbol{\varepsilon} \mathbf{D}, \qquad \mathbf{B} = \boldsymbol{\mu}\, \mathbf{H},$$

where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are the electric permittivity and the magnetic permeability , respectively, that we assume to be *scalar constants.* Finally we have **boundary conditions**. Here we consider the *perfect conducting* case:

$$\mathbf{E} \times \mathbf{n} = 0 \qquad \mathbf{B} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega.$$

where $\mathbf{n}$ denotes the outward unit vector on $\partial\Omega$.

The *low frequency* approximations

We consider now some typical simplified cases of Maxwell equations, that can be obtained in the so-called "low frequency" case. The fields are assumed to be slowly varying in time and the terms with time derivatives are dropped. In this case, the equations decouple:

$$\nabla \times \mathbf{E} = 0, \qquad \nabla \times \mathbf{H} = \mathbf{J},$$

$$\nabla \cdot \mathbf{D} = \rho, \qquad \nabla \cdot \mathbf{B} = 0,$$

$$\mathbf{E} = \boldsymbol{\varepsilon}\mathbf{D}, \qquad \mathbf{B} = \boldsymbol{\mu}\,\mathbf{H},$$

$$\mathbf{E} \times \mathbf{n} = 0. \qquad \mathbf{B} \cdot \mathbf{n} = 0.$$

Electrostatics

We assume that the electric charge density ρ is given. From $\nabla \times \mathbf{E} = 0$ we have that the electric field can be represented in terms of a scalar potential p:

$$\mathbf{E} = -\nabla p.$$

We then have the following equations, called **the mixed formulation of the electrostatic problem**:

$$\mathrm{div}\mathbf{D} = \rho, \quad \mathbf{D} = -\boldsymbol{\varepsilon}\nabla p \quad \text{in } \Omega; \qquad p = 0 \text{ on } \partial\Omega$$

and, after elimination of the the electric induction $\mathbf{D}$, we obtain the so-called **primal formulation of the electrostatic problem**

$$-\mathrm{div}(\boldsymbol{\varepsilon}\nabla p) = \rho \text{ in } \Omega; \qquad p = 0 \text{ on } \partial\Omega.$$

Magnetostatics

We assume that we are given a *divergence free* current density $\mathbf{J}$. Then the equations and boundary conditions are

$$\operatorname{curl}\mathbf{H} = \mathbf{J}, \quad \mathbf{B} = \boldsymbol{\mu}\,\mathbf{H}, \quad \operatorname{div}\mathbf{B} = 0 \text{ in } \Omega; \qquad \mathbf{B}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega.$$

This time the absence of magnetic charges ($\operatorname{div}\mathbf{B} = 0$) implies that $\mathbf{B}$ can be represented in terms of a magnetic *vector potential* $\mathbf{u}$:

$$\mathbf{B} = \operatorname{curl}\mathbf{u} \text{ in } \Omega; \qquad \operatorname{curl}\mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega.$$

We then have the following equations in Ω

$$\operatorname{curl}\mathbf{u} = \boldsymbol{\mu}\,\mathbf{H}, \quad \operatorname{curl}\mathbf{H} = \mathbf{J},$$

and, after elimination of the magnetic field $\mathbf{H}$, we find:

$$\operatorname{curl}\boldsymbol{\mu}^{-1}\operatorname{curl}\mathbf{u} = \mathbf{J} \;\text{ in } \Omega, \qquad \mathbf{u}\times\mathbf{n} = 0 \;\text{ on } \partial\Omega.$$

Functional spaces in 3 dimensions

Let Ω be a Lipschitz continuous polyhedral domain. We will need the following spaces in order to write the variational formulation of the previous equations.

- $L^2(\Omega)$ and $(L^2(\Omega))^3$, that we assume to be known.
- $H(\mathrm{div};\Omega) := \{\boldsymbol{\tau} \in \ (L^2(\Omega))^3 \text{ such that } \mathrm{div}\,\boldsymbol{\tau} \in L^2(\Omega)\}$
- $H(\mathrm{curl};\Omega) := \{\varphi \in \ (L^2(\Omega))^3 \text{ such that } \mathrm{curl}\,\varphi \in (L^2(\Omega))^3\}$
- $H(\mathrm{grad};\Omega) := \{v \in \ L^2(\Omega) \text{ such that } \mathrm{grad}\,v \in (L^2(\Omega)^3\} \equiv H^1(\Omega)$

Polynomial spaces

The following polynomial spaces are typically used, element by element, in order to approximate the above spaces:

- $\mathbb{P}_0 := \{\text{ constants}\}$ (1 d.o.f.)
- $RT_0 := \{\boldsymbol{\tau} = \mathbf{a} + c\mathbf{x}\}$ with $\mathbf{a} \in \mathbb{R}^3$ and $c \in \mathbb{R}$ (4 d.o.f.)
- $N_0 := \{\varphi = \mathbf{a} + \mathbf{c} \wedge \mathbf{x}\}$ with $\mathbf{a} \in \mathbb{R}^3$ and $\mathbf{c} \in \mathbb{R}^3$ (6 d.o.f.)
- $\mathbb{P}_1 := \{\text{polynomials of degree} \leq 1\}$ (4 d.o.f.)

Note that we could have written, as well,

- $\mathbb{P}_1 := \{v = a + \mathbf{c} \cdot \mathbf{x}\}$ with $a \in \mathbb{R}$ and $\mathbf{c} \in \mathbb{R}^3$ (4 d.o.f.)

Note also that $\mathbb{P}_0^3 \subset RT_0 \subset \mathbb{P}_1^3$ and $\mathbb{P}_0^3 \subset N_0 \subset \mathbb{P}_1^3$

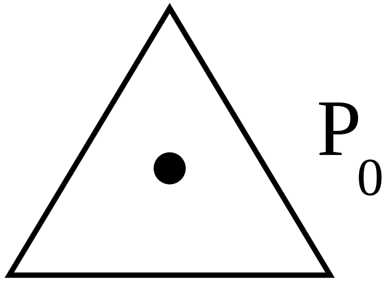
Finite Element Spaces in 3 dimensions

Let $\mathcal{T}_h$ be a decomposition of Ω in tetrahedra. We consider the following finite element approximations.

.

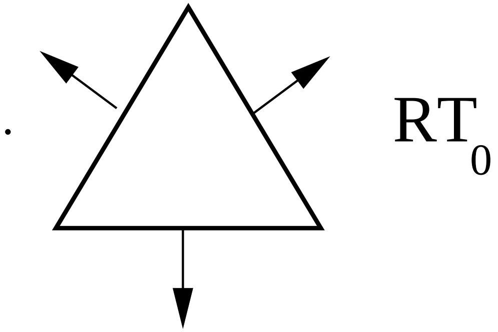
- $L^2(\Omega) \sim \{b \in L^2(\Omega) \text{ such that } b_{|T} \in \mathbb{P}_0 \quad \forall T \in \mathcal{T}_h\}$
- $H(\mathrm{div};\Omega) \sim \{\boldsymbol{\tau} \in H(\mathrm{div};\Omega) \text{ such that } \boldsymbol{\tau}_{|T} \in RT_0 \quad \forall T \in \mathcal{T}_h\}$
- $H(\mathrm{curl};\Omega) \sim \{\varphi \in H(\mathrm{curl};\Omega) \text{ such that } \varphi_{|T} \in N_0 \quad \forall T \in \mathcal{T}_h\}$
- $H(\mathrm{grad};\Omega) \sim \{v \in H(\mathrm{grad};\Omega) \text{ such that } v_{|T} \in \mathbb{P}_1 \quad \forall T \in \mathcal{T}_h\}$

Degrees of Freedom for discrete n-forms



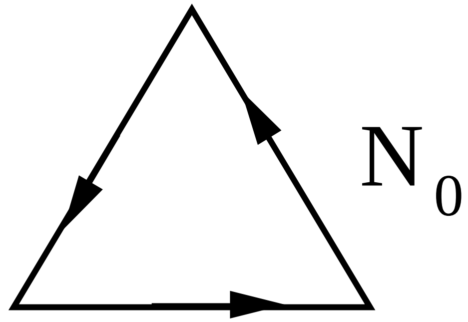
Integral over the element T

Degrees of Freedom for discrete $(n-1)$-forms



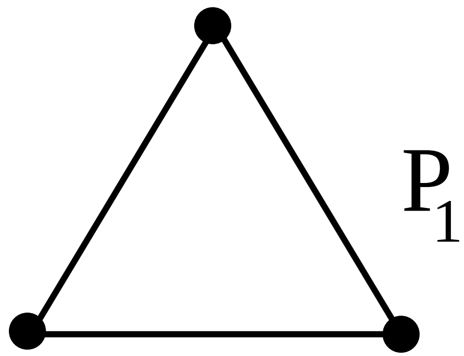
Integral on each face of T

Degrees of Freedom for discrete 1-forms



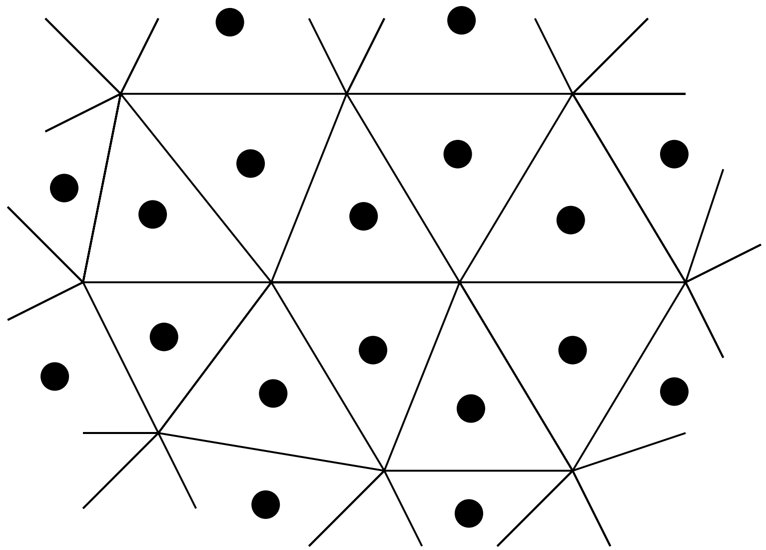
Integral on each edge of T

Degrees of Freedom for discrete 0-forms



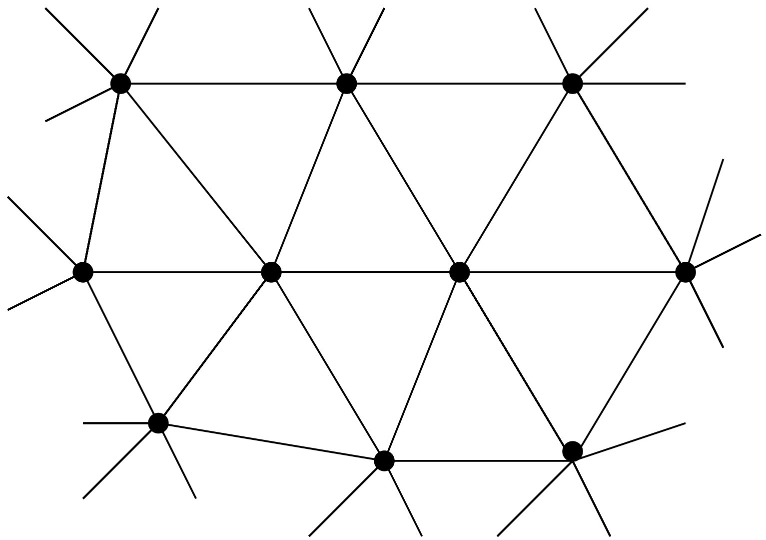
Value at each vertex of T

Discrete n-forms on a triangular decomposition



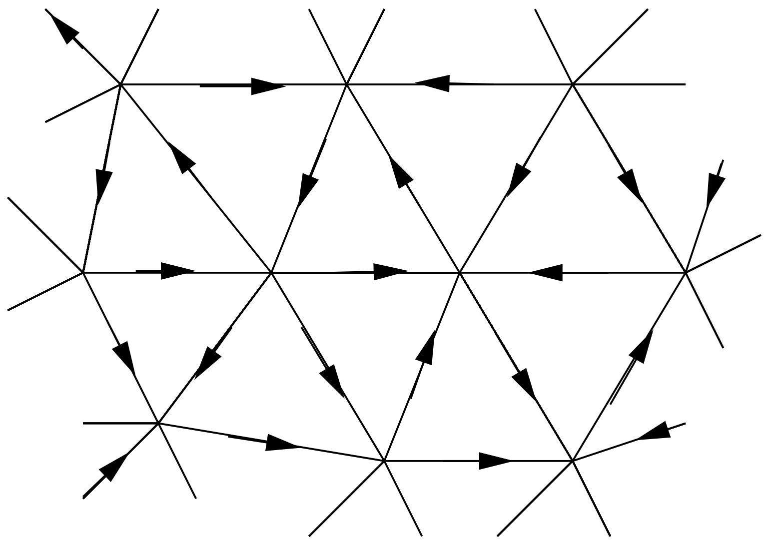
Piecewise-constant functions

Discrete 0-forms on a triangular decomposition



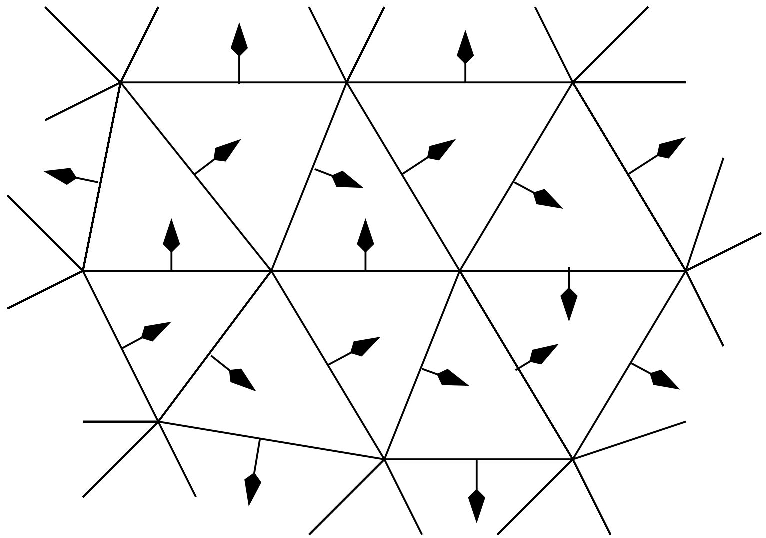
Piecewise-linear functions

Discrete 1-forms on a triangular decomposition



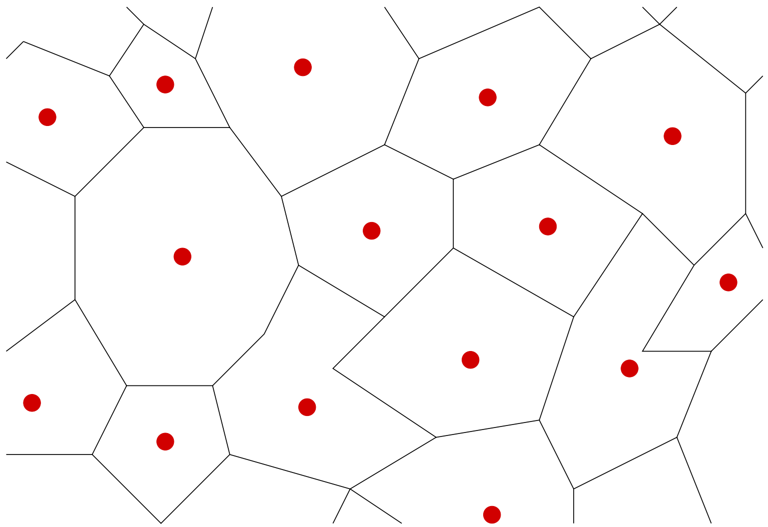
Piecewise-N_0 vector valued functions

Discrete $(n-1)$-forms on a triangular decomposition



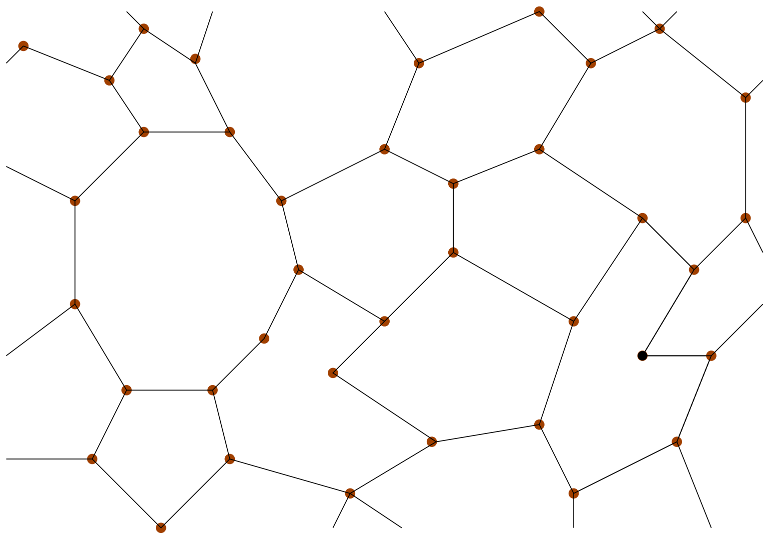
Piecewise-RT_0 vector valued functions

Discrete n-forms on a general decomposition



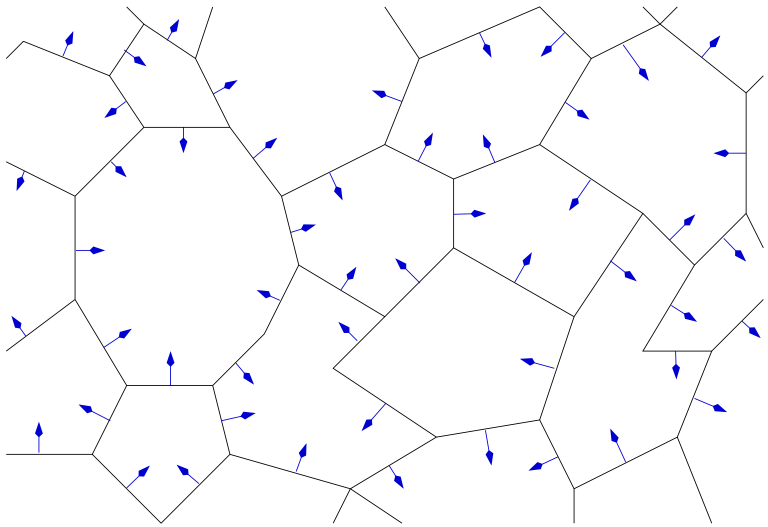
Piecewise-constant functions

Discrete 0-forms on a general decomposition



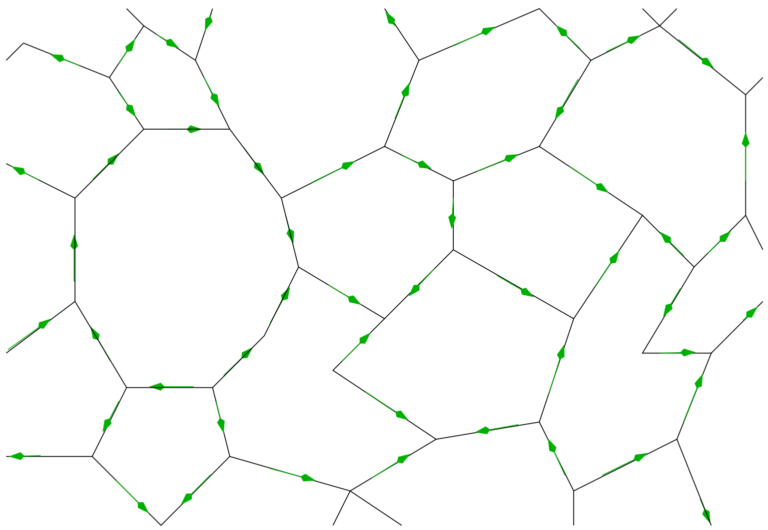
No shape-functions inside the elements

Discrete $(n-1)$-forms on a general decomposition



No shape-functions inside the elements

Discrete 1-forms on a general decomposition



No shape-functions inside the elements

Variational formulations of the electrostatic problem

The electrostatic problem in **mixed** formulation can now be written as:

$$\begin{cases} \text{Find } \mathbf{D} \in H(\mathrm{div};\Omega) \text{ and } p \in L^2(\Omega) \text{ such that} \\ (\boldsymbol{\varepsilon}^{-1}\mathbf{D}, \delta\mathbf{D}) + (p, \mathrm{div}\,\delta\mathbf{D}) = 0 & \forall\, \delta\mathbf{D} \in H(\mathrm{div};\Omega) \\ \qquad\qquad (\mathrm{div}\mathbf{D}, \delta p) = (\rho, \delta p) & \forall\, \delta p \in L^2(\Omega) \end{cases}$$

whereas the variational form of the **primal** formulation reads:

$$\begin{cases} \text{Find } p \in H^1_0(\Omega) \text{ such that} \\ (\boldsymbol{\varepsilon}\nabla p, \nabla\delta p) = (\rho, \delta p) \qquad \forall\, \delta p \in H^1_0(\Omega). \end{cases}$$

Variational formulation of the magnetostatic problem

The variational formulation of the magnetostatic problem, using the *vector potential* $\mathbf{u}$ with the gauge $\operatorname{div}\mathbf{u} = 0$, assuming for simplicity that $\boldsymbol{\varepsilon}$ is constant, is:

$$
\left\{
\begin{array}{ll}
\text{Find } \mathbf{u} \in H_0(\text{curl}, \Omega) \text{ and } p \in H_0^1(\Omega) \text{ such that :} & \\
(\boldsymbol{\mu}^{-1}\text{curl}\,\mathbf{u}, \text{curl}\,\delta\mathbf{u}) - (\nabla p, \boldsymbol{\varepsilon}\delta\mathbf{u}) = (\mathbf{J}, \delta\mathbf{u}) & \forall \delta\mathbf{u} \in H_0(\text{curl}; \Omega) \\
\qquad\qquad\qquad\qquad (\boldsymbol{\varepsilon}\mathbf{u}, \nabla\delta p) = 0 & \forall\, \delta p \in H_0^1(\Omega).
\end{array}
\right.
$$

Given the domain Ω, we consider a partition $\mathcal{T}_h$ of Ω into polyhedra (with the usual nondegeneracy assumptions) having, in total,

N vertices $V_1, V_2, ...V_{\mathsf{N}}$,

E edges $e_1, e_2, ..., e_{\mathsf{E}}$,

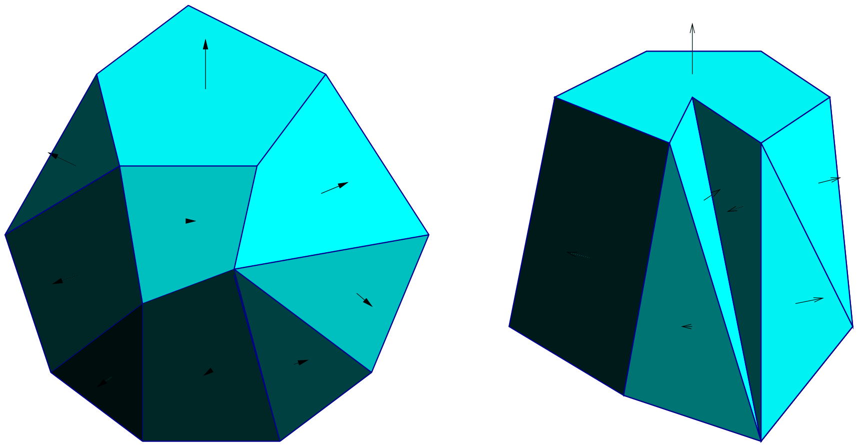
F faces $f_1, f_2, ..., f_{\mathsf{F}}$,

and P elements $P_1, \ldots, P_{\mathsf{P}}$.

Note that the same element can have faces with a different number of vertices from one another. Similarly, two different elements can have a different number of faces, and so on.

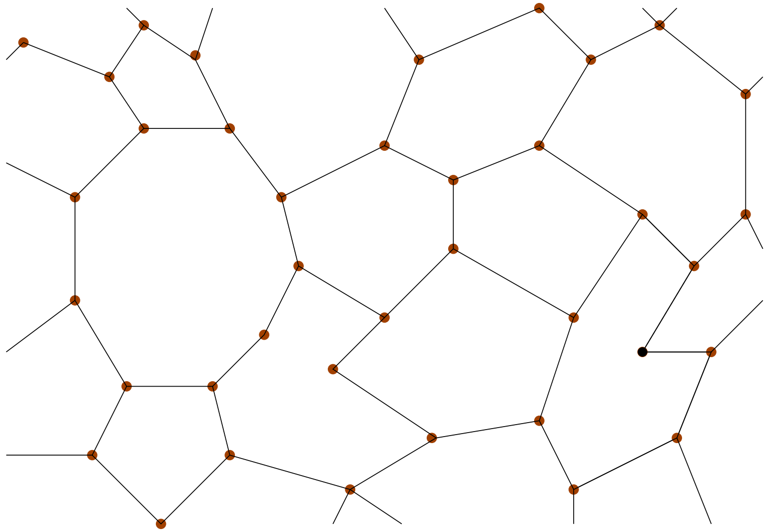
Generality of the approach

In the previous and in the following discussion, many *figures* will be 2-dimensional. This is due to a *limitation of the speaker*, *not of the method*. Indeed, the method works in *very general* situations, and has actually been conceived in a three-dimensional framework.



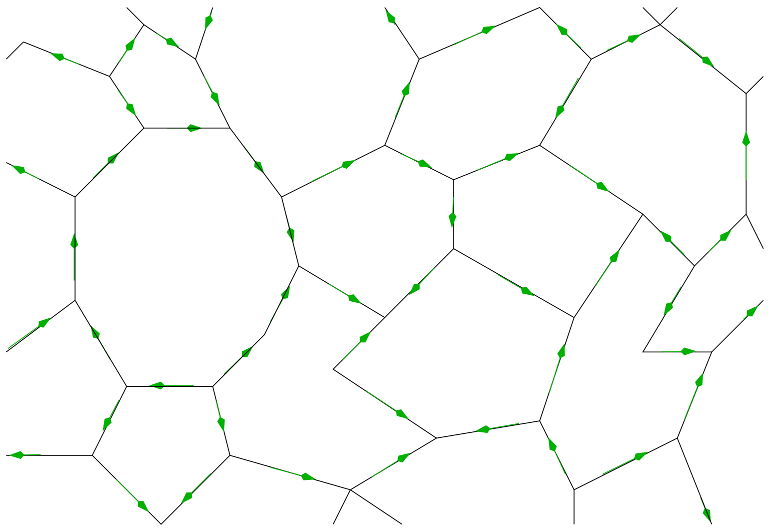
Here you can see two possible *elements* that are perfectly allowed in our theory. In our decomposition we shall then, in a natural way, consider four types of unknowns:

• node unknowns, whose values are attached to vertices and are to be interpreted as the *value of a scalar function at each node*



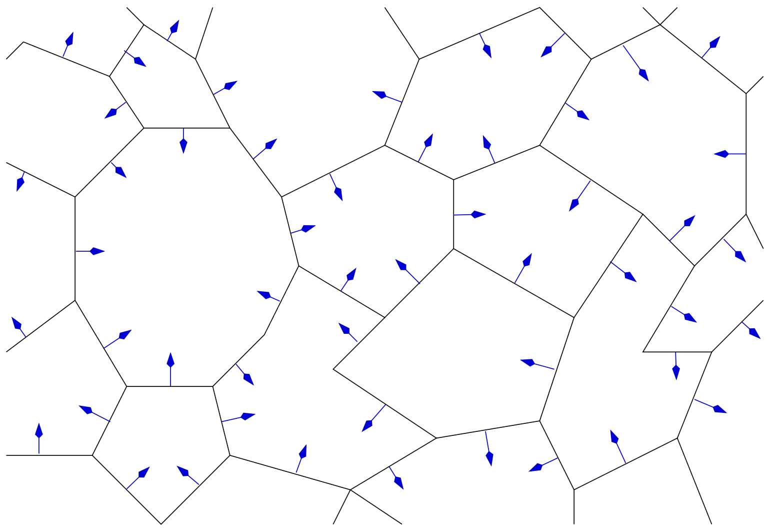
No shape-functions inside the elements

• edge unknowns, whose values are attached to edges and are to be interpreted as the *work of a vector field along each edge*;



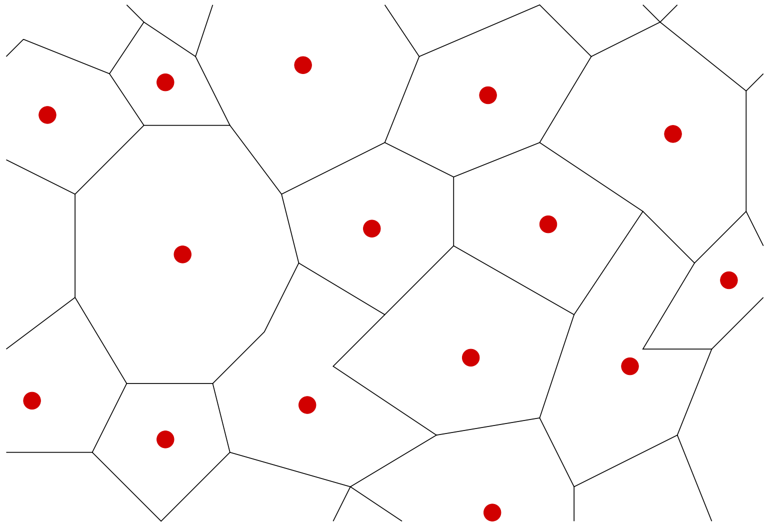
No shape-functions inside the elements

• face unknowns, whose values are attached to faces and are to be interpreted as the *flux of a vector field across each face*



No shape-functions inside the elements

• element unknowns, whose values are attached to elements and are to be interpreted as the *integral of a scalar function over each element*.



Piecewise constants

Accordingly, we denote by

• $\mathcal{N}$ the space of all node unknowns. Its dimension will be equal to the total number of vertices N.

• $\mathcal{E}$ the space of all edge unknowns. Its dimension will be equal to the total number of edges E.

• $\mathcal{F}$ the space of all face unknowns. Its dimension will be equal to the total number of faces F.

• $\mathcal{P}$ the space of all element unknowns. Its dimension will be equal to the total number of elements P.

The **sign** of the elements in $\mathcal{E}$ and $\mathcal{F}$ will depend on the **orientation** of edges and faces, respectively. We will consider that such an orientation is **fixed once and for all**.

Cochains as Discrete differential Forms

- $\mathcal{N}$ is the natural discretization space for $0-$forms as the scalar potential p .

- $\mathcal{E}$ is the natural discretization space for $1-$forms, as the magnetic field $\mathbf{H}$, or the electric field $\mathbf{E}$.

- In its turn, $\mathcal{F}$ is the natural discretization space for $2-$forms as the electric displacement $\mathbf{D}$ or the magnetic induction $\mathbf{B}$.

- Finally, the right candidate to discretize $3-$forms (as the charge density ρ), is clearly $\mathcal{P}$.

In our formulations, however, **these spaces are also used in a different way**. For instance $\mathcal{P}$ is also used as a discretization space for the electric potential p in the mixed formulation of the electrostatic problem. To be precise, however, p is approximated by the *dual space* of $\mathcal{P}$. Along the same lines, **we might use**

- the dual of $\mathcal{N}$ to discretize $3-$forms (as the charge density ρ);

- the dual of $\mathcal{E}$ to discretize $2-$forms as the electric displacement $\mathbf{D}$ or the magnetic induction $\mathbf{B}$;

- the dual of $\mathcal{F}$ to discretize $1-$forms, as the magnetic field $\mathbf{H}$ or the electric field $\mathbf{E}$,

- the dual of $\mathcal{P}$ to discretize $0-$forms as the scalar potential p.

From the point of view of algebraic topology, $\mathcal{N}$, $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{P}$ are **cochain spaces** and form a complex (a cochain complex) together with the co-boundary operator. (See also Matiussi, Tonti, Christiansen, Bochev-Hyman, etc.)

The co-boundary operator is a collection of operators linking our spaces one to the other. When cochains are interpreted as discrete differential forms, then the co-boundary operator can be seen as a discretization of the standard differential operator d, that is, in our simplified setting, as grad, curl, or div depending on the space on which it acts. Here we adopt a self evident notation (as it is nowadays standard in MFD):

- The $\mathcal{GRAD}^h$ operator, from $\mathcal{N}$ to $\mathcal{E}$, defined as follows: for each edge e with vertices (V_1, V_2) and oriented from V_1 to V_2

$$\left(\mathcal{GRAD}^h u\right)\Big|_e = u|_{V_2} - u|_{V_1}.$$

• The $\mathcal{CURL}^h$ operator, from $\mathcal{E}$ to $\mathcal{F}$, defined as follows:
for each element $\varphi \in \mathcal{E}$ and for each face f we denote by $e_1, e_2, ..., e_{\mathsf{E}_f}$ the edges sharing the face f and we suppose they are endowed with the orientation induced by the orientation of f. We consider the corresponding values $\varphi_1, \varphi_2, ..., \varphi_{\mathsf{E}_f}$ of φ with the sign corresponding to the orientation just chosen. Then $\mathcal{CURL}^h\varphi$ *on the face* f *is defined as*

$$\left(\mathcal{CURL}^h\varphi\right)\Big|_f = \sum_{i=1}^{\mathsf{E}_f} \varphi_i.$$

• The $\mathcal{DIV}^h$ operator, from $\mathcal{F}$ to $\mathcal{P}$, defined as follows:
let $f_1, ..., f_{\mathsf{F}_P}$ be all the faces of an element P, and for each $\sigma \in \mathcal{F}$ let $\sigma_1, ..., \sigma_{\mathsf{F}_P}$ be its values on each face that we assume to be oriented outward with respect to P. Then $\mathcal{DIV}^h\sigma$ *is defined as*

$$\left(\mathcal{DIV}^h\sigma\right)|_P = \sum_{k=1}^{\mathsf{F}_P} \sigma_k.$$

It is interesting to note that, taking in the spaces $\mathcal{N}$, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{P}$ the obvious canonical basis (after choosing an orientation of the edges, faces and elements in an arbitrary way, but once and for all), then the matrices associated with the operators $\mathcal{GRAD}^h$, $\mathcal{CURL}^h$, and $\mathcal{DIV}^h$ are the incidence matrices (and their elements are either 0 or 1 or -1).

We shall now define interpolation operators $\Pi_{\mathcal{N}}$, $\Pi_{\mathcal{E}}$, $\Pi_{\mathcal{F}}$, and $\Pi_{\mathcal{P}}$ from spaces of smooth enough scalar or vector valued functions to the discrete spaces $\mathcal{N}$, $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{P}$, respectively. In particular for each smooth scalar function u and for each smooth vector valued function $\boldsymbol{\theta}$ we can set

- $\Pi_{\mathcal{N}} u \in \mathcal{N}$ defined by: $(\Pi_{\mathcal{N}} u)|_V = u(V) \quad \forall$ vertex V;
- $\Pi_{\mathcal{E}} \boldsymbol{\theta} \in \mathcal{E}$ defined by: $(\Pi_{\mathcal{E}} \boldsymbol{\theta})|_e = \int_e \boldsymbol{\theta} \cdot \mathbf{t} \mathrm{d}s \quad \forall$ edge e
  where the unit tangent vector $\mathbf{t}$ indicates the orientation of e;
- $\Pi_{\mathcal{F}} \boldsymbol{\theta} \in \mathcal{F}$ defined by: $(\Pi_{\mathcal{F}} \boldsymbol{\theta})|_f = \int_f \boldsymbol{\theta} \cdot \mathbf{n} \mathrm{d}S \quad \forall$ face f
  where the unit normal outward vector $\mathbf{n}$ indicates the orientation of f;
- $\Pi_{\mathcal{P}} u \in \mathcal{P}$ defined by: $(\Pi_{\mathcal{P}} u)|_P = \int_P u \mathrm{d}P \quad \forall$ element P.

Note that the interpolation operators and the differential operators introduced above have interesting commuting properties. Namely

$$\mathcal{GRAD}^h \Pi_{\mathcal{N}} = \Pi_{\mathcal{E}}\,\mathrm{grad}$$

$$\mathcal{CURL}^h \Pi_{\mathcal{E}} = \Pi_{\mathcal{F}}\,\mathrm{curl}$$

$$\mathcal{DIV}^h \Pi_{\mathcal{F}} = \Pi_{\mathcal{P}}\,\mathrm{div}.$$

These properties reproduce, on general polyhedral meshes, the commuting properties linking finite element spaces which are fundamental for a correct discretization of mixed formulations. In particular they can be represented by a commuting diagram.

The commuting diagram

$$
\begin{array}{ccccccc}
C^{\infty}(\mathbb{P}) & \xrightarrow{\text{grad}} & \Big(C^{\infty}(\mathbb{P})\Big)^3 & \xrightarrow{\text{curl}} & \Big(C^{\infty}(\mathbb{P})\Big)^3 & \xrightarrow{\text{div}} & C^{\infty}(\mathbb{P}) \\
\downarrow \Pi_{\mathcal{N}} & & \downarrow \Pi_{\mathcal{E}} & & \downarrow \Pi_{\mathcal{F}} & & \downarrow \Pi_{\mathcal{P}} \\
\mathcal{N} & \xrightarrow{\mathcal{GRAD}^h} & \mathcal{E} & \xrightarrow{\mathcal{CURL}^h} & \mathbf{F} & \xrightarrow{\mathcal{DIV}^h} & \mathcal{P}
\end{array}
$$

Scalar products in $\mathcal{N}$, $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{P}$

When using the variational formulation of a PDE problem, however, we typically encounter integrals to be computed on the domain Ω, that, in a finite-element-like approach, we write as a sum over the elements of integrals on each element.

Most of these integrals actually involve *material dependent coefficients* (like $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ in the Maxwell equations).

Hence we need to define, in a proper way, the right *material dependent scalar product* in each of the spaces $\mathcal{N}$, $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{P}$.

Scalar products and *Hodge operators

The problem of finding the right *material dependent scalar product* in each of the spaces $\mathcal{N}$, $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{P}$ can obviously be seen as the problem of *finding a suitable discretization of the *Hodge operator* (which maps k-*forms* ($k = 0, 1, .., n$) into ($n - k$)-*forms*).

The only trivial case is that of $\mathcal{P}$: in each element, you know the integral over the element of your 0-*form*. It is immediate to associate a piecewise constant *function* having, in each element, the prescribed integral.

Then you easily pass (in an almost *canonical way*) from *discretized 0-forms* to *piecewise constant functions* (and we actually did that already). Once you have functions, you can easily integrate them as you like.

Scalar Products in $\mathcal{F}$

To start with, consider first the case when the decomposition is made of tetrahedra. Hence, in each element, we can use the values of the integral of the normal component on each face in order to determine a *reconstruction* of the vector field inside the element, *assuming* that it is an RT_0 polynomial. More precisely for each element P and for each $\mathbf{G} \in \mathcal{F}_P$ we *reconstruct* a vector field $R_{\mathcal{F}}(\mathbf{G})$ in P such that:

1) $R_{\mathcal{F}}(\mathbf{G}) \in RT_0$

2) On each face f we have $\int_f R_{\mathcal{F}}(\mathbf{G}) \cdot \mathbf{n}_f \equiv \mathbf{G}_f$

3) For every constant vector $\mathbf{c}$, setting for all faces $\mathbf{G}^{\mathbf{c}}_{|f_i} := \int_{f_i} \mathbf{c} \cdot \mathbf{n}_{|f_i}$ (that is, taking $\mathbf{G}^{\mathbf{c}} := \Pi_{\mathcal{F}} \mathbf{c}$) we have that $R_{\mathcal{F}P}(\mathbf{G}^{\mathbf{c}}) \equiv \mathbf{c}$

Scalar products in RT spaces

Once you have a reconstruction of the fluxes, you can introduce the inner product in $\mathcal{F}$,

$$[\mathbf{F}, \mathbf{G}]_{\mathcal{F}} := \int_{\Omega} R_{\mathcal{F}}(\mathbf{F}) \cdot R_{\mathcal{F}}(\mathbf{G}) \mathrm{d}V.$$

In most applications however, as we have seen already, we will have a *material dependent* tensor $\mathbb{K}$. To fix the ideas, imagine that (in three dimensions) $\mathbb{K}$ is a given tensor that physically maps 1-*forms* into 2-*forms* (as for instance $\mathbf{E} \rightarrow \mathbf{D}$). It will then be convenient to use a *material dependent scalar product* of the form

$$[\mathbf{F}, \mathbf{G}]_{\mathcal{F}} := \int_{\Omega} \mathbb{K}^{-1} R_{\mathcal{F}}(\mathbf{F}) \cdot R_{\mathcal{F}}(\mathbf{G}) \mathrm{d}V.$$

In a similar way you deal with $\mathcal{E}$ or $\mathcal{N}$ using N_0 or $\mathbb{P}_1$ (respectively).

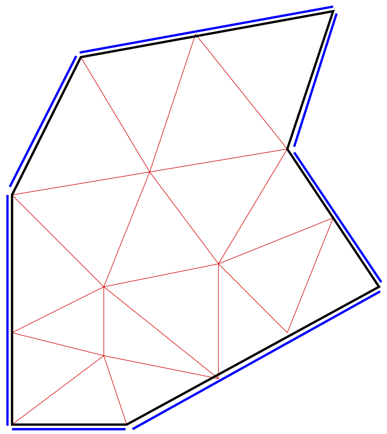
WHAT TO DO FOR COMPLEX GEOMETRIES

In more complex geometries, simple spaces (as RT) to be used for the reconstruction are *not available*. Hence to build a suitable reconstruction operator becomes *cumbersome*.

A very good idea to deal with the problem was proposed by Y. Kuznetsov-S. Repin (2004) and generalized by S.H. Christiansen (2006). It amounts to construct a *subgrid* made of triangles or tetrahedra and reconstruct the vector according to the following rules

- In each triangle/tetrahedron the vector is an RT_0 field.
- The divergence of the vector is *constant on the whole element*.

RECONSTRUCTION USING A SUBGRID



For each $\mathbf{G} \in \mathcal{F}$ and for each element P, we use the subgrid to construct a MFE (RT_0-$\mathbb{P}_0$) approximate solution $(\boldsymbol{\tau}_h, \phi_h)$ of the Neumann problem

$$-\mathrm{div}\mathbb{K}\nabla\phi = \overline{\mathcal{DIV}^h\mathbf{G}_P} \quad \text{in } P \qquad -\mathbb{K}\nabla\phi \cdot \mathbf{n}_{ext} = \overline{(\mathbf{G}_P)}_{ext} \quad \text{on } \partial P$$

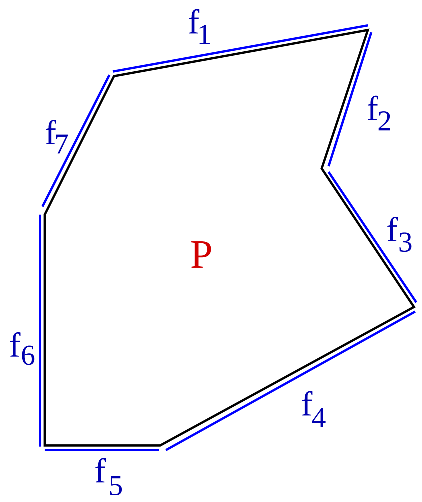
where $\overline{\mathcal{DIV}^h\mathbf{G}_P}$ is a constant function over P and $\overline{(\mathbf{G}_P)}$ is a piecewise constant function over $\partial\Omega$. Finally we set $R_{\mathcal{F}P}(\mathbf{G}_P) := \boldsymbol{\tau}_h$.

THE BASIC "ALTERNATIVE" IDEA

The idea of reconstructing a piecewise polynomial function (or vector-valued function) can obviously be applied to the spaces $\mathcal{E}$ and $\mathcal{N}$ as well (*mutatis mutandis*). It is a good idea, and it has several advantages. Here, however, we consider a different (and often *cheaper*) strategy.

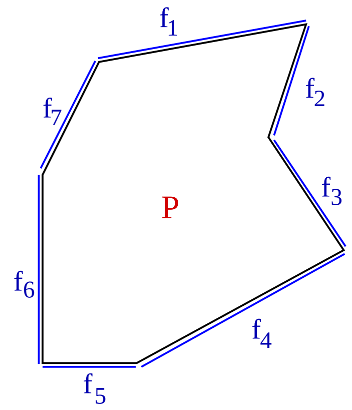
The name of the game is to **guess** how a "scalar product based on reconstructions" should be, and then **invent** (or rather, *cook up*) a scalar product **without** actually building a reconstruction operator. And **to get away with it**.

Our scalar product in $\mathcal{F}$ will be defined as the sum of scalar products $[\cdot,\cdot]_{\mathcal{F}_P}$ on individual elements P. To fix ideas, we assume that we are in 2 dimensions, that P has 7 edges, and that $\mathbb{K}_P$ is constant in P.



Assume that you represent the elements of $\mathcal{F}_P$ in the canonical basis $\mathbf{E}^{(1)}, \mathbf{E}^{(2)}, ...\mathbf{E}^{(7)}$ by prescribing $\mathbf{E}^{(i)}{}_{|f_j} = \delta_{i,j}$. Then every element $\mathbf{G}$ in $\mathcal{F}_{|P}$ will be represented as an element of $\mathbb{R}^7$ with $\mathbf{G} = \sum_{i=1}^{7} \mathbf{G}_i \mathbf{E}^{(i)}$.

ASSOCIATED MATRIX



Every possible reconstruction $R_{\mathcal{F}P}$ will produce a scalar product

$$[\mathbf{F}, \mathbf{G}]_{\mathcal{F}_P} = \int_P \mathbb{K}_P^{-1} R_{\mathcal{F}P}(\mathbf{F}) \cdot R_{\mathcal{F}P}(\mathbf{G}) \mathrm{d}V$$

which, in turn, will be representable as a 7×7 matrix $\mathbb{M}_P$, namely $[\mathbf{F}, \mathbf{G}]_{\mathcal{F}_P} = \sum_{i,j} \mathbb{M}_{P\,i,j} \mathbf{F}_i \mathbf{G}_j$, with

$$\mathbb{M}_{P\,i,j} := \int_P \mathbb{K}_P^{-1} R_{\mathcal{F}P}(\mathbf{E}^{(i)}) \cdot R_{\mathcal{F}P}(\mathbf{E}^{(j)}) \mathrm{d}V.$$

$\mathbb{P}_0$-COMPATIBLE RECONSTRUCTIONS IN $\mathcal{F}$

We shall now restrict our attention to *reasonable reconstructions* (that we shall call $\mathbb{P}_0$-*compatible reconstructions*). These are linear mappings $R_{\mathcal{F}P}$ defined on $\mathcal{F}_P$ and having the following properties:

- For every $\mathbf{G} \in \mathcal{F}_P$, we have that $R_{\mathcal{F}P}(\mathbf{G}) \in (H^1(P))^3$.
- For every $\mathbf{G} \in \mathcal{F}_P$, we have that $\mathrm{div} R_{\mathcal{F}P}(\mathbf{G})$ is constant in P.
- For every $\mathbf{G} \in \mathcal{F}_P$ and for every face f_i of ∂P, we have $\int_{f_i} R_{\mathcal{F}P}(\mathbf{G})_{|f_i} \cdot \mathbf{n}_{|f_i} = \mathbf{G}_{|f_i}$ (hence $\mathrm{div} R_{\mathcal{F}P}(\mathbf{G}) = \mathcal{DIV}^h \mathbf{G}$).
- For every constant vector $\mathbf{c}$, taking for all faces $\mathbf{G}^{\mathbf{c}}_{|f_i} := \int_{f_i} \mathbf{c} \cdot \mathbf{n}_{|f_i}$ (that is, taking $\mathbf{G}^{\mathbf{c}} := \Pi_{\mathcal{F}} \mathbf{c}$) we have that $R_{\mathcal{F}P}(\mathbf{G}^{\mathbf{c}}) \equiv \mathbf{c}$. Note the difference between $\mathbf{c} \in \mathbb{R}^2$ and $\mathbf{G}^{\mathbf{c}} \in \mathbb{R}^7$!!!

SCALAR PRODUCTS ASSOCIATED WITH $\mathbb{P}_0$-COMPATIBLE RECONSTRUCTIONS

We **claim** now that: **if** $\mathbf{c}$ is a constant vector and $\mathbf{G}^{\mathbf{c}} \equiv \Pi_{\mathcal{F}}\mathbf{c}$ has been constructed as before, and **if** $R_{\mathcal{F}P}$ is a $\mathbb{P}_0$-compatible reconstruction, **then** for every $\mathbf{G} \in \mathcal{F}_P$ the result of

$$\int_P \mathbb{K}_P^{-1} R_{\mathcal{F}P}(\mathbf{G}^{\mathbf{c}}) \cdot R_{\mathcal{F}P}(\mathbf{G}) \mathrm{d}V$$

depends on P, $\mathbb{K}$, $\mathbf{c}$ and $\mathbf{G}$, but **not** on the choice of the reconstruction (among all possible $\mathbb{P}_0$-compatible reconstructions).
Indeed...

Set $q^1(\mathbf{x}) := (\mathbb{K}_P^{-1}\mathbf{c}) \cdot (\mathbf{x} - \mathbf{x}_B)$ (where $\mathbf{x_B}$ is the barycenter of P).
Then we have

$$
\begin{aligned}
&\int_P \mathbb{K}_P^{-1} R_{\mathcal{F}\,P}(\mathbf{G}^{\mathbf{c}}) \cdot R_{\mathcal{F}\,P}(\mathbf{G}) \mathrm{d}V = \\
&\qquad \int_P \mathbb{K}_P^{-1}\mathbf{c} \cdot R_{\mathcal{F}\,P}(\mathbf{G}) \mathrm{d}V = \int_P \nabla q^1 \cdot R_{\mathcal{F}\,P}(\mathbf{G}) \mathrm{d}V = \\
&\qquad -\int_P \operatorname{div} R_{\mathcal{F}\,P}(\mathbf{G})\, q^1 \mathrm{d}V + \int_{\partial P} q^1\, R_{\mathcal{F}\,P}(\mathbf{G}) \cdot \mathbf{n}_{ext}\, \mathrm{d}S = \\
&\qquad -\int_P \mathcal{DIV}^h \mathbf{G}\, q^1 \mathrm{d}V + \int_{\partial P} q^1\, \mathbf{G}_{ext}\, \mathrm{d}S \\
&\qquad = 0 + \int_{\partial P} (\mathbb{K}_P^{-1}\mathbf{c}) \cdot (\mathbf{x} - \mathbf{x}_B)\, \mathbf{G}_{ext}\, \mathrm{d}S.
\end{aligned}
$$

Let us summarize the previous result. We found that if the scalar product in $\mathcal{F}_P$ is obtained through a $\mathbb{P}_0$-compatible reconstruction, and if $\mathbf{G}^{\mathbf{c}}$ is associated to *a constant vector*, $\mathbf{c}$, then

$$[\mathbf{G}^{\mathbf{c}}, \mathbf{G}]_{\mathcal{F}_P} = \int_{\partial P} (\mathbb{K}_P^{-1}\mathbf{c}) \cdot (\mathbf{x} - \mathbf{x}_B)\, \mathbf{G}_{ext}\, \mathrm{d}S.$$

It is also simple to check that taking two constant vectors in the canonical basis of $\mathbb{R}^2$, $\mathbf{e}^1 = (1, 0)$ and $\mathbf{e}^2 = (0, 1)$, then we *must* have

$$[\mathbf{G}^{\mathbf{e}^i}, \mathbf{G}^{\mathbf{e}^j}]_{\mathcal{F}_P} = \int_P \mathbb{K}_P^{-1} R_{\mathcal{F}P}(\mathbf{G}^{\mathbf{e}^i}) \cdot R_{\mathcal{F}P}(\mathbf{G}^{\mathbf{e}^j})\mathrm{d}V = \int_P \mathbb{K}_P^{-1}\mathbf{e}^i \cdot \mathbf{e}^j \mathrm{d}V = (\mathbb{K}_P^{-1})_{i,j}|P|.$$

It seems now natural to *change the basis in $\mathcal{F}_P$*. We take

$$\widetilde{\mathbf{E}}^1 := \mathbf{G}^{\mathbf{e}^1}, \quad \widetilde{\mathbf{E}}^2 := \mathbf{G}^{\mathbf{e}^2},$$

and then we complete the basis with vectors in $\mathbb{R}^7$

$$\widetilde{\mathbf{E}}^3, \quad \widetilde{\mathbf{E}}^4, \ldots \quad \widetilde{\mathbf{E}}^7$$

such that

$$[\mathbf{G}^{\mathbf{e}^i}, \widetilde{\mathbf{E}}^j]_{\mathcal{F}_P} = \int_{\partial P} (\mathbb{K}_P^{-1}\mathbf{e}^i)\cdot(\mathbf{x}-\mathbf{x}_B)\, \widetilde{\mathbf{E}}^j_{ext}\, \mathrm{d}S = 0 \quad (i = 1, 2 \quad j = 3, .., 7)$$

Note that all this *does not* depend on the choice of the reconstruction.

In the new basis $\widetilde{\mathbf{E}^1}, \ldots, \widetilde{\mathbf{E}^7}$ the matrix associated to **any** scalar product obtained with **any** $\mathbb{P}_0$-compatible reconstruction will **always** have the form

$$\begin{array}{|c|c|}\hline K^{-1}|P| & 0 \\ \hline 0 & ? \\ \hline \end{array}$$

with the 5×5 diagonal block "**?**" depending on the reconstruction.

Theorem. There exists an $\alpha_0 > 0$ such that: for every symmetric and positive definite 5×5 matrix $\mathbf{S}$ with smallest eigenvalue $\geq \alpha_0$ there exists a $\mathbb{P}_0$-compatible reconstruction whose associated scalar product corresponds, in the basis $\widetilde{\mathbf{E}^1}, \ldots, \widetilde{\mathbf{E}^7}$, to the matrix

$$\begin{array}{|c|c|}\hline \mathrm{K}^{-1}|\mathrm{P}| & 0 \\ \hline 0 & \mathrm{S} \\ \hline \end{array}$$

In other words: I know that the matrix comes from **a** reconstruction. I don't care to know "which one".

In all our experiments we took the matrix as

$$\left[\begin{array}{c|c} \mathrm{K}^{-1}|\mathrm{P}| & 0 \\ \hline 0 & \alpha \mathrm{I} \end{array}\right]$$

with $\alpha = |P|\mathrm{trace}(\mathbb{K}^{-1})$, and we got very good results.

$\mathbb{P}_0$-COMPATIBLE RECONSTRUCTIONS IN $\mathcal{F}$

We got the previous *miracle* by restricting our attention to $\mathbb{P}_0$-*compatible reconstructions*. In order to generalize the idea, we must however re-phrase their definition: They are linear mappings $R_{\mathcal{F}P}$ defined on $\mathcal{F}_P$ and having the following properties:

- For every $\mathbf{G} \in \mathcal{F}_P$, we have that $R_{\mathcal{F}P}(\mathbf{G}) \in (H^1(P))^3$.
- For every $\mathbf{G} \in \mathcal{F}_P$ and for every constant vector $\mathbf{c}$, we have that $\mathrm{div} R_{\mathcal{F}P}(\mathbf{G})$ is orthogonal to $\mathbf{c} \cdot (\mathbf{x} - \mathbf{x}_B)$ in P.
- For every $\mathbf{G} \in \mathcal{F}_P$ and for every face f_i of ∂P, we have $\int_{f_i} R_{\mathcal{F}P}(\mathbf{G})_{|f_i} \cdot \mathbf{n}_{|f_i} = \mathbf{G}_{|f_i}$ (hence $\mathrm{div} R_{\mathcal{F}P}(\mathbf{G}) = \mathcal{DIV}^h \mathbf{G}$).
- For every constant vector $\mathbf{c}$, taking for all faces $\mathbf{G}^{\mathbf{c}}_{|f_i} := \int_{f_i} \mathbf{c} \cdot \mathbf{n}_{|f_i}$ (that is, taking $\mathbf{G}^{\mathbf{c}} := \Pi_{\mathcal{F}} \mathbf{c}$) we have that $R_{\mathcal{F}P}(\mathbf{G}^{\mathbf{c}}) \equiv \mathbf{c}$.

$\mathbb{P}_0$-COMPATIBLE RECONSTRUCTIONS IN $\mathcal{E}$

In a similar way we can define $\mathbb{P}_0$-*compatible reconstructions* in $\mathcal{E}$. These will be linear mappings $R_{\mathcal{E}\,P}$ defined on $\mathcal{E}_P$ and having the following properties:

- For every $\mathbf{\Psi} \in \mathcal{E}_P$, we have that $R_{\mathcal{E}\,P}(\mathbf{\Psi}) \in (H^2(P))^3$.
- For every $\mathbf{\Psi} \in \mathcal{E}_P$ and for every constant vector $\mathbf{c}$, we have that $\mathrm{curl} R_{\mathcal{E}\,P}(\mathbf{\Psi})$ is orthogonal to $\mathbf{c} \wedge (\mathbf{x} - \mathbf{x}_B)$ in P.
- For every $\mathbf{\Psi} \in \mathcal{E}_P$ and for every edge e_i of ∂P, we have $\int_{e_i} R_{\mathcal{E}\,P}(\mathbf{\Psi})_{|e_i} \cdot \mathbf{t}_{|e_i} = \mathbf{\Psi}_{|e_i}$.
- For every constant vector $\mathbf{c}$, taking $\mathbf{\Psi}^{\mathbf{c}} := \Pi_{\mathcal{E}} \mathbf{c}$ we have that $R_{\mathcal{E}\,P}(\mathbf{\Psi}^{\mathbf{c}}) \equiv \mathbf{c}$.

$\mathbb{P}_0$-COMPATIBLE RECONSTRUCTIONS IN $\mathcal{N}$

Finally we can define $\mathbb{P}_0$-*compatible reconstructions* in $\mathcal{N}$. These will be linear mappings $R_{\mathcal{N}\,P}$ defined on $\mathcal{N}_P$ and having the following properties:

- For every $\mathbf{U} \in \mathcal{N}_P$, we have that $R_{\mathcal{N}\,P}(\mathbf{U}) \in H^2(P)$.
- For every $\mathbf{U} \in \mathcal{N}_P$ and for every constant c, we have that $\mathrm{grad} R_{\mathcal{N}\,P}(\mathbf{U})$ is orthogonal to $c(\mathbf{x} - \mathbf{x}_B)$ in P.
- For every $\mathbf{U} \in \mathcal{N}_P$ and for every vertex V_i of P, we have $R_{\mathcal{N}\,P}(\mathbf{U})_{|V} = \mathbf{U}_{|V_i}$.
- For every constant c, setting $\mathbf{U}^c := \Pi_{\mathcal{N}} c$ we have that $R_{\mathcal{N}\,P}(\mathbf{U}^c) \equiv c$.

General Philosophy - 1

Assume that we consider one of the spaces $\mathcal{N}, \mathcal{E}, \mathcal{F}$ and a material dependent constant (or *symmetric tensor*) $\mathbb{A}$. On each element P we consider the scalar product

$$(*) \qquad (u, v)_P := \int_P \mathbb{A}\, R_P u \; R_P u \mathrm{d}P$$

where R_P is any $\mathbb{P}_0$-compatible reconstruction operator (that can be either an $R_{\mathcal{N}\,P}$, or an $R_{\mathcal{E}\,P}$ or an $R_{\mathcal{F}\,P}$ according with the space we are considering). Afterwards we will use the scalar product on Ω obtained by summing the scalar products (*) over P. Then

- if u is constant on P, then: the scalar product $(*)$ depends on P, on $\mathbb{A}$, and on u and v, but is *independent of the choice of the reconstruction operator* R among all the (infinitely many) $\mathbb{P}_0$-compatible reconstructions.

General Philosophy - 2

- As a consequence, the rows and columns (of the local "stiffness" matrix) corresponding to constants (or constant vectors) are *known*, independently of the choice of the reconstruction operator (i.e. without knowing the *shape functions*).
- We then *complete* the local stiffness matrix in an (almost) arbitrary way (provided it is SPD...).
- Then *there exists* a (**guardian angel**)-reconstruction R such that: the corresponding scalar product $(*)$ is exactly the one we chose at the previous point (i.e the **g. a.** provides some shape functions of his own)
- This is enough to ensure *linear convergence* (and we **got away** without the shape functions).

Summary

- Cochain approximations of differential forms can be introduced on almost arbitrary decompositions.
- However they require scalar products (or ”discrete *Hodge operators”) in order to be usable for PDE problems
- The *canonical way* to get *Hodge operators is through the construction of *shape functions*, that provide *reconstruction operators* in a natural way.
- Shape functions are easily available only on very simple geometries.
- However we can easily construct local scalar products that are generated by *some* shape functions that ”reproduce the constants”.
- This is sufficient to insure some kind of *patch test*.

Conclusions

- Cochain approximations are a valuable numerical instrument, allowing the use of very general decompositions.
- The local material properties can be described with a certain freedom.
- The best use of such freedom is, in most cases, still to be determined.
- The extension to cochains of most properties satisfied by compatible FE discretizations is still to be done.
- The passage from De Rham complex to Bernstein - Gelfand - Gelfand complex seems feasible, but so far it has been done only in very special cases.
- Please note that the *general philosophy* presented before is indeed *a philosophy* and not *a theorem*.