A WORLD OF BINOMIALS

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ORIGINAL PLAN OF THE TALK

BASICS ON BINOMIALS
ORIGINAL PLAN OF THE TALK

- BASICS ON BINOMIALS
- COUNTING SOLUTIONS TO BINOMIAL SYSTEMS
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- BASICS ON BINOMIALS
- COUNTING SOLUTIONS TO BINOMIAL SYSTEMS
- DISCRIMINANTS (DUALS OF BINOMIAL VARIETIES)
ORIGINAL PLAN OF THE TALK

- Basics on Binomials
- Counting Solutions to Binomial Systems
- Discriminants (Duals of Binomial Varieties)
- Binomials and Recurrence of Hypergeometric Coefficients
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- **BASICS ON BINOMIALS**
- **COUNTING SOLUTIONS TO BINOMIAL SYSTEMS**
- **DISCRIMINANTS (DUALS OF BINOMIAL VARIETIES)**
- **BINOMIALS AND RECURRENCE OF HYPERGEOMETRIC COEFFICIENTS**
- **BINOMIALS AND MASS ACTION KINETICS DYNAMICS**
RESCHEDULED PLAN OF THE TALK

BASICS ON BINOMIALS - How binomials “sit” in the polynomial world and the main “secrets” about binomials
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- BASICS ON BINOMIALS - How binomials “sit” in the polynomial world and the main “secrets” about binomials
- COUNTING SOLUTIONS TO BINOMIAL SYSTEMS - with a touch of complexity
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- Brief summary of what we won’t have time to talk about today
1. BASICS ON BINOMIALS

What is a binomial?

A polynomial with two terms
1. BASICS ON BINOMIALS

What is a binomial?

A polynomial with two terms

\[ ax^\alpha + bx^\beta \in k[x_1, \ldots, x_n] \]

\[ x = (x_1, \ldots, x_n), \quad \alpha \neq \beta \in \mathbb{N}^n, \quad a, b \in k \]
1. BASICS ON BINOMIALS

Linear systems: LINEAR ALGEBRA
1. BASICS ON BINOMIALS

One step before wilderness

- Linear systems: LINEAR ALGEBRA

- Monomials: COMBINATORICS
1. BASICS ON BINOMIALS

- Linear systems: LINEAR ALGEBRA
- Monomials: COMBINATORICS
- Binomials: LINEAR ALGEBRA AND COMBINATORICS
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- Linear systems: LINEAR ALGEBRA
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- Binomials: LINEAR ALGEBRA AND COMBINATORICS
- Trinomials: THE WHOLE WILD WORLD
1. BASICS ON BINOMIALS

One step before wilderness

- Linear systems: LINEAR ALGEBRA
- Monomials: COMBINATORICS
- Binomials: LINEAR ALGEBRA AND COMBINATORICS
- Trinomials: THE WHOLE WILD WORLD

Any system is equivalent to a system with at most trinomials

\[ m_1 + m_2 + m_3 + m_4 = 0 \iff m_1 + m_2 - z_1 = m_3 + m_4 - z_2 = z_1 + z_2 = 0 \]
Given any system of binomial equations = any binomial ideal,

\[ a_j x^{\alpha_j} + b_j x_j^{\beta_j} = 0, \quad j = 1, \ldots, r, \]
1. BASICS ON BINOMIALS

First main fact

Given any system of binomial equations = any binomial ideal,

\[ a_j x^{\alpha_j} + b_j x_j^{\beta_j} = 0, \quad j = 1, \ldots, r, \]

If there exists a solution \( c \) in the torus

\[ T_n = \{(c_1, \ldots, c_n), \quad c_i \neq 0, \quad i = 1, \ldots n\}, \]

in new coordinates \( y_i = x_i / c_i, \quad i = 1, \ldots, n \) the system looks (up to multiplying by non-zero constants)

\[ y^{\alpha_j} - y^{\beta_j} = 0, \quad j = 1, \ldots, r. \]
Given any system of binomial equations = any binomial ideal,

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1. BASICS ON BINOMIALS  

Second main fact

Given any system of binomial equations = any binomial ideal,

\[ a_j x^{\alpha_j} + b_j x^{\beta_j} = 0, \quad j = 1, \ldots, r, \]

There exists such a solution \( c \in T_n \) if and only if (assuming that not both of \( a_j \) and \( b_j \) are 0) \( a_j, b_j \neq 0 \) and for each linear relation

\[ \sum_{j=1}^{r} \lambda_j (\alpha_j - \beta_j) = 0, \quad \lambda_j \in \mathbb{Z}, \]

it holds that

\[ \prod_{j=1}^{r} \left( \frac{-b_j}{a_j} \right)^{\lambda_j} = 1. \]
On the other hand, any sparse polynomial system on the torus $T_n$ is equivalent to a system of binomials in linear forms:

$$ f_i = \sum_{j=1}^{N} c^i_j x^m_j = 0, \quad i = 1, \ldots, r \quad (*) $$

Given $y = (y_1, \ldots, y_N) \in T_N$, there exists $x \in T_n$ such that

$y = (x^{m_1}, \ldots, x^{m_N})$ if and only if for any $\lambda$ in the integer kernel $I$ of the $n \times N$-integer matrix $M$ with columns $m_1, \ldots, m_N$ it holds that

$$ y^\lambda = 1 \quad \text{or} \quad y^{\lambda+} - y^{\lambda-} = 0 \quad (**) $$

So $(*)$ is equivalent to the system of linear forms and binomials

$$ \sum_{i=1}^{N} c^i_j y_j = 0, \quad i = 1, \ldots, r, \quad y^{\lambda+} - y^{\lambda-} = 0, \lambda \in I \quad (***) $$
1. BASICS ON BINOMIALS  

Gale Duality

\[ f_i = \sum_{j=1}^{N} c_j^i x^{m_j} = 0, \quad i = 1, \ldots r \quad (*) \]

\[ \sum_{i=1}^{N} c_i^j y_j = 0, \quad i = 1, \ldots r, \quad y^{\lambda^+} - y^{\lambda^-} = 0, \quad \lambda \in I \quad (***) \]

If the columns of the matrix \( V \) give a basis of the kernel \( K \) of the \( n \times N \) matrix with entries \( c_j^i \), write any \( N \)-tuple \( y \in K \) as

\[ y = (\langle b_1, t \rangle, \ldots \langle b_s, t \rangle) = \langle b, t \rangle, \]

where \( b_1, \ldots, b_s \) are the row vectors of \( V \) and \( t = (t_1, \ldots, t_{\dim K}) \).
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where \( b_1, \ldots, b_s \) are the row vectors of \( V \) and \( t = (t_1, \ldots, t_{\dim K}) \).

Then, finding \( x \in T_n \) satisfying \((*)\) is equivalent to finding \( t \) with \( \langle b, t \rangle \in T^N \) such that for any \( \lambda \) in \( I \) (i.e. \( \sum_i \lambda_i m_i = 0 \)),

\[
\langle b, t \rangle^\lambda^+ - \langle b, t \rangle^\lambda^- = 0, \quad (****)
\]
2. COUNTING SOLUTIONS Third main fact + complexity

Given any square system of $n$ binomial equations in $n$ variables,

$$a_j x^{\alpha_j} + b_j x^{\beta_j} = 0, \quad j = 1, \ldots, n, \quad a_j, b_j \neq 0$$

call $M \in \mathbb{Z}^{n \times n}$ the matrix with rows $\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n$. 

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Set $\delta := |\det(M)|$. When $\delta \neq 0$, the number of solutions in the torus $T_n$ equals $\delta > 0$, independently of the value of the coefficients [BKK].

Can decide in polynomial time if the system has a finite number of solutions. [Cattani-D., JofC’07].
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Can decide in polynomial time if the system has a finite number of solutions. [Cattani-D., JofC’07].

When \( \delta = 0 \), it is possible to decide in polynomial time (in the size of the sparse input) whether for generic coefficients the system has no solutions in the torus. Likewise, it is possible to determine in polynomial time whether the zero set of the system in affine space \( k^n \) is empty or not [follows from ibid., thanks to J.M Rojas for posing this question] [Ex.].
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2. COUNTING SOLUTIONS Complexity

\[ a_j x^{\alpha_j} + b_j x^{\beta_j} = 0, \ j = 1, \ldots, n, \quad a_j, b_j \neq 0 \]

- For “generic” exponents the (finite) number of solutions can be computed in \textit{polynomial time} (precise combinatorial conditions identified through commutative algebra).

However,

- If we allow \( M \) to be any integer matrix (even if \( \det(M) \neq 0 \)) counting the number of solutions to a square binomial system \textit{with or without multiplicity} is \#P-complete (thanks to P. Bürgisser).

- We give a “nice” combinatorial formula. The main complexity is based on deciding which are the possible (zero and non zero coordinates) of the solutions.

- In some sense, this problem is “orthogonal” to numerical analysis (pure structure vs. behaviour of coefficients)
Given any bipartite digraph $G = (V, E)$, $V = V_1 \cup V_2$, $E \subset V_1 \times V_2$, $V = \{1, \ldots, n\}$, we define $n$ binomials in $n$ variables defining a complete intersection

$$p_i = x_i - x_i^2, \quad i \in V_1 \quad p_j = x_j - \left( \prod_{(i,j) \in E} x_i \right) x_j^2, \quad j \in V_2.$$

$V(p_1, \ldots, p_n) \subset \{0, 1\}^n$ and its cardinal equals the number of independent sets of $G$ (all roots are simple and determined by its support).

A universal Gröbner basis of the ideal $\langle p_1, \ldots, p_n \rangle$ equals $x_i - x_i^2$ ($i = 1, \ldots, n$); $x_j - x_i x_j$ ($\forall (i, j) \in E$) [E. Tobis’07]
3. DISCRIMINANTS

What is a (mixed) discriminant

Given finite sets $A_1, \ldots, A_n \subset \mathbb{Z}^n$ and sparse polynomials $f_1, \ldots, f_n$ with these supports,

$$f_i(c^{(i)}, x) = \sum_{\alpha \in A_i} c^{(i)}_\alpha x^\alpha,$$

there exists (under some conditions) an irreducible integer polynomial $\Delta_A$ in the vector of coefficients $c = (c^{(1)}, \ldots, c^{(n)}) \in \mathbb{C}^\ell$ which vanishes whenever there exists $x \in T_n$ which is not a simple zero of $f_1, \ldots, f_n$ (where the Jacobian vanishes) [Gelfand-Kapranov-Zelevinsky’94]
3. DISCRIMINANTS What is a (mixed) discriminant

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That is, $\Delta_A = 0$ describes the variety of ill-posed systems, and the distance of a coefficient vector to it is basic for numerical continuation and numerical stability [M. Shub, J.P. Dedieu, C. Beltran, G. Malajovich, ...].
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- $\Delta_A$ is called the mixed discriminant associated to the support sets $A_1, \ldots, A_n$. 
3. DISCRIMINANTS

What is a (mixed) discriminant

Computing $\Delta_A$ is an elimination problem:

$\pi_1$\hfill $\pi_2$
\begin{align*}
Z & \quad \text{coefficient space} \\
\quad \quad \text{where } Z \text{ is the incidence variety of tuples } (x, t, c), x, t \in T_n \text{ such that} \\
& \quad \frac{\partial}{\partial x_j} (f_i(c^{(1)}, x)) t_i = 0, j = 1, \ldots, n.
\end{align*}

We are interested in the closure of the image $\{\Delta_A = 0\}$ of $\pi_1$. But $\pi_2$ is much easier to understand and allows us to find a rational parametrization of the discriminant variety.
Mixed discriminants are a particular case of general sparse discriminant (aka $A$-discriminants) and define the dual variety of the toric (binomial) variety associated to the given supports.
3. MIXED DISCRIMINANTS

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- When $A_i = d_i \Delta_n \cap \mathbb{Z}^n$ are the lattice points of a dilate of the standard $n$-simplex, $f_i$ is just a generic polynomial with degree $d_i$. 
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- When $A_i = d_i \Delta_n \cap \mathbb{Z}^n$ are the lattice points of a dilate of the standard $n$-simplex, $f_i$ is just a generic polynomial with degree $d_i$.

- The well known numerical unstability of the Wilkinson polynomial

$$W_{20} = \prod_{i=1}^{20}(x + i) = \sum_{j=0}^{20} c_j x^j,$$

can be explained by the fact that its vector of coefficients

$c = (20!, \ldots, 1)$ is very close to a singular point of the discriminant variety $\Delta_A = 0$, where $A = \{0, 1, \ldots, 20\}$.  

A. Dickenstein - FoCM 2008, Hong Kong – p.15/39
3. MIXED DISCRIMINANT

Consider the matrix

\[
A := \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
6 & 0 & 0 & 0 & 3 & 1 \\
0 & 3 & 1 & 6 & 0 & 0
\end{pmatrix}.
\]

\(A\) is the Cayley matrix associated to 2 planar configurations, and the \(A\)-discriminant \(\Delta_A(y_1, \ldots, y_6)\) is the \textit{mixed discriminant} of the family of polynomials

\[
\begin{align*}
h_1(y; t, s) & := y_1 t^6 + y_2 s^3 + y_3 s^1 \\
h_2(y; t, s) & := y_4 s^6 + y_5 t^3 + y_6 t^1
\end{align*}
\]

\(\Delta_A(y) = 0\) whenever there exists a common zero \((s, t) \in (k^*)^2\) which is not simple.
3. MIXED DISCRIMINANTS

An example

Consider the matrix

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1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
6 & 0 & 0 & 0 & 3 & 1 \\
0 & 3 & 1 & 6 & 0 & 0
\end{pmatrix}.
\]

\[\Delta_A(y) = 0\] whenever there exists a common zero \((s, t) \in (k^*)^2\) which is not simple.

The Horn-Kapranov parametrization of \(X_A^* = (\Delta_A(y) = 0)\) is given by

\[
y_1 = (-2\lambda_1 + \lambda_2) t_1 t^6, \quad y_2 = (\lambda_1 - 6\lambda_2) t_1 s^3, \\
y_3 = (-3\lambda_1 + 6\lambda_2) t_1 s, \quad y_4 = 2\lambda_2 t_2 s^6, \\
y_5 = (-6\lambda_1 + \lambda_2) t_2 t^3, \quad y_6 = (6\lambda_1 - 3\lambda_2) t_2 t.
\]
3. MIXED DISCRIMINANT

... and $\Delta_A(1, a, -1, 1, b, -1)$ equals

\[
\begin{align*}
& 82754024941868680778822139064668229594467072 \times a^{47} \times b^{33} + \\
& 24519711093887016527058411574716512472434688 \times a^{46} \times b^{39} - \\
& 24519711093887016527058411574716512472434688 \times b^{46} \times a^{39} + \\
& 236627403090264575474785219707184968001345670463360 \times a^{28} \times b^7 + \\
& 17631004810327637966335552676449435712814331054687500 \times a^4 \times b^{11} + \\
& 53 \text{ additional monomial terms of comparable size}
\end{align*}
\]

It is a polynomial of degree 90 with 58 monomials and huge integer coefficients!
A-discriminants are in general complicated polynomials which carry a lot of combinatorial information.

In principle, we can compute $\Delta_A$ by standard methods in elimination … but in practice we reach the limits of the current computations very easily.

So, instead, we can try to get a first combinatorial approximation, which can nonetheless give us the information about discrete invariants as dimension and degree (and asymptotics), by computing its Newton polytope $N(\Delta_A)$ or its tropicalization $\tau(X^*_A)$.

This is obtained from the tropicalization of an homogeneous version of the Horn-Kapranov parametrization, by monomials in linear forms.

Tropicalization is an operation that turns complex projective varieties into polyhedral fans.

[D.-Feichtner-Sturmfels, JAMS’07]
3. DISCRIMINANTS

The tropicalization \( \tau(Y) \) of a variety \( Y \) is (as a set)

\[
\tau(Y) = \{ w \in \mathbb{R}^n : \text{in}_w(I_Y) \text{ does not contain a monomial} \},
\]

where for \( w \in \mathbb{R}^n \) and \( f = \sum_{\gamma \in C} \gamma_c x^c, \gamma_c \neq 0, C \subset \mathbb{Z}^n \), define

\[
\text{in}_w f = \sum_{w \cdot c \min} \gamma_c x^c \quad \text{initial term of } f,
\]

\[
\text{in}_w(I_Y) = \langle \text{in}_w f | f \neq 0 \in I_Y \rangle \quad \text{initial ideal of } I_Y.
\]

... plus intersection theory information attached to each of the cones in the polyhedral fan \( \tau(Y) \) \([\text{Sturmfels-Tevelev '07}]\)

\( \tau(Y) \) can also be defined via valuations \([\text{Bieri-Groves’84, Einsidler-Kapranov-Lind’06, Sturmfels-Speyer’06}]\).

In the hypersurface case, \( \tau(\{\Delta_A = 0\}) \) is the codimension one skeleton of the normal fan of \( \Delta_A \).
The discriminant of a cubic polynomial in 1 variable

\[ A := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \]

\( X_A \) is the twisted cubic.

\[ f_A(x; t) = x_1 t^0 + x_2 t^1 + x_3 t^2 + x_4 t^3 \]

\[ \Delta_A = 27 x_1^2 x_4^2 - 18 x_1 x_2 x_3 x_4 + 4 x_1 x_3^3 + 4 x_2 x_4^3 - x_2^2 x_3^2 \]

\[ \text{in}_{(-1, -1, -1, 0)}(\Delta_A) = 4 x_1 x_3^3 - x_2^2 x_3^2 \]

\[ \text{in}_{(1, 0, 1, 0)}(\Delta_A) = 4 x_2^3 x_4 \]

\((-1, -1, -1, 0) \in \tau(X_A^*)\)

\((1, 0, 1, 0) \notin \tau(X_A^*)\)
Newton polygon, tropicalization and extreme monomials of the discriminant of a degree 3 polynomial

\[ \Delta_A = 27x_1^2x_4^2 - 18x_1x_2x_3x_4 + 4x_1x_3^3 + 4x_2^3x_4 - x_2^2x_3^2 \]

\[ x_2^3x_4, \ x_2^2x_3, \ x_1x_3^3, \ x_1^2x_4^2 \]
3. DISCRIMINANTS

Tropical information

Discriminant in \( b, c \) space of \( f := x^4 + bx^2 + cx + 1 \)

\[ -4b^3c^2 - 27c^4 + 16b^4 - 128b^2 + 144bc^2 + 256 = 0 \]

Green: \( 4b^3 + 27c^2 = 0 \), Black: \( 16b^4 - 128b^2 + 256 = 0 \), Magenta: \( -4c^2 + 16b = 0 \)

The discriminant of the quartic equation \( x^4 + bx^2 + cx + 1 \) and its asymptotes
The discriminant of the quartic equation \( x^4 + bx^2 + cx - 1 \) and its asymptotes.
3. TWO THEOREMS

Theorem: The tropical $A$-discriminant is the Minkowski sum of the tropicalization of the kernel $B(A)$ and the (classical) row space of the $d \times N$-matrix $A$.

$$\tau(X^*_A) = \{ w + vA, w \in B(A), v \in \mathbb{R}^d \}.$$
3. TWO THEOREMS

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\[ \tau(X^*_A) = \{ w + vA, w \in B(A), v \in \mathbb{R}^d \}. \]

$L(A) =$ geometric lattice whose elements are the sets of zero-entries of the vectors in $\text{kernel}(A)$, ordered by inclusion.

$C(A) =$ set of proper maximal chains in $L(A)$.

We represent these chains as $(N - d - 1)$-element subsets $\sigma = \{ \sigma_1, \ldots, \sigma_{N-d-1} \}$ of $\{0, 1\}^N$.

The tropicalization of the kernel of $A$ equals $B(A) := \tau(\text{kernel}(A)) = \bigcup_{\sigma \in C(A)} \mathbb{R}_{\geq 0} \sigma$.

This tropical linear space is a subset of $\mathbb{R}^N$. 
3. TWO THEOREMS

DATA: $A \in \mathbb{Z}^{d \times N}$ (e.g. $A =$ Cayley matrix of $A_1, \ldots, A_n$, $d = 2n$), $w \in \mathbb{R}^N$ generic.
3. TWO THEOREMS

**DATA:** $A \in \mathbb{Z}^{d \times N}$ (e.g. $A = $ Cayley matrix of $A_1, \ldots, A_n$, $d = 2n$), $w \in \mathbb{R}^N$ generic.

**Theorem:** The exponent of $x_i$ in the initial monomial $w(\Delta_A)$ equals the number of intersection points of the halfray

$$w + \mathbb{R}_{>0} e_i$$

with the tropical discriminant $\tau(X^*_A)$, counting multiplicities:

$$\deg_{x_i}(\text{in}_w(\Delta_A)) = \sum_{\sigma \in \mathcal{B}(\ker A)_{i,w}} \left| \det (A^T, \sigma_1, \ldots, \sigma_{N-d-1}, e_i) \right| .$$

where $\mathcal{B}(\ker A)_{i,w}$ is the subset of $\mathcal{C}(A)$ consisting of all chains such that the row space of the matrix $A$ has non-empty intersection with the cone $\mathbb{R}_{>0}\{\sigma_1, \ldots, \sigma_{N-d-1}, -e_i, -w\}$.

**Click** Smooth case: [Katz, Kleiman, Holme], [GKZ’94]
Descartes’ theorem (1637) for univariate polynomials allows to bound the number of real solutions in terms of the number of monomials independently of the degree.

e.g. $x^d - a, \, 0 \neq a \in \mathbb{R}$ has $d$ complex solutions but at most 2 real solutions (and only one positive).

A generalization to the multivariate setting is currently an open problem.
3. AN APPLICATION TO COUNTING REAL ROOTS

Khovanskii (1980): There exists a (huge, non sharp) bound for the number of real solutions of a system of multivariate real polynomials in terms of the number of monomials which are present.

Better bounds: only a few partial results (Li-Rojas-Wang, Bihan-Sottile, after 2002)

There exists a (false) conjecture by Koushnirenko, which in particular would imply that the number of positive simple real roots of a system of two trinomials in two variables is at most 4.

There exists a counterexample by Haas (2002), with polynomials of degree $10^6$ and 5 positive simple real solutions. In fact, 5 is the correct bound.

“It is hard to find real sparse polynomials systems with many real solutions”. 
3. AN APPLICATION TO COUNTING REAL ROOTS

We could prove that the two parameter family of real bivariate trinomials

\[ H(a,b) := \begin{cases} h_1(x,y) := x^6 + ay^3 - y \\ h_2(x,y) := y^6 + bx^3 - x \end{cases} \]

gives a far simpler family of counter-examples to Kushnirenko’s Conjecture for \( a = b = \frac{44}{31} \). [D.-Rojas-Rusek-Shih, MMJ’07]

In fact, the area of the set of points \((a, b) \in \mathbb{R}^2\) such that the system has 5 positive real simple roots is smaller than \(5.701 \times 10^{-7}\).

This is a dehomogenization of the generic family associated to the configuration

\[ A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 6 & 0 & 0 & 0 & 3 & 1 \\ 0 & 3 & 1 & 6 & 0 & 0 \end{pmatrix} \].
3. AN APPLICATION TO COUNTING REAL ROOTS

Two points in the same chamber (connected component) of the complement in \( \mathbb{R}^2 \) of the zero set of the dehomogenized discriminant \( \nabla(a, b) = \Delta(1, a, -1, 1, b, -1) \) have the same number of real roots (also the same number of positive real roots in this case).

In particular, \( \nabla_A(\frac{44}{31}, \frac{44}{31}) \neq 0 \), and this implies that \( H_{(44/31,44/31)} \) has no degenerate roots.

An implicit plot of \( \nabla_A = 0 \) has very poor quality, but instead we can draw it efficiently using the dehomogenized version of the Horn-Kapranov parametrization!
Below is a sequence of 4 plots, drawn on a logarithmic scale and successively magnified up to a factor of about 1700, of the real part of the discriminant variety ($\Delta_A = 0$)
3. ANOTHER REAL APPLICATION

We moreover get an explicit good upper bound on the number of chambers of the complement of the real points in a dehomogenized $A$-discriminant (not just a mixed discriminant) for general configurations $A$ of codimension two (i.e. $n + 3$ general lattice points in $\mathbb{Z}^n$).

The Horn-Kapranov parametrization of dehomogenizations of $A$-discriminants gives a (multivalued) inverse of the logarithmic Gauss map.

We get a bound for the number of chambers smaller than $\frac{26}{5} (n + 4)^6$, which is completely independent of the coordinates of $A$. 
SOME OPEN QUESTIONS ABOUT DISCRIMINANTS

- Intrinsic formula for the degree in the singular case

- How to estimate $d(c, \{ \Delta_A = 0 \})$? i.e., how to assemble the combinatorial description and the numerical aspects (with a condiment of number theory)?

- Discriminantal matrices, i.e. describe $\Delta_A(c) = 0$ as the rank drop of a matrix. In particular, can we then estimate $d(c, \{ \Delta_A = 0 \})$ with such a matrix?

- Precise description of the singularities of the discriminant locus (Weyman-Zelevinsky’96: hyperdeterminant case; D’Andrea-Chipalkatti’07: univariate case)
WHAT WE MISSED TODAY  Hypergeometric functions

- Main “yoga” of hypergeometry:

Hypergeometric recurrences in \((a_\alpha)\): \(\frac{a_\alpha + e_i}{a_\alpha}\) rational function of \(\alpha\).

= 

Hypergeometric differential equations satisfied by \(f = \sum_\alpha a_\alpha x^\alpha\)

- Classical hypergeometric series and differential equations

vs

Binomial differential equations + Euler operators, and homogeneous \(\Gamma\)-series. The singular locus of the system is described by the vanishing of discriminants.

[Following: Gelfand-Kapranov-Zelevinsky’89,’90 - Saito-Sturmfels-Takayama’00]
Holonomic rank, (explicit) particular solutions, recurrences with finite support

translated from

A chemical reaction network consists of \( n \) complexes that are comprised of \( s \) species.

Represent reactions by a digraph \( G \) with \( n \) nodes, one for each complex, labeled by monomials.

**Triangle Example:** \( s = 2 \) species \( c_1 \) and \( c_2 \),
\( n = 3 \) complexes \( c_1^2, c_1 c_2, c_2^2 \), with all possible six reactions among them.

In this system we have \( c_1 + c_2 = \text{const} \) (i.e. \( dc_1/dt + dc_2/dt = 0 \)):

\[
\frac{dc_1}{dt} = 2 \cdot (c_1 c_2 \kappa_{21} + c_2^2 \kappa_{31} - c_1^2 (\kappa_{12} + \kappa_{13})) \\
+ (c_1^2 \kappa_{12} + c_2^2 \kappa_{32} - c_1 c_2 (\kappa_{21} + \kappa_{23})) = \]

\[
(\kappa_{21} c_1 c_2 - \kappa_{12} c_1^2) + 2 \cdot (\kappa_{31} c_2^2 - \kappa_{13} c_1^2) + (c_2^2 \kappa_{32} - c_1 c_2 \kappa_{23})
\]
The mathematical foundation for this model of chemical reactions was set by Horn, Jackson and Feinberg (70').

Dynamics of the concentrations is given by an autonomous system of ODE's of the form \( \frac{dc}{dt} = f(c) \), where \( f \) is a real polynomial with a lot of combinatorial structure coming from the digraph of reactions, with many unknown parameters (which makes numerical simulations practically unfeasible).

Binomial equations characterize the “best” models in the rate constant space and give equations for the steady states.

In these cases, the dynamic behaviour seems to be independent of the chosen constants and there is a (very partially studied) “global attractor conjecture”.
Many thanks for your attention!!
...Even if it might require polynomials $g_i$ of degree exponential in $n$ to write 1 in terms of the given binomials, as in

\[
f_1 := x_1^d, \]
\[
f_2 := x_1x_n^{d-1} - x_2^d, \]
\[
\ldots \]
\[
f_{n-1} := x_{n-2}x_n^{d-1} - x_{n-1}^d, \]
\[
f_n := x_{n-1}x_n^{d-1} - 1
\]

\[1 = \sum_{i=1}^{n} g_i f_i\]

[among many other examples, thanks to Teresa]