Smoothed Analysis of Condition Numbers

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Outline

Condition Numbers
  Linear Equations
  Linear Inequalities
  Condition-based Analysis

Smoothed Analysis

Conic Condition Numbers

A General Result

Applications

Proof of General Result
Condition Numbers
General Definition

Suppose we have a numerical computation problem

\[ f: \mathbb{R}^p \rightarrow \mathbb{R}^q, \ x \mapsto y = f(x). \]

We fix norms \( \| \| \) on \( \mathbb{R}^p, \mathbb{R}^q \).
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- Formal definition for differentiable \( f \):

\[ \kappa(f, x) := \| Df(x) \| \frac{\| x \|}{\| f(x) \|} \]

where \( \| Df(x) \| \) denotes the operator norm of the Jacobian of \( f \) at \( x \).
Consider matrix inversion

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- \( \kappa(A) \) was introduced by A. Turing in 1948.
Geometric Interpretation

- We call the set of singular matrices $\Sigma \subseteq \mathbb{R}^{m \times m}$ the set of ill-posed instances for matrix inversion. Clearly, $A \in \Sigma \iff \det A = 0$.  

The Eckart-Young Theorem from 1936 states that $\|A - 1\| = \text{dist}(A, \Sigma)$, where dist either refers to operator or Frobenius norm.

Hence $\kappa(A) = \|A\| \|A - 1\| = \|A\| \text{dist}(A, \Sigma)$. 
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- Hence

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For $A \in \mathbb{R}^{n \times m}$, $n > m$, consider the system of linear inequalities

$$\exists x \in \mathbb{R}^m \ Ax < 0$$  \hspace{1cm} (P)

and its dual problem $\exists y \in \mathbb{R}^n \ y^T A = 0, y > 0$.  \hspace{1cm} (D)
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Let $\mathcal{I}_P$ and $\mathcal{I}_D$ denote the set of instances where $P$ and $D$ is solvable, respectively.

It is well known that we have a disjoint union

$$\mathbb{R}^{n \times m} = \mathcal{I}_P \cup \mathcal{I}_D \cup \Sigma_{n,m},$$

where the set of ill-posed instances $\Sigma_{n,m}$ is the common boundary of $\mathcal{I}_P$ and $\mathcal{I}_D$. 
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- The Linear Programming Feasibility problem (LPF) is to decide for given $A$, whether $A \in \mathcal{I}_P$ or $A \in \mathcal{I}_D$. 
J. Renegar defined the condition number of the linear programming feasibility problem corresponding to $A \in \mathbb{R}^{n \times m}$ as

$$C_R(A) := \frac{\|A\|}{\text{dist}(A, \Sigma_{n,m})}.$$
Condition Number for Linear Programming

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**Theorem (Renegar '95)**

The linear programming feasibility problem on input $A \in \mathbb{R}^{n \times m}, n \geq m$, can be solved by an interior point method with

$$\mathcal{O}\left(\sqrt{n} \log(nC_R(A))\right)$$

iterations (one requiring to solve a system of linear equations).
Condition-based Complexity Analysis

- L. Khachian: for an integer matrix $A$, LPF can be solved in polynomial time (in the bit size of $A$).
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- Consequence: LPF can be solved in polynomial time for an integer matrix $A$, counting bit operations.
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Obvious: Condition numbers are a crucial issue for designing “numerically stable” algorithms.
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Smoothed Analysis of Condition Numbers

Smoothed Analysis
Average-Case Analysis

- An **average-case analysis** of the running time of a numerical algorithm assumes a certain probability distribution on the set of inputs.
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- This general methodology was pioneered in an influential paper by S. Smale (BAMS 1981).
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There are various papers by L. Blum, J. Demmel, A. Edelman, E. Kostlan, J. Renegar, M. Shub, S. Smale and others elaborating this approach.
Examples for Average-Case Analysis

Let $A \in \mathbb{R}^{n \times m}$ be a Gaussian random matrix, $n \geq m$.

**Classical condition number** ($n = m$)

A. Edelman (1992) determined the exact distribution of a scaled variant of the classical condition number $\kappa(A)$. 

*Strictly speaking, this holds for a scaled variant of Renegar’s condition number, called GCC condition number.*

Sharpest results known so far due to B, F. Cucker and M. Lotz, extending previous work by D. Cheung, F. Cucker, R. Hauser, M. Wschebor. In particular $E(\log \kappa(A)) = O(\log m)$. 
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- Dunagan, Spielman, Teng (2003) gave a smoothed analysis of Renegar’s condition number of linear programming.
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Worst case analysis

Average case analysis

Smoothed analysis

$\sup_{a \in \mathbb{R}^p} T(a)$

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$\sup_{a \in \mathbb{R}^p} \mathbb{E}_{a \in N(a, \sigma^2)} T(a)$

$D$ distribution on $\mathbb{R}^p$, $N(a, \sigma^2)$ Gaussian distribution centered at $a$. 
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<table>
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<tr>
<th>Worst case analysis</th>
<th>Average case analysis</th>
<th>Smoothed analysis</th>
</tr>
</thead>
<tbody>
<tr>
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$\mathcal{D}$ distribution on $\mathbb{R}^p$, $N(\bar{a}, \sigma^2)$ Gaussian distribution centered at $\bar{a}$. 
Conic Condition Numbers
Conic Condition Numbers: Definition

- Consider an abstract setting with input space $\mathbb{R}^{p+1}$, together with a symmetric cone $\Sigma \subseteq \mathbb{R}^{p+1}$: $a \in \Sigma \Rightarrow \lambda a \in \Sigma$ for all $\lambda \in \mathbb{R}$.
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- If $\text{dist}_S$ denotes the angular distance on $S^p$:

$$C(a) = \frac{1}{\text{dist}(a, \Sigma)} = \frac{1}{\sin \text{dist}_S(a, \Sigma_S)}.$$
Conic Condition Numbers: Examples

- $\Sigma = \{ A \in \mathbb{R}^{m \times m} \mid \det A = 0 \}$ is a symmetric cone. The scaled condition number $\kappa_F(A)$ is conic by the Eckart-Young:

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- $\kappa_F(A)$ differs from $\kappa(A)$ at most by a factor of $\sqrt{m}$.

- The set $\Sigma_{n,m}$ of ill-posed instances for LPF is a symmetric cone. Instead of Renegar’s condition number we may consider its conic modification:

$$C_F(A) := \frac{\|A\|_F}{\text{dist}_F(A, \Sigma)}.$$
Uniform Smoothed Analysis

Choose \( a \) uniformly at random in the ball \( B(\bar{a}, \sigma) \subseteq S^p \) with center \( \bar{a} \) and radius \( \arcsin \sigma \).
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Uniform smoothed analysis of conic condition number $C$: provide good upper bounds on

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- \( \sigma = 0 \) yields worst-case analysis
- \( \sigma = 1 \) yields average-case analysis
Uniform Smoothed Analysis (2)

Let $T(\Sigma_S, \varepsilon)$ denote the neighborhood (or tube) of $\Sigma_S$ of radius $\arcsin \varepsilon$. 
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Uniform smoothed analysis means to provide relative bounds on the volume of tubes intersected with small balls!
A General Result for Uniform Smoothed Analysis
A General Result

Theorem (B, F. Cucker, M. Lotz, Math. Comp. 2008)

Let $\mathcal{C}$ be a conic condition number with set $\Sigma$ of ill-posed inputs. Assume that $\Sigma$ is contained in a real algebraic hypersurface, given as the zero set of a homogeneous polynomial of degree $d$. Then, for all $\sigma \in (0, 1]$ and all $t \geq (2d + 1) \frac{p}{\sigma}$,

$$
\sup_{\tilde{a} \in S^p} \sup_{a \in B(\tilde{a}, \sigma)} \text{Prob}\{\mathcal{C}(a) \geq t\} \leq 26 dp \frac{1}{\sigma t}.
$$

$$
\sup_{\tilde{a} \in S^p} \mathbb{E}_{a \in B(\tilde{a}, \sigma)}(\ln \mathcal{C}(a)) \leq 2 \ln(dp) + 2 \ln \frac{1}{\sigma} + 4.7.
$$

For average-case analysis ($\sigma = 1$), similar bounds have been given by J. Demmel (1988) and, over $\mathbb{C}$, by C. Beltrán & L.M. Pardo (2005).
A General Result

Theorem (B, F. Cucker, M. Lotz, Math. Comp. 2008)

Let $\mathcal{C}$ be a conic condition number with set $\Sigma$ of ill-posed inputs. Assume that $\Sigma$ is contained in a real algebraic hypersurface, given as the zero set of a homogeneous polynomial of degree $d$. Then, for all $\sigma \in (0, 1]$ and all $t \geq (2d + 1) \frac{p}{\sigma}$,

$$
\sup_{a \in \mathcal{S}^p} \mathrm{Prob} \{ \mathcal{C}(a) \geq t \} \leq 26 \, dp \, \frac{1}{\sigma t}.
$$

$$
\sup_{a \in \mathcal{S}^p} \mathbb{E}_{a \in B(\overline{a}, \sigma)} (\ln \mathcal{C}(a)) \leq 2 \ln(dp) + 2 \ln \frac{1}{\sigma} + 4.7.
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Applications
Linear Equations

- **Problem:** Solving the system of equations $Ax = b$, $A \in \mathbb{R}^{m \times m}$
- **Set of ill-posed inputs:** $\Sigma = \{A \in \mathbb{R}^{m \times m} | \det A = 0\}$ is the zero set of the determinant polynomial of degree $d = m$
- **Condition number:** $\kappa_F(A) = \|A\|_F \|A^{-1}\|$
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**Corollary**

For all $\overline{A} \in \mathbb{R}^{m \times m}$ of Frobenius norm one and $0 < \sigma \leq 1$ we have

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▷ **Problem:** Compute the (complex) eigenvalues of a matrix $A \in \mathbb{R}^{m \times m}$

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For all $\overline{A} \in \mathbb{R}^{n \times n}$ of Frobenius norm one and $0 < \sigma \leq 1$ we have

$$\mathbb{E}_{A \in B(\overline{A}, \sigma)} (\ln \kappa_{\text{eigen}}(A)) \leq 8 \ln m + 2 \ln \frac{1}{\sigma} + 5.1.$$
Complex Polynomial Systems

- Fix $d_1, \ldots, d_n \in \mathbb{N} \setminus \{0\}$. We denote by $\mathcal{H}_d$ the vector space of polynomial systems $f = (f_1, \ldots, f_n)$ with $f_i \in \mathbb{C}[X_0, \ldots, X_n]$ homogeneous of degree $d_i$. $\mathcal{H}_d$ carries a Hermitian product invariant under the action of the unitary group.
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Corollary

For all $f \in \mathcal{H}_d$ of norm one we have

$$E_f \in B(f, \sigma)(\ln \mu_{\text{norm}}(f)) \leq 2 \ln N + 4 \ln D + 2 \ln n + 2 \ln \frac{1}{\sigma} + 6.$$ 

where $N = \dim \mathcal{H}_d - 1$ and $D = d_1 \cdots d_n$ is the Bézout number. 

S. Smale and M. Shub obtained similar estimates for average complexity.
Fix $d_1, \ldots, d_n \in \mathbb{N} \setminus \{0\}$. We denote by $\mathcal{H}_d$ the vector space of polynomial systems $f = (f_1, \ldots, f_n)$ with $f_i \in \mathbb{C}[X_0, \ldots, X_n]$ homogeneous of degree $d_i$. $\mathcal{H}_d$ carries a Hermitian product invariant under the action of the unitary group.

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Linear Inequalities

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**Theorem (B, Amelunxen)**

For all $\bar{A} \in (S^m)^n$ and $0 < \sigma \leq 1$ we have

$$\mathbb{E}_{A \in B(\bar{A},\sigma)}(\ln C(A)) = O(\ln \frac{nm}{\sigma}).$$

Dunagan, Spielman, Teng got similar bounds for Gaussian perturbations.
Proof of the General Result
Main Tools

$T(\Sigma_S, \varepsilon)$ is the tube of radius $\arcsin \varepsilon$ around $\Sigma_S$ in $S^p$.

$$\text{Prob}_{a \in B(\bar{a}, \sigma)}\{C(a) \geq \varepsilon^{-1}\} = \frac{\text{vol}(T(\Sigma_S, \varepsilon) \cap B(\bar{a}, \sigma))}{\text{vol}(B(\bar{a}, \sigma))}$$

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Ingredients:

- Bézout's theorem
- H. Weyl's formula on the volume of tubes
- Integral geometry:
  the principal kinematic formula (W. Blaschke, S.S. Chern)
On the Volume of Tubes

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- Let $K \subset \mathbb{R}^p$ be a compact convex set with boundary $\partial K$. Denote by $T(\partial K, \epsilon)$ the $\epsilon$-tube w.r.t. euclidean distance. **J. Steiner’s formula** says

$$\text{vol}(T(\partial K, \epsilon)) = \sum_{i=1}^{p} \mu_{i-1}(K) \epsilon^i$$

where the coefficients $\mu_i(K)$ are, up to a scaling factor, **H. Minkowski’s cross-sectional measures (Quermassintegrale)**. For instance, $\mu_0(K) = 2 \text{vol}_{p-1}(\partial K)$. 

If $\partial K$ is a smooth hypersurface, the cross-sectional measures can be expressed as integrals of mean curvature. 

H. Weyl (1939) extended Steiner’s formula for the volume of tubes to arbitrary smooth hypersurfaces $M$ of euclidean space or spheres.
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Crofton’s Formula from Integral Geometry

Integral geometry allows to reduce the estimation of volumes to counting points and thus to “degree arguments”.

We denote by $dG$ the invariant volume element on the orthogonal group $G = O(p+1)$ (compact Lie group), normalized such that the volume of $G$ equals one. $G$ operates on $S^p$ in the natural way.

Crofton’s formula

Let $M$ be a submanifold of $S^p$ with $\dim M = p-1$. Then

$$\vol^{p-1}(M) \vol(S^{p-1}) = \frac{1}{2} \int_{g \in G} \#(M \cap gS^1) \, dG(g).$$

If $M$ is given by a homogeneous equation of degree $d$, then the right hand side is bounded by $d$. 
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- Suppose we want to apply Weyl’s tube formula of the form

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For this, one studies intersections of $M$ with random linear subspaces of a certain dimension.
Technical Difficulties

For implementing the above plan for bounding the volume of patches of tubes, several problems have to be addressed:

▶ Weyl's formula requires a smooth hypersurface, but our sets of ill-posed instances usually have singularities. This difficulty can be dealt with by a perturbation argument.

▶ Weyl's formula only holds for sufficiently small radius $\varepsilon$. However, one can upper bound the volume of tubes by using larger coefficients, the so-called integrals of absolute curvature.

▶ The principle kinematic formula does not hold for integrals of absolute curvature. All of these technical difficulties can be overcome with some effort!
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Thank you!