Distributing many points in the two-dimensional sphere

Carlos Beltrán

Universidad de Cantabria, Santander, Spain

Foundations of Computational Mathematics, Budapest 2011

A good way to distribute points in a sphere?

An easier question: a bad way to distribute points in the sphere?

[Kuijlaars–Saff] [Bendito et al.] [The web]

Points satisfying some "extremal" property

Look for N points $x_1, \ldots, x_N$ in the sphere $\mathbb{S}$ such that...

- the volume of the convex envelope of $x_1, \ldots, x_N$ is maximized.
- the separation distance

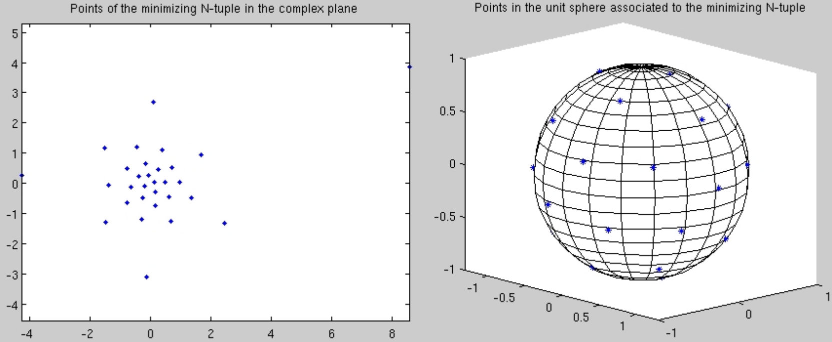
$$d_{\mathrm{sep}}(x_1, \ldots, x_N) = \min_{i<j} \|x_i - x_j\| \quad \text{is maximized,}$$

  Tammes Problem or "hard spheres problem".
- the function

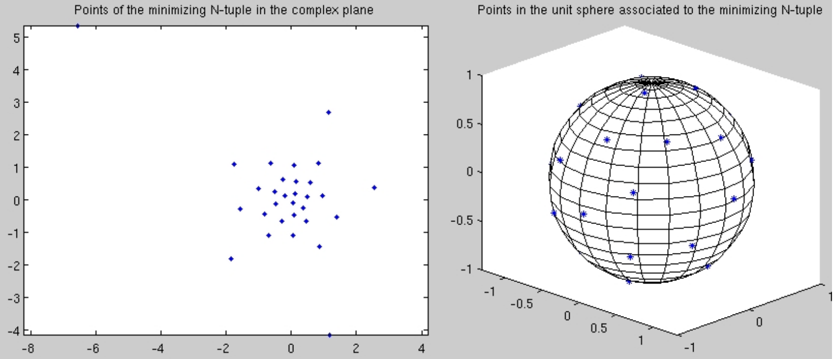
$$E_u(x_1, \ldots, x_N) = \sum_{i<j} \|x_i - x_j\| \quad \text{is maximized,}$$

  a classical open problem in discrete geometry.

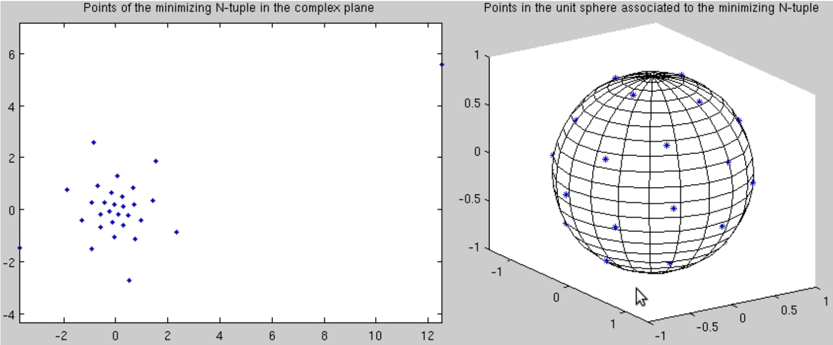
Maximize volume of convex envelope ($N = 30$)



Maximize the separation distance ($N = 30$)



Maximize sum of $\|x_i - x_j\|$ $(N = 30)$



Ipomoea purpurea pollen, flu virus, dessert

March,october, february

Points satisfying some “extremal” property

Look for N points $x_1, \ldots, x_N$ in the sphere $\mathbb{S}$ such that...

- the s–energy (some fixed $s \geq 0$)

$$\sum_{i<j} \|x_i - x_j\|^{-s} \quad \text{is minimized,}$$

a classical problem in physics for $s = 1$, that is Thomson's problem.

Points satisfying some "extremal" property

Look for N points $x_1, \ldots, x_N$ in the sphere $\mathbb{S}$ such that...

- the s–energy (some fixed $s \geq 0$)

$$\sum_{i<j} \|x_i - x_j\|^{-s} \quad \text{is minimized,}$$

  a classical problem in physics for $s = 1$, that is Thomson's problem.

- the logarithmic energy (aka logarithmic potential)

$$\mathcal{E}(x_1, \ldots, x_N) = \log \prod_{i<j} \|x_i - x_j\|^{-1} = -\sum_{i<j} \log \|x_i - x_j\|$$

  is minimized.

Points satisfying some "extremal" property

Look for N points $x_1, \ldots, x_N$ in the sphere $\mathbb{S}$ such that...

- the s–energy (some fixed $s \geq 0$)

$$\sum_{i<j} \|x_i - x_j\|^{-s} \quad \text{is minimized,}$$

  a classical problem in physics for $s = 1$, that is Thomson's problem.

- the logarithmic energy (aka logarithmic potential)

$$\mathcal{E}(x_1, \ldots, x_N) = \log \prod_{i<j} \|x_i - x_j\|^{-1} = -\sum_{i<j} \log \|x_i - x_j\|$$

  is minimized.
  A set of N points in $\mathbb{S}$ minimizing $\mathcal{E}$ (i.e. maximizing the product of their mutual distances) is called a set of **Elliptic Fekete Points**.

Elliptic Fekete points

Early works by Fekete, Szegö, Whyte, Hille, Tsuji, etc.

For $X = (x_1, \ldots, x_N) \in \mathbb{S}^N$ where $x_i \in \mathbb{S}$, $1 \leq i \leq N$, ellipic Fekete points minimize the logaritmic energy

$$\mathcal{E}(X) = \mathcal{E}(x_1, \ldots, x_N) = \log \prod_{i<j} \|x_i - x_j\|^{-1} = -\sum_{i<j} \log \|x_i - x_j\|$$

Let $\Sigma \subseteq \mathbb{S}^N$ be the set of points such that $\mathcal{E} = +\infty$. Let

$$m_N = \min\{\mathcal{E}(X) : X \in \mathbb{S}^N\}.$$

Smale's 7th problem: can one find $X \in \mathbb{S}^N$ such that

$$\mathcal{E}(X) - m_N \leq c \log N?$$

Elliptic Fekete points

Early works by Fekete, Szegö, Whyte, Hille, Tsuji, etc.

For $X = (x_1, \ldots, x_N) \in \mathbb{S}^N$ where $x_i \in \mathbb{S}$, $1 \leq i \leq N$, ellipic Fekete points minimize the logaritmic energy

$$\mathcal{E}(X) = \mathcal{E}(x_1, \ldots, x_N) = \log \prod_{i<j} \|x_i - x_j\|^{-1} = -\sum_{i<j} \log \|x_i - x_j\|$$

Let $\Sigma \subseteq \mathbb{S}^N$ be the set of points such that $\mathcal{E} = +\infty$. Let

$$m_N = \min\{\mathcal{E}(X) : X \in \mathbb{S}^N\}.$$

Smale's 7th problem: can one find $X \in \mathbb{S}^N$ such that

$$\mathcal{E}(X) - m_N \leq c \log N?$$

"Can one find" means...

Elliptic Fekete points

Early works by Fekete, Szegö, Whyte, Hille, Tsuji, etc.

For $X = (x_1, \ldots, x_N) \in \mathbb{S}^N$ where $x_i \in \mathbb{S}$, $1 \leq i \leq N$, ellipic Fekete points minimize the logaritmic energy

$$\mathcal{E}(X) = \mathcal{E}(x_1, \ldots, x_N) = \log \prod_{i<j} \|x_i - x_j\|^{-1} = -\sum_{i<j} \log \|x_i - x_j\|$$

Let $\Sigma \subseteq \mathbb{S}^N$ be the set of points such that $\mathcal{E} = +\infty$. Let

$$m_N = \min\{\mathcal{E}(X) : X \in \mathbb{S}^N\}.$$

Smale's 7th problem: can one find $X \in \mathbb{S}^N$ such that

$$\mathcal{E}(X) - m_N \leq c \log N?$$

"Can one find" means...can one describe a polynomial time algorithm (BSS model)?

A beginner's problems with Smale's 7th problem.

Probably also fair to say "An expert's problems..."

1 Problem 1: the value of m_N is not known, even to $O(N)$.

Theorem (Wagner,Rakhmanov–Saff–Zhou)

$$m_N = -\frac{N^2}{4}\ln\frac{4}{e} - \frac{N\ln(N)}{4} - R_N,$$

where

$$-0.112768770\ldots \leq \liminf_{N\to\infty} R_{N,0} \leq \limsup_{N\to\infty} R_N \leq -0.0234973\ldots$$

A beginner's problems with Smale's 7th problem.

Probably also fair to say "An expert's problems..."

2 Problem 2: We need to solve a global minimization problem, not just a local minimization problem. Moreover, usual minimization algorithms will likely fall into "traps": experiments find many local minima of $\mathcal{E}$.

A beginner's problems with Smale's 7th problem.

Probably also fair to say "An expert's problems..."

2 Problem 2: We need to solve a global minimization problem, not just a local minimization problem. Moreover, usual minimization algorithms will likely fall into "traps": experiments find many local minima of $\mathcal{E}$.

Regular Polyhedra seem to be an answer for $N = 4, 6, 8, 12, 20$. Are they really?

Wikipedia picture

Regular Polyhedra seem to be an answer for $N = 4, 6, 8, 12, 20$. Are they really?

Ashmolean Museum de Oxford. $\approx$ 2500 B.C.

Regular Polyhedra seem to be an answer for $N = 4, 6, 8, 12, 20$. Are they really?

Not all of them! Föppl, Fejes, Rutishauser.

For $N = 5$ solved by [Dragnev–Legg-Townsend]

Euler's characteristic

Dirichlet cells

The Dirichlet cell of x_i is the set of points $x \in \mathbb{S}$ such that

$$\|x - x_i\| = \min_{1 \leq j \leq N} \|x - x_j\|.$$

The Dirichlet cell of x_i is the set of points $x \in \mathbb{S}$ such that

$$\|x - x_i\| = \min_{1 \leq j \leq N} \|x - x_j\|.$$

What should they look like when $x_1, \ldots, x_N$ is a set of elliptic Fekete points? Intuitively, pretty regular hexagons and pentagons.

Euler's characteristic

Dirichlet cells

The Dirichlet cell of x_i is the set of points $x \in \mathbb{S}$ such that

$$\|x - x_i\| = \min_{1 \leq j \leq N} \|x - x_j\|.$$

What should they look like when $x_1, \ldots, x_N$ is a set of elliptic Fekete points? Intuitively, pretty regular hexagons and pentagons. Say P pentagons and H hexagons, each vertex having three edges touching it.

The Dirichlet cell of x_i is the set of points $x \in \mathbb{S}$ such that

$$\|x - x_i\| = \min_{1 \leq j \leq N} \|x - x_j\|.$$

What should they look like when $x_1, \ldots, x_N$ is a set of elliptic Fekete points? Intuitively, pretty regular hexagons and pentagons. Say P pentagons and H hexagons, each vertex having three edges touching it.Then,

$$2 = \chi(\mathbb{S}) = \frac{6H + 5P}{3} + H + P - \frac{6H + 5P}{2} = \frac{P}{6}.$$

Thus, such a tessalation must have $P = 12$ pentagons.

The soccer ball and the buckminsterfullerene

20 hexagons and 12 pentagons. C60 discovered by Curl, Kroto, Smalley 1985

The soccer ball and the buckminsterfullerene

Diego Forlan and Harry Kroto with their respective belowed objects

Euler's characteristic

For greater N other figures appear in the (numerical) minima. [Hardin–Saff]

(Minimization of s–Energy, $s = 1, 4$, $N = 1600$.)

Euler's characteristic

Approximation to 1000 elliptic Fekete points by Bendito, Carmona, Encinas, Gesto, Gómez, Mouriño, Sánchez

We do know some things

Separation distance

- Theorem (Toth, Habicht– van der Waerden)

  For the Tammes problem (maximize separation distance)

$$\sqrt{\frac{8\pi}{\sqrt{3}N}} - CN^{-2/3} \leq d_{\mathrm{sep}}(X_{Tammes}) \leq \sqrt{\frac{8\pi}{\sqrt{3}N}} \approx \frac{3.8093}{\sqrt{N}}.$$

- Theorem (Rakhmanov–Saff–Zhou,Dubickas,Dragnev)

  For the elliptic Fekete points,

$$\frac{2}{\sqrt{N-1}} \leq d_{\mathrm{sep}}(X_{Fekete}) \leq \frac{3.8093}{\sqrt{N}}.$$

We do know some things

Baricenter [Bergersen-Boal-Palffy Muhoray], [Dragnev-Legg-Townsend]. True for any critical point of $\mathcal{E}$

Let $x_1, \ldots, x_N$ be a set of elliptic Fekete points.

- The baricenter of $x_1, \ldots, x_N$ is the center of the sphere.
- For each i,

$$\sum_{j \neq i} \frac{x_i - x_j}{\|x_i - x_j\|^2} = \frac{N-1}{2} x_i,$$

  and

$$\sum_{j \neq i} \|x_i - x_j\|^2 = 2N.$$

Back to the Problems about Smale's 7th Problem

Minimum value vs. *Expected value*

- Recall we have said

$$m_N = -\frac{N^2}{4}\ln\frac{4}{e} - \frac{N\ln(N)}{4} + O(N)$$

- Expectation if we choose $x_1, \ldots, x_N$ just randomly and uniformly in $\mathbb{S}$:

$$\mathbb{E}_{uniform} = -\frac{N^2}{4}\ln\frac{4}{e} + \frac{N}{4}\ln\frac{4}{e}.$$

Back to the Problems about Smale's 7th Problem

Minimum value vs. *Expected value*

- Recall we have said

$$m_N = -\frac{N^2}{4}\ln\frac{4}{e} - \frac{N\ln(N)}{4} + O(N)$$

- Expectation if we choose $x_1, \ldots, x_N$ just randomly and uniformly in $\mathbb{S}$:

$$\mathbb{E}_{uniform} = -\frac{N^2}{4}\ln\frac{4}{e} + \frac{N}{4}\ln\frac{4}{e}.$$

Theorem (Armentano-B.-Shub)

Expectation for points comming from the zeros of random polynomials (Bombieri–Weyl distribution):

$$\mathbb{E}_{B-W} = -\frac{N^2}{4}\ln\frac{4}{e} - \frac{N\ln(N)}{4} + \frac{N}{4}\ln\frac{4}{e}.$$

Back to the Problems about Smale's 7th Problem

Minimum value vs. *Expected value*

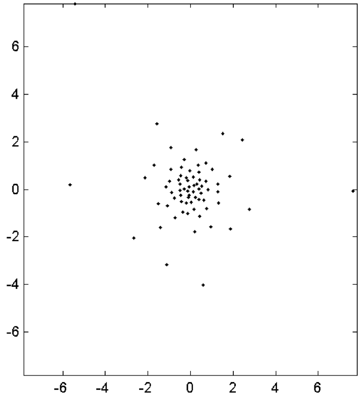
$$m_N = -\frac{N^2}{4}\ln\frac{4}{e} - \frac{N\ln(N)}{4} + O(N)$$

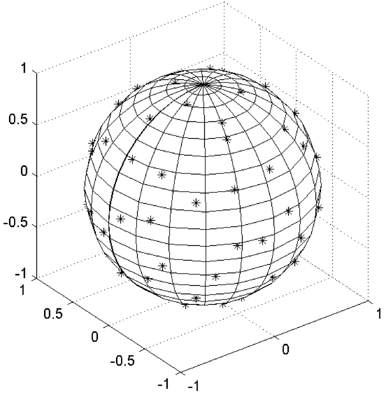
$$\mathbb{E}_{uniform} = -\frac{N^2}{4}\ln\frac{4}{e} + \frac{N}{4}\ln\frac{4}{e}$$

$$\mathbb{E}_{B-W} = -\frac{N^2}{4}\ln\frac{4}{e} - \frac{N\ln(N)}{4} + \frac{N}{4}\ln\frac{4}{e}$$

Roots of a randomly chosen (B–W) polynomial

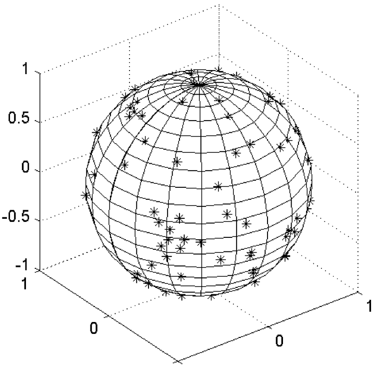
Just one try

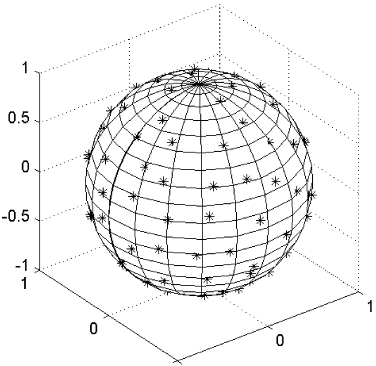




Roots of a random polynomial vs. random points

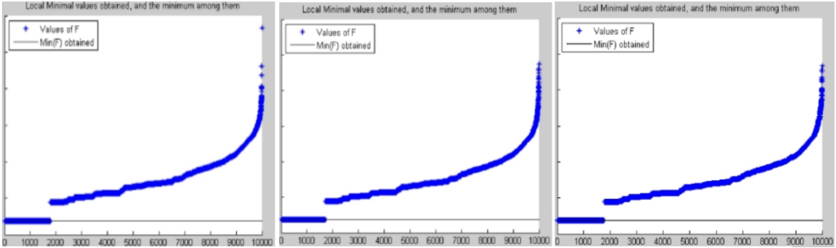
Maybe 4 or 5 tries





Comparison of the end–game starting at...

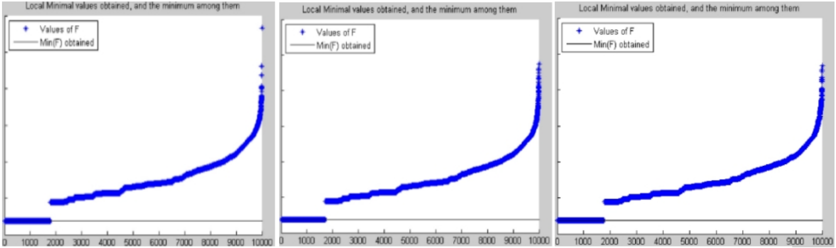
Random eigenvalues, random zeros of B-W polynomials, uniform points in $\mathbb{S}$



The distribution of the values of $\mathcal{E}$ at the end–points of the gradient flow seem not to depend on the distribution of the initial data.

Comparison of the end–game starting at...

Random eigenvalues, random zeros of B-W polynomials, uniform points in $\mathbb{S}$



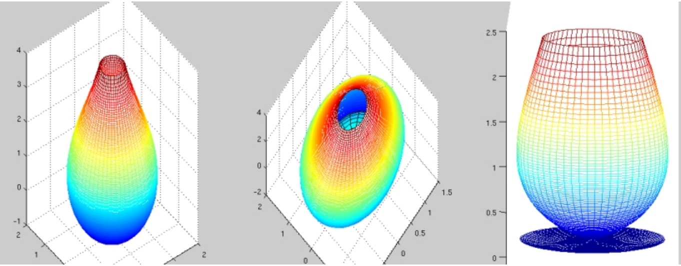
The distribution of the values of $\mathcal{E}$ at the end–points of the gradient flow seem not to depend on the distribution of the initial data. **Which is clearly imposible.**

The logarithm function in the sphere

The graphic corresponds to the function $x \mapsto -\log \|x - (0,0,1)\|$

Notational abuse: from now on, $\mathbb{S}$ *is the Riemann sphere, that is the sphere of diameter* 1. Let

$$\begin{array}{rccc} F_q : & \mathbb{S} \setminus \{q\} & \to & \mathbb{R} \\ & p & \mapsto & \log \|p - q\|^{-1} \end{array}$$



The logarithm function in the sphere

Notational abuse: from now on, $\mathbb{S}$ *is the Riemann sphere, that is the sphere of diameter* 1. Let

$$\begin{array}{rccc} F_q: & \mathbb{S}\setminus\{q\} & \to & \mathbb{R} \\ & p & \mapsto & \log\|p-q\|^{-1} \end{array}$$

The (Riemannian) Laplacian of this function is constant:

$$\Delta F_q(p) = 2 \;\; \forall\, p \in \mathbb{S}\setminus\{q\}.$$

Let $\mathcal{M}$ be a Riemannian manifold and let $f : \mathcal{M} \to \mathbb{R}$. The Hessian of f at $p \in \mathcal{M}$ is a bilinear form

$$\begin{array}{rccll} \mathrm{Hess}(f)(p): & T_p\mathcal{M} \times T_p\mathcal{M} & \to & \mathbb{R} & \\ & (v, w) & \mapsto & \mathrm{Hess}(f)(p)(v, w) & = w^t(h_{ij}(x))v, \end{array}$$

where

$$h_{ij}(x) = \frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} \Gamma_{ij}^k$$

and Γ_{ij}^k are the Christoffel symbols.

Let $\mathcal{M}$ be a Riemannian manifold and let $f : \mathcal{M} \to \mathbb{R}$. The Hessian of f at $p \in \mathcal{M}$ is a bilinear form such that

$$\mathrm{Hess}(f)(p)(v,v) = \frac{d^2}{dt^2}\mid_{t=0} f\left(\gamma_{p,v}(t)\right),$$

where $\gamma_{p,v}(t)$ is the geodesic in $\mathcal{M}$ such that

$$\gamma_{p,v}(0) = p, \qquad \dot{\gamma}_{p,v}(0) = v.$$

Thus, if you know geodesics then you can compute the Hessian

Let $\mathcal{M}$ be a Riemannian manifold and let $f : \mathcal{M} \to \mathbb{R}$. The Hessian of f at $p \in \mathcal{M}$ is a bilinear form
such that

$$\mathrm{Hess}(f)(p)(v, v) = \frac{d^2}{dt^2} \mid_{t=0} f\left(\gamma_{p,v}(t)\right).$$

Then, the Laplacian of f at p is

$$\Delta f(p) = \sum_{i=1}^{k} \mathrm{Hess}(f)(p)(v_i, v_i),$$

where the v_i are a orthonormal basis of $T_p\mathcal{M}$. **Thus, if you know geodesics then you can compute the Laplacian**

A function $f : \mathcal{M} \to \mathbb{R}$ is harmonic if $\Delta f(p) = 0$ for $p \in \mathcal{M}$.
A manifold $\mathcal{M}$ is harmonic if for every $p \in \mathcal{M}$ and small enough $\varepsilon > 0$ all the Riemannian spheres

$$S(x, \varepsilon) = \{q \in \mathcal{M} : d_R(p, q) = \varepsilon\}$$

have constant mean curvature.

Theorem (Willmore)

If $\mathcal{M}$ is harmonic and $f : \mathcal{M} \to \mathbb{R}$ is harmonic then the mean value equality holds:

$$\fint_{S(x,\varepsilon)} f = f(p), \qquad \textit{for small enough } \varepsilon > 0.$$

A function $f : \mathcal{M} \to \mathbb{R}$ is harmonic if $\Delta f(p) = 0$ for $p \in \mathcal{M}$.
A manifold $\mathcal{M}$ is harmonic if for every $p \in \mathcal{M}$ and small enough $\varepsilon > 0$ all the Riemannian spheres

$$S(x, \varepsilon) = \{q \in \mathcal{M} : d_R(p, q) = \varepsilon\}$$

have constant mean curvature.
If $\mathcal{M}$ is harmonic and $f : \mathcal{M} \to \mathbb{R}$ satisfies $\Delta f \equiv C$ then the mean value equality holds:

$$\fint_{S(x,\varepsilon)} f = f(p) + C \int_0^\varepsilon \frac{Vol(B(p,s))}{Vol(S(p,s))}\, ds, \qquad B_p(s) \text{ the Riemannian ball.}$$

A brief survey on harmonic analysis on manifolds

Harmonic functions, functions with constant Laplacian

A function $f : \mathcal{M} \to \mathbb{R}$ is harmonic if $\Delta f(p) = 0$ for $p \in \mathcal{M}$.
A manifold $\mathcal{M}$ is harmonic if for every $p \in \mathcal{M}$ and small enough $\varepsilon > 0$ all the Riemannian spheres

$$S(x, \varepsilon) = \{q \in \mathcal{M} : d_R(p, q) = \varepsilon\}$$

have constant mean curvature.
The sphere is clearly a harmonic manifold and hence

$$\fint_{S(x,\varepsilon)} F_q = F_q(p) + 2 \int_0^{\varepsilon} \frac{\pi \sin^2 s}{\pi \sin(2s)} \, ds$$

A function $f : \mathcal{M} \to \mathbb{R}$ is harmonic if $\Delta f(p) = 0$ for $p \in \mathcal{M}$.
A manifold $\mathcal{M}$ is harmonic if for every $p \in \mathcal{M}$ and small enough $\varepsilon > 0$ all the Riemannian spheres

$$S(x, \varepsilon) = \{q \in \mathcal{M} : d_R(p, q) = \varepsilon\}$$

have constant mean curvature.
The sphere is clearly a harmonic manifold and hence

$$\fint_{S(x,\varepsilon)} F_q = F_q(p) + \int_0^{\varepsilon} \tan s \, ds$$

A brief survey on harmonic analysis on manifolds

Harmonic functions, functions with constant Laplacian

A function $f : \mathcal{M} \to \mathbb{R}$ is harmonic if $\Delta f(p) = 0$ for $p \in \mathcal{M}$.
A manifold $\mathcal{M}$ is harmonic if for every $p \in \mathcal{M}$ and small enough $\varepsilon > 0$ all the Riemannian spheres

$$S(x, \varepsilon) = \{q \in \mathcal{M} : d_R(p, q) = \varepsilon\}$$

have constant mean curvature.
The sphere is clearly a harmonic manifold and hence

$$⨍_{S(x,\varepsilon)} F_q = F_q(p) - \log\cos\varepsilon$$

A function $f : \mathcal{M} \to \mathbb{R}$ is harmonic if $\Delta f(p) = 0$ for $p \in \mathcal{M}$.
A manifold $\mathcal{M}$ is harmonic if for every $p \in \mathcal{M}$ and small enough $\varepsilon > 0$ all the Riemannian spheres

$$S(x, \varepsilon) = \{q \in \mathcal{M} : d_R(p, q) = \varepsilon\}$$

have constant mean curvature.
We can also write

$$\fint_{B(x,\varepsilon)} F_q = F_q(p) + \frac{1}{2} + \frac{\log \cos \varepsilon}{\tan^2 \varepsilon}.$$

Harmonic properties of the logarithmic energy in the sphere

Laplacian and a Mean Value Theorem

Recall: for $X = (x_1, \ldots, x_N) \in \mathbb{S}^N$, where $\mathbb{S}^N$ has the product Riemannian structure,

$$\mathcal{E}(x_1, \ldots, x_N) = \log \prod_{i<j} \|x_i - x_j\|^{-1} = -\sum_{i<j} \log \|x_i - x_j\|.$$

Then:

- $\Delta \mathcal{E} = 2N(N-1)$.

Recall: for $X = (x_1, \ldots, x_N) \in \mathbb{S}^N$, where $\mathbb{S}^N$ has the product Riemannian structure,

$$\mathcal{E}(x_1, \ldots, x_N) = \log \prod_{i<j} \|x_i - x_j\|^{-1} = -\sum_{i<j} \log \|x_i - x_j\|.$$

Then:

- $\Delta \mathcal{E} = 2N(N-1)$.
- Thus, there exist no local maxima of $\mathcal{E}$.

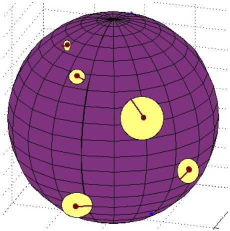
Harmonic properties of the logarithmic energy in the sphere

Laplacian and a Mean Value Theorem

Recall: for $X = (x_1, \ldots, x_N) \in \mathbb{S}^N$, where $\mathbb{S}^N$ has the product Riemannian structure,

$$\mathcal{E}(x_1, \ldots, x_N) = \log \prod_{i<j} \|x_i - x_j\|^{-1} = -\sum_{i<j} \log \|x_i - x_j\|.$$

Then:



The average value of the logarithmic energy on the yellow area is equal to the logarithmic energy at the centers of the circles, plus a constant. This constant depends only on the values of the radios.

(for non-overlapping circles!)

Theorem (B.)

Let $\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_N) \in [0, \pi/2)^N$ *and let*

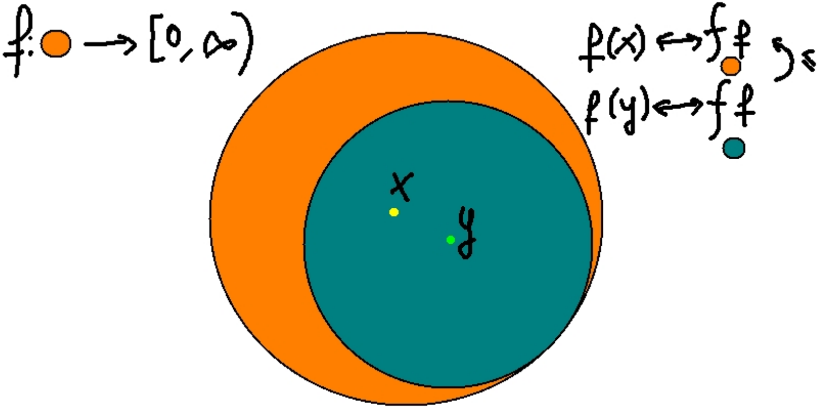
$$B_\infty(X, \vec{\varepsilon}) = B(x_1, \varepsilon_1) \times \cdots \times B(x_N, \varepsilon_N).$$

Then, if $B_\infty(X, \vec{\varepsilon}) \cap \Sigma = \emptyset$*,*

$$\fint_{B_\infty(X,\vec{\varepsilon})} \mathcal{E} = \mathcal{E}(X) + (N-1) \sum_{i=1}^{N} \overbrace{\left(\frac{1}{2} + \frac{\log \cos \varepsilon_i}{\tan^2 \varepsilon_i} \right)}^{\approx \varepsilon_i^2/4}$$

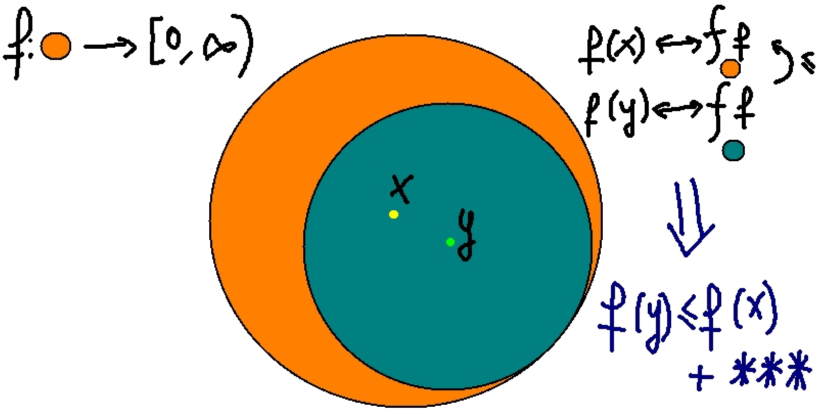
Harmonic properties of the logarithmic energy in the sphere

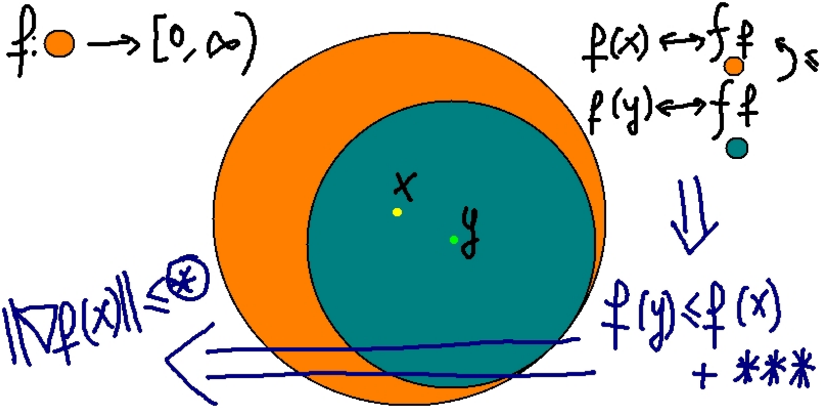
Mean value theorem and classical Harnack's theorem

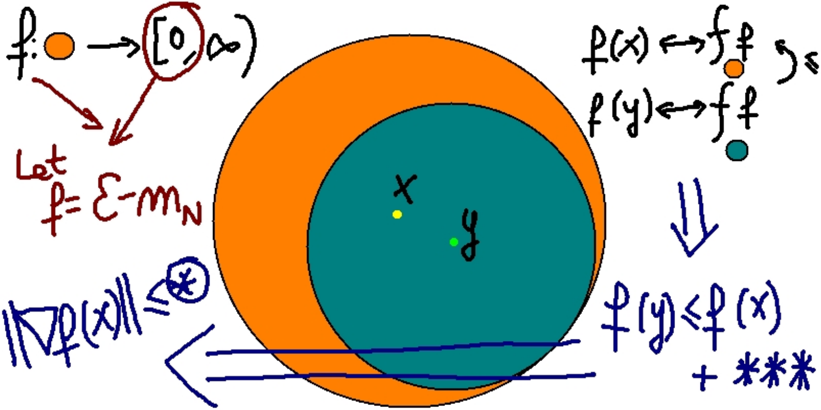


Harmonic properties of the logarithmic energy in the sphere

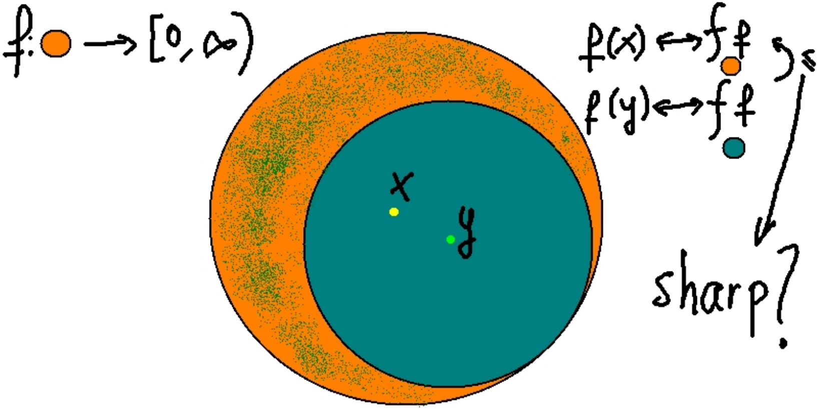
Mean value theorem and classical Harnack's theorem




f: ● → [0, ∞)
f(x) ↔ ⨍ f
f(y) ↔ ⨍ f
x
y
⇓
f(y) ≤ f(x) + ***
‖∇f(x)‖ ≤ *

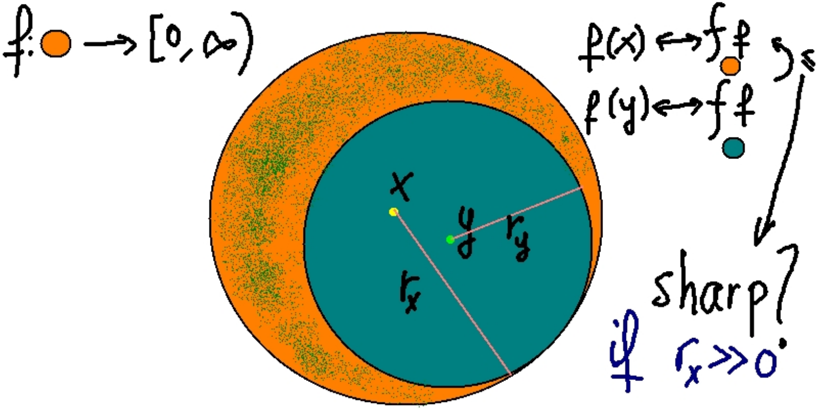


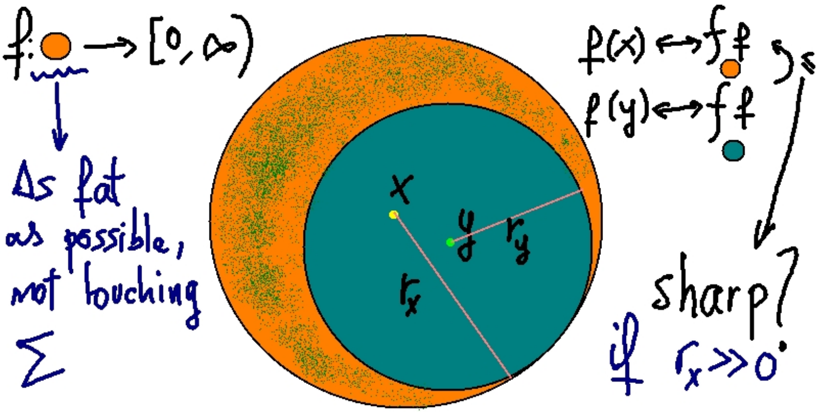
f: ● → [0, ∞)
Let f = E − m_N
||∇f(x)|| ≤ ⊛
x
y
f(x) ⟷ ⨍ f ●
f(y) ⟷ ⨍ f ●
≤
⇓
f(y) ≤ f(x) + ***


f: → [0, ∞)
f(x) ⟷ ∫f
f(y) ⟷ ∫f
≤
x
y
sharp?

Harmonic properties of the logarithmic energy in the sphere

Mean value theorem and classical Harnack's theorem




f: ● → [0, ∞)
As flat as possible, not touching Σ
x
y
r_x
r_y
f(x) ↔ ∫ f ●
f(y) ↔ ∫ f ●
≤
sharp?
if $r_x \gg 0$

Recall: m_N is the minimum of $\mathcal{E}$ in $\mathbb{S}^N \setminus \Sigma$.

Theorem (B.)

Let $X \in \mathbb{S}^N$ be such that

$$B_\infty(X, \vec{\varepsilon}) \subseteq \mathbb{S}^N \setminus \Sigma, \text{ where } \vec{\varepsilon} = (\varepsilon, \ldots, \varepsilon), \varepsilon = \sqrt{\frac{2(\mathcal{E}(X) - m_N)}{N-1}}.$$

Then,

$$\|D\mathcal{E}(X)\| \leq 2\sqrt{2N(N-1)(\mathcal{E}(X) - m_N)}.$$

Harmonic properties of the logarithmic energy...

... combined with the separation distance results, yield:

Let $X = (x_1, \ldots, x_N) \in \mathbb{S}^N$ be a N–tuple minimizing $\mathcal{E}$. Let $Y = (y_1, \ldots, y_N) \in \mathbb{S}^N$ be such that

$$d_R(x_i, y_i) \leq \frac{1/6}{N\sqrt{N-1}}, \qquad 1 \leq i \leq N.$$

Then,

$$\mathcal{E}(Y) \leq m_N + \frac{1}{18}$$

But, there exists $Y = (y_1, \ldots, y_N) \in \mathbb{S}^N$ such that

$$d_R(x_i, y_i) \leq \frac{1/3}{\sqrt{N(N-1)}}, \qquad 1 \leq i \leq N,$$

and

$$\mathcal{E}(Y) > m_N + \frac{1}{18}.$$

Admissible error function

This may have another name in a general context, I just made this one up.

The *admissible error function* $\mathbf{e} : (0,\infty)\to(0,\infty)$ is the function defined as

$$\mathbf{e}(t) = \sup\{\varepsilon : Y \in B_\infty(X,\vec{\varepsilon}), \text{ implies } \mathcal{E}(Y) \le m_N + t\},$$

We have just claimed:

$$\mathbf{e}\left(\frac{1}{18}\right) \in \left[\frac{1/6}{N\sqrt{N-1}}, \frac{1/3}{\sqrt{N(N-1)}}\right].$$

Admissible error function

This may have another name in a general context, I just made this one up.

The *admissible error function* $\mathbf{e} : (0, \infty) \to (0, \infty)$ is the function defined as

$$\mathbf{e}(t) = \sup\{\varepsilon : Y \in B_\infty(X, \vec{\varepsilon}), \text{ implies } \mathcal{E}(Y) \le m_N + t\},$$

One can also bound $\mathbf{e}(t)$ for general t:

Corollary (B.)

Let $N \ge 3$. The admissible error function satisfies

$$\mathbf{e}(t) \in \left[\sqrt{\frac{t}{2N^2(N-1)}}, \sqrt{\frac{2t}{N(N-1)}}\right], \qquad 0 \le t \le \frac{N^2(N-1)d_N^2}{2(1+2N)^2}.$$

For any $t > 0$ a less precise estimate also follows.

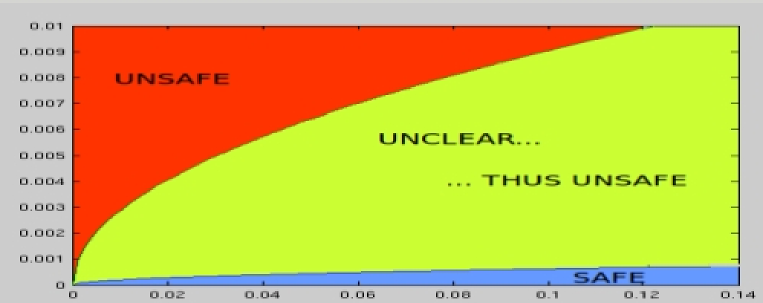
Admissible error function

This may have another name in a general context, I just made this one up.

The *admissible error function* $\mathbf{e} : (0,\infty)\to(0,\infty)$ is the function defined as

$$\mathbf{e}(t) = \sup\{\varepsilon : Y \in B_\infty(X,\vec{\varepsilon}), \text{ implies } \mathcal{E}(Y) \leq m_N + t\},$$

One can also bound $\mathbf{e}(t)$ for general t. For $N = 50$:



The set of points in $\mathbb{S}$ whose coordinates are rational numbers is dense in $\mathbb{S}$.
Moreover, precise bounds are known on the size of spherical rational points [Schmutz]: Given $z \in \mathbb{R}^3$, $\|z\| = 1$, there exists $\tilde{z}$ such that

$$\tilde{z} = \left(\frac{\tilde{p}^{(1)}}{\tilde{q}^{(1)}}, \frac{\tilde{p}^{(2)}}{\tilde{q}^{(2)}}, \frac{\tilde{p}^{(3)}}{\tilde{q}^{(3)}} \right) \in \mathbb{Q}^3 \cap \mathbb{S},$$

such that

$$\left| \frac{\tilde{p}^{(j)}}{\tilde{q}^{(j)}} - \tilde{x}^{(j)} \right| \leq \varepsilon, \qquad 0 \leq |\tilde{p}^{(j)}| \leq \tilde{q}^{(j)} \leq \left(\frac{128}{\varepsilon^2} \right)^2.$$

Computability of elliptic Fekete points

There are rational solutions (bounded bit length) to Smale's 7th problem

The set of points in $\mathbb{S}$ whose coordinates are rational numbers is dense in $\mathbb{S}$.

There is a universal constant $c \geq 0$ ($c = 11$ suffices) with the following property: for every $N \geq 2$ there exists $Z = (z_1, \ldots, z_N) \in \mathbb{S}^N$ such that:

1. $\mathcal{E}(Z) \leq m_N + 1/18$.
2. For $1 \leq i \leq N$,

$$z_i = \left(\frac{p_i^{(1)}}{q_i^{(1)}}, \frac{p_i^{(2)}}{q_i^{(2)}}, \frac{p_i^{(3)}}{q_i^{(3)}} \right) \in \mathbb{S} \cap \mathbb{Q}^3, \qquad p_i^{(j)}, q_i^{(j)} \in \mathbb{Z}, \ 1 \leq j \leq 3,$$

   where

$$0 \leq |p_i^{(j)}| \leq q_i^{(j)} \leq (cN)^6, \qquad 1 \leq i \leq N, \ 1 \leq j \leq 3,$$

The set of points in $\mathbb{S}$ whose coordinates are rational numbers is dense in $\mathbb{S}$.

Corollary (B.)

There exists a Turing machine which, on input $N \in \{2, 3, \dots\}$, outputs $X = (x_1, \dots, x_N) \in \mathbb{S}^N \cap \mathbb{Q}^{3N}$ satisfying

$$\mathcal{E}(X) \leq m_N + \frac{1}{18},$$

and such that the running time is simply exponential in N. More precisely: polynomial$(N) \cdot (11N)^{36N}$.

The soccer ball and the buckminsterfullerene

20 hexagons and 12 pentagons. C60 discovered by Curl, Kroto, Smalley 1985

But why BuckminsterFullerene

Richard Buckminster Fuller and his Spaceship Earth. First such a "geodesic dome" was designed by Walther Bauersfeld for his 1912 planetarium.

The ball beneath the lion paw. Again 12 pentagons

A chinesse temple in Beijing. Bronze lion-dogs flank the entrances to the halls. This lion has a ball under his paw symbolizing control of the empire. Math Intelligencer 17, n. 3.

Origami spheres

Recommended movie: *between the folds*

Equal area of Dirichelt cells, 122 electrons in nearly optimal configuration and dual.

The 400 pieces in the first picture [Rakhmanov–Saff–Zhou], [Kuijlaars–Saff] have area $\frac{\pi}{100}$ diameter $\leq \frac{7}{2\sqrt{10}}$

Other aspects of the problem

Locating good points in the sphere is studied in other contexts

- Packing and covering radius.
- Location problems.
- Quadrature formulas (spherical N–designs).
- Spherical harmonics and interpolation.

and many others.

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree N and let $\zeta \in \mathbb{C}$ be a zero of f. Let

$$\mu(f, \zeta) = \frac{N^{1/2}(1 + \|\zeta\|^2)^{\frac{N-2}{2}}}{|f'(\zeta)|}\|f\|_{B-W}.$$

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree N and let $\zeta \in \mathbb{C}$ be a zero of f. Let

$$\mu(f, \zeta) = \frac{N^{1/2}(1 + \|\zeta\|^2)^{\frac{N-2}{2}}}{|f'(\zeta)|} \|f\|_{B-W}.$$

This is the *condition number*, which actually controls the sensibility of the zero ζ to perturbations of f. Let

$$\mu(f) = \max(\mu(f, \zeta) : f(\zeta) = 0).$$

Let $f:\mathbb{C}\to\mathbb{C}$ be a polynomial of degree N and let $\zeta\in\mathbb{C}$ be a zero of f. Let

$$\mu(f,\zeta)=\frac{N^{1/2}(1+\|\zeta\|^2)^{\frac{N-2}{2}}}{|f'(\zeta)|}\|f\|_{B-W}.$$

This is the *condition number*, which actually controls the sensibility of the zero ζ to perturbations of f. Let

$$\mu(f)=\max(\mu(f,\zeta):f(\zeta)=0).$$

Theorem (Shub–Smale)

For every polynomial f, we have $\mu(f)\geq 1$. For random f, with probability at least $1/2$ we have $\mu(f)\leq N$.

So, for many polynomials, $\mu(f) \leq N$.

So, for many polynomials, $\mu(f) \leq N$. Can we find one f with that property? **not easy!** even changing N to N^c, c a constant.

Theorem (Shub–Smale)

Let $x_1, \dots, x_N \in \mathbb{S}$ be a set of elliptic Fekete points. Let $z_1, \dots, z_N \in \mathbb{C}$ be the preimage of $x_1, \dots, x_N$ under the stereographic projection. Let f be the polynomial which has zeros $z_1, \dots, z_N$. Then,

$$\mu(f) \leq \sqrt{N(N+1)}.$$

So, for many polynomials, $\mu(f) \leq N$. Can we find one f with that property? **not easy!** even changing N to N^c, c a constant.

Theorem (Shub–Smale)

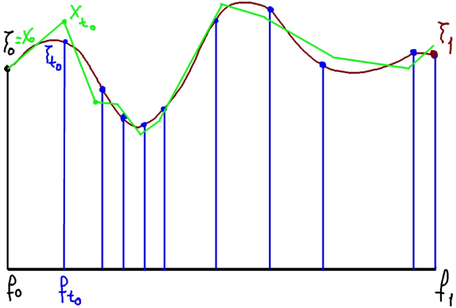
Let $x_1, \ldots, x_N \in \mathbb{S}$ be a set of elliptic Fekete points. Let $z_1, \ldots, z_N \in \mathbb{C}$ be the preimage of $x_1, \ldots, x_N$ under the stereographic projection. Let f be the polynomial which has zeros $z_1, \ldots, z_N$. Then,

$$\mu(f) \leq \sqrt{N(N+1)}.$$

Experiments suggest $\mu(f) \leq \sqrt{N}/2$.

Theorem (Burgisser–Cucker)

Any polynomial whose zeros correspond to a set of elliptic Fekete points is a good starting point for homotopy methods that solve polynomials.



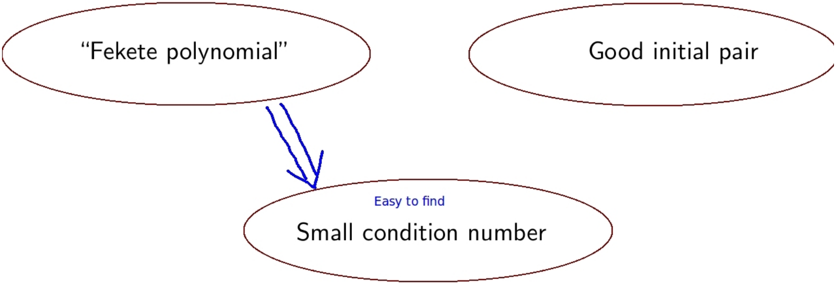
Elliptic Fekete points and the condition number of polynomials

Best conditioned polynomials and homotopy methods

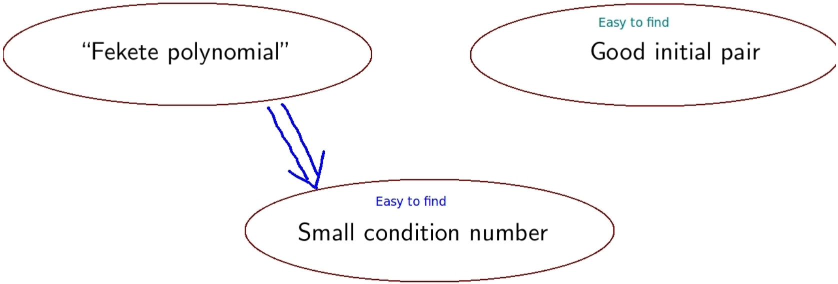
Shub-Smale

B.-Pardo

Burgisser-Cucker



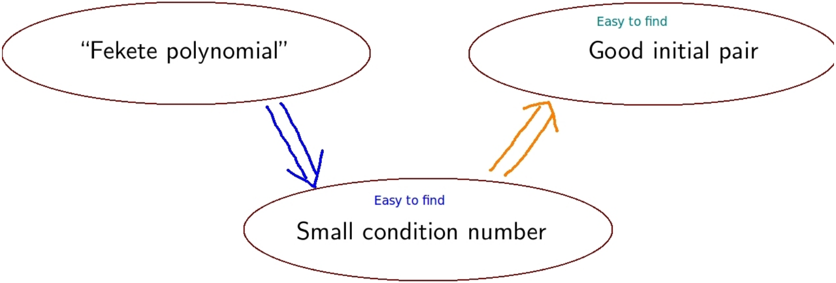
Shub-Smale
B.-Pardo
Burgisser-Cucker



Elliptic Fekete points and the condition number of polynomials

Best conditioned polynomials and homotopy methods

Shub-Smale
B.-Pardo
Burgisser-Cucker



Elliptic Fekete points and the condition number of polynomials

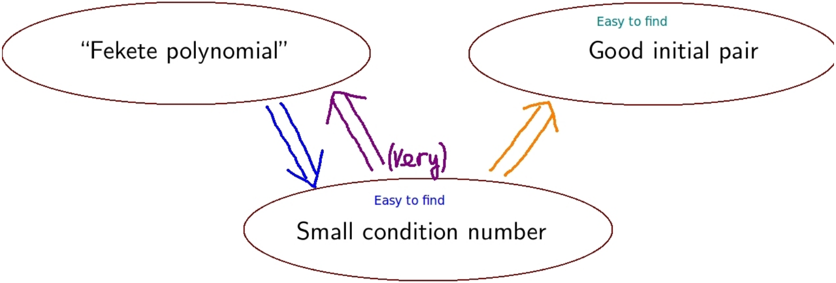
Best conditioned polynomials and homotopy methods

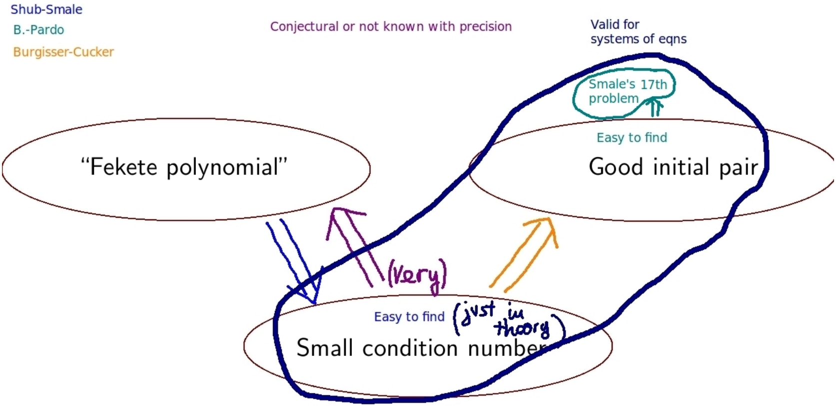
Shub-Smale

B.-Pardo

Burgisser-Cucker

Conjectural or not known with precision




Shub-Smale
B.-Pardo
Burgisser-Cucker
Conjectural or not known with precision
Valid for
systems of eqns
Smale's 17th
problem
Easy to find
Good initial pair
"Fekete polynomial"
(very)
Easy to find
(just in theory)
Small condition number

Let $x_1, \ldots, x_N \in \mathbb{S}$ and associated $z_1, \ldots, z_N \in \mathbb{C}$. Let

$$f(Z) = (Z - z_1) \cdots (Z - z_N).$$

Let $x_1, \ldots, x_N \in \mathbb{S}$ and associated $z_1, \ldots, z_N \in \mathbb{C}$. Let

$$f(Z) = (Z - z_1) \cdots (Z - z_N).$$

Then,

$$\mathcal{E}(x_1, \ldots, x_N) = \frac{1}{2} \sum_{i=1}^{N} \ln \mu(f, z_i) + \frac{N}{2} \ln \frac{\prod_{i=1}^{N} \sqrt{1 + |z_i|^2}}{\|f\|} - \frac{N}{4} \ln N,$$

$$\mathcal{E}(x_1,\ldots,x_N)=\frac{1}{2}\sum_{i=1}^{N}\ln\mu(f,z_i)+\frac{N}{2}\ln\frac{\prod_{i=1}^{N}\sqrt{1+|z_i|^2}}{\|f\|}-\frac{N}{4}\ln N,$$

$$f(z)=\prod(z-z_i) = g_i(z)$$

$$\left(\prod_{i=1}^{N}\|g_i\|\right)\Big/\|f\|$$

Elliptic Fekete points and the condition number of polynomials

Condition number, logarithmic potential and Bombieri–Weyl norm

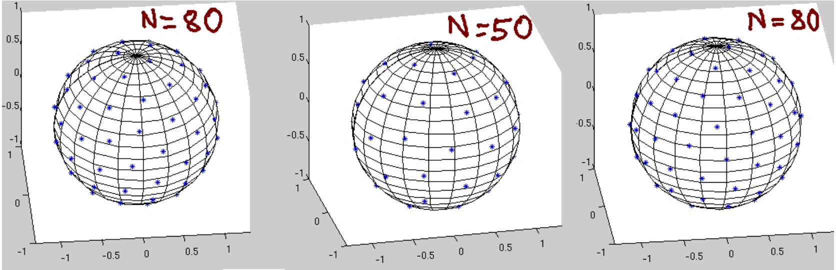
P. theory | Complexity | Alg. Geometry & Number theory

$$\mathcal{E}(x_1,\dots,x_N) = \frac{1}{2}\sum_{i=1}^{N} \ln \mu(f,z_i) + \frac{N}{2}\ln \frac{\prod_{i=1}^{N}\sqrt{1+|z_i|^2}}{\|f\|} - \frac{N}{4}\ln N,$$

Min | Min | Max

Elliptic Fekete points and the condition number of polynomials

Condition number, logarithmic potential and Bombieri–Weyl norm



Escher's sphere.

This is my last picture.