Pathwise convergence of numerical schemes for stochastic differential equations

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joint work with Arnulf Jentzen and Andreas Neuenkirch.

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Differential equations with noise

Langevin used "noisy" differential equations to model the particle dynamics of Brownian motion in the early 1900s:

$$\frac{dx}{dt} = f(t,x) + g(t,x)\,\eta_t, \qquad \text{noise: } \eta_t$$

• Gaussian white noise: many mathematical problems finally resolved by Itô in the 1940s with the introduction of stochastic calculus

$\Longrightarrow$ stochastic differential equations (SODEs)

$$dX_t = f(t,X_t) + g(t,X_t)\,dW_t, \qquad \text{Wiener process: } W_t$$

• more regular noise $\Longrightarrow$ random ordinary differential equations (RODEs)

Stochastic Differential Equations

Consider an Itô SDE in $\mathbb{R}^d$ for $t \in [0, T]$

$$dX_t = a(X_t)\, dt + \sum_{j=1}^{m} b^j(X_t) dW_t^j,$$

with

- drift and diffusion coefficients a, $b^j : \mathbb{R}^d \to \mathbb{R}^d$ for $j = 1, \ldots, m$
- an m-dimensional Wiener process $W_t = (W_t^1, \ldots, W_t^m)$

This is really an Itô stochastic integral equation

$$X_t = X_0 + \int_0^t a(X_s)\, ds + \sum_{j=1}^{m} \int_0^t b^j(X_s) dW_s^j.$$

Numerical approximation of solutions of SDE

Consider a partition $0 = t_0 < t_1 < \ldots < t_{N_T} = T$ of $[0, T]$ with step sizes $\Delta_n := t_{n+1} - t_n > 0$ and maximum step size $\Delta := \max_n \Delta_n$.

Let $Y_n^{(\Delta)}$ be an approximation of X_{t_n} for a solution X_t of an SDE.

In the literature one mainly considers <u>average error criteria</u>
Weak approximation of order β

$$\left| \mathbb{E}\phi(X_T) - \mathbb{E}\phi(Y_{N_T}^{(\Delta)}) \right| \leq K_{\phi,T}\, \Delta^{\beta}$$

for smooth test functions $\phi : \mathbb{R}^d \to \mathbb{R}$.
Strong approximation of order γ (usually just $p = 1$ or 2)

$$\left(\mathbb{E} \sup_{i=0,\ldots,N_T} \left| X_n - Y_n^{(\Delta)} \right|^p \right)^{1/p} \leq K_{p,T}\, \Delta^{\gamma}$$

The chain rule w.r.t. a solution $x(t)$ of an ODE $\dfrac{dx}{dt} = f(t,x)$ is

$$U(t,x(t)) = U(0,x(0)) + \int_0^t LU(s,x(s))\,ds, \qquad LU := \frac{\partial U}{\partial t} + f\frac{\partial U}{\partial x}$$

Apply this to the integrand of the ODE in integral form

$$\begin{aligned} x(t) &= x(0) + \int_0^t f(s,x(s))\,ds && \Leftarrow U(t,x) = x \\ &= x(0) + \int_0^t \left[f(0,x(0)) + \int_0^s Lf(r,x(r))\,dr \right] ds && \Leftarrow U(t,x) = f(t,x) \\ &= x(0) + f(0,x(0)) \int_0^t ds + \int_0^t \int_0^s Lf(r,x(r))\,dr\,ds \end{aligned}$$

Itô-Taylor Schemes

- Differential operators:

$$L^0 = \sum_{k=1}^{d} a^k \frac{\partial}{\partial x^k} + \frac{1}{2} \sum_{k,l=1}^{d} \sum_{j=1}^{m} b^{k,j} b^{l,j} \frac{\partial^2}{\partial x^k \partial x^l}, \qquad L^j = \sum_{k=1}^{d} b^{k,j} \frac{\partial}{\partial x^k}$$

for $j = 1, \ldots, m$,

where a^k, $b^{k,j}$ are the k-th components of a and b^j

- Set of all multi-indices

$$\mathcal{M}_m = \left\{ \alpha = (j_1, \ldots, j_l) \in \{0, 1, 2, \ldots, m\}^l : l \in \mathbb{N} \right\} \bigcup \{\emptyset\}$$

where

$l(\alpha)$: length of α, $\qquad n(\alpha)$: number of zero entries of α

$\emptyset$: multi-index of length 0

- Iterated integrals and coefficient functions:

$$I_\alpha(s,t) = \int_s^t \cdots \int_s^{\tau_2} dW_{\tau_1}^{j_1} \ldots dW_{\tau_l}^{j_l}$$

$$f_\alpha(x) = L^{j_1} \cdots L^{j_{l-1}} b^{j_l}(x)$$

  with $\alpha = (j_1, \ldots, j_l)$ with the notation $dW_t^0 = dt$, $b^0 = a$

<u>Examples</u>

$$I_{(0)}(t_n, t_{n+1}) = \int_{t_n}^{t_{n+1}} dW_s^0 = \Delta_n, \qquad I_{(1)}(t_n, t_{n+1}) = \int_{t_n}^{t_{n+1}} dW_s^1 = \Delta W_n^1,$$

and

$$I_{(1,1)}(t_n, t_{n+1}) = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW_\tau^1 \, dW_s^1 = \frac{1}{2}\left[(\Delta W_n^1)^2 - \Delta_n\right]$$

Itô-Taylor scheme of strong order $\gamma = \frac{1}{2},\ 1,\ \frac{3}{2},\ 2,\ \ldots$

$$Y_{n+1}^{\gamma} = Y_n^{\gamma} + \sum_{\alpha \in \mathcal{A}_\gamma \setminus \{\emptyset\}} f_\alpha(Y_n^{\gamma}) \cdot I_\alpha(t_n, t_{n+1})$$

for $n = 0, 1, 2, \ldots,$, where $Y_0^{\gamma} = X_0$ and

$$\mathcal{A}_\gamma = \{\alpha \in \mathcal{M}_m : l(\alpha) + n(\alpha) \leq 2\gamma \text{ or } l(\alpha) = n(\alpha) = \gamma + 1/2\}$$

Itô-Taylor scheme of weak order $\beta = 1,\ 2,\ 3, \ldots$

$$Y_{n+1}^{\beta} = Y_n^{\beta} + \sum_{\alpha \in \mathcal{A}_\beta \setminus \{\emptyset\}} f_\alpha(Y_n^{\beta}) \cdot I_\alpha(t_n, t_{n+1})$$

for $n = 0, 1, 2, \ldots,$, where $Y_0^{\gamma} = X_0$ and $\mathcal{A}_\beta = \{\alpha \in \mathcal{M}_m : l(\alpha) \leq \beta\}$

Examples for a scalar SDE

$$dX_t = a(X_t)\,dt + b(X_t)\,dW_t$$

Euler-Maruyama scheme strong order $\gamma = \frac{1}{2}$, weak order $\beta = 1$

$$\boxed{Y_{n+1} = Y_n + a(Y_n)\,\Delta_n + b(Y_n)\,\Delta W_n}$$

Milstein scheme strong order $\gamma = 1$, weak order $\beta = 1$

$$\boxed{Y_{n+1} = Y_n + a(Y_n)\,\Delta_n + b(Y_n)\,\Delta W_n + \frac{1}{2} b'(Y_n) b(Y_n)\left[(\Delta W_n)^2 - \Delta_n\right]}$$

Proofs in the literature

e.g. in the monographs

P.E. Kloeden and E. Platen, *The Numerical Solution of Stochastic Differential Equations*, Springer, 1992.

G.N. Milstein, *Numerical Integration of Stochastic Differential Equations*, Kluwer, 1995

assume that the coefficient functions f_α in the Taylor scheme are uniformly bounded on $\mathbb{R}^d$

i.e., the partial derivatives of appropriately high order of the SDE coefficient functions a, b^1, $\ldots$, b^m are uniformly bounded on $\mathbb{R}^d$

This assumption is **not satisfied** in many important applications.

Duffing-van der Pol oscillator with multiplicative noise

$$dX_t^1 = X_t^2 \, dt$$
$$dX_t^2 = \left[-X_t^1 + \beta X_t^2 - (X_t^1)^3 - (X_t^1)^2 X_t^2\right] \, dt + \sigma X_t^2 \, dW_t$$

Fisher-Wright type diffusions with $X_t \in [0,1]$

$$dX_t = \left[\kappa_1 (1 - X_t) - \kappa_2 X_t\right] \, dt + \sqrt{X_t(1 - X_t)} \, dW_t$$

Cox-Ingersoll-Ross models in finance with $V_t \geq 0$

$$dV_t = \kappa \left(\vartheta - V_t\right) \, dt + \mu \sqrt{V_t} \, dW_t$$

Restrictions on the dynamics

e.g. dissipativity, ergodicity

J.C. Mattingly, A.M. Stuart and D.J. Higham,
Ergodicity for SDEs and approximations: locally Lipschitz vector fields and degenerate noise, *Stochastic Processes Applns.* **101** (2002), 185–232.

D.J. Higham, X. Mao and A.M. Stuart,
Strong convergence of Euler-type methods for nonlinear stochastic differential equations, *SIAM J. Num Anal.*, **40** (2002), 1041-1063.

G.N. Milstein and M.V. Tretjakov,
Numerical integration of stochastic differential equations with nonglobally Lipschitz coefficients.*SIAM J. Numer. Anal.* **43** (2005), 1139-1154.

$\Longrightarrow$ order estimates without bounded derivatives of coefficients

Difficulties still with square roots in coefficients, positivity of solutions $\cdots$

Pathwise convergence

$$\sup_{n=0,\ldots,N_T} \left| X_{t_n}(\omega) - Y_n^{(\Delta)}(\omega) \right| \longrightarrow 0 \text{ as } \Delta \to 0, \qquad \omega \in \Omega$$

Why?

- Numerical calculation of the approximation $Y_n^{(\Delta)}$ is carried out path by path
- The theory of random dynamical systems is of pathwise nature

  e.g. random attractors, stochastic bifurcations
- Solutions of the SDE may be non-integrable, i.e. $\mathbb{E}|X_t| = \infty$ for some $t \geq 0$
- ......

BUT recall that Ito calculus is a mean-square, i.e. L^2, calculus !

Known results for pathwise approximation

- Milstein scheme for SDE with a scalar Wiener process (Talay, 1983):

$$\sup_{n=0,\ldots,N_T} \left| X_{t_n}(\omega) - Y_n^{(\Delta)}(\omega) \right| \leq K_{\epsilon,T}^{(M)}(\omega)\, \Delta^{\frac{1}{2}-\epsilon},$$

  for all $\epsilon > 0$ and almost all $\omega \in \Omega$

- Euler scheme for a general SDE under weak assumptions (Gyöngy 1998, Fleury, 2005):

$$\sup_{n=0,\ldots,N_T} \left| X_{t_n}(\omega) - Y_n^{(\Delta)}(\omega) \right| \leq K_{\epsilon,T}^{(E)}(\omega)\, \Delta^{\frac{1}{2}-\epsilon},$$

  for all $\epsilon > 0$ and almost all $\omega \in \Omega$

Wiener process paths are Hölder continuous with exponent $\frac{1}{2} - \epsilon$.

Is the convergence order $\frac{1}{2} - \epsilon$ "sharp" for pathwise approximation ?

An arbitrary pathwise order is possible

P.E. Kloeden and A. Neuenkirch, The pathwise convergence of approximation schemes for stochastic differential equations, *LMS J. Comp. Math.* **10** (2007), 235-253.

Theorem 1 *Under classical assumptions an Itô-Taylor scheme of strong order* $\gamma > 0$ *converges pathwise with order* $\gamma - \epsilon$ *for all* $\epsilon > 0$, *i.e.*

$$\sup_{i=0,\ldots,N_T} \left| X_{t_n}(\omega) - Y_n^{(\Delta)}(\omega) \right| \leq K_{\epsilon,T}^{\gamma}(\omega) \cdot \Delta^{\gamma-\epsilon}$$

for almost all $\omega \in \Omega$.

$\Rightarrow$ The Milstein scheme has pathwise order $1 - \epsilon$

The proof is based on

1) **Burkholder-Davis-Gundy inequality**

$$\mathbb{E} \sup_{s \in [0,t]} \left| \int_0^s X_\tau \, dW_\tau \right|^p \leq C_p \cdot \mathbb{E} \left| \int_0^t X_\tau^2 \, d\tau \right|^{p/2}$$

2) a Borel-Cantelli argument in the following Lemma

Lemma 1 *Let* $\gamma > 0$ *and* $c_p \geq 0$ *for* $p \geq 1$. *If* $\{Z_n\}_{n\in\mathbb{N}}$ *is a sequence of random variables with*

$$(\mathbb{E}|Z_n|^p)^{1/p} \leq c_p \cdot n^{-\gamma}$$

for all $p \geq 1$ *and* $n \in \mathbb{N}$, *then for each* $\epsilon > 0$ *there exists a finite non-negative random* K_ϵ *such that*

$$|Z_n(\omega)| \leq K_\epsilon(\omega) \cdot n^{-\gamma+\varepsilon} \qquad a.s.$$

for all $n \in \mathbb{N}$.

SDE without uniformly bounded coefficients

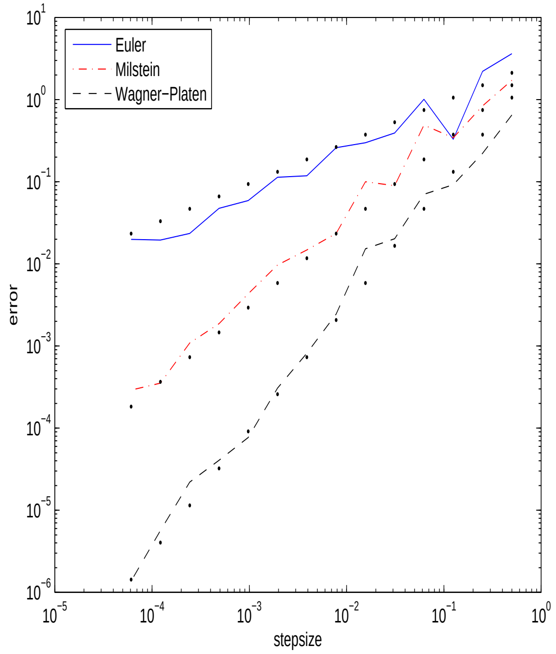
A. Jentzen, P.E. Kloeden and A. Neuenkirch, Convergence of numerical approximations of SDE under nonstandard assumptions, *Numerische Mathematik* (to appear)

Theorem 1 remains true if the SDE coefficients a, b^1, …,$b^m \in C^{2\gamma+1}(\mathbb{R}^d;\mathbb{R}^d)$, i.e., without uniform bounded derivatives, using a localization argument.

e.g. Theorem 1 applies to the Duffing-van der Pol oscillator with multiplicative noise.

Numerical Example I Duffing-van der Pol oscillator with multiplicative noise

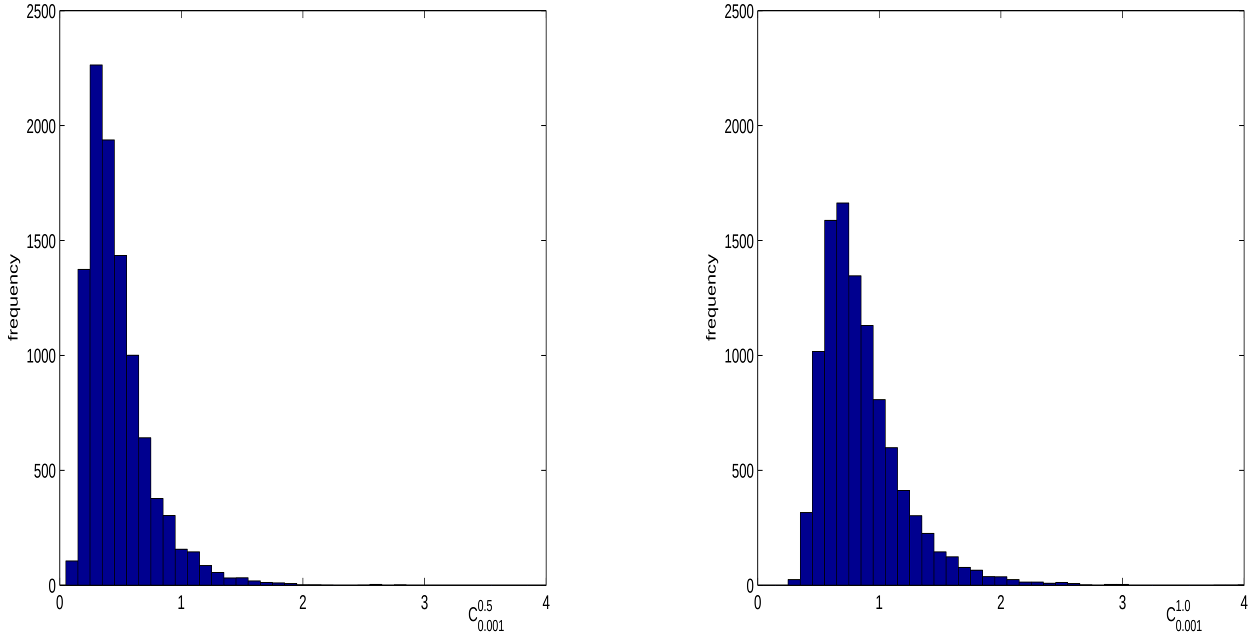
$$\beta = 3, \sigma = 2, X_1(0) = X_2(0) = 1, T = 1$$



pathwise maximum error vs. stepsize for two sample paths

Numerical Example II

$$dX_t = -(1+X_t)(1-X_t^2)\,dt + (1-X_t^2)\,dW_t, \qquad t \in [0,1], \quad X(0) = 0$$



empirical distribution of $K_{0.001}^{0.5}$ and $K_{0.001}^{1.0}$ (sample size: $N = 10^4$)

SDE on restricted regions

e.g. Fisher-Wright and Cox-Ingersoll-Ross SDEs with square-root coefficients

- the numerical iterations may leave the restricted region, the algorithm may crash

Consider a domain $D \subseteq \mathbb{R}^d$ and suppose that the SDE coefficients a, b^1, ..., b^m are r-times continuously differentiable on D

Define $\quad E := \{x \in \mathbb{R}^d \,:\, x \notin \overline{D}\}$.

Choose auxiliary functions f, g^1, ..., $g^m \in C^s(E;\mathbb{R}^d)$ for $s \in \mathbb{N}$ and define

$$\widetilde{a}(x) = a(x) \cdot \mathbf{1}_D(x) + f(x) \cdot \mathbf{1}_E(x), \qquad x \in \mathbb{R}^d,$$

$$\widetilde{b}^j(x) = b^j(x) \cdot \mathbf{1}_D(x) + g^j(x) \cdot \mathbf{1}_E(x), \qquad x \in \mathbb{R}^d, \quad j = 1, \ldots, m.$$

For $x \in \partial D$ define

$$\widetilde{a}(x) = \lim_{y \to x;\, y \in D} \widetilde{a}(y), \qquad \widetilde{b}^j(x) = \lim_{y \to x; y \in D} \widetilde{b}^j(y), \quad j = 1, \ldots, m.$$

if these limits exist. Otherwise, define $\widetilde{a}(x) = 0$, respectively $\widetilde{b}^j(x) = 0$ for $x \in \partial D$.

Finally, define the "modified" derivative of a function $h : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\partial_{x^l} h(x) = \frac{\partial}{\partial x^l} h(x), \quad x \in D \cup E, \qquad l = 1, \ldots, d,$$

and for $x \in \partial D$ define

$$\partial_{x^l} h(x) = \lim_{y \to x;\, y \in D} \partial_{x^l} h(x).$$

if this limit exists — otherwise set $\partial_{x^l} h(x) = 0$ for $x \in \partial D$.

A **modified Itô-Taylor** scheme is the corresponding Itô-Taylor scheme for the SDE with modified coefficients

$$dX_t = \widetilde{a}(X_t)\, dt + \sum_{j=1}^{m} \widetilde{b}^j(X_t) dW_t^j,$$

with differential operators $\widetilde{L}^0$, $\widetilde{L}^1$, ..., $\widetilde{L}^m$ using the above modified derivatives.

Note that this method is well defined as long as the coefficients of the equation are $(2\gamma - 1)$-times differentiable on D and the auxiliary functions are $(2\gamma - 1)$-times differentiable on E.

• Theorem 1 adapts to modified Itô-Taylor schemes for SDEs on domains in $\mathbb{R}^d$

A. Jentzen, P.E. Kloeden and A. Neuenkirch, Convergence of numerical approximations of SDE under nonstandard assumptions, *Numerische Mathematik* (to appear)

Theorem 2 *Assume that*

$$a, b^1, \ldots, b^m \in C^{2\gamma+1}(D; \mathbb{R}^d) \bigcap C^{2\gamma-1}(E; \mathbb{R}^d),$$

and let $Y_n^{mod,\gamma}$ *be the modified Itô-Taylor scheme for* $\gamma = \frac{1}{2}, 1, \frac{3}{2}, \ldots$.

Then for all $\epsilon > 0$ *there exists a finite, non-negative random variable* $K_{\gamma,\epsilon}^{f,g}$ *such that*

$$\sup_{i=0,\ldots,n} \left| X_{t_n}(\omega) - Y_n^{mod,\gamma}(\omega) \right| \leq K_{\gamma,\epsilon}^{f,g}(\omega) \cdot \Delta^{\gamma-\epsilon}$$

for almost all $\omega \in \Omega$ *and all* $n = 1, \ldots, N_T$.

Remark: The auxiliary functions can be chosen to be zero.

Wright-Fisher type diffusions

$$dX_t = [\kappa_1(1 - X_t) - \kappa_2 X_t]\ dt + \sqrt{X_t(1 - X_t)}\, dW_t$$

If $\min\{\kappa_1, \kappa_2\} \geq \frac{1}{2}$ and $X_0 \in (0, 1)$, then

$$\mathbf{P}(X(t) \in (0, 1) \text{ for all } t \geq 0) = 1$$

- However, standard Itô-Taylor schemes may leave $[0, 1]$, so we use a modified scheme:

  (1) choose new coefficients outside $[0, 1]$, e.g.

$$\text{auxiliary drift:} \quad f(x) = \kappa_1(1 - x) - \kappa_2 x, \qquad x \notin [0, 1]$$

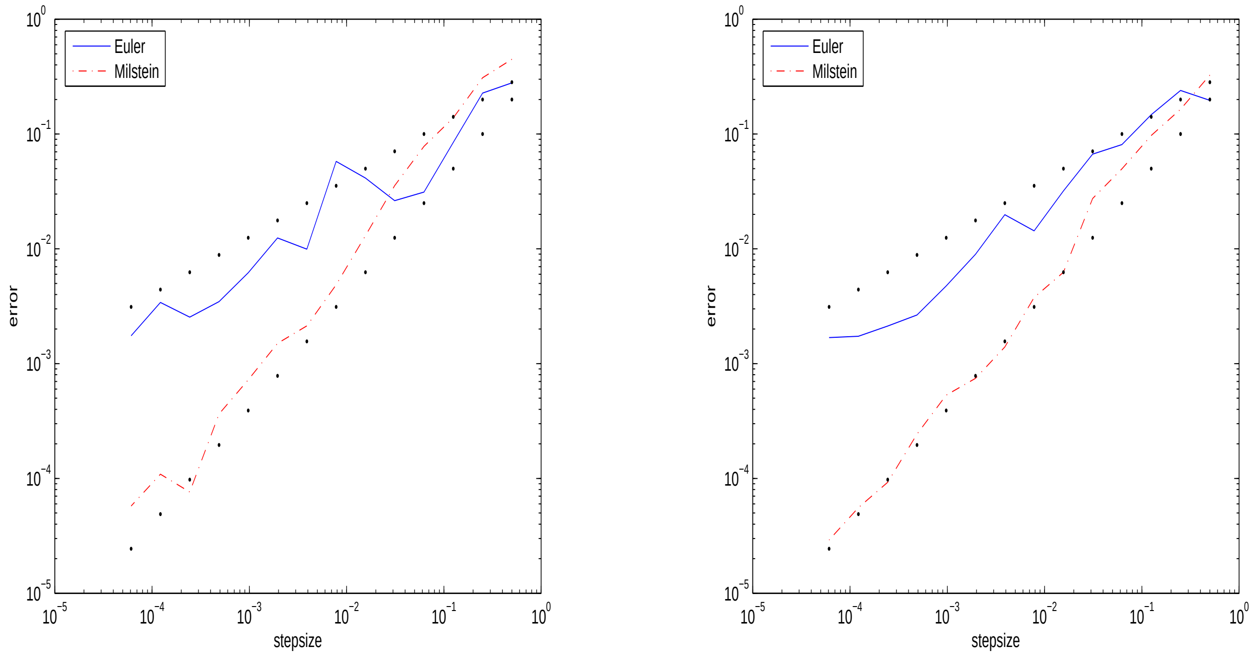
$$\text{auxiliary diffusion:} \quad g(x) = 0, \qquad x \notin [0, 1]$$

  (2) define the coefficients of the Itô-Taylor scheme "appropriately" for $x \in \{0, 1\}$

- the modified Itô-Taylor scheme of order γ converges pathwise with order $\gamma - \epsilon$

Numerical Example III

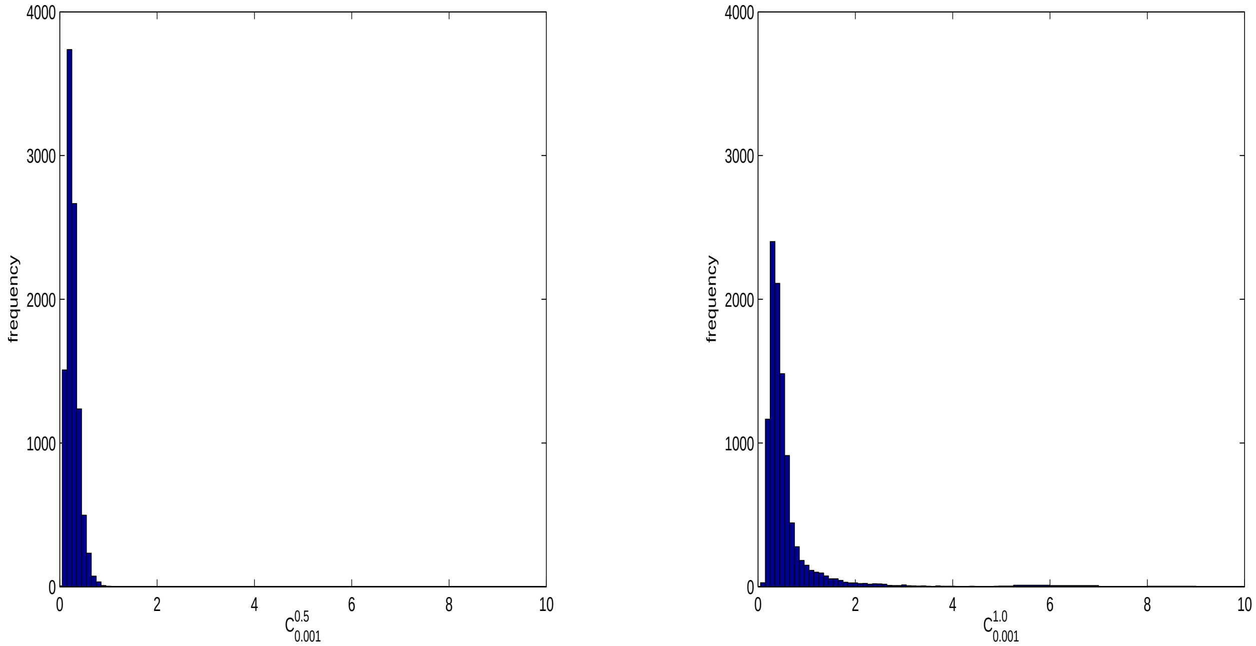
$$\kappa_1 = 0.5, \kappa_2 = 1, X_0 = 0.1, T = 1$$



pathwise maximum error vs. stepsize for two sample paths

Numerical Example III (cont'd)

$$\kappa_1 = 0.5, \kappa_2 = 1, X_0 = 0.1, T = 1$$



empirical distribution of $K^{0.5}_{0.001}$ and $K^{1.0}_{0.001}$ (sample size: $N = 10^4$)

Random ordinary differential equations (RODEs)

Let ζ_t be an m-dimensional stochastic process

Let $f : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^d$ be smooth

A **random ordinary differential equation** on $\mathbb{R}^d$

$$\boxed{\frac{dx}{dt} = f(\zeta_t, x)}$$

is pathwise an ordinary differential equation (ODE) on $\mathbb{R}^d$

$$\frac{dx}{dt} = F_\omega(t, x) := f(\zeta_t(\omega), x), \qquad \omega \in \Omega.$$

The mapping $t \mapsto F_\omega(t, x)$ is usually only continuous but not differentiable — no matter how smooth the function f – since the paths of the stochastic process ζ are often at most Hölder continuous

Example: $$\frac{dx}{dt} = -x + \sin W_t(\omega)$$

Why are RODEs interesting?

- RODEs occur in many applications

- RODEs may be more realistic than SDE with their idealized noise

i.e. noise in physical systems usual has a wide band spectrum, i.e. a Δ-correlated stationary Gaussian process $\zeta_t^{(\Delta)}$ with a white noise limit as $\Delta \to 0$

RODE $\quad \dfrac{dx}{dt} = a(x)+b(x)\,\zeta_t^{(\Delta)} \quad \Longrightarrow \quad$ Stratonovich SDE $\quad dX_t = a(X_t)+b(X_t)\circ dW_t$

e.g Wong & Zakai (1965), Godin & Molchanov (2007)

- RODEs with a Wiener process can be rewritten as stochastic differential equations

$$\frac{dx}{dt} = -x + \sin W_t(\omega) \qquad \Leftrightarrow \qquad d\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} -X_t + \sin Y_t \\ 0 \end{pmatrix} dt + \begin{pmatrix} 0 \\ 1 \end{pmatrix} dW_t$$

- stochastic differential equations can be rewritten as RODEs

$$dX_t = f(X_t)\,dt + dW_t \qquad \Leftrightarrow \qquad \frac{dz}{dt} = f(z + O_t) + O_t$$

where O_t is the stochastic stationary Ornstein-Uhlenbeck process satisfying the linear SDE

$$dO_t = -O_t\,dt + dW_t \qquad \text{and} \quad z(t) = X_t - O_t$$

To see this, note that by continuity and the fundamental theorem of calculus

$$\begin{aligned} z(t) = X_t - O_t &= X_0 - O_0 + \int_0^t \left[f(X_s) + O_s \right] ds \\ &= z(0) + \int_0^t \left[f(z(s) + O_s) + O_s \right] ds \end{aligned}$$

is pathwise differentiable

Doss, Sussmann (1970s), Imkeller, Lederer, Schmalfuß (2000s)

We can use <u>deterministic calculus pathwise</u> for RODEs

$$dX_t = f(X_t)\,dt + dW_t \qquad \Leftrightarrow \qquad \frac{dz}{dt} = f(z + O_t) + O_t$$

- Suppose that f satisfies a one-sided dissipative Lipschitz condition $(L > 0)$

$$\langle x - y, f(x) - f(y) \rangle \leq -L|x - y|^2, \qquad \forall x, y$$

Then for any two solutions $z_1(t)$ and $z_2(t)$ of the RODE

$$\begin{aligned}
\frac{d}{dt}|z_1(t) - z_2(t)|^2 &= 2\left\langle z_1(t) - z_2(t), \frac{dz_1}{dt} - \frac{dz_2}{dt} \right\rangle \\
&= 2\left\langle z_1(t) - z_2(t), f(z_1(t) + O_t) - f(z_2(t) + O_t) \right\rangle \\
&\leq -2L\,|z_1(t) - z_2(t)|^2
\end{aligned}$$

$$\Longrightarrow \qquad |z_1(t) - z_2(t)|^2 \leq e^{-2Lt}|z_1(0) - z_2(0)|^2 \to 0 \quad \text{as } t \to \infty \qquad \text{(pathwise)}$$

Hence there exists a pathwise asymptotically stable stochastic stationary solution

Numerical schemes for RODEs

- we can solve RODEs pathwise as ODEs with Runge-Kutta schemes

- BUT these not attain their traditional order since the vector field $F_\omega(t,x)$ is not smooth enough in t

The Euler scheme attains order $\theta - \epsilon$ when applied to the RODE

$$\frac{dx}{dt} = -x + \zeta_t(\omega) \qquad \Longrightarrow \qquad \boxed{Y_{n+1}(\omega) = (1-\Delta_n)\, Y_n(\omega) + \zeta_{t_n}(\omega)\,\Delta_n}$$

However, one can do better by using the pathwise averaged Euler scheme

$$\boxed{Y_{n+1}(\omega) = (1-\Delta_n)\, Y_n(\omega) + \int_{t_n}^{t_{n+1}} \zeta_t(\omega)\, dt}$$

which was proposed in

L. Grüne and P. E. Kloeden, Pathwise approximation of random ordinary differential equations, *BIT*, **12** (2001), 6-81

The averaged Euler scheme attains pathwise order $1-\epsilon$ provided the integral is approximated with Riemann sums

$$\boxed{\int_{t_n}^{t_{n+1}} \zeta_t(\omega)\,dt \approx \sum_{j=1}^{J_{\Delta_n}} \zeta_{t_n+j\delta}(\omega)\,\delta}$$

with step size $\delta^\theta \approx \Delta_n$ and $\delta \cdot J_{\Delta_n} = \Delta_n$

For a general RODE this suggests averaging the whole vectorfield.

Less expensive computationally is to use the average of the noise

$$I_n(\omega) := \frac{1}{\Delta_n}\int_{t_n}^{t_{n+1}} \zeta_s(\omega)\,ds.$$

in the vectorfield, e.g., as in the the explicit *averaged Euler scheme*

$$\boxed{Y_{n+1} = Y_n + f\left(I_n, Y_n\right)\Delta_n.} \qquad \text{order } \min\{1, 2\theta\}$$

- B-stable schemes include the *implicit averaged Euler scheme*

$$\boxed{Y_{n+1} = Y_n + f\left(I_n, Y_{n+1}\right)\,\Delta_n} \qquad \text{order } \min\{1, 2\theta\}$$

and the *implicit averaged midpoint scheme*

$$\boxed{Y_{n+1} = Y_n + f\left(I_n, \frac{1}{2}\left(Y_n + Y_{n+1}\right)\right)\,\Delta_n.} \qquad \text{order } 2\theta$$

A. Jentzen and P.E. Kloeden, Stable time integration of spatially discretized random and stochastic PDEs, *IMA J. Numer. Anal.* (submitted).

- A systematic derivation of higher order numerical schemes for RODEs involving multiple integrals of the noise are given in

A. Jentzen and P.E. Kloeden, Pathwise convergent higher order numerical schemes for random ordinary differential equations, *Proc. Roy. Soc. London A* **463** (2007), 2929–2944.

A. Jentzen and P.E. Kloeden, Pathwise Taylor schemes for random ordinary differential equations, *BIT* (submitted)

A. Jentzen, A. Neuenkirch and A. Rößler, Runge-Kutta type schemes for random ordinary differential equations, *LMS J. Comp. Math.* (submitted)

Stochastic and random partial differential equations

Consider a bounded spatial domain $\mathcal{D}$ in $\mathbb{R}^d$ and a Dirichlet boundary condition

- RPDE $$\boxed{\frac{\partial u}{\partial t} = \Delta u + f(\zeta_t, u)}$$ with noise ζ_t.

- SPDE $$\boxed{dU = \Delta U + f(U) + g(U)\, dW}$$

where W is an infinite dimensional Wiener process

- in both time and spatial variables (Brownian sheet)

or

- of the form $W(t,x) = \sum_{j=1}^{\infty} c_j W_t^j \phi_j(x)$ with mutually independent scalar Wiener

processes W_t^j and the ϕ_j a basis system in e.g. $L^2(\mathcal{D})$ form by the Laplace operator on $\mathcal{D}$ with Dirichlet boundary condition.

- In simple cases, e.g. additive noise, we can transform an SPDE to an RPDE

(Doss–Sussmann)

Numerical methods

All of the difficulties encountered for deterministic PDE plus more due to the noise
e.g. nature, approximation and simulation of the noise

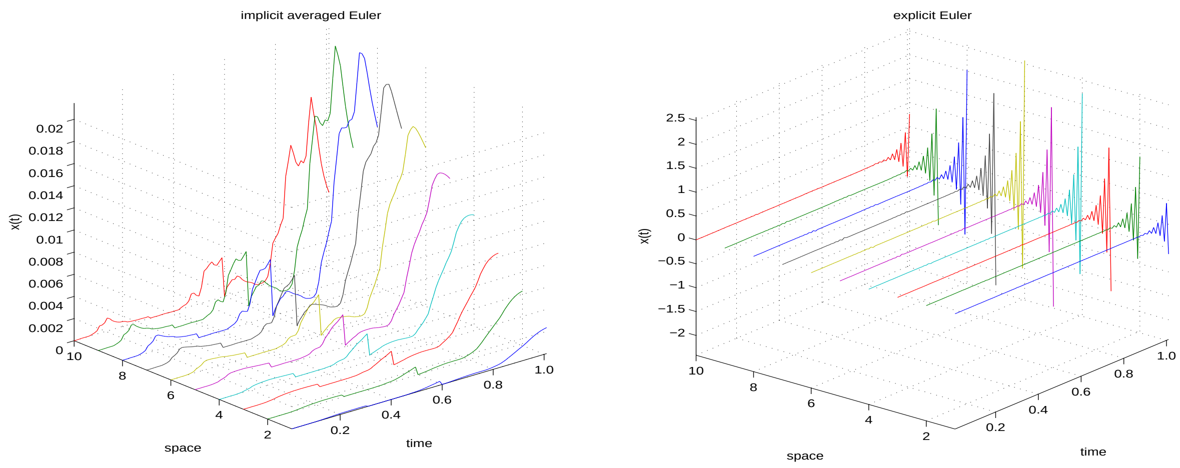
low order due to the roughness of the noise if only simple increments are used

- A.M. Davie and J.G. Gaines, Convergence of numerical schemes for the solution of parabolic stochastic partial differential equations, *Math. Computat.* **70** (2000), no. 233, 123–134.
- T. Müller-Gronbach and K. Ritter, Lower bounds and nonuniform time discretization for approximation of stochastic heat equations. *Found. Computat. Math.*, **7** (2007), no. 2, 135–181.

A higher order is possible if multiple integrals of the noise are used

- W. Grecksch and P.E. Kloeden, Time–discretized Galerkin approximations of parabolic stochastic PDEs, *Bulletin Austral. Math. Soc.* **54** (1996), 79–84.
- E. Hausenblas, Numerical analysis of semilinear stochastic evolution equations in Banach spaces. *J. Computat. Appl. Math.* **147** (2002), 485–516.
- E. Hausenblas, Approximation of semilinear stochastic evolution equations. *Potential Anal.* **18** (2003), 141–186.

Also: Gyöngy, Krylov, Millet, Nualart, Rosovskii, Sanz-Sole, etc



Method of lines for the random PDE with a scalar noise Wiener process

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u - (u + W_t)^3$$

on the interval $0 \leq x \leq 1$ with Dirichlet boundary condition