Smoothed Analysis of Condition Numbers

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Outline

Condition Numbers

General Definition

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- Formal definition for differentiable f:

$$\kappa(f, x) \ := \ \|Df(x)\| \ \frac{\|x\|}{\|f(x)\|}$$

  where $\|Df(x)\|$ denotes the operator norm of the Jacobian of f at x.

Linear Equations

- Consider matrix inversion

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- Note that $\kappa(\lambda A) = \kappa(A)$ for $\lambda \in \mathbb{R}$.
- $\kappa(A)$ was introduced by A. Turing in 1948.

Geometric Interpretation

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- Hence

$$\kappa(A) = \|A\|\,\|A^{-1}\| = \frac{\|A\|}{\operatorname{dist}(A, \Sigma)}.$$

Linear Inequalities

- For $A \in \mathbb{R}^{n \times m}$, $n > m$, consider the system of linear inequalities

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- It is well known that we have a disjoint union

$$\mathbb{R}^{n \times m} = \mathcal{I}_P \cup \mathcal{I}_D \cup \Sigma_{n,m},$$

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  where the set of ill-posed instances $\Sigma_{n,m}$ is the common boundary of $\mathcal{I}_P$ and $\mathcal{I}_D$.
- The Linear Programming Feasibility problem (LPF) is to decide for given A, whether $A \in \mathcal{I}_P$ or $A \in \mathcal{I}_D$.

Condition Number for Linear Programming

J. Renegar defined the condition number of the linear programming feasibility problem corresponding to $A \in \mathbb{R}^{n \times m}$ as

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Theorem (Renegar '95)

The linear programming feasibility problem on input $A \in \mathbb{R}^{n \times m}$, $n \geq m$, can be solved by an interior point method with

$$\mathcal{O}\left(\sqrt{n}\,\log(n\,\mathcal{C}_R(A))\right)$$

iterations (one requiring to solve a system of linear equations).

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  - dimension n of the problem
  - logarithm of its condition number.
- $\log \mathcal{C}_R(A)$ is polynomially bounded in bitsize of A for integer matrices $A \notin \Sigma_{n,m}$.
- Consequence: LPF can be solved in polynomial time for an integer matrix A, counting bit operations.

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- conjugate gradient method for solving linear equations
- M. Shub and S. Smale's Newton homotopy method to solve systems of polynomial equations

Smoothed Analysis

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- This general methodology was pioneered in an influential paper by S. Smale (BAMS 1981).
- There are various papers by L. Blum, J. Demmel, A. Edelman, E. Kostlan, J. Renegar, M. Shub, S. Smale and others elaborating this approach.

Examples for Average-Case Analysis

Let $A \in \mathbb{R}^{n \times m}$ be a Gaussian random matrix, $n \geq m$.

Classical condition number $(n = m)$
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- Strictly speaking, this holds for a scaled variant of Renegar's condition number, called GCC condition number.
- Sharpest results known so far due to B, F. Cucker and M. Lotz, extending previous work by D. Cheung, F. Cucker, R. Hauser, M. Wschebor. In particular $\mathbb{E}(\log \mathcal{C}_R(A)) = O(\log m)$.

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- Dunagan, Spielman, Teng (2003) gave a smoothed analysis of Renegar's condition number of linear programming.

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Worst case analysis	Average case analysis	Smoothed analysis
$\sup_{a \in \mathbb{R}^p} T(a)$	$\mathbb{E}_{a \in \mathcal{D}} T(a)$	$\sup_{\bar{a} \in \mathbb{R}^p} \mathbb{E}_{a \in N(\bar{a}, \sigma^2)} T(a)$

$\mathcal{D}$ distribution on $\mathbb{R}^p$, $N(\bar{a}, \sigma^2)$ Gaussian distribution centered at $\bar{a}$.

Conic Condition Numbers

Conic Condition Numbers: Definition

- Consider an abstract setting with input space $\mathbb{R}^{p+1}$, together with a symmetric cone $\Sigma \subseteq \mathbb{R}^{p+1}$: $a \in \Sigma \Rightarrow \lambda a \in \Sigma$ for all $\lambda \in \mathbb{R}$.
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- If dist_S denotes the angular distance on S^p:

$$\mathscr{C}(a) = \frac{1}{\operatorname{dist}(a, \Sigma)} = \frac{1}{\sin \operatorname{dist}_S(a, \Sigma_S)}.$$

Conic Condition Numbers: Examples

- $\Sigma = \{A \in \mathbb{R}^{m \times m} \mid \det A = 0\}$ is a symmetric cone. The scaled condition number $\kappa_F(A)$ is conic by the Eckart-Young:

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where dist refers to the Frobenius norm $\|A\|_F := (\sum_{ij} a_{ij}^2)^{1/2}$.
- $\kappa_F(A)$ differs from $\kappa(A)$ at most by a factor of $\sqrt{m}$.
- The set $\Sigma_{n,m}$ of ill-posed instances for LPF is a symmetric cone. Instead of Renegar's condition number we may consider its conic modification:
$$\mathcal{C}_F(A) := \frac{\|A\|_F}{\mathrm{dist}_F(A,\Sigma)}.$$

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Uniform smoothed analysis of conic condition number $\mathscr{C}$: provide good upper bounds on

$$\sup_{\bar{a} \in S^p} \operatorname*{Prob}_{a \in B(\bar{a}, \sigma)} \{\mathscr{C}(a) \geq \varepsilon^{-1}\}.$$

This was proposed by B, F. Cucker and M. Lotz (2006). Note

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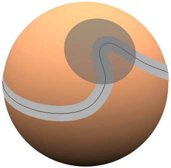
- $\sigma = 0$ yields worst-case analysis
- $\sigma = 1$ yields average-case analysis

Uniform Smoothed Analysis (2)

Let $T(\Sigma_S, \varepsilon)$ denote the neighborhood (or tube) of Σ_S of radius $\arcsin \varepsilon$.



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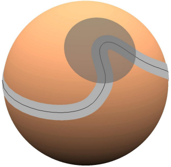

$$\operatorname*{Prob}_{a \in B(\bar{a},\sigma)} \{\mathscr{C}(a) \geq \varepsilon^{-1}\} = \operatorname*{Prob}_{a \in B(\bar{a},\sigma)} \{\sin \mathrm{dist}_S(a, \Sigma_S) \leq \varepsilon\}$$

$$= \operatorname*{Prob}_{a \in B(\bar{a},\sigma)} \{a \in T(\Sigma_S, \varepsilon)\} = \frac{\mathrm{vol}(T(\Sigma_S, \varepsilon) \cap B(\bar{a}, \sigma))}{\mathrm{vol}(B(\bar{a}, \sigma))}$$

Uniform smoothed analysis means to provide relative bounds on the volume of tubes intersected with small balls!

A General Result for Uniform Smoothed Analysis

A General Result

Theorem (B, F. Cucker, M. Lotz, Math. Comp. 2008)

Let $\mathscr{C}$ be a conic condition number with set Σ of ill-posed inputs.

Assume that Σ is contained in a real algebraic hypersurface, given as the zero set of a homogeneous polynomial of degree d.

Then, for all $\sigma \in (0,1]$ and all $t \geq (2d+1)\frac{p}{\sigma}$,

$$\sup_{\overline{a}\in S^p} \operatorname*{Prob}_{a\in B(\overline{a},\sigma)} \{\mathscr{C}(a) \geq t\} \leq 26\, dp\, \frac{1}{\sigma t}.$$

$$\sup_{\overline{a}\in S^p} \mathbb{E}_{a\in B(\overline{a},\sigma)}(\ln \mathscr{C}(a)) \leq 2\ln(dp) + 2\ln\frac{1}{\sigma} + 4.7.$$

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For average-case analysis ($\sigma = 1$), similar bounds have been given by J. Demmel (1988) and, over $\mathbb{C}$, by C. Beltrán & L.M. Pardo (2005).

Applications

Linear Equations

- Problem: Solving the system of equations $Ax = b$, $A \in \mathbb{R}^{m \times m}$
- Set of ill-posed inputs: $\Sigma = \{A \in \mathbb{R}^{m \times m} \mid \det A = 0\}$ is the zero set of the determinant polynomial of degree $d = m$
- Condition number: $\kappa_F(A) = \|A\|_F \, \|A^{-1}\|$

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M. Wschebor (2004) and A. Sankar, D. Spielman and S.-H. Teng (2006) derived similar bounds for Gaussian perturbations by direct methods (2004).

Eigenvalue Computations

- Problem: Compute the (complex) eigenvalues of a matrix $A \in \mathbb{R}^{m \times m}$
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$$\mathbb{E}_{A \in B(\overline{A},\sigma)}(\ln \kappa_{\mathrm{eigen}}(A)) \leq 8 \ln m + 2 \ln \frac{1}{\sigma} + 5.1.$$

Complex Polynomial Systems

- Fix $d_1, \ldots, d_n \in \mathbb{N} \setminus \{0\}$. We denote by $\mathcal{H}_{\mathbf{d}}$ the vector space of polynomial systems $f = (f_1, \ldots, f_n)$ with $f_i \in \mathbb{C}[X_0, \ldots, X_n]$ homogeneous of degree d_i. $\mathcal{H}_{\mathbf{d}}$ carries a Hermitian product invariant under the action of the unitary group.

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For all $\overline{f} \in \mathcal{H}_{\mathbf{d}}$ of norm one we have

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S. Smale and M. Shub obtained similar estimates for average complexity.

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Theorem (B, Amelunxen)

For all $\overline{A} \in (S^m)^n$ and $0 < \sigma \leq 1$ we have

$$\mathbb{E}_{A \in B(\overline{A},\sigma)}(\ln \mathscr{C}(A)) = O(\ln \frac{nm}{\sigma}).$$

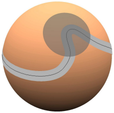
Dunagan, Spielman, Teng got similar bounds for Gaussian perturbations.

Proof of the General Result

Main Tools

$T(\Sigma_S, \varepsilon)$ is the tube of radius $\arcsin \varepsilon$ around Σ_S in S^p.

$$\mathrm{Prob}_{a \in B(\bar{a},\sigma)}\{\mathscr{C}(a) \geq \varepsilon^{-1}\} = \frac{\mathrm{vol}(T(\Sigma_S,\varepsilon) \cap B(\bar{a},\sigma))}{\mathrm{vol}(B(\bar{a},\sigma))}$$



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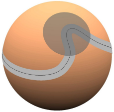

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Ingredients:

- Bézout's theorem
- H. Weyl's formula on the volume of tubes
- Integral geometry:
  the principal kinematic formula (W. Blaschke, S.S. Chern)

On the Volume of Tubes

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$$\mathrm{vol}(T(\partial K, \epsilon)) = \sum_{i=1}^{p} \mu_{i-1}(K)\, \varepsilon^i$$

where the coefficients $\mu_i(K)$ are, up to a scaling factor, H. Minkowski's cross-sectional measures (Quermassintegrale). For instance, $\mu_0(K) = 2\,\mathrm{vol}_{p-1}(\partial K)$.

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- If ∂K is a smooth hypersurface, the cross-sectional measures can be expressed as integrals of mean curvature.
- H. Weyl (1939) extended Steiner's formula for the volume of tubes to arbitrary smooth hypersurfaces M of euclidean space or spheres.

Crofton's Formula from Integral Geometry

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If M is given by a homogeneous equation of degree d, then the right hand side is bounded by d.

Estimating the volume of tubes

- Suppose we want to apply Weyl's tube formula of the form

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- For this, one studies intersections of M with random linear subspaces of a certain dimension.

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All of these technical difficulties can be overcome with some effort!

Thank you!