Linear and Non-Linear Subdivision Schemes in Geometric Modeling

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Outline of the talk

- linear subdivision schemes refining control points

  schemes generating curves

  few words on schemes generating surfaces

- three families of non-linear subdivision schemes

  subdivision schemes on surfaces

  geometrically parametrized subdivision schemes

  subdivision schemes refining curves

Linear subdivision schemes for the refinement of control points

Efficient computational methods for the generation of smooth curves/surfaces from discrete sets of points with topological relations

Subdivision schemes for curves:

- the data is a polygonal line called the control polygon $\mathcal{P}^0$

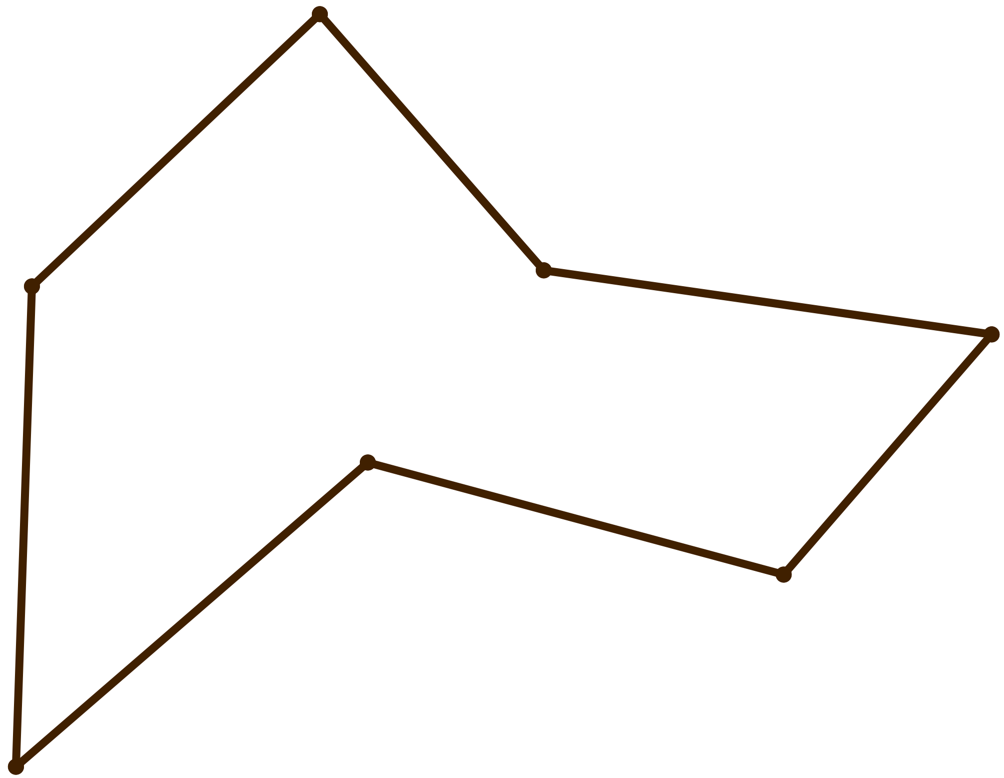
- the scheme generates a sequence of finer control polygons

$$\mathcal{P}^0 \xrightarrow[\text{step}]{\text{refinement}} \mathcal{P}^1 \xrightarrow[\text{step}]{\text{refinement}} \cdots \xrightarrow[\text{step}]{\text{refinement}} \mathcal{P}^k \xrightarrow[\text{step}]{\text{refinement}} \cdots$$

$$\mathcal{P}^{k+1} = S\mathcal{P}^k$$

- the limit of the sequence $\{\mathcal{P}^k\}$ is the curve generated by the scheme

Corner Cutting



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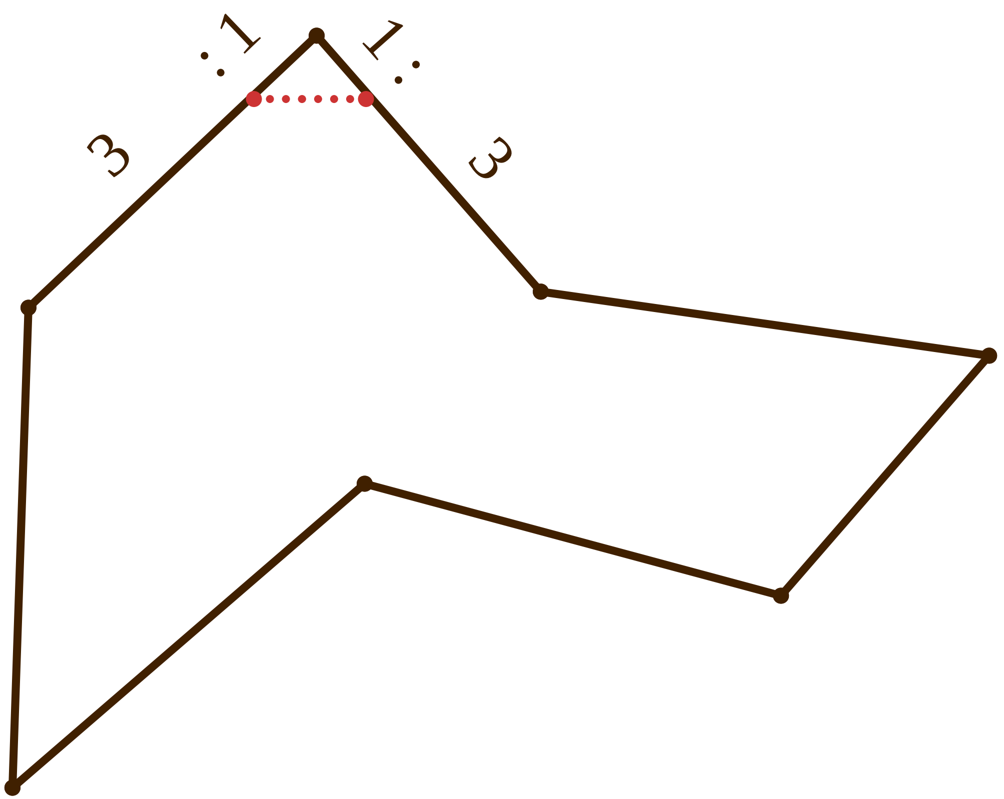

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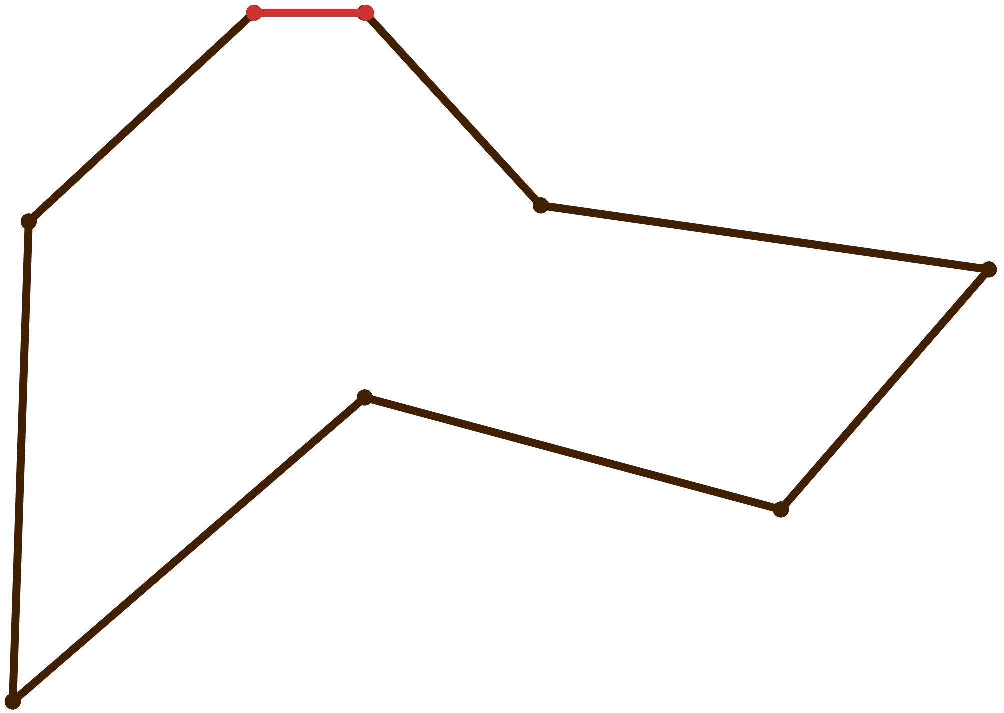

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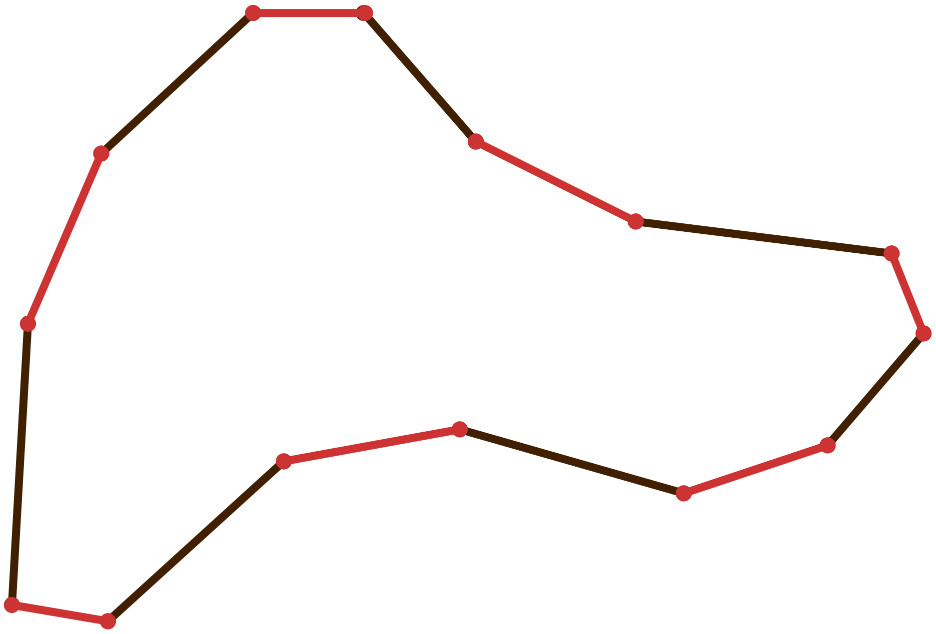

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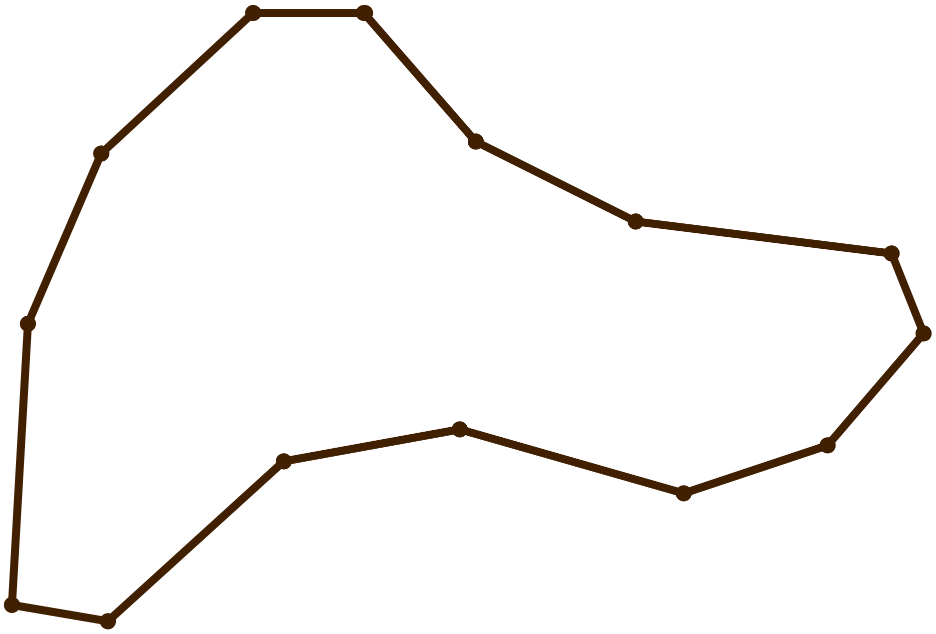

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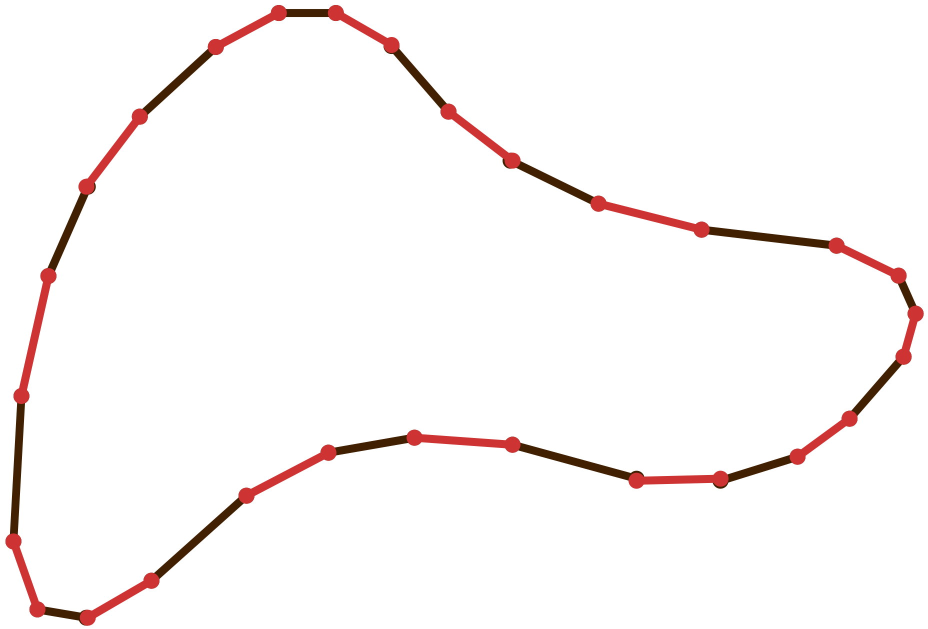

Corner Cutting

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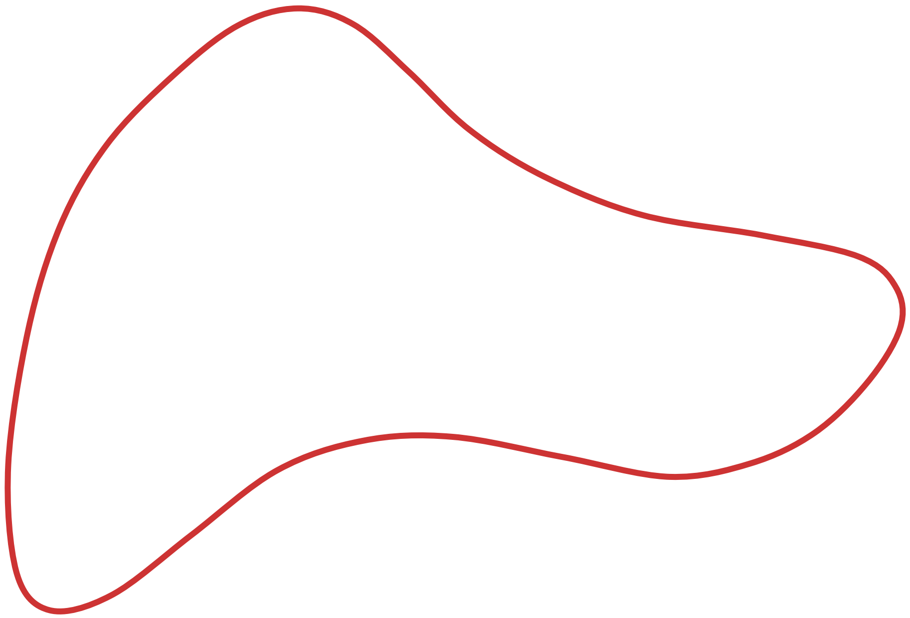
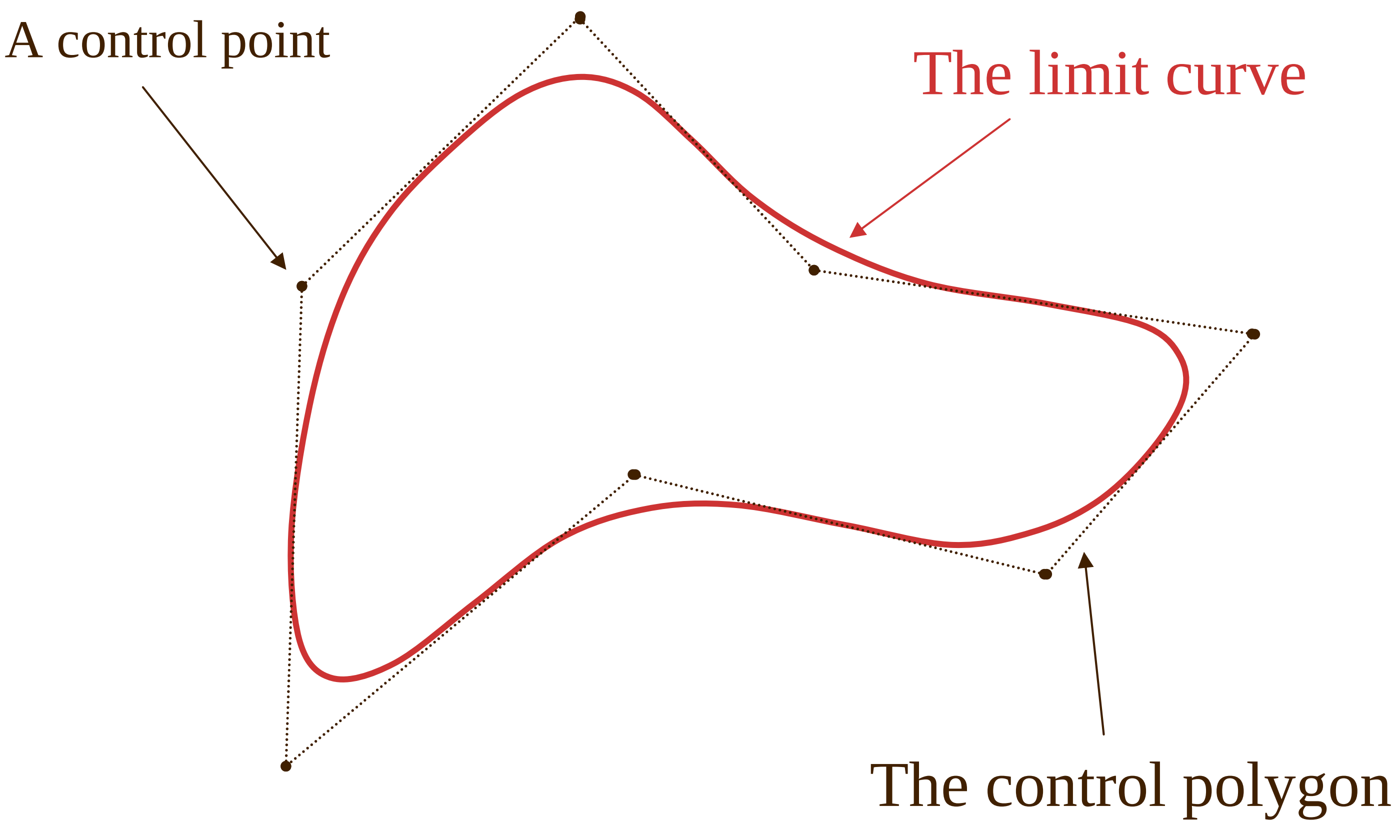


Corner Cutting



- the subdivision scheme is defined by the operator S

- S is linear and local

$$\mathcal{P}^k = \{P_i^k \in \mathbb{R}^d : i \in I_k\} \ , \ |I_k| \sim 2^k$$

$$\mathcal{P}^{k+1} = S\mathcal{P}^k \Longleftrightarrow P_i^{k+1} = \sum_j a_{i-2j} P_j^k$$

$\mathbf{a} = \{a_i : i \in \sigma(\mathbf{a})\},\ |\sigma(\mathbf{a})| < \infty$, is the mask of the scheme

- two rules:

$$P_{2i}^{k+1} = \sum_j a_{2j} P_{i-j}^k, \quad P_{2i+1}^{k+1} = \sum_j a_{2j+1} P_{i-j}^k$$

$S_{\mathbf{a}}$ – the subdivision with the mask $\mathbf{a}$

Convergence

- for geometrical applications – uniform convergence

  $S_{\mathbf{a}}^{\infty}\mathcal{P}^0$ denotes the limit curve

- analysis of convergence by parametrization of the control polygons

$\mathcal{P}^k(t)$ is the piecewise linear interpolant to the data $\{(i2^{-k}, P_i^k) : i \in I_k\}$

the convergence of the components of $\{\mathcal{P}^k(t)\}$ is investigated

- $d = 1$: special initial data $\boldsymbol{\delta} = \{\delta_i = \delta_{i,0}\}$

  For a convergent scheme $S_{\mathbf{a}}$

$$\phi_{\mathbf{a}} = S_{\mathbf{a}}^{\infty}\boldsymbol{\delta}, \quad S_{\mathbf{a}}^{\infty}\mathcal{P}^0 = \sum_{i \in I_0} P_i^0 \phi_{\mathbf{a}}(\bullet - i)$$

B-spline curves

a B-spline curve:

$$C(t) = \sum_{i \in I} P_i B_m(t - i)$$

- $B_m(t)$ – B-spline of degree m with integer knots:

  $$\text{supp }(B_m) = [0, m+1], \qquad B_m\Big|_{[i,i+1]} \in \pi_m, \qquad B_m \in C^{m-1}(\mathbb{R})$$

- by properties of B_m, $C(t)$ "has the shape" of $\mathcal{P} = \{P_i : i \in I\}$

- B-spline curves are a design tool

- B-spline curves are rendered by B-spline subdivision schemes

B-spline subdivision schemes

- the refinement step for a B-spline curve of degree m $P^{k+1} = \sum_j a^{[m]}_{i-2j} P^k_j$, can be decomposed into one trivial refinement step followed by m repeated averaging steps (Lane and Riesenfeld, 1980)

$$
\begin{aligned}
&P^{k+1,0}_{2i} = P^{k+1,0}_{2i+1} = P^k_i \ , \\
&P^{k+1,\ell+1}_i = \frac{1}{2}\left(P^{k+1,\ell}_{i-1} + P^{k+1,\ell}_i\right), \ \ell = 0, \ldots, m-1 \\
&P^{k+1}_i = P^{k+1,m}_i
\end{aligned}
$$

- the case $m = 2$ corresponds to Chaikin's algorithm (Chaikin, 1974)

$$
P^{k+1}_{2i} = \frac{3}{4}P^k_{i-1} + \frac{1}{4}P^k_i, \quad P^{k+1}_{2i+1} = \frac{1}{4}P^k_{i-1} + \frac{3}{4}P^k_i
$$

Construction of subdivision schemes by local approximation

A local polynomial approximation

$$(\mathcal{A}f)(x) = \sum_{i=-\ell+1}^{\ell} f(i)w_i(x) \ , \quad x \in [0,1] \ , \quad w_i(x) \in \pi_n$$

$\mathcal{A}$ is shift invariant and scale invariant

- interpolatory scheme:

$$P_{2i}^{k+1} = P_i^k, \qquad P_{2i+1}^{k+1} = \sum_{j=-\ell+1}^{\ell} w_j(\frac{1}{2})P_{i+j}^k$$

- "approximating" scheme:

$$P_{2i}^{k+1} = \sum_{j=-\ell+1}^{\ell} w_j(\frac{1}{4})P_{i+j}^k \ , \quad P_{2i+1}^{k+1} = \sum_{j=-\ell+1}^{\ell} w_j(\frac{3}{4})P_{i+j}^k$$

- Chaikin's algorithm: $\mathcal{A}f(x)$ the linear interpolant ($\ell = 1$)

Approximation Result

A convergent subdivision scheme S, based on local polynomial approximation $(\mathcal{A}f)(x) = \sum_{i=-\ell+1}^{\ell} f(i)w_i(x)$,
with the property $\mathcal{A}p = p$, $p \in \pi_n$

- reconstructs polynomials of degree $\leq n$ from their samples

- has approximation order $n+1$,

$$\mathcal{P}^0 = \{P_i^0 = f(ih)\} \ , \quad f \in C^{n+1}$$

$$\left|(S^\infty \mathcal{P}^0)(t) - f(t)\right| \leq M h^{n+1}$$

Examples

(I) $\mathcal{A}f$ interpolation polynomial of degree $2\ell - 1$, based on 2ℓ points

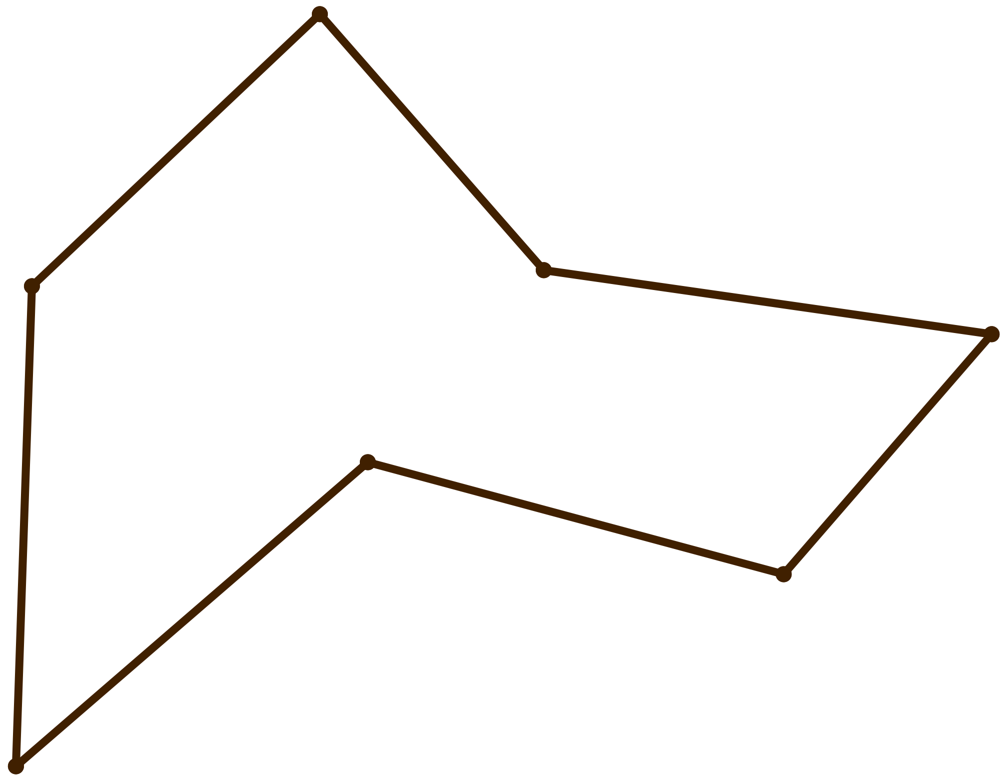
- interpolatory schemes (Deslauriers and Dubuc, 1986, 1989) closely related to compactly supported, orthonormal wavelets (Daubechies, 1988)
- approximating schemes (Dyn, Floater and Hormann, 2005) higher smoothness for the same ℓ, $\ell \leq 5$

(II) $\mathcal{A}f$ convex combination of linear and cubic interpolants
the interpolatory 4-point scheme

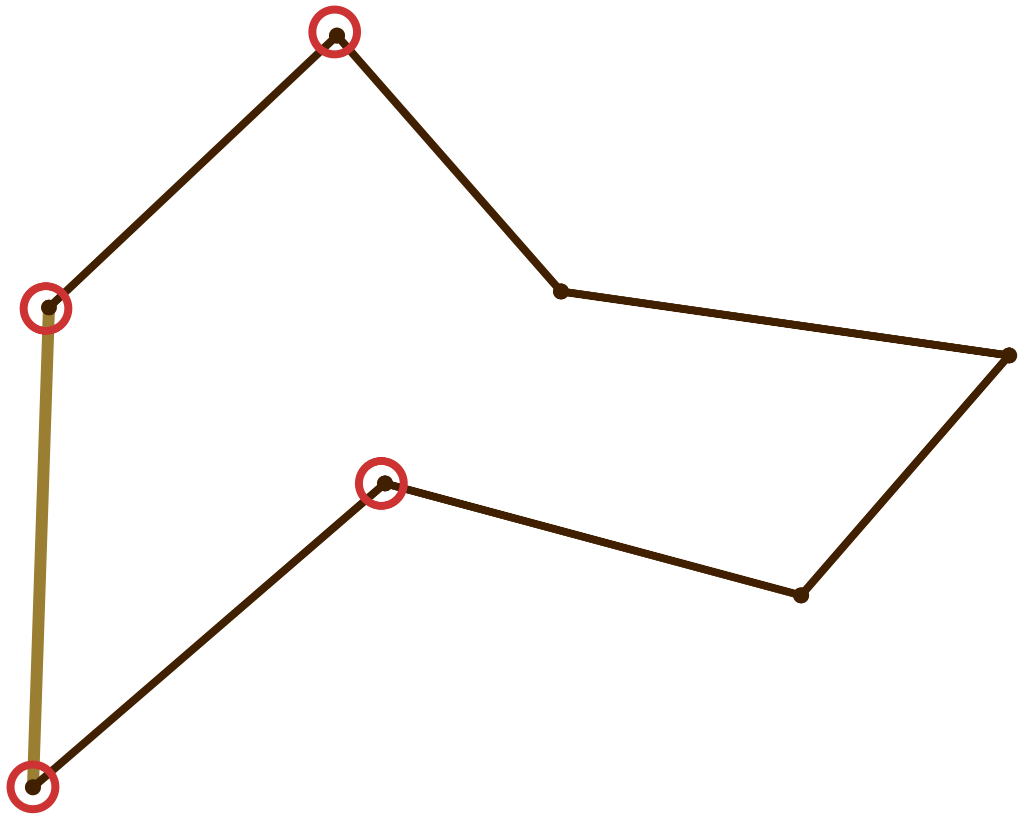
$$P_{2i}^{k+1} = P_i^k \;, \quad P_{2i+1}^{k+1} = -w(P_{i-1}^k + P_{i+2}^k) + \left(\frac{1}{2} + w\right)(P_i^k + P_{i+1}^k)$$

(Dyn, Gregory and Levin, 1987)
w is a shape parameter
convergence for $|w| < \frac{1}{4}$, C^1 limit curves for $0 < w < \frac{1}{8}$

The 4-point scheme



The 4-point scheme



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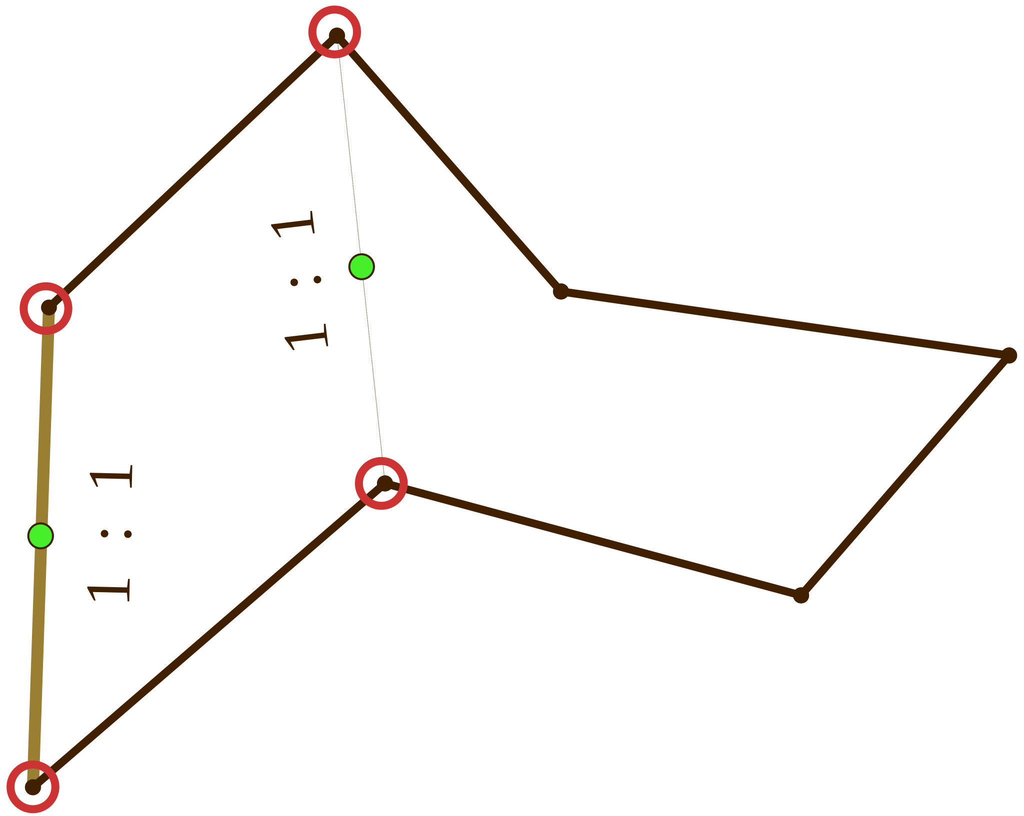

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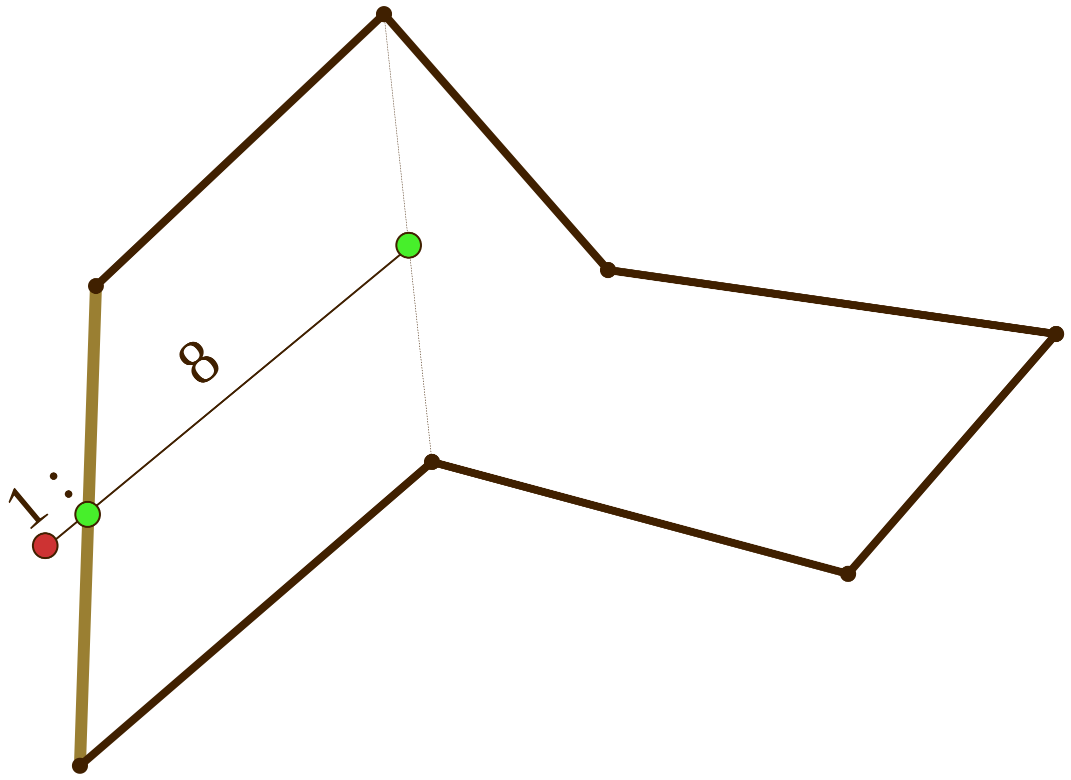
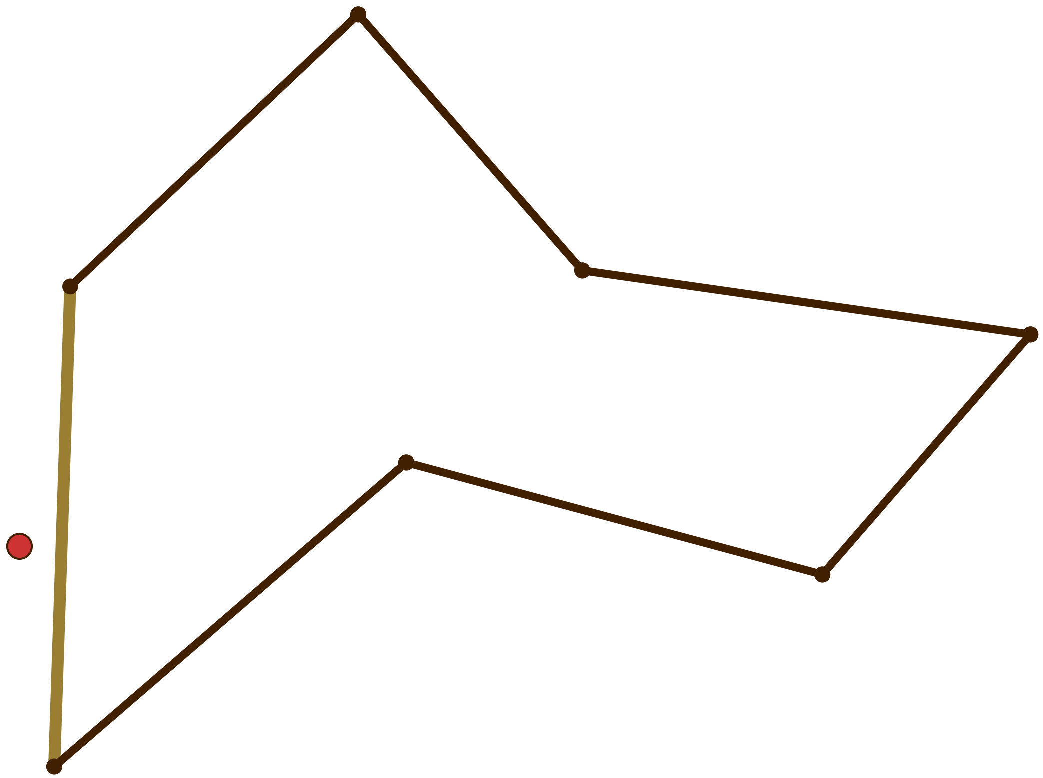
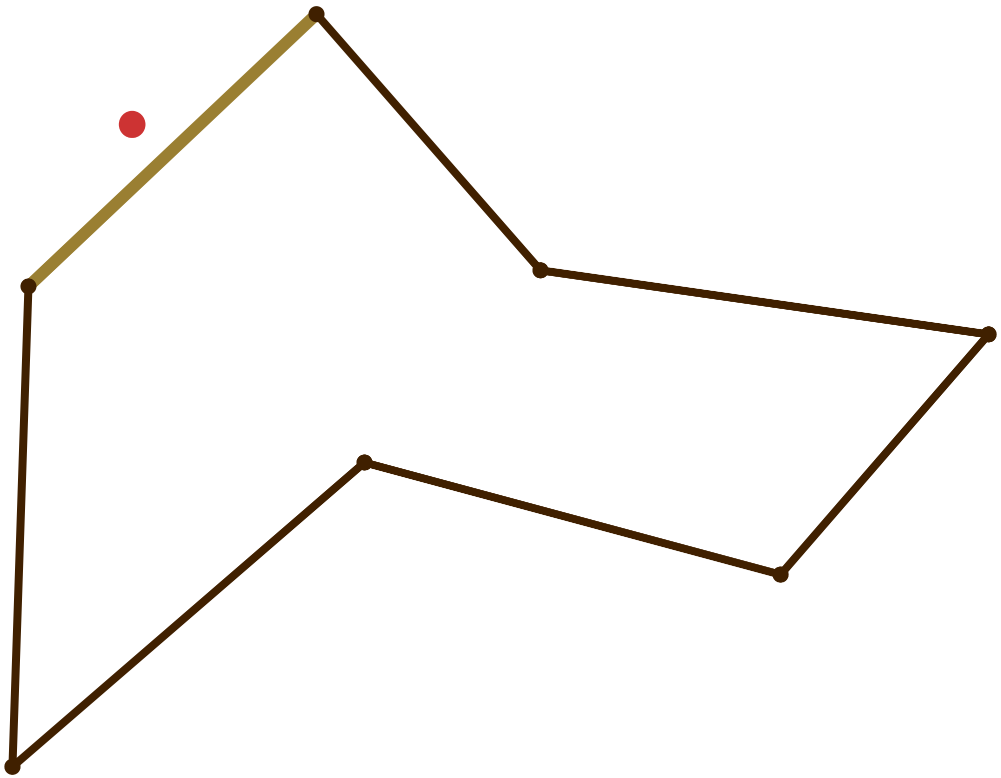


The 4-point scheme



The 4-point scheme



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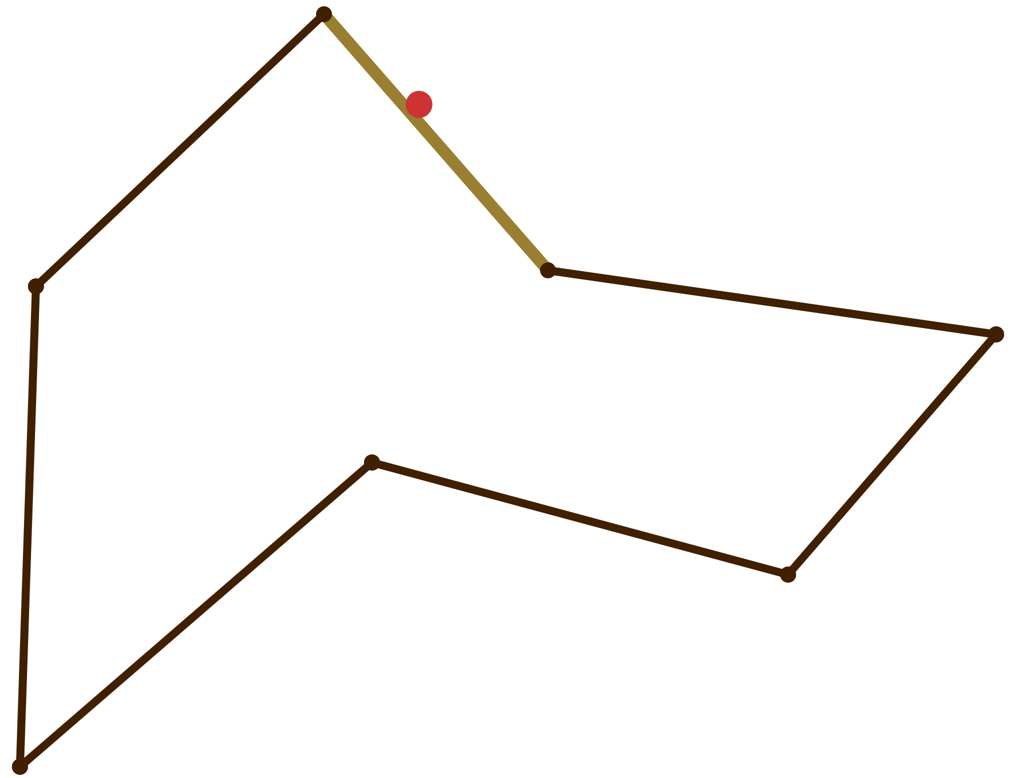

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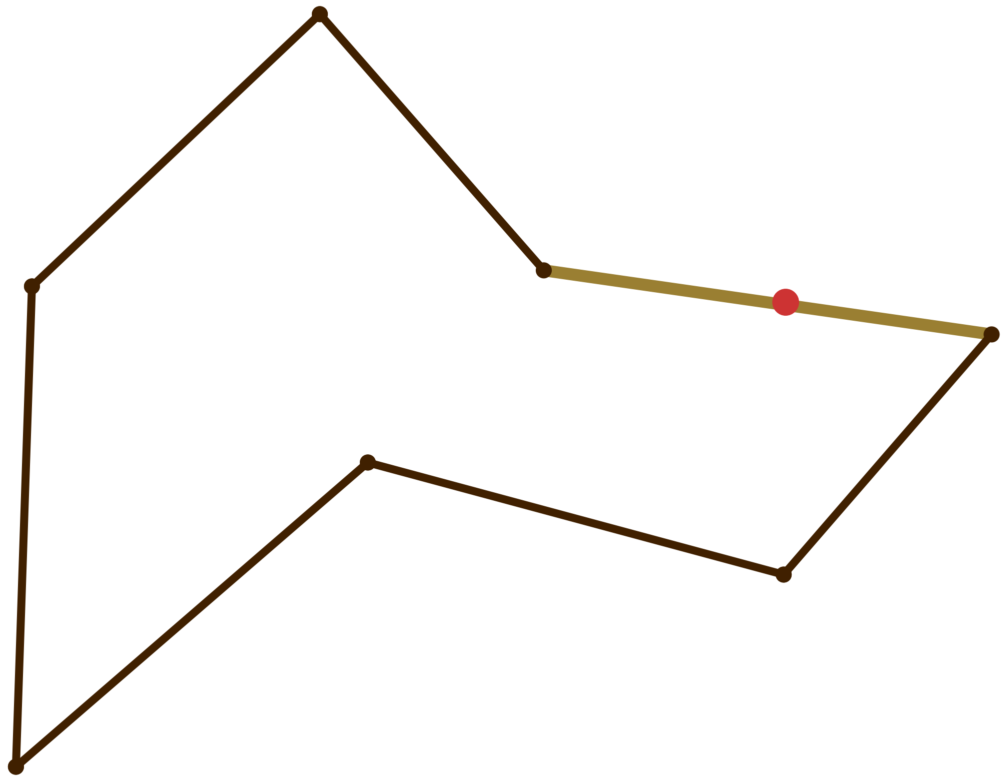

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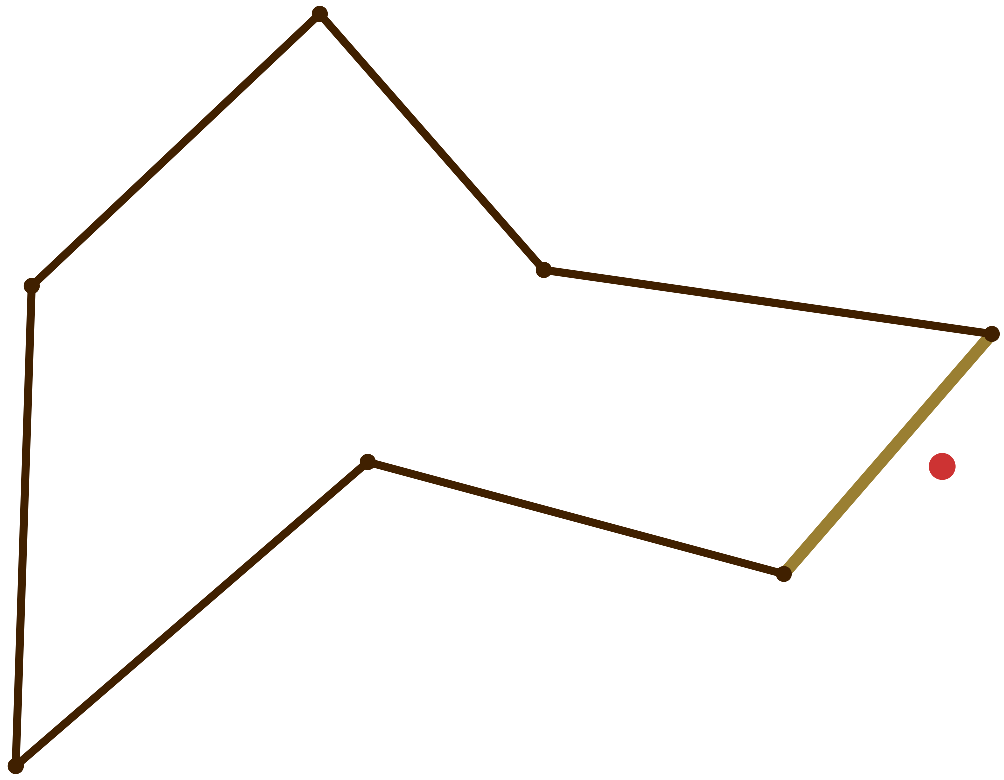

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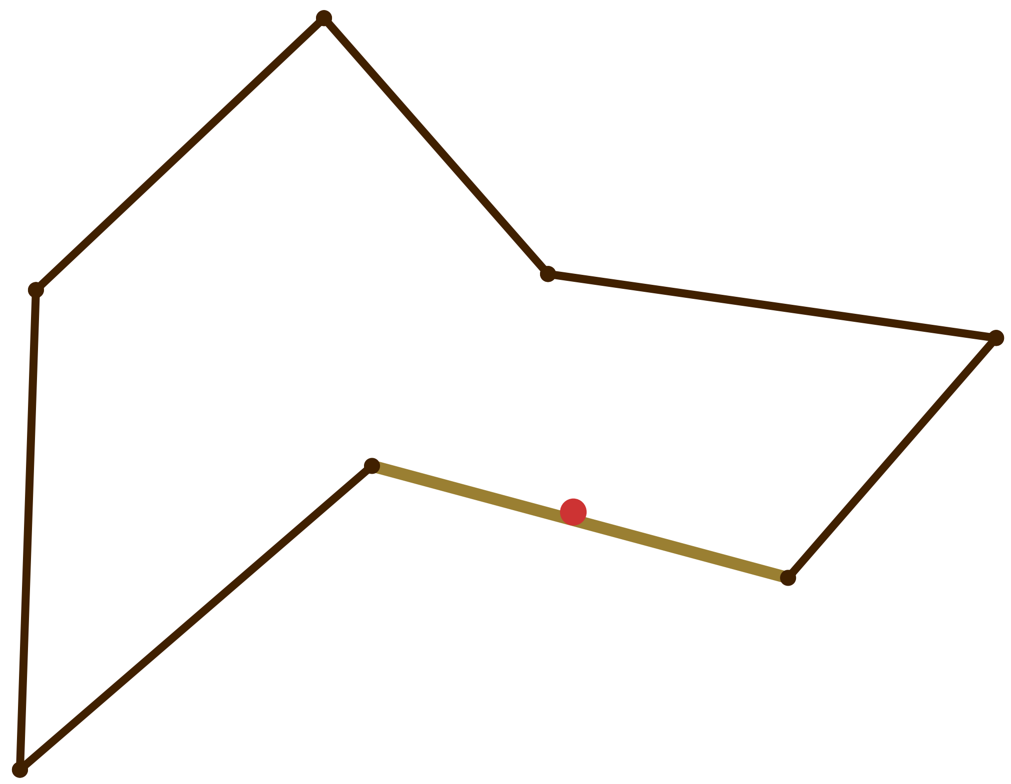

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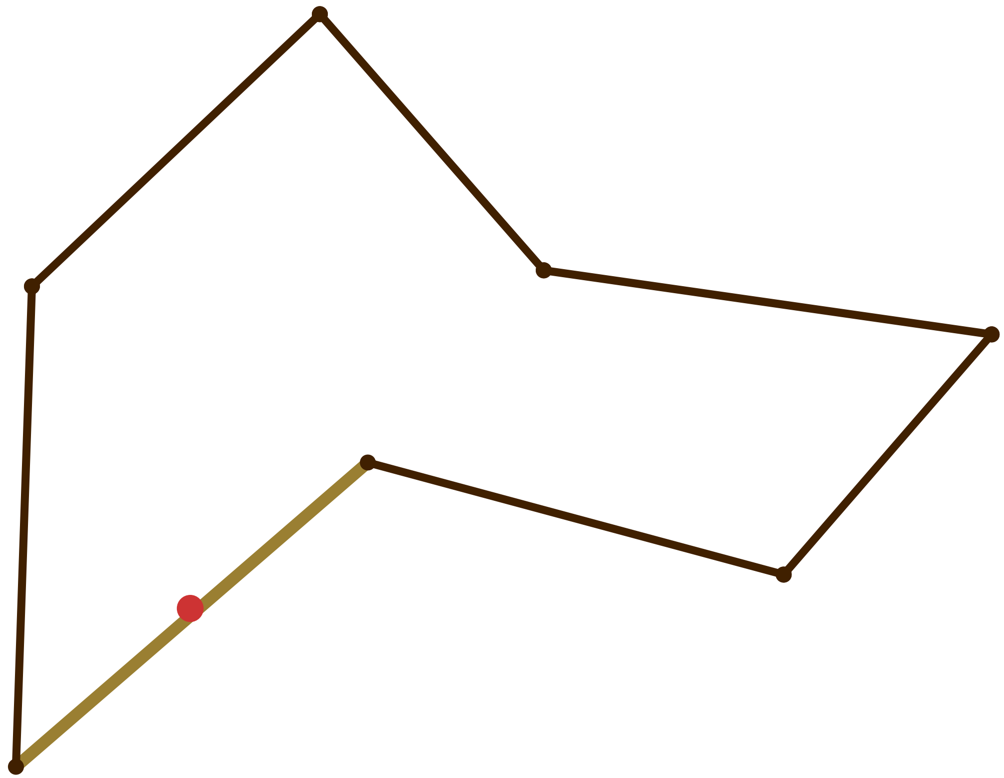

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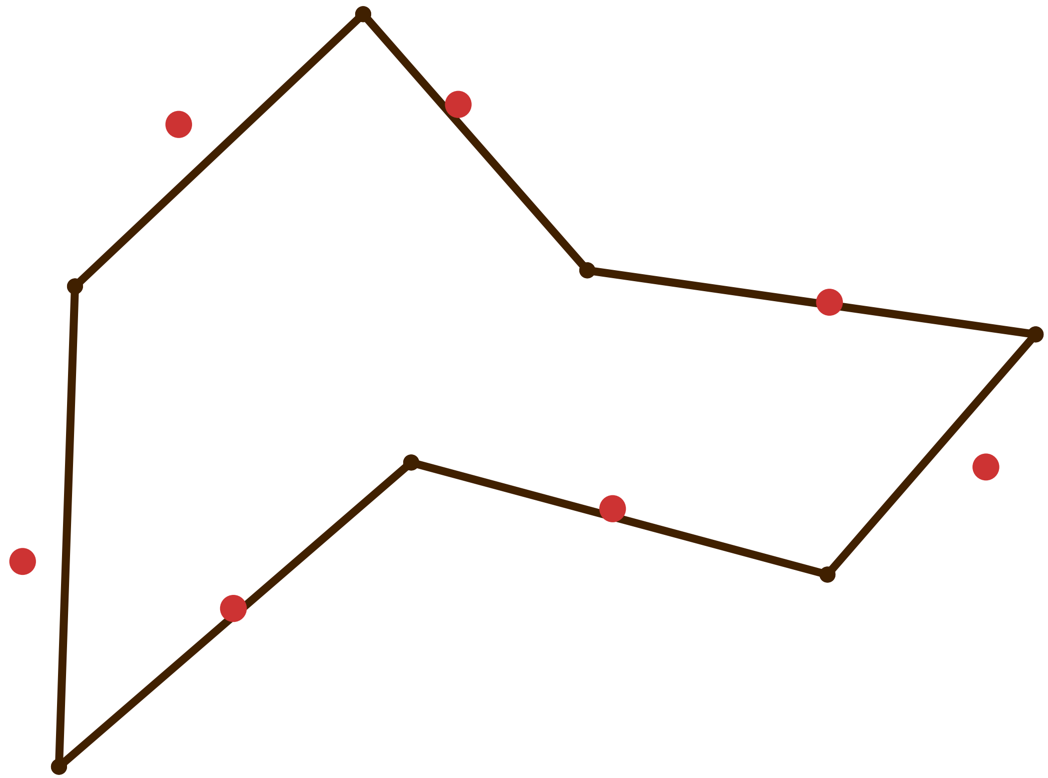

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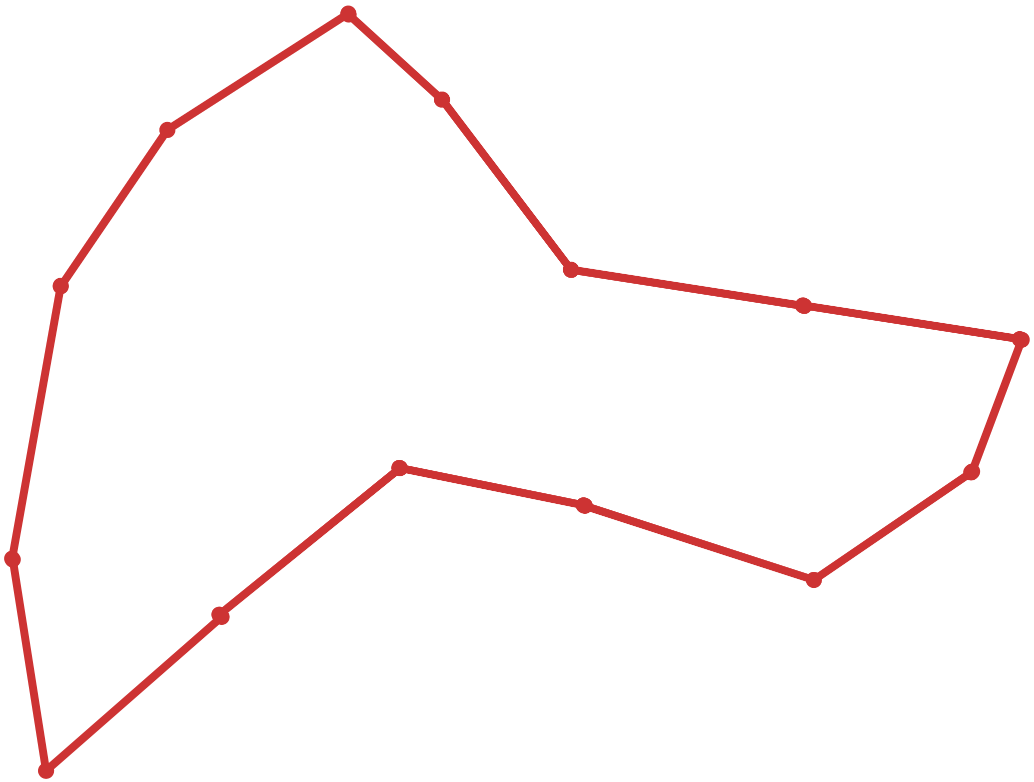

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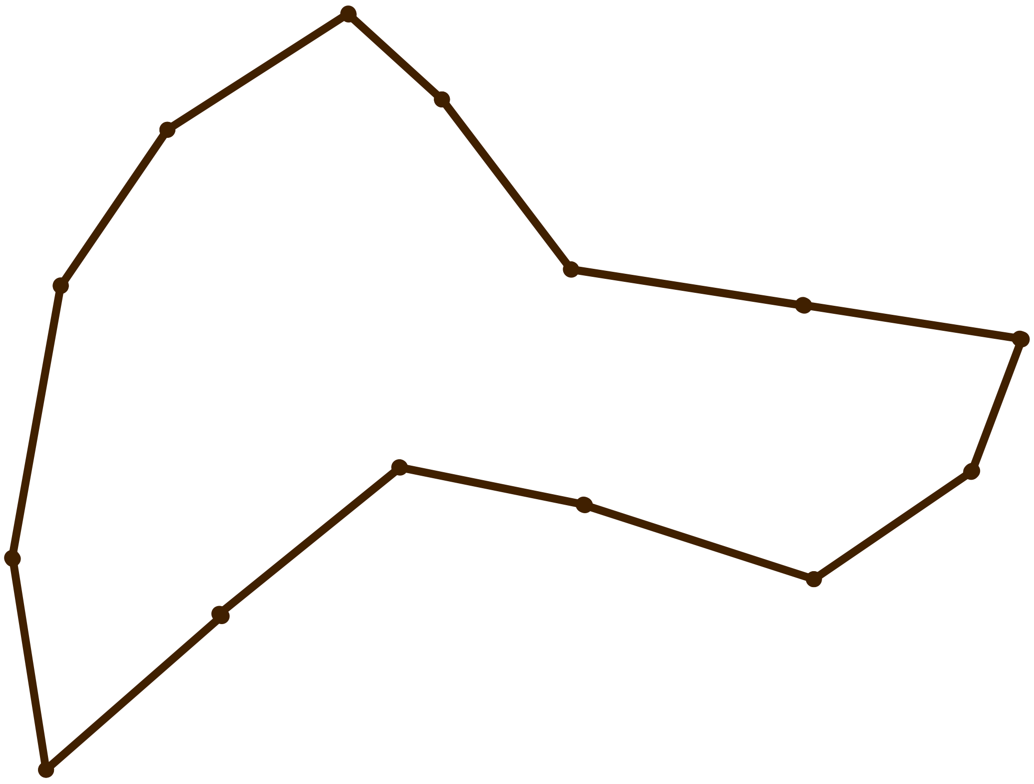

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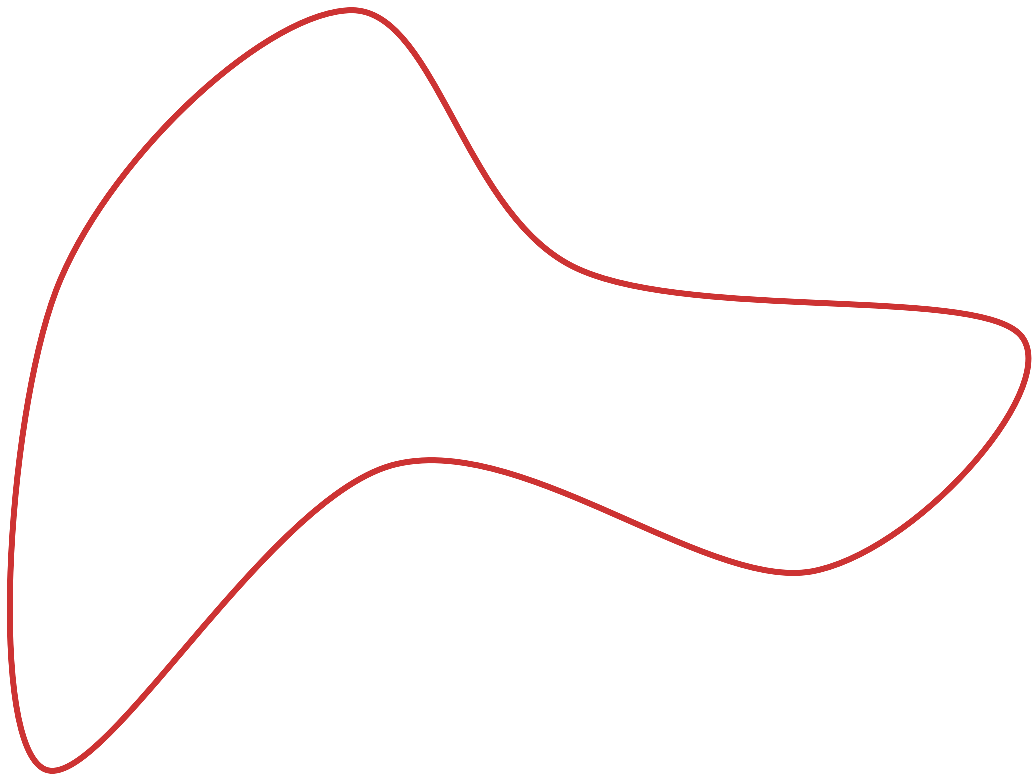

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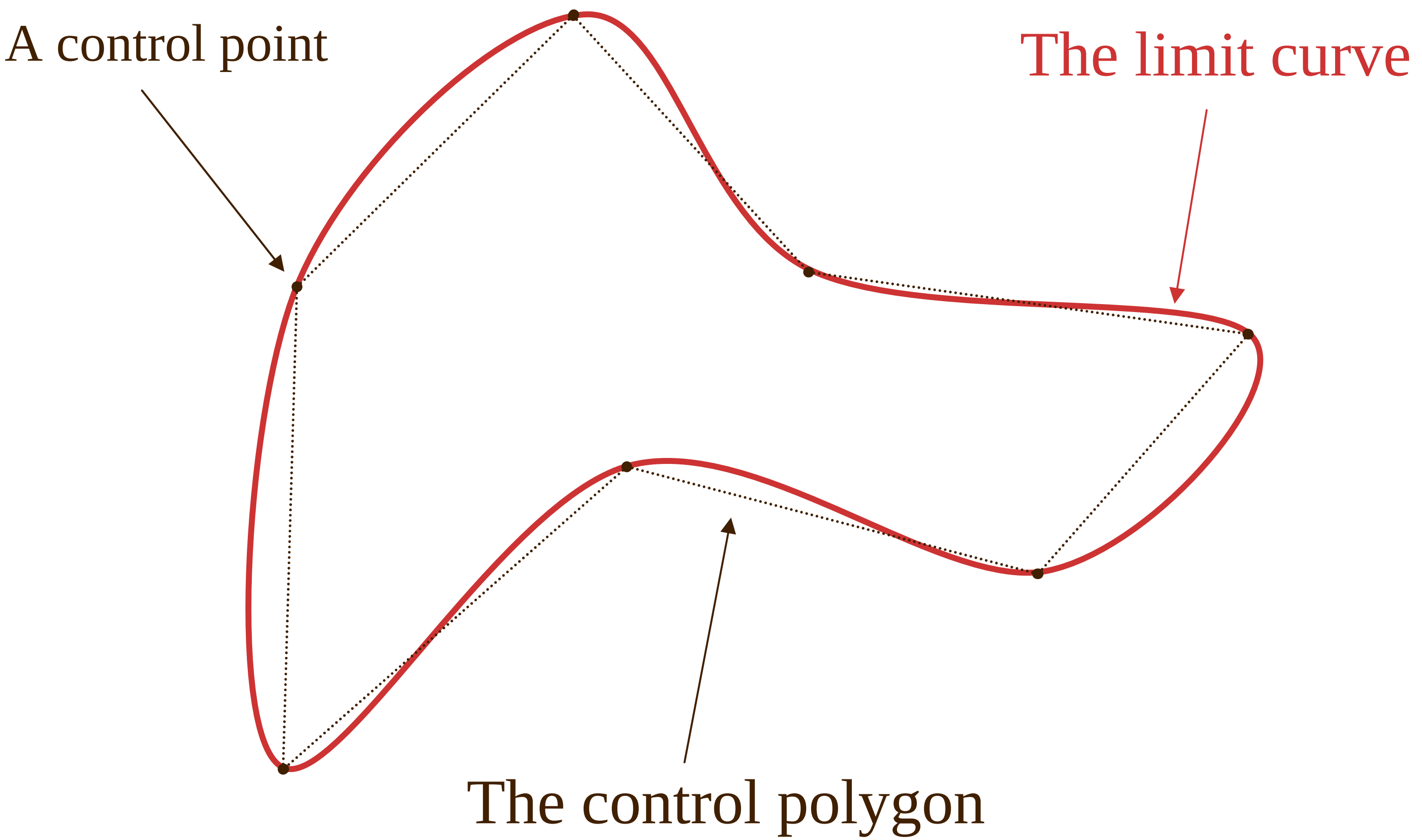
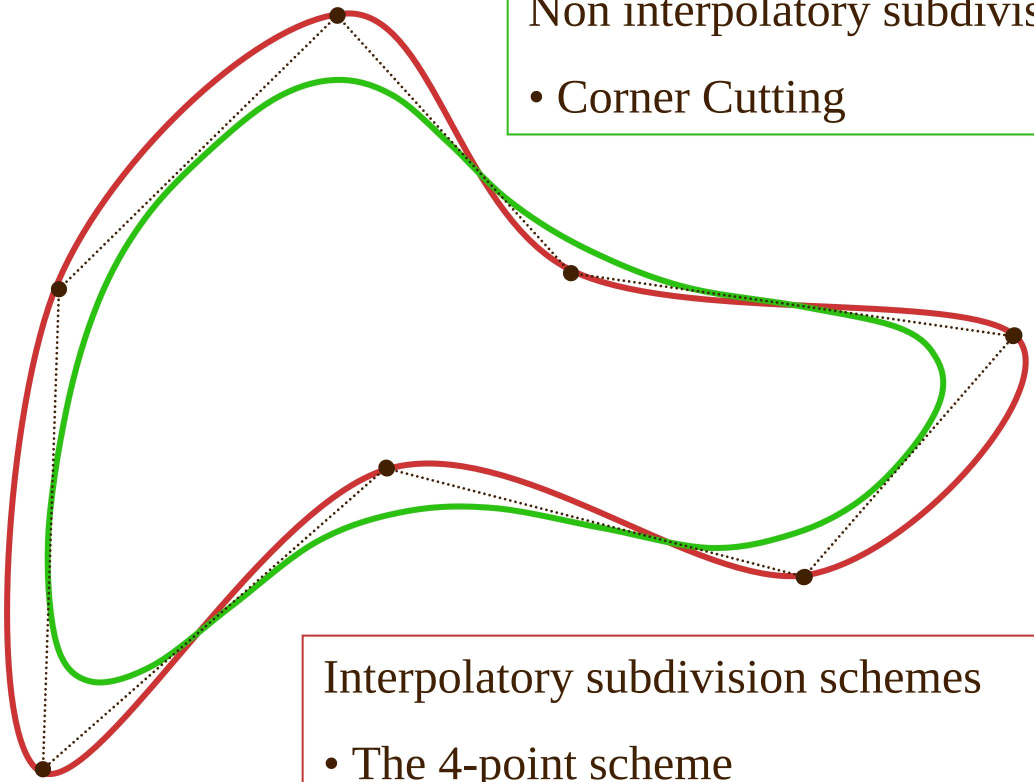


Subdivision curves



Non interpolatory subdivision schemes

- Corner Cutting

Interpolatory subdivision schemes

- The 4-point scheme

Subdivision schemes and the construction of wavelets

The subdivision refinement rule: $f_i^{k+1} = \sum_j a_{i-2j} f_j^k$

The basic limit function $\phi_{\mathbf{a}} = S_{\mathbf{a}}^{\infty} \delta$

For any initial data $\mathbf{f}^0 = \{f_i^0 \ : \ i \in \mathbb{Z}\}$

$$(S_{\mathbf{a}}^{\infty} \mathbf{f}^0)(x) = \sum_i f_i^0 \phi_{\mathbf{a}}(x - i) \ \subseteq V_0$$

$$V_0 = \mathrm{span}\{\phi_{\mathbf{a}}(\cdot - i) \ : \ i \in \mathbb{Z}\}$$

$$(S_{\mathbf{a}}\delta)_i = \sum_j a_{i-2j}\delta_j = a_i$$

$$\phi_{\mathbf{a}}(x) = (S_{\mathbf{a}}^{\infty}\delta)(x) = S_{\mathbf{a}}^{\infty}(S_{\mathbf{a}}\delta) = \sum_i a_i \phi_{\mathbf{a}}(2x-i)$$

The function $\phi_{\mathbf{a}}$ is a scaling function

The function $\phi_{\mathbf{a}}$ defines a sequence of nested spaces

$$V_j = \text{span}\{\phi_{\mathbf{a}}(2^j(x-i) \ : \ i \in \mathbb{Z}\}, \quad j \in \mathbb{Z},$$

$$\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots$$

Convergence analysis

$$P_{2i}^{k+1} = \sum_j a_{2j} P_{i-j}^k, \quad P_{2i+1}^{k+1} = \sum_j a_{2j+1} P_{i-j}^k, \quad \mathcal{P}^k(2^{-k} i) = P_i^k$$

A necessary condition for convergence to non-zero limits

$$\sum_j a_{2j} = \sum_j a_{2j+1} = 1$$

$\Rightarrow$ the symbol $a(z) = \sum_{j \in \sigma(\mathbf{a})} a_j z^j$ vanishes at $z = -1$

$$a(z) = (1+z)q(z)$$

$$S_{\mathbf{a}} \quad \text{converges} \quad \Longleftrightarrow \quad \lim_{k \to \infty} S_{\mathbf{q}}^k = 0$$

The condition $\lim_{k \to 0} S_{\mathbf{q}}^k = 0$ can be checked by algebraic manipulations on $q(z)$

$$S_{\mathbf{a}} \quad \text{converges} \quad \Longrightarrow \quad S_{\mathbf{a}}^\infty \mathcal{P} \quad \text{is continuous}$$

Smoothness analysis

- if $a(z) = \frac{1+z}{2}\, b(z)$ and $S_{\mathbf{b}}$ is a convergent scheme then $S_{\mathbf{a}}$ generates C^1 curves: $S_{\mathbf{a}}^{\infty}\mathcal{P} \in C^1$

Example

B-spline subdivision scheme of degree $m \geq 1$

$$a^{[m]}(z) = \frac{(1+z)^{m+1}}{2^m}$$

$$q(z) = \left(\frac{1+z}{2}\right)^m, \qquad \|S_{\mathbf{q}}\|_{\infty} = \frac{1}{2} \quad \Longrightarrow \text{ convergence}$$

$$a^{[m]}(z) = \left(\frac{1+z}{2}\right)^{m-1} a^{[1]}(z) \Longrightarrow \quad S_{\mathbf{a}^{[m]}}^{\infty}\mathcal{P} \in C^{m-1}$$

Subdivision schemes for surfaces

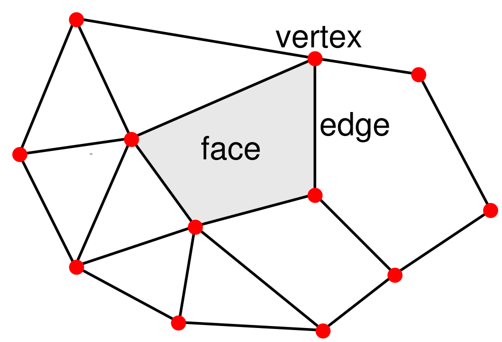
- the topological relations between the control points are richer than in the curve case
- refinement of control nets $N(V, E, F)$

  V–the vertices

  E–the edges (pairs of vertices)

  F–the faces (cyclic lists of edges)

$$N(V, E, F) \xrightarrow[\text{refinement step}]{} N'(V', E', F')$$

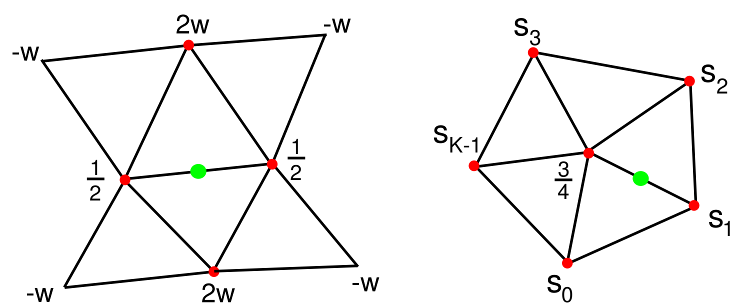


a schematical control net

The “butterfly” scheme: an interpolatory scheme

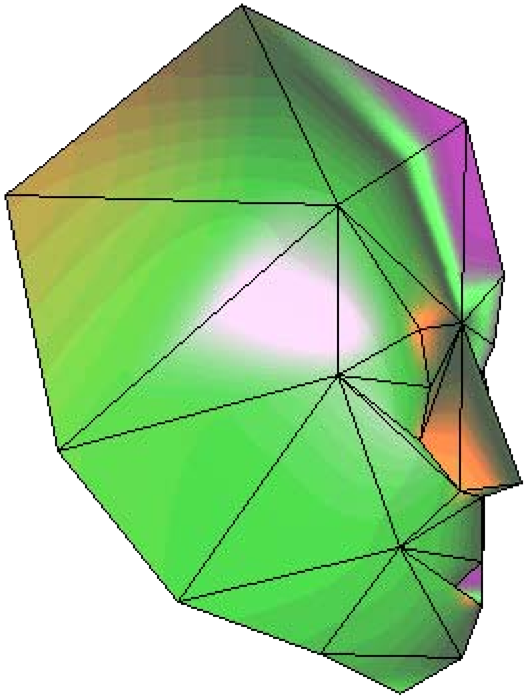
$$V' = V \cup \{v(e) : e \in E\}$$

one rule for inserting a new e-vertex
(Dyn, Gregory and Levin, 1990)

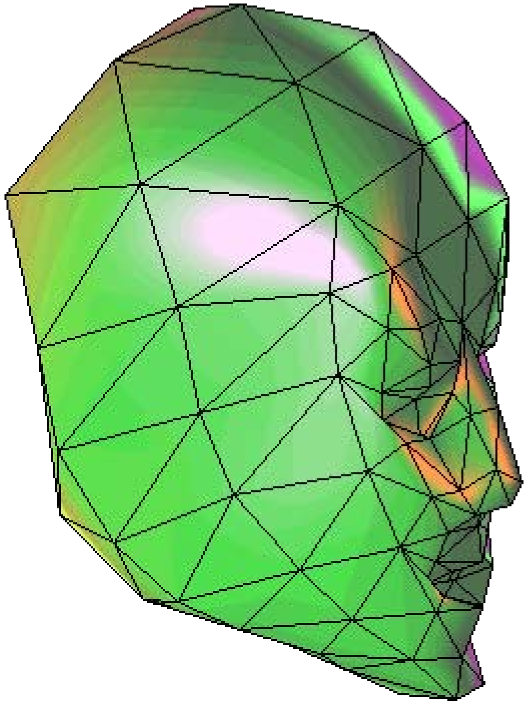


The butterfly scheme generates C^1 surfaces on general triangular nets except at irregular vertices of valency 3 or of valency greater than 7, for $w \in (0, w^*)$, $w^* > \frac{1}{16}$.

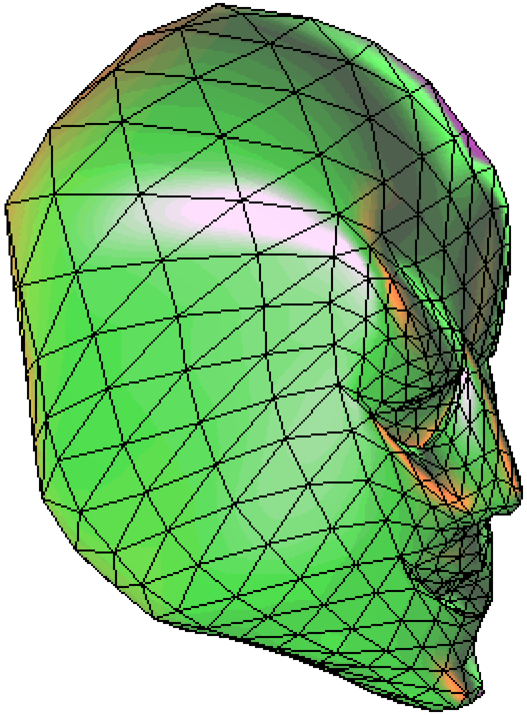
The original control net



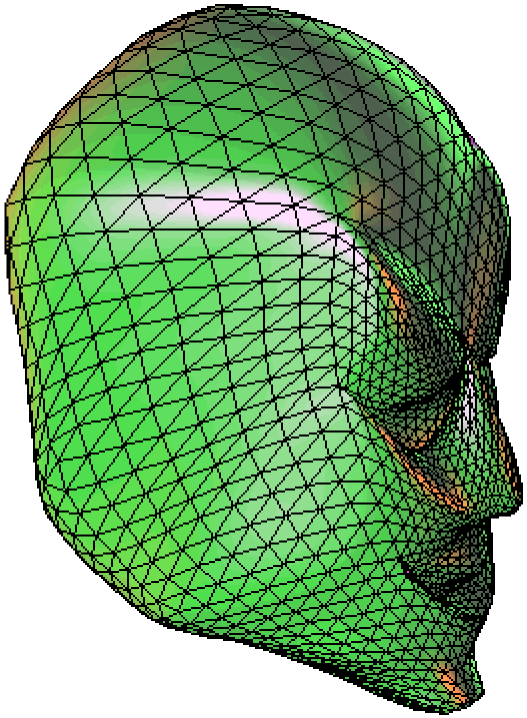
After 1st iteration



After 2nd iteration



After 3rd iteration



The limit surface

The limit surfaces of the Butterfly subdivision are smooth but are nowhere twice differentiable.

Nonlinear subdivision schemes on surfaces

Methods apply also to manifold-valued data.

Requirement: to generate refined control points on a surface from a given set of control points on the surface

Idea: to modify "good" linear schemes to refine points on a surface

First approach (Donoho et al. 2005)

Given control points $\{P_i\}$, execute a linear refinement step of a scheme $S_{\boldsymbol{a}}$ on the projections of the points $\{P_j : a_{i-2j} \neq 0\}$ to a tangent plane at a chosen point P_i^*, and map it back to the surface:

$$(T\mathcal{P})_i = \psi_{P_i^*}^{-1}\Big(\sum_{j\in\mathbb{Z}} a_{i-2j}\psi_{P_i^*}(P_j)\Big)$$

P_i^* is some chosen "center" of the points $\{P_j : a_{i-2j} \neq 0\}$

Analysis of convergence and smoothness (Yu et al. 2008).

Other modifications (Wallner and Dyn, 2005)

- expressing the linear subdivision scheme in terms of repeated linear binary averages

$$av_\lambda(P,Q) = (1-\lambda)P + \lambda Q\,, \quad \lambda \in \left(-\frac{1}{2}, \frac{3}{2}\right)$$

- replacing all the binary linear averages by

(i) geodesic averages on the surface

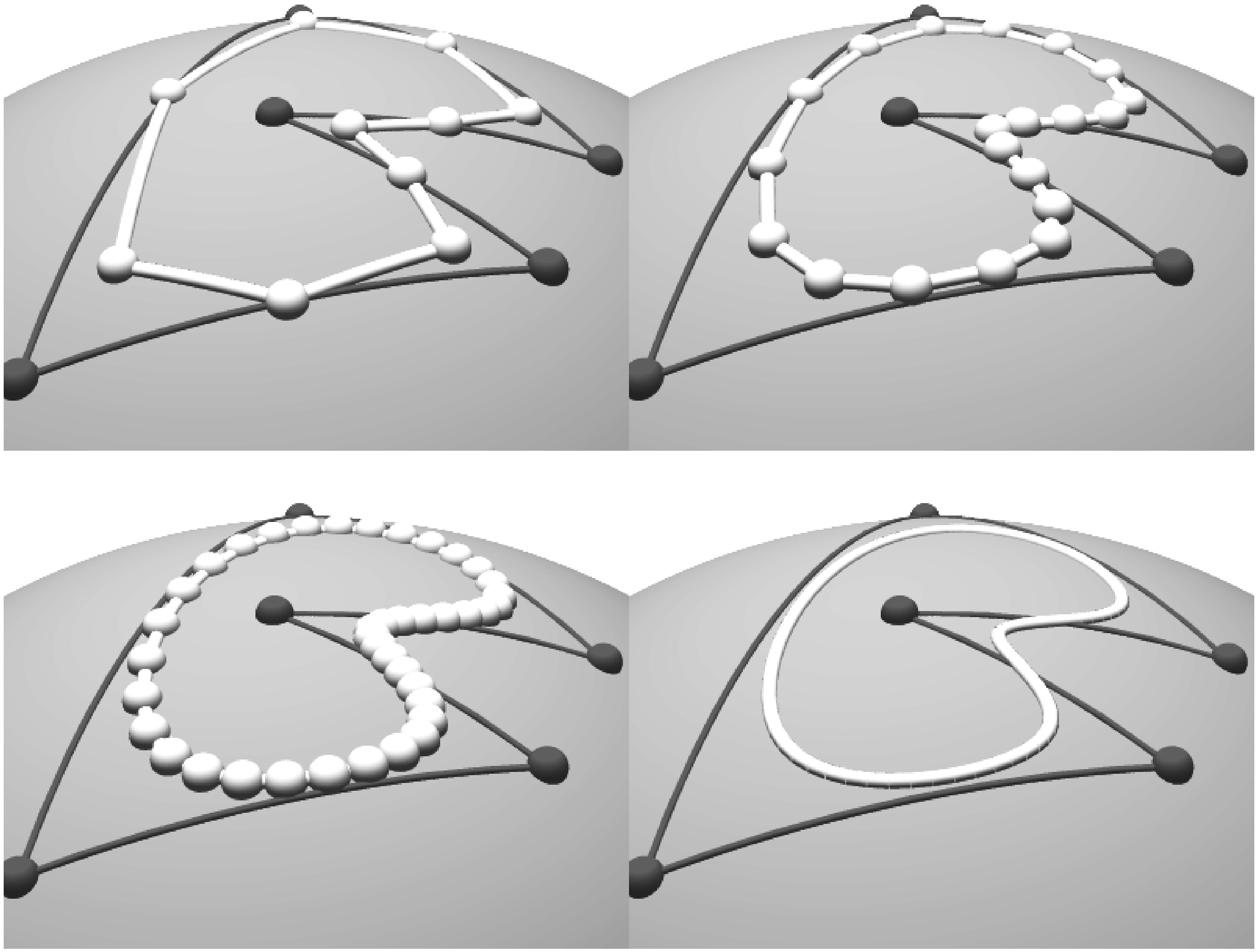
or

(ii) the projection of the linear binary averages to the surface

A geodesic average

$$gav_\lambda(P,Q) = C(\lambda\tau)$$

$C(t)$ is the geodesic curve on the surface from P to Q, such that $C(0) = P$, $C(\tau) = Q$.



geodesic cubic B-spline scheme on a sphere

Example: the 4-point scheme with $w = \frac{1}{16}$

$$(S\mathcal{P})_{2i} = P_i \qquad (S\mathcal{P})_{2i+1} = -\frac{1}{16}(P_{i-1} + P_{i+2}) + \frac{9}{16}(P_i + P_{i+1})$$

$$(S\mathcal{P})_{2i+1} = av_{\frac{1}{2}}\Big(av_{-\frac{1}{8}}(P_i, P_{i+2}), av_{-\frac{1}{8}}(P_{i+1}, P_{i-1})\Big)$$

$$(S\mathcal{P})_{2i+1} = av_{-\frac{1}{8}}\Big(av_{\frac{1}{2}}(P_i, P_{i+1}), av_{\frac{1}{2}}(P_{i-1}, P_{i+2})\Big)$$

The symbol of the scheme is factorizable into real linear factors:

$$a(z) = z^{-3}\left(\frac{\alpha + \beta z}{2}\right)\left(\frac{\beta + \alpha z}{2}\right)\left(\frac{1 + z}{2}\right)^3 (1 + z)$$

$$\alpha = 1 + \sqrt{3}, \quad \beta = 1 - \sqrt{3}$$

One refinement step is equivalent to the following global elementary steps:

$$
\begin{array}{lcll}
(1+z) & \Rightarrow & Q_{2i,0} = Q_{2i+1,0} = P_i & \text{(elementary refinement)} \\
\left(\frac{1+z}{2}\right)^3 & \Rightarrow & Q_{i,j+1} = \frac{1}{2}(Q_{i,j} + Q_{i-1,j}), & j = 0,1,2 \\
 & & & \text{(repeated symmetric averages)} \\
\frac{\alpha+\beta z}{2} & \Rightarrow & Q_{i,4} = \frac{\beta}{2}Q_{i-1,3} + \frac{\alpha}{2}Q_{i,3}, & \\
 & & & \text{(non-symmetric averages)} \\
\frac{\beta+\alpha z}{2} & \Rightarrow & Q_{i,5} = \frac{\alpha}{2}Q_{i-1,4} + \frac{\beta}{2}Q_{i,4}, & \\
z^{-3} & \Rightarrow & (S\mathcal{P})_i = Q_{i+3,5}, & \text{(shift)}
\end{array}
$$

$$
\frac{\alpha}{2} = \frac{1}{2} + \frac{\sqrt{3}}{2} \cong 1.366, \quad \frac{\beta}{2} = \frac{1}{2} - \frac{\sqrt{3}}{2} \cong -0.366
$$

Analysis by proximity

- under the proximity condition:

$$\|S\mathcal{P} - T\mathcal{P}\|_\infty \le C\|P_{i+1} - P_i\|_\infty^2$$

  for $\mathcal{P} = \{P_i\}$ with $\|P_{i+1} - P_i\|_\infty$ small enough

  (a) if the linear scheme S generates C^0 limits so does T

  (b) if S generates C^1 limits in a "strong sense" then T generates C^1 limits

- a proximity condition guaranteeing C^2 of T for S generating C^2-limits in a "strong sense" exists (Wallner, 2006)

- for smooth surfaces with bounded normal curvatures, the proximity conditions are satisfied by the modifications of linear schemes, based on repeated averaging

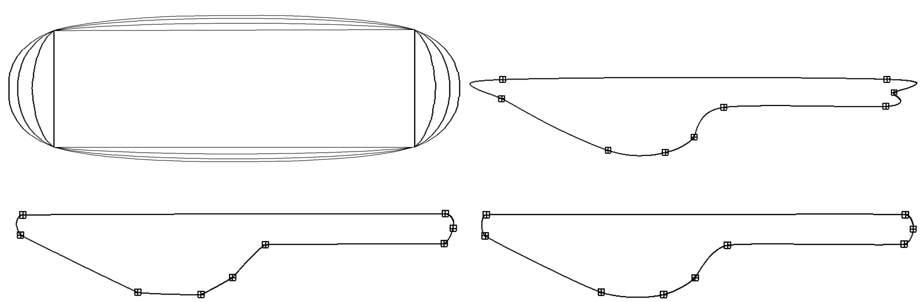
Why nonlinear schemes in $\mathbb{R}^3$?

The aim in the design of curves from control polygons is to obtain curves without artifacts.

> **Artifacts** are geometric features of a designed curve which do not exist in the given control polygon, such as self-intersections and inflection points.

Linear schemes generate curves without artifacts from initial polygons with edges of similar lengths.

Linear schemes generate curves with artifacts from control polygons with edges of significantly different lengths.



curves generated by the 4-point scheme

Geometric subdivision schemes

Linear schemes are applied separately to each component of the Points in $\mathbb{R}^3$,

$$P_i^{k+1} = \sum_j a_{i-2j} P_j^k.$$

Geometric schemes are data dependent. Each refinement step $\mathcal{P}^{k+1} = T\mathcal{P}^k$ depends on the points $\{P_i^k\}$.

Example: 4-point subdivision scheme based on iterated centripetal parametrization (Dyn, Floater and Hormann, preprint).

A good parametrization of a sequence of points $\{P_i\}$ for interpolation by a spline curve is the centripetal parametrization (Floater, 2006):

$$t_0 = 0, \qquad t_i = t_{i-1} + \|P_i - P_{i-1}\|_2^{1/2}.$$

The scheme

- for $\{P_i^k\}$ define the parametrization

$$t_0^k = 0, \quad t_i^k = t_{i-1}^k + \|P_i^k - P_{i-1}^k\|_2^{1/2}$$

- the refinement step:

$$P_{2i}^{k+1} = P_i^k, \quad P_{2i+1}^{k+1} = \pi_{k,i}\left(\frac{t_i^k + t_{i+1}^k}{2}\right)$$

  $\pi_{i,k}(t)$ – a cubic polynomial interpolating the data $\{(t_{i+j}^k, P_{i+j}^k),\, j = -1, 0, 1, 2\}$

Results:

- the scheme is well defined: $P_{2i+1}^{k+1} \neq P_i^k, \quad P_{2i+1}^{k+1} \neq P_{i+1}^k$

- uniform convergence of $\{\mathcal{P}^k\}$ to a continuous curve passing through the points $\{P_i^0\}$

- tightness of the limit curve C relative to $\{\mathcal{P}^0\}$

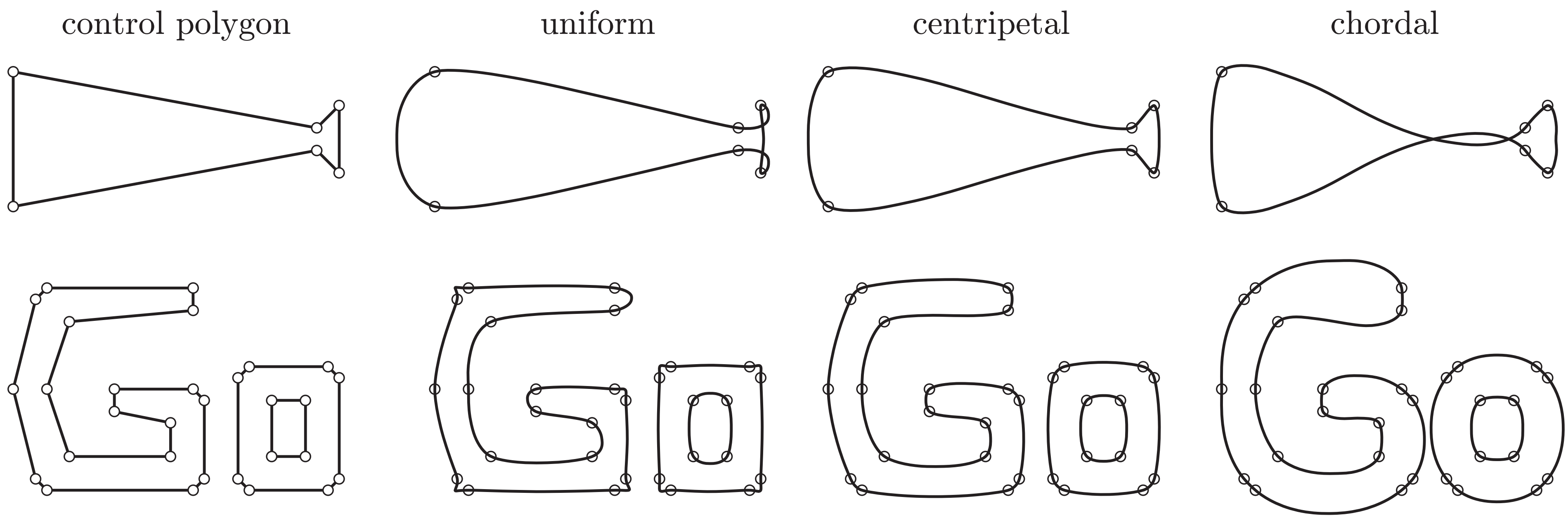
$$\text{haus}\Big(C|_{\{P_i^0, P_{i+1}^0\}}, [P_i^0, P_{i+1}^0]\Big) \leq \frac{5}{7} \|P_{i+1}^0 - P_i^0\|_2$$

  in comparison with the bound

$$\frac{3}{13} \max\Big\{\|P_{\ell+1}^0 - P_\ell^0\|_2 : i-2 \leq \ell \leq i+2\Big\}$$

  for the linear 4-point scheme

Ad-hoc methods of analysis



comparison between 4-point schemes based on different parametrizations

Geometric refinement of curves

Aim: given a finite sequence of curves, refine it repeatedly to generate a surface

The linear approach:

- parametrize each curve (e.g. by arc length): $\{C_i(s)\}_{i=1}^n$
- apply $S_{\boldsymbol{a}}^\infty$ to the n points $\{C_i(s)\}$
- the limit curve is $\sum_{i=1}^n C_i(s)\phi_{\boldsymbol{a}}(t)$
- $\sum_{i=1}^n C_i(s)\phi_a(t)$ is the limit surface

A geometric approach based on a correspondence between curves
(work in progress with Elber and our joint student Itai)

A map $t = t(C, \widetilde{C})$ is a correspondence between the curves $C, \widetilde{C}$,
if it maps C continuously onto $\widetilde{C}$, and if it is one-to-one
$T(C, \widetilde{C})$ is the collection of all correspondences between C and $\widetilde{C}$

The correspondence used is a geometric correspondence:

$$t^*(C, \widetilde{C}) = \arg \min_{\tau \in T(C, \widetilde{C})} \max\{\|\tau(P) - P\|_2 : P \in C\}$$

An extension of the linear Chaikin algorithm to curves

One refinement step of a set of curves $\{C_i\}_{i=1}^n$ (contained in a compact subset of $\mathbb{R}^3$)

- for $i = 1, \ldots, n-1$,

  1. compute the geometric correspondence $t^*(C_i, C_{i+1})$ (computed by dynamical programming)

  2. for each $P \in C_i$, define

  $$Q_i(P) = \tfrac{3}{4}P + \tfrac{1}{4}t^*(C_i, C_{i+1})(P), \quad R_i(P) = \tfrac{1}{4}P + \tfrac{3}{4}t^*(C_i, C_{i+1})(P)$$

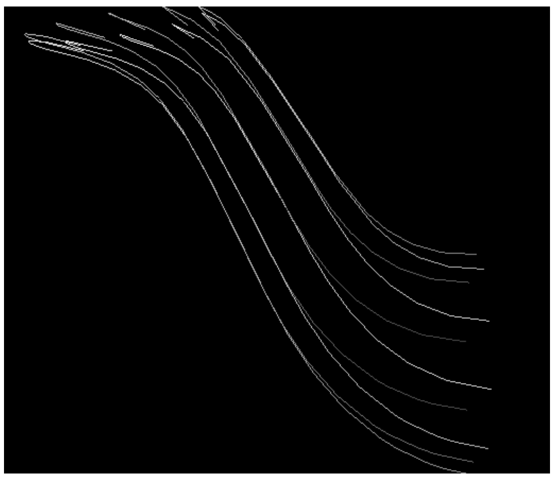
  3. define two refined curves

  $$\overline{C}_{2i} = \{Q_i(P) : P \in C_i\}, \quad \overline{C}_{2i+1} = \{R_i(P) : P \in C_i\}$$

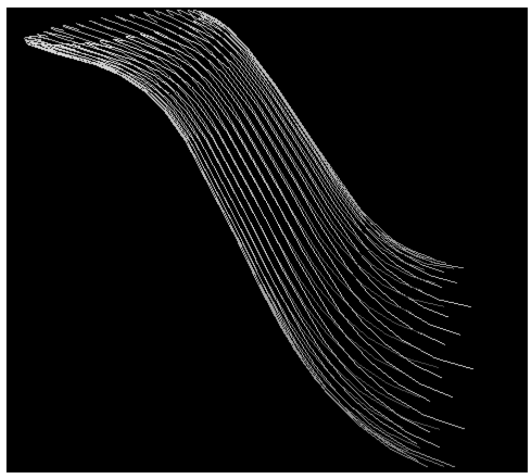
- the refined curves are $\{\overline{C}_i\}_{i=1}^{2(n-1)}$

Under mild conditions on the initial curves, the scheme is well defined and convergent (also with $\frac{1}{4}, \frac{3}{4}$ replaced by $\mu, 1-\mu$ with $0 < \mu < \frac{1}{2}$).

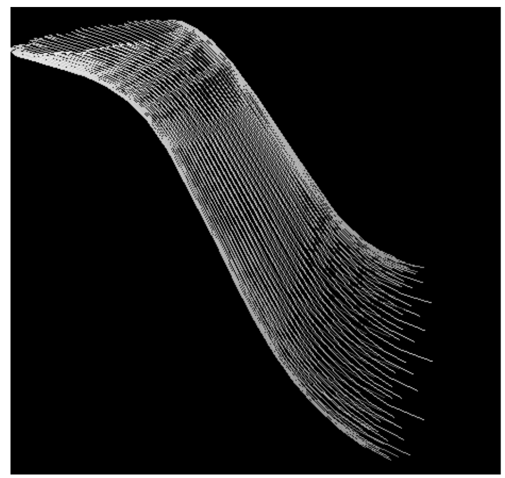
Example:



initial curves



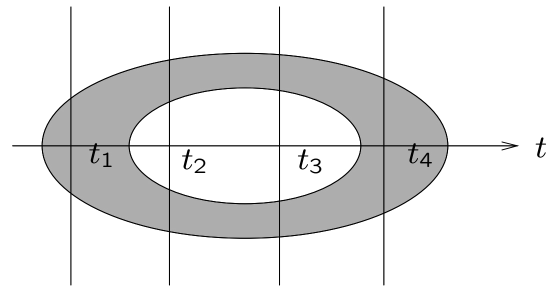
curves after 2 refinement steps



curves after 3 refinement steps

Subdivision schemes for compact sets

Motivation: approximating a 3D object from its parallel cross-sections.



4 parallel cross-sections

approach: refinement of the cross-sections

the cross-sections are 2D sets.

Refinement of compact sets in $\mathbb{R}^n$ based on Minkowski average

- A, B compact sets in $\mathbb{R}^n$

$$\frac{1}{2}A + \frac{1}{2}B = \{\frac{1}{2}(a+b) : a \in A, b \in B\}$$

- extension of B-spline subdivision schemes to compact sets

  The refinement step

$$F^{k+1} = S_m F^k \ , \qquad F^k = \{F_i^k\}$$

  consists of a trivial refinement followed by m repeated averages

$$F_{2i}^{k+1,0} = F_{2i+1}^{k+1,0} = F_i^k,$$

$$F_i^{k+1,\ell+1} = \frac{1}{2} F_i^{k+1,\ell} + \frac{1}{2} F_{i+1}^{k+1,\ell} \ , \ \ell = 0, 1, ..., m-1$$

$$F_i^{k+1} = F_{i-[\frac{m}{2}]}^{k+1,m}$$

Results Dyn N., Farki E. (2000, 2004)

- the subdivision converges to the limit
$$S_m^\infty F^0(t) = \sum_i \langle F_i^0 \rangle B_m(t-i)$$
with $\langle A \rangle$ denoting the convex hull of A.

- the limit has convex sets as images.

- approximation result:
for $F_i^0 = G(ih)$, $(h > 0)$ with G a Lipschitz continuous set-valued function $(haus(G(t), G(t+\Delta)) \le L\Delta)$, with convex images
$$haus(G(t),\ S_m^\infty F^0(t)) \le Ch$$

- no approximation result for G with general compact sets as images!

Refinement of compact sets based on the metric average

$$A \oplus_{\frac{1}{2}} B = \{\frac{1}{2}(a+b) : a \in A, b \in B,\ \|a-b\| = \text{dist }(a,B) \text{ or dist }(b,A)\}$$

Results Dyn, N., Farkhi E. (2001)

- convergence

- approximation result for a Lipschitz continuous G with general compact sets as images.

- $S_m^\infty F^0$ is not known explicitly, it is a limit of a Cauchy sequence in a complete metric space