Biorthogonal polynomials and zero-mapping transformations

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Abstract The authors have presented in [6] a technique to generate transformations $\mathcal{T}$ of the set P_n of nth degree polynomials to itself such that if $p \in P_n$ has all its zeros in (c,d) then $\mathcal{T}\{p\}$ has all its zeros in (a,b), where (a,b) and (c,d) are given real intervals. The technique rests upon the derivation of an explicit form of biorthogonal polynomials whose Borel measure is strictly sign consistent and such that the ratio of consecutive generalized moments is a rational [1/1] function of the parameter. Specific instances of strictly sign consistent measures that have been debated in [6] include $x^\mu \, \mathrm{d}\psi(x)$, $\mu^x \, \mathrm{d}\psi(x)$ and $x^{\log_q \mu} \, \mathrm{d}\psi(x)$, $q \in (0,1)$. In this paper we identify all measures ψ such that their consecutive generalized moments have a rational [1/1] quotient, thereby characterizing all possible zero-mapping transformations of this kind.

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1 Zero-mapping transformations

In the present paper we wish to return to a theme that has been already deliberated in [6, 9]. Let (a, b) and (c, d) be two nonempty real intervals and denote by P_n the set of nth degree polynomials. We are interested in linear transformations $\mathcal{T} : P_n \to P_n$ such that any polynomial with all its zeros in (c, d) is mapped into a polynomial with all its zeros in (a, b).

A trivial instance of such a transformation is

$$\mathcal{T}\left\{\sum_{k=0}^{n} a_k x^k\right\} = \sum_{k=0}^{n} k a_k x^k,$$

which maps real zeros into real zeros. Another example, with ubiquitous applications, is

$$\mathcal{T}\left\{\sum_{k=0}^{n} a_k x^k\right\} = \sum_{k=0}^{n} \tfrac{1}{k!} a_k x^k$$

– it maps positive zeros to positive zeros. Both follow from the theory of *multiplier sequences*, well known since the pioneering work of E. Laguerre [10] and of G. Pólya and I. Schur [12]. The transformation

$$\mathcal{T}\left\{\sum_{k=0}^{n} a_k x^k\right\} = \sum_{k=0}^{n} a_k T_k(x),$$

where T_k stands for the kth Chebyshev polynomial, maps positive zeros to positive zeros – the proof is elementary (although perhaps surprising), takes just few lines of undergraduate mathematics and we challenge the reader to find it without reference to [8].

More examples of 'zero-mapping' transformations are available in literature, e.g. [2, 11]. A powerful technique for the generation of such constructs has been presented by the authors in [6]. For every $\mu \in (c, d)$ we let $\mathrm{d}\varphi(x, \mu)$ be a Borel measure supported by $x \in (a, b)$. Following [5], we say that $p \in P_n \times C[(c, d)^n]$, $p \not\equiv 0$, is a *biorthogonal polynomial* if

$$\int_a^b p(x; \mu_1, \ldots, \mu_n)\, \mathrm{d}\varphi(x, \mu_\ell) = 0, \qquad \ell = 1, 2, \ldots, n.$$

Biorthogonal polynomials exist and are unique (up to a nonzero multiplicative constant) if and only if $\mathrm{d}\varphi$ is *regular* [5], that is, for distinct $\mu_1, \mu_2, \ldots, \mu_n$,

$$\det \begin{bmatrix} \int_a^b \mathrm{d}\varphi(x, \mu_1) & \int_a^b \mathrm{d}\varphi(x, \mu_2) & \cdots & \int_a^b \mathrm{d}\varphi(x, \mu_n) \\ \int_a^b x\, \mathrm{d}\varphi(x, \mu_1) & \int_a^b x\, \mathrm{d}\varphi(x, \mu_2) & \cdots & \int_a^b x\, \mathrm{d}\varphi(x, \mu_n) \\ \vdots & \vdots & & \vdots \\ \int_a^b x^{n-1}\, \mathrm{d}\varphi(x, \mu_1) & \int_a^b x^{n-1}\, \mathrm{d}\varphi(x, \mu_2) & \cdots & \int_a^b x^{n-1}\, \mathrm{d}\varphi(x, \mu_n) \end{bmatrix} \neq 0, \quad n = 1, 2, \ldots.$$

We henceforth assume regularity.

Provided that φ is a *strictly sign consistent (SSC)* function, i.e. that

$$\det \begin{bmatrix} \varphi(x_1, \mu_1) & \varphi(x_1, \mu_2) & \cdots & \varphi(x_1, \mu_n) \\ \varphi(x_2, \mu_1) & \varphi(x_2, \mu_2) & \cdots & \varphi(x_2, \mu_n) \\ \vdots & \vdots & & \vdots \\ \varphi(x_n, \mu_1) & \varphi(x_n, \mu_2) & \cdots & \varphi(x_n, \mu_n) \end{bmatrix} \neq 0$$

for every $a < x_1 < x_2 < \cdots < x_n < b$, $c < \mu_1 < \mu_2 < \cdots < \mu_n < d$, it can be proved that all the zeros of p_n reside in the set (a,b) [5]. Likewise, zeros of p_n live in (a,b) if $\mathrm{d}\varphi(x,\mu) = \omega(x,\mu)\,\mathrm{d}\psi(x)$, where $\mathrm{d}\psi(x)$ is a Borel measure and the function ω is SSC. Biorthogonal polynomials – and, with greater generality, biorthogonal functions – feature in a wide variety of interesting applications, mainly in numerical analysis [3, 7].

Suppose that φ (or ω) is indeed an SSC function. Given any $\mu_1, \ldots, \mu_n \in (c,d)$, we define

$$\mathcal{T}\left\{\prod_{\ell=1}^{n}(x-\mu_\ell)\right\} = p_n(x;\mu_1,\mu_2,\ldots,\mu_n). \tag{1.1}$$

Given that the range of the linear operator $\mathcal{T}$ in (1.1) can be extended to all of P_n, it is a zero-mapping transformation. Specifically, it maps a polynomial with all its zeros in (c,d) into a polynomial with all its zeros in (a,b).

For the transformation (1.1) to be of any interest, beyond the most formal, we need to know *explicitly* the form of $p_n(x;\mu_1,\mu_2,\ldots,\mu_n)$. Fortunately, it is demonstrated in [6] that p_n can be described in a closed form in an important special case. Thus, let $\vec{\rho} := \{\rho_k\}_{k\in\mathbb{Z}^+}$ be an infinite sequence of monic polynomials, such that

$$\begin{aligned}
\rho_0(x) &\equiv 1, \\
\rho_1(x) &= (x-\sigma_1), \\
\rho_2(x) &= (x-\sigma_1)(x-\sigma_2) \\
&\vdots \\
\rho_k(x) &= (x-\sigma_1)(x-\sigma_2)\cdots(x-\sigma_k) = \rho_{k-1}(x)(x-\sigma_k), \qquad k = 1,2,\ldots,
\end{aligned}$$

and consider the *generalized moments*

$$I_k(\mu) = \int_a^b \rho_k(x)\,\mathrm{d}\varphi(x,\mu), \qquad k = 0,1,\ldots.$$

Note that $\mathrm{d}\varphi$ is regular if and only if

$$\det\begin{bmatrix} I_0(\mu_1) & I_0(\mu_2) & \cdots & I_0(\mu_n) \\ I_1(\mu_1) & I_1(\mu_2) & \cdots & I_1(\mu_n) \\ \vdots & \vdots & & \vdots \\ I_{n-1}(\mu_1) & I_{n-1}(\mu_2) & \cdots & I_{n-1}(\mu_n) \end{bmatrix} \neq 0, \qquad n = 1,2,\ldots, \tag{1.2}$$

for all $c < \mu_1 < \mu_2 < \cdots < \mu_n < d$.

If the sequence $\{I_{k+1}(\mu)/I_k(\mu)\}_{k\in\mathbb{Z}^+}$ consists of rational [1/1] functions, i.e.

$$\frac{I_{k+1}(\mu)}{I_k(\mu)} = \frac{g_k(\mu)}{h_k(\mu)}, \tag{1.3}$$

where $g_k, h_k \in P_1$ then, subject to $g_k h_k' - g_k' h_k \neq 0$, $x \in (c,d)$, $k \in \mathbb{Z}^+$,

$$p_n(x;\mu_1,\mu_2,\ldots,\mu_n) = \sum_{k=0}^{n} d_k \rho_k(x),$$

where

$$\prod_{\ell=1}^{n}(x-\mu_\ell)=\sum_{k=0}^{n} d_k \prod_{j=0}^{k-1} g_j(x) \prod_{j=k}^{n-1} h_j(x).$$

In other words, (1.1) becomes

$$T\left\{\sum_{k=0}^{m} d_k \prod_{j=0}^{k-1} g_j(x) \prod_{j=k}^{m-1} h_j(x)\right\}=\sum_{k=0}^{m} d_k \rho_k(x). \tag{1.4}$$

In particular, if ω, say, is SSC then (1.4) maps polynomials with all their zeros in (c,d) into polynomials with all their zeros in (a,b) [6].

Fifteen examples of transformation of this form have been presented in [6]. For example, letting $\mathrm{d}\varphi(x,\mu)=(\Gamma(\mu))^{-1}x^{\mu-1}\mathrm{e}^{-x}\,\mathrm{d}x$, $(a,b)=(c,d)=(0,\infty)$, and $\sigma_k\equiv 0$, we have $I_k(\mu)=(\mu)_k$, $k\in\mathbb{Z}^+$. Here $(z)_k$ is the *Pochhammer symbol* [13],

$$(z)_0=1 \qquad \text{and} \qquad (z)_k=(z)_{k-1}(z+k-1)=\prod_{\ell=0}^{k-1}(z+\ell), \quad k=1,2,\ldots.$$

Therefore $g_k(\mu)=k+\mu$, $h_k(\mu)\equiv 1$, $g_k h_k'-g_k h_k'\equiv 1$ and, x^μ being SSC [5], it follows from (1.4) that the *Laguerre transformation*

$$\mathcal{T}\left\{\sum_{k=0}^{n} d_k (x)_k\right\}=\sum_{k=0}^{n} d_k x^k$$

maps polynomials with positive zeros into polynomials with positive zeros.

All the transformations in [6] follow from eight strictly sign consistent choices of ω, namely (a) x^μ, $x,\mu>0$, (b) μ^x, $x\in\mathbb{R}$, $\mu>0$, (c) $x^{\log_q \mu}$, $x,\mu>0$, $q\in(0,1)$, (d) $x^{\log_q\mu}$, $x,\mu>0$, $q>1$, (e) $\Gamma(x+\mu)$, $x,\mu>0$, (f) $1/\Gamma(x+\mu)$, $x,\mu>0$, (g) $1/(x\mu;q)_\infty$, $x,\mu,q\in(0,1)$, (h) $(-x\mu;q)_\infty$, $x>0$, $\mu,q\in(0,1)$. The notation $(z;q)_k$ stands for the *Gauß–Heine symbol* [14],

$$(z;q)_0=1 \qquad \text{and} \qquad (z;q)_k=(z;q)_{k-1}(1-q^{k-1}z)=\prod_{\ell=0}^{k-1}(1-q^\ell z), \quad k=1,2,\ldots,\infty.$$

In other words, each transformation is obtained by choosing one of the above functions ω, in tandem with a specific choice of $\mathrm{d}\psi$ and $\vec{\rho}$ which is consistent with (1.3).

In the present paper we adopt a complementary approach. Thus, given ω, we attempt to identify *all* Borel measures $\mathrm{d}\psi$ and sets $\vec{\rho}$ such that (1.3) is true. Specifically, we consider the three choices (a)–(c). There are in [6] seven transformations corresponding to these choices. We prove in the sequel that, up to linear mapping of x and μ, this almost exhausts the list of all possible transformations – just a single transformation has been missed in [6]!

We expect to return to this issue in a future paper, characterizing all transformations associated with the choices (d)–(h) and consistent with (1.3).

2 The measure $\mathrm{d}\varphi(x,\mu) = x^\mu\,\mathrm{d}\psi(x)$, $\mu > 0$

Let $E = (a,b) \subseteq (0,\infty)$ be the least real interval that contains the essential support of $\mathrm{d}\psi$, $\{\sigma_\ell\}_{\ell=1}^\infty$ be a set of arbitrary real numbers and define $\rho_k(x) = \prod_{\ell=1}^k (x-\sigma_\ell)$, $k = 0,1,\ldots$. The *moments* of $\vec{\rho} = \{\rho_k\}_{k=0}^\infty$ are

$$I_k(\mu,\vec{\rho}) := \int_E x^\mu \rho_k(x)\,\mathrm{d}\psi(x), \qquad k = 0,1,\ldots,$$

therefore

$$\begin{aligned} I_k(\mu,\vec{\rho}) &= \int_E x^{\mu-1}(x-\sigma_{k+1}+\sigma_{k+1})\rho_k(x)\,\mathrm{d}\psi(x) \\ &= I_{k+1}(\mu-1,\vec{\rho}) + \sigma_{k+1} I_k(\mu-1,\vec{\rho}), \qquad k = 0,1,\ldots. \end{aligned} \tag{2.1}$$

The following trivial observation will be repeatedly used in the sequel.

Proposition 1 Suppose that there exists $k \in \mathbb{Z}^+$ such that

$$\frac{I_{k+1}(\mu)}{I_k(\mu)} \equiv \text{const.} \tag{2.2}$$

Then $\mathrm{d}\varphi$ is not regular.

Proof. Follows at once from (1.2), since (2.2) implies that two columns in the matrix are proportional and the determinant must therefore vanish for $k \geq n$. □

We wish to characterize all $\mathrm{d}\psi$ and $\{\sigma_\ell\}$ so that

$$\frac{I_{k+1}(\mu,\vec{\rho})}{I_k(\mu,\vec{\rho})} = \frac{\alpha_k + \beta_k\mu}{\gamma_k + \delta_k\mu}, \qquad k = 0,1,\ldots, \tag{2.3}$$

for arbitrary real constants $\alpha_k, \beta_k, \gamma_k, \delta_k$ such that $|\alpha_k| + |\beta_k|, |\gamma_k| + |\delta_k| > 0$, $k = 0,1,\ldots$.

Substituting (2.1) into (2.3), we have for all $k = 0,1,\ldots$

$$\begin{aligned} \frac{\alpha_k+\beta_k\mu}{\gamma_k+\delta_k\mu} &= \frac{I_{k+2}(\mu-1,\vec{\rho}) + \sigma_{k+2} I_{k+1}(\mu-1,\vec{\rho})}{I_{k+1}(\mu-1,\vec{\rho}) + \sigma_{k+1} I_k(\mu-1,\vec{\rho})} \\ &= \frac{\dfrac{I_{k+2}(\mu-1,\vec{\rho})}{I_{k+1}(\mu-1,\vec{\rho})} + \sigma_{k+2}}{1 + \sigma_{k+1}\dfrac{I_k(\mu-1,\vec{\rho})}{I_{k+1}(\mu-1,\vec{\rho})}} = \frac{\dfrac{\alpha_{k+1}+\beta_{k+1}(\mu-1)}{\gamma_{k+1}+\delta_{k+1}(\mu-1)} + \sigma_{k+2}}{1 + \sigma_{k+1}\dfrac{\gamma_k+\delta_k(\mu-1)}{\alpha_k+\beta_k(\mu-1)}} \\ &= \frac{(\alpha_k-\beta_k)+\beta_k\mu}{(\gamma_{k+1}-\delta_{k+1})+\delta_{k+1}\mu} \\ &\qquad \times \frac{(\alpha_{k+1}-\beta_{k+1}+\sigma_{k+2}(\gamma_{k+1}-\delta_{k+1})) + (\beta_{k+1}+\sigma_{k+2}\delta_{k+1})\mu}{(\alpha_k-\beta_k+\sigma_{k+1}(\gamma_k-\delta_k)) + (\beta_k+\sigma_{k+1}\delta_k)\mu}. \end{aligned}$$

We shift $\mu \to (\mu+1)$, thus obtaining the cubic identity

$$\begin{aligned} & [(\alpha_k+\beta_k)+\beta_k\mu][\gamma_{k+1}+\delta_{k+1}\mu][(\alpha_k+\sigma_{k+1}\gamma_k)+(\beta_k+\sigma_{k+1}\delta_k)\mu] \\ = \; & [\alpha_k+\beta_k\mu][(\gamma_k+\delta_k)+\delta_k\mu][(\alpha_{k+1}+\sigma_{k+2}\gamma_{k+1})+(\beta_{k+1}+\sigma_{k+2}\delta_{k+1})\mu]. \end{aligned} \tag{2.4}$$

We distinguish between the following cases:

Case I: $(\alpha_k+\beta_k)+\beta_k\mu$ is a constant multiple of $\alpha_k+\beta_k\mu$.
Hence $\beta_k=0$ and, without loss of generality, $\alpha_k=1$. Therefore (2.4) reduces to

$$\begin{aligned} & [\gamma_{k+1}+\delta_{k+1}\mu][(1+\sigma_{k+1}\gamma_k)+\sigma_{k+1}\delta_k\mu] \\ = \; & [(\gamma_k+\delta_k)+\delta_k\mu][(\alpha_{k+1}+\sigma_{k+2}\gamma_{k+1})+(\beta_{k+1}+\sigma_{k+2}\delta_{k+1})\mu]. \end{aligned} \tag{2.5}$$

There are thus two possible subcases:

Subcase I.1: $\gamma_{k+1}+\delta_{k+1}\mu=C(\gamma_k+\delta_k+\delta_k\mu)$ for some $C\neq 0$.
We thus deduce that

$$\begin{aligned} \alpha_{k+1} &= C(1+\sigma_{k+1}\gamma_k-\sigma_{k+2}\gamma_{k+1}), \\ \beta_{k+1} &= C(\sigma_{k+1}\delta_k-\sigma_{k+2}\delta_{k+1}), \\ \gamma_{k+1} &= C(\gamma_k+\delta_k), \\ \delta_{k+1} &= C\delta_k. \end{aligned}$$

However, since the numerator and the denominator of a rational function can be rescaled by a nonzero constant, we may assume without loss of generality that $C=1$. Therefore

$$\left.\begin{aligned} \beta_k &= 0, \\ \alpha_{k+1} &= 1+(\sigma_{k+1}-\sigma_{k+2})\gamma_k-\sigma_{k+2}\delta_k, \\ \beta_{k+1} &= (\sigma_{k+1}-\sigma_{k+2})\delta_k, \\ \gamma_{k+1} &= \gamma_k+\delta_k, \\ \delta_{k+1} &= \delta_k. \end{aligned}\right\} \tag{2.6}$$

Subcase I.2: $1+\sigma_{k+1}\gamma_k+\sigma_{k+1}\delta_k\mu=C(\gamma_k+\delta_k+\delta_k\mu)$, $C\neq 0$.
Thus, either $\delta_k=0$ or $C=\sigma_{k+1}\neq 0$. In the first case I_{k+1}/I_k is constant as a function of μ and regularity is lost. In the second case (2.5) yields

$$\begin{aligned} \alpha_{k+1} &= (\sigma_{k+1}-\sigma_{k+2})\gamma_{k+1}, \\ \beta_{k+1} &= (\sigma_{k+1}-\sigma_{k+2})\delta_{k+1}. \end{aligned}$$

Since $|\alpha_{k+1}|+|\beta_{k+1}|\neq 0$, we deduce that $\sigma_{k+2}\neq\sigma_{k+1}$ and I_{k+2}/I_{k+1} is a constant. This, again, contradicts regularity.
Thus, we deduce that **Case I** necessarily implies (2.6)

Case II: $(\alpha_k+\beta_k)+\beta_k\mu$ is a constant nonzero multiple of $(\alpha_{k+1}+\sigma_{k+2}\gamma_{k+1})+(\beta_{k+1}+\sigma_{k+2}\delta_{k+1})\mu$.
Thus, there exists $C_1\neq 0$ such that

$$\begin{aligned} \alpha_{k+1} &= C_1(\alpha_k+\beta_k)-\sigma_{k+2}\gamma_{k+1}, \\ \beta_{k+1} &= C_1\beta_k-\sigma_{k+2}\delta_{k+1}. \end{aligned}$$

Again, there are two possibilities in (2.4).

Subcase II.1: There exists $C_2 \neq 0$ such that

$$\begin{aligned} \gamma_{k+1} + \delta_{k+1}\mu &= C_2(\gamma_k + \delta_k + \delta_k\mu), \\ C_2((\alpha_k + \sigma_{k+1}\gamma_k) + (\beta_k + \sigma_{k+1}\delta_k)\mu) &= C_1(\alpha_k + \beta_k\mu). \end{aligned}$$

Therefore $\delta_{k+1} = C_2\delta_k$, $\gamma_{k+1} = C_2(\gamma_k + \delta_k)$ and

$$\begin{aligned} (C_2 - C_1)\alpha_k &= -C_2\sigma_{k+1}\gamma_k, \\ (C_2 - C_1)\beta_k &= -C_2\sigma_{k+1}\delta_k. \end{aligned}$$

If $C_1 \neq C_2$ then, again, I_{k+1}/I_k is a constant and regularity is lost. Hence necessarily $C_2 = C_1$ and, since $\gamma_k = \delta_k = 0$ is impossible, we have $\sigma_{k+1} = 0$. Thus, we may assume without loss of generality that $C_1 = C_2 = 1$ and obtain

$$\left.\begin{aligned} \sigma_{k+1} &= 0, \\ \alpha_{k+1} &= \alpha_k + \beta_k - \sigma_{k+2}(\gamma_k + \delta_k), \\ \beta_{k+1} &= \beta_k - \sigma_{k+2}\delta_k, \\ \gamma_{k+1} &= \gamma_k + \delta_k, \\ \delta_{k+1} &= \delta_k. \end{aligned}\right\} \tag{2.7}$$

Subcase II.2: $C_2 \neq 0$ exists so that

$$\begin{aligned} \gamma_{k+1} + \delta_{k+1}\mu &= C_2(\alpha_k + \beta_k\mu), \\ C_2((\alpha_k + \sigma_{k+1}\gamma_k) + (\beta_k + \sigma_{k+1}\delta_k)\mu) &= C_1(\gamma_k + \delta_k + \delta_k\mu). \end{aligned}$$

We deduce that $\gamma_{k+1} = C_2\alpha_k$, $\delta_{k+1} = C_2\beta_k$ and

$$\begin{aligned} \alpha_{k+1} &= (C_1 - \sigma_{k+2}C_2)\alpha_k, \\ \beta_{k+1} &= (C_1 - \sigma_{k+2}C_2)\beta_k. \end{aligned}$$

Therefore I_{k+2}/I_{k+1} is a constant multiple of I_{k+1}/I_k, and this cannot coexist with regularity – the proof is identical to that of Proposition 1. Hence, this subcase is impossible.

Case III: $(\alpha_k + \beta_k) + \beta_k\mu$ is a constant multiple of $(\gamma_k + \delta_k) + \delta_k\mu$.
It is obvious in that case that I_{k+1}/I_k is a constant and this is ruled out by regularity.

The above three cases exhaust all possibilities. Therefore, we deduce that for every $k = 0, 1, \ldots$ either (2.6) or (2.7) must hold. This, in particular, implies that

$$\delta_k \equiv \delta_0, \quad \gamma_k = \gamma_0 + k\delta_0, \qquad k = 0, 1, \ldots. \tag{2.8}$$

Suppose first that $\delta_0 = 0$. Hence, without loss of generality, $\gamma_k \equiv 1$ and

$$\frac{I_{k+1}(\mu)}{I_k(\mu)} = \alpha_k + \beta_k\mu.$$

Regularity thus requires $\beta_k \neq 0$ and this rules out (2.6). We deduce that $\delta_0 = 0$ implies (2.7) for all $k = 0, 1, \ldots$. This results in the explicit form

$$\left.\begin{array}{rcl} \sigma_{k+1} &=& 0, \\ \alpha_k &=& \alpha_0 + k\beta_0, \\ \beta_k &\equiv& \beta_0, \\ \gamma_k &\equiv& 1, \\ \delta_k &\equiv& 0. \end{array}\right\} \tag{2.9}$$

Next we consider the case $\delta_0 \neq 0$ and assume without loss of generality that $\delta_0 = 1$. Either (2.6) or (2.7) must hold for each $k \in \mathbb{Z}^+$ and we commence by assuming that integers

$$0 = m_0 < n_0 < m_1 < n_1 < \cdots$$

exist so that for all $\ell = 0, 1, \ldots$

$$\begin{array}{rcl} k \in \{m_\ell, m_\ell + 1, \ldots, n_\ell - 1\} & \Rightarrow & (2.6), \\ k \in \{n_\ell, n_\ell + 1, \ldots, m_{\ell+1} - 1\} & \Rightarrow & (2.7). \end{array}$$

Since $\beta_{k+1} = 0$ implies $\sigma_{k+1} = \sigma_{k+2}$ in (2.6), we deduce that

$$\sigma_k = \sigma_{m_\ell+1}, \qquad k = m_\ell + 1, \ldots, n_\ell. \tag{2.10}$$

By applying similar argument to (2.7) we deduce

$$\sigma_k = 0, \qquad k = n_\ell, n_\ell + 1, \ldots, m_{\ell+1}. \tag{2.11}$$

Thus, (2.7) implies that $\beta_k = \beta_{n_\ell}$, $k = n_\ell, n_\ell+1, \ldots, m_{\ell+1}-1$ and, to obtain $\beta_{m_{\ell+1}} = 0$, we need $\beta_{n_\ell} = \sigma_{m_{\ell+1}+1}$. But $\beta_{n_\ell} = \sigma_{m_\ell}$ and we deduce that $\sigma_{m_\ell} \equiv \sigma$, say, for all $\ell = 0, 1, \ldots$. Therefore, letting $g_k(\mu) = \alpha_k + \beta_k\mu$,

$$\begin{array}{rl} k = m_\ell, \ldots, n_\ell - 1: & g_k(\mu) \equiv \alpha_0 - \ell\sigma, \\ k = n_\ell, \ldots, m_{\ell+1} - 1: & g_k(\mu) = \alpha + \sigma(\gamma_0 + k - \ell - 1) + \sigma\mu. \end{array}$$

This formula is consistent with (2.3), but we need to check whether it is also consistent with the fact that $\{I_k(\mu)\}_{k=0}^\infty$ corresponds to a Hamburger moment sequence for every $\mu > 0$. Let

$$\tilde{I}_k(\mu) := \int_E x^{k+\mu}\,\mathrm{d}\psi(x) = I_k(\mu, \{x^j\}_{j=0}^\infty), \qquad k = 0, 1, \ldots.$$

Then, by [1], we need

$$\Delta_n := \det \begin{bmatrix} \tilde{I}_0(\mu) & \tilde{I}_1(\mu) & \cdots & \tilde{I}_n(\mu) \\ \tilde{I}_1(\mu) & \tilde{I}_2(\mu) & \cdots & \tilde{I}_{n+1}(\mu), \\ \vdots & \vdots & & \vdots \\ \tilde{I}_n(\mu) & \tilde{I}_{n+1}(\mu) & \cdots & \tilde{I}_{2n}(\mu) \end{bmatrix} > 0, \qquad n = 0, 1, \ldots, \qquad \mu > 0.$$

Let us first assume that $n_0 \geq 2$. Then, letting $h_k(\mu) = \gamma_k + \delta_k\mu$, and bearing in mind that $I_k = I_k(\,\cdot\,, \{(x-\sigma)^\ell\}_{\ell=0}^\infty)$,

$$\begin{aligned} I_1 &= \frac{g_0}{h_0} I_0, \quad I_2 = \frac{g_0 g_1}{h_0 h_1} I_0, \\ \tilde{I}_0 &= I_0, \quad \tilde{I}_1 = I_1 + \sigma I_0, \quad \tilde{I}_2 = I_2 + 2\sigma I_1 + \sigma^2 I_0 \end{aligned}$$

imply

$$\tilde{I}_1 = \left(\frac{\alpha_0}{\gamma_0 + \mu} + \sigma\right) \tilde{I}_0$$

and

$$\tilde{I}_2 = \left(\frac{\alpha_0^2}{(\gamma_0+\mu)(\gamma_0+\mu+1)} + 2\frac{\sigma\alpha_0}{\gamma_0+\mu} + \sigma^2\right) \tilde{I}_0,$$

therefore

$$\Delta_1 = -\frac{\alpha_0^2}{(\gamma_0+\mu)^2(\gamma_0+\mu+1)} \tilde{I}_0^2 < 0$$

for sufficiently large μ. We deduce that $n_0 \geq 2$ is impossible.

The remaining case is $n_0 = 1$. $\tilde{I}_1$ remains intact, whereas $\tilde{I}_2 = I_2 + \sigma I_1 + \sigma^2 I_0$ and

$$\Delta_1 = -\frac{\alpha_0(\alpha_0 + \sigma(\gamma_0+\mu))}{(\gamma_0+\mu)^2(\gamma_0+\mu+1)} \tilde{I}_0^2 < 0$$

for $\mu \gg 0$. We deduce that $0 = m_0 < n_0 < m_1 < \cdots$ is impossible.

Finally, we check the case whereby there exist

$$0 = n_0 < m_1 < n_1 < m_2 < \cdots$$

so that

$$\begin{aligned} k \in \{n_\ell, n_\ell + 1, \ldots, m_{\ell+1} - 1\} \quad &\Rightarrow \quad (2.7), \\ k \in \{m_\ell, m_\ell + 1, \ldots, n_\ell - 1\} \quad &\Rightarrow \quad (2.6). \end{aligned}$$

(2.10) and (2.11) follow as before and, in addition,

$$\begin{aligned} k = n_\ell, \ldots, m_{\ell+1} - 1: \qquad & g_k(\mu) = \alpha_0 + \sigma(k-\ell) + \sigma\mu, \\ k = m_\ell, \ldots, n_\ell - 1: \qquad & g_k(\mu) = \alpha_0 - \sigma(\gamma_0 + \ell - 1). \end{aligned}$$

Suppose that such $m_1 \geq 1$ exists and let

$$\mathrm{d}\varphi^*(x,\mu) := x^{m_1}\,\mathrm{d}\varphi(x,\mu).$$

Therefore

$$\tilde{I}_\ell^*(\mu) := \int_E x^\ell\,\mathrm{d}\varphi^*(x,\mu) = \tilde{I}_{\ell+m_1}(\mu), \qquad \ell = 0, 1, \ldots,$$

and, proceeding as before (replacing $\tilde{I}_0$ by $\tilde{I}_{m_1}$ etc.) we can prove that $\{\tilde{I}_\ell^*\}_{\ell=0}^\infty$ cannot be a moment sequence.

We conclude that no such m_1 exists, hence, necessarily, (2.7) *is true for all* $k = 0, 1, \ldots$. We obtain

$$\left.\begin{array}{rcl} \sigma_k & \equiv & 0, \\ \alpha_k & = & \alpha_0 + k\beta_0, \\ \beta_k & \equiv & \beta_0, \\ \gamma_k & = & \gamma_0 + k\delta_0, \\ \delta_k & \equiv & \delta_0. \end{array}\right\} \tag{2.12}$$

Note that (2.9) is a special case.

If $\delta_0 = 0$ we obtain (with $\gamma_0 = 1$)

$$g_k(\mu) = \alpha_0 + k\beta_0 + \beta_0\mu, \quad h_k(\mu) \equiv 1, \qquad k = 0, 1, \ldots$$

and this, in the terminology of [6], is the *Laguerre transformation.* On the other hand, if $\delta_0 \neq 0$ then, letting $\delta_0 = 1$,

$$g_k(\mu) = \alpha_0 + k\beta_0 + \beta_0\mu, \quad h_k(\mu) = \gamma_0 + k + \mu, \qquad k = 0, 1, \ldots,$$

namely the *Jacobi transformation* [6].

Theorem 2 The only transformations consistent with the asserted form of $\mathrm{d}\varphi$ are the Laguerre transformation and the Jacobi transformation. □

3 The measure $\mathrm{d}\varphi(x, \mu) = \mu^x \,\mathrm{d}\psi(x)$, $\mu > 0$

E is now a measurable subset of $(-\infty, \infty)$, otherwise we employ the notation from the previous section. Thus, differentiating

$$I_k(\mu, \vec{\rho}) = \int_E \rho_k(x)\mu^x \,\mathrm{d}\psi(x), \qquad k = 0, 1, \ldots,$$

with respect to μ we obtain

$$I_k'(\mu) = \int_E \rho_k(x)x\mu^{x-1} \,\mathrm{d}\psi(x) = \mu^{-1}\int_E \rho_k(x)((x - \sigma_{k+1}) + \sigma_{k+1})\mu^x \,\mathrm{d}\psi(x)$$

(here and elsewhere we abbreviate $I_k(\mu) = I_k(\mu, \vec{\rho})$) and deduce the equation

$$\mu I_k'(\mu) = I_{k+1}(\mu) + \sigma_{k+1}I_k(\mu), \qquad k = 0, 1, \ldots. \tag{3.1}$$

Assuming (2.3), we hence obtain from (3.1) the linear differential equation

$$I_k'(\mu) = \frac{1}{\mu}\left\{\sigma_{k+1} + \frac{\alpha_k + \beta_k\mu}{\gamma_k + \delta_k\mu}\right\} I_k(\mu) \tag{3.2}$$

We distinguish among the following cases:

Case I: $\gamma_k = 0$ and, without loss of generality, $\delta_k = 1$.

Therefore, by (3.2),

$$I_k'(\mu) = \left\{(\sigma_{k+1} + \beta_k)\frac{1}{\mu} + \frac{\alpha_k}{\mu^2}\right\} I_k(\mu)$$

and integration yields the explicit form

$$I_k(\mu) = I_k(\mu_0)\left(\frac{\mu}{\mu_0}\right)^{\sigma_{k+1}+\beta_k} \exp\left(-\alpha_k(\mu^{-1}-\mu_0^{-1})\right),$$

where $\mu_0 > 0$. In particular, choosing without loss of generality $\mu_0 = 1$, we obtain

$$I_k(\mu) = I_k(1)\mathrm{e}^{\alpha_k}\mu^{\sigma_{k+1}+\beta_k}\mathrm{e}^{-\alpha_k/\mu}. \tag{3.3}$$

Case II: $\delta_k = 0$ and, without loss of generality, $\gamma_k = 1$.

Again, we solve (3.2) explicitly with an initial condition at $\mu_0 = 1$, say, to obtain

$$I_k(\mu) = I_k(1)\mathrm{e}^{-\beta_k}\mu^{\sigma_{k+1}+\alpha_k}\mathrm{e}^{\beta_k\mu}. \tag{3.4}$$

Case III: $\gamma_k, \delta_k \neq 0$ and, without loss of generality, $\delta_k = 1$.

Therefore

$$I_k'(\mu) = \left\{\left(\sigma_{k+1} + \frac{\alpha_k}{\gamma_k}\right)\frac{1}{\mu} - \left(\frac{\alpha_k}{\gamma_k} - \beta_k\right)\frac{1}{\mu+\gamma_k}\right\} I_k(\mu),$$

with the solution

$$I_k(\mu) = I_k(1)(1+\gamma_k)^{\alpha_k/\gamma_k-\beta_k}\mu^{\sigma_{k+1}+\alpha_k/\gamma_k}(\mu+\gamma_k)^{-\alpha_k/\gamma_k+\beta_k}. \tag{3.5}$$

Let us suppose that we have case I for k and case II for $k+1$. Therefore

$$\frac{I_{k+1}(\mu)}{I_k(\mu)} = \frac{I_{k+1}(1)}{I_k(1)}\mathrm{e}^{\alpha_k+\beta_{k+1}}\mu^{\sigma_{k+1}-\sigma_{k+2}+\beta_k-\alpha_{k+1}}\mathrm{e}^{-(\alpha_k/\mu+\beta_{k+1}\mu)}.$$

But, since $\gamma_k = 0$, $\delta_k = 1$, we have

$$\frac{I_{k+1}(\mu)}{I_k(\mu)} = \frac{\alpha_k}{\mu} + \beta_k$$

and, necessarily, $\alpha_k/\mu + \beta_{k+1}\mu \equiv \text{const}$. We deduce that $\alpha_k = 0$, but this, in tandem with $\gamma_k = 0$, contradicts regularity.

Similar contradiction is obtained if case I follows case II.

Now, if case I is followed by case III then, to get rid of the exponential term, we require $\alpha_k = 0$, and this again contradicts regularity. Similarly, case II cannot be followed by case III. Finally, if case III is followed by case I (case II) then again we need to eliminate an exponential term, this requires $\alpha_{k+1} = 0$ ($\beta_{k+1} = 0$) and is inconsistent with regularity.

We deduce that if any of cases I–III holds for one k then it must hold for all $k = 0, 1, \ldots$.

Case I for all $k = 0, 1, \ldots$:

Therefore

$$\frac{I_{k+1}(\mu)}{I_k(\mu)} = \frac{I_{k+1}(1)}{I_k(1)}\mu^{\sigma_{k+2}-\sigma_{k+1}+\beta_{k+1}-\beta_k}\mathrm{e}^{(\alpha_k-\alpha_{k+1})(\mu-1)} = \frac{\alpha_k}{\mu} + \beta_k, \qquad k = 0, 1, \ldots.$$

We deduce that
$$\alpha_{k+1}=\alpha_k=\cdots=\alpha_0.$$
Moreover, there are exactly two possibilities. Either $\beta_k=0$ and $\sigma_{k+2}-\sigma_{k+1}+\beta_{k+1}-\beta_k=-1$ or $\alpha_0=0$. In the second case I_{k+1}/I_k is a constant and regularity is violated, hence necessarily $\beta_k=0$ for all $k=0,1,\ldots$. We hence deduce that $\sigma_{k+2}-\sigma_{k+1}=-1$, thus
$$\left.\begin{array}{rcl}\sigma_k&=&\sigma_1-k+1,\\ \alpha_k&\equiv&\alpha_0\neq 0,\\ \beta_k&\equiv&0.\end{array}\right\}\tag{3.6}$$
In other words,
$$I_k(\mu)=\left(\frac{\alpha_0}{\mu}\right)^k I_0(\mu),\qquad k=0,1,\ldots,$$
and $\rho_k(x)=(x-\sigma_1)_k$. Under a substitution $x\to -x+\sigma_1$, $\mu\to\alpha_0/\mu$ this yields *precisely* the Charlier transformation from [6].

Case II for all $k=0,1,\ldots$:

Since
$$\frac{I_{k+1}(\mu)}{I_k(\mu)}=\frac{I_{k+1}(1)}{I_k(1)}\mu^{\sigma_{k+2}-\sigma_{k+1}+\alpha_{k+1}-\alpha_k}\mathrm{e}^{(\beta_{k+1}-\beta_k)(\mu-1)}=\alpha_k+\beta_k\mu,$$
we deduce that $\beta_{k+1}=\beta_k=\cdots=\beta_0$. Again, there are two alternatives – either $\beta_0=0$ and this contradicts regularity (since I_{k+1}/I_k becomes a constant) or $\alpha_k=0$ and $\alpha_{k+1}=\sigma_{k+1}-\sigma_{k+2}+1$. We deduce that $\alpha_k\equiv 0$ and $\sigma_k=\sigma_1+k-1$. Therefore
$$\left.\begin{array}{rcl}\sigma_k&=&\sigma_1+k-1,\\ \alpha_k&\equiv&0,\\ \beta_k&\equiv&\beta_0\neq 0.\end{array}\right\}\tag{3.7}$$
We conclude that
$$I_k(\mu)=(\beta_0\mu)^k I_0(\mu),\qquad k=0,1,\ldots,$$
and $\rho_k(x)=(-1)^k(-x+\sigma_1)_k$. Therefore $x\to x-\sigma_1$, $\mu\to\mu/\beta_0$ yields, again, the Charlier transformation.

Case III for all $k=0,1,\ldots$.

We now have
$$\frac{I_{k+1}(\mu)}{I_k(\mu)}=C\mu^{\sigma_{k+2}-\sigma_{k+1}+\alpha_{k+1}/\gamma_{k+1}-\alpha_k/\gamma_k}\frac{(\mu+\gamma_k)^{\alpha_{k+1}/\gamma_{k+1}-\beta_k}}{(\mu+\gamma_{k+1})^{\alpha_{k+1}/\gamma_{k+1}-\beta_{k+1}}}=\frac{\alpha_k+\beta_k\mu}{\mu+\gamma_k},$$
where $C\neq 0$. Regularity implies that $\alpha_j/\gamma_j\neq\beta_j$ for all $j\in\mathbb{Z}^+$ (otherwise I_{k+1}/I_k is constant). Therefore, letting $\mu=-\gamma_k$ in
$$C\mu^{\sigma_{k+2}-\sigma_{k+1}+\alpha_{k+1}/\gamma_{k+1}-\alpha_k/\gamma_k}(\mu+\gamma_k)^{\alpha_k/\gamma_k-\beta_k+1}=(\alpha_k+\beta_k\mu)(\mu+\gamma_{k+1})^{\alpha_{k+1}/\gamma_{k+1}-\beta_{k+1}}$$
demonstrates at once that $\gamma_{k+1}=\gamma_k\equiv\gamma_0\neq 0$ and that either $\alpha_k=0$ or $\beta_k=0$. Thus, either
$$\alpha_k=0,\qquad\frac{\alpha_{k+1}}{\gamma_0}=1-\sigma_{k+2}+\sigma_{k+1},\qquad\beta_{k+1}-\sigma_{k+2}+\sigma_{k+1}\tag{3.8}$$

or

$$\frac{\alpha_{k+1}}{\gamma_0} = \frac{\alpha_k}{\gamma_0} + \sigma_{k+1} - \sigma_{k+2}, \qquad \beta_k = 0, \qquad \beta_{k+1} = \sigma_{k+1} - \sigma_{k+2} - 1. \tag{3.9}$$

Suppose that integers

$$0 = m_0 < n_0 < m_1 < n_1 < \cdots$$

exist so that (3.8) holds for all $k = m_\ell, \ldots, n_\ell - 1$ and (3.8) holds for $k = n_\ell, \ldots, m_{\ell+1} - 1$. The range of ℓ might be infinite or finite – in the latter case we assume that the largest value of m_ℓ or n_ℓ, as the case might be, is ∞. Thus, for example, if (3.8) is satisfied by all $k \in \mathbb{Z}^+$ then this corresponds to $m_0 = 0$, $n_0 = \infty$.

There is no loss of generality in assuming that (3.8) holds for $k = 0$, otherwise we map $x \to -x$. Hence, only the present case need be considered. Moreover, we may assume without loss of generality that $\sigma_1 = 0$, otherwise we replace σ_k by $\sigma_k - \sigma_1$ and shift $x \to x + \sigma_1$.

Let

$$M_\ell = \sum_{j=0}^{\ell} m_j, \qquad N_\ell = \sum_{j=0}^{\ell} n_j.$$

Long and tedious calculation affirms that

$$\left.\begin{array}{rcl} \alpha_k & \equiv & 0, \quad \beta_k = \beta_0 - k, \quad \gamma_k \equiv 0, \quad \delta_k \equiv 1, \\ \sigma_{k+1} & = & N_{\ell-1} - M_\ell + k, \\ \rho_k(x) & = & (-1)^{N_{\ell-1}-M_\ell+k}(-x)_{N_{\ell-1}-M_\ell+k}(x-\beta_0)_{M_\ell-N_{\ell-1}} \end{array}\right\} \; k = m_\ell, \ldots, n_\ell - 1; \tag{3.10}$$

$$\left.\begin{array}{rcl} \alpha_k & = & (k-\beta_0)\gamma_0, \quad \beta_k \equiv 0, \quad \gamma_k \equiv \gamma_0, \quad \delta_k \equiv 1, \\ \sigma_{k+1} & = & \beta_0 + N_\ell - M_\ell - k, \\ \rho_k(x) & = & (-1)^{N_\ell - M_\ell}(-x)_{N_\ell-M_\ell}(x-\beta_0)_{M_\ell-N_\ell+k} \end{array}\right\} \; k = n_\ell, \ldots, m_{\ell+1} - 1 \tag{3.11}$$

and

$$I_k(\mu) = (-1)^{N_{\ell-1}-M_\ell+k}\frac{(-\beta_0)_k}{(\gamma_0+\mu)^k}\mu^{N_{\ell-1}-M_\ell+k}\gamma_0^{M_\ell-N_{\ell-1}}, \qquad k = m_\ell, \ldots, n_\ell; \tag{3.12}$$

$$I_k(\mu) = (-1)^{N_\ell-M_\ell}\frac{(-\beta_0)_k}{(\gamma_0+\mu)^k}\mu^{N_\ell-M_\ell}\gamma_0^{M_\ell-N_\ell+k}, \qquad k = n_\ell, \ldots, m_{\ell+1}. \tag{3.13}$$

Note that both definitions match at $k = m_\ell$ and $k = n_\ell$ and that they are indeed consistent – as they should – with

$$\frac{I_{k+1}(\mu)}{I_k(\mu)} = \frac{\alpha_k + \beta_k\mu}{\gamma_0 + \mu}, \qquad k = 0, 1, \ldots.$$

We will now single out a measure $\mathrm{d}\varphi$ of the desired form that results in the required value of $\{I_k\}_{k\in\mathbb{Z}^+}$. Thus, we let

$$\mathrm{d}\varphi(x,\mu) = \mu^x \,\mathrm{d}\psi(x), \qquad x > 0,$$

where $\mathrm{d}\psi$ is an atomic measure with jumps of

$$(-1)^j\left(1+\frac{\mu}{\gamma_0}\right)^{-\beta_0}\frac{(-\beta_0)_j}{j!\gamma_0^j}$$

at $j=0,1,\ldots$. In other words, for every appropriate function f we have

$$\int_0^\infty f(x)\,\mathrm{d}\varphi(x,\mu)=\sum_{j=0}^\infty(-1)^j\frac{(-\beta_0)_j f(j)}{j!}\left(\frac{\mu}{\gamma_0}\right)^j.$$

In particular, substituting the values of ρ_k from (3.10) and (3.11) respectively, we obtain

$$\begin{aligned}I_k(\mu) &= \left(1+\frac{\mu}{\gamma_0}\right)^{-\beta_0}\sum_{j=N_{\ell-1}-M_\ell+k}^\infty(-1)^j\frac{(j-\beta_0)_{M_\ell-N_{\ell-1}}(-\beta_0)_j}{(j-N_{\ell-1}+M_\ell-k)!}\left(\frac{\mu}{\gamma_0}\right)^j,\\ &\qquad k=m_\ell,\ldots,n_\ell, \qquad (3.14)\\ I_k(\mu) &= \left(1+\frac{\mu}{\gamma_0}\right)^{-\beta_0}\sum_{j=N_\ell-M_\ell}^\infty(-1)^j\frac{(j-\beta_0)_{M_\ell-N_\ell+k}(-\beta_0)_j}{(j-N_\ell+M_\ell)!}\left(\frac{\mu}{\gamma_0}\right)^j,\\ &\qquad k=n_\ell,\ldots,m_{\ell+1}. \qquad (3.15)\end{aligned}$$

Our goal is to demonstrate that (3.12) is identical with (3.14) and (3.13) coincides with (3.15).

We now prove that (3.14) is the same as (3.12). Letting $s=M_\ell-N_{\ell-1}>0$ and commencing from (3.14), we have for every $k=m_\ell,\ldots,n_\ell$

$$\begin{aligned}I_k(\mu) &= \left(1+\frac{\mu}{\gamma_0}\right)^{-\beta_0}\sum_{j=k-s}^\infty(-1)^j\frac{(-\beta_0)_{j+s}}{(j+s-k)!}\left(\frac{\mu}{\gamma_0}\right)^j\\ &= (-1)^{k-s}(-\beta_0)_k\left(\frac{\mu}{\gamma_0}\right)^{k-s}\left(1+\frac{\mu}{\gamma_0}\right)^{-\beta_0}\sum_{j=0}^\infty(-1)^j\frac{(-\beta_0+k)_j}{j!}\left(\frac{\mu}{\gamma_0}\right)^j\\ &= (-1)^{k-s}(-\beta_0)_k\left(\frac{\mu}{\gamma_0}\right)^{k-s}\left(1+\frac{\mu}{\gamma_0}\right)^{-\beta_0}\times\left(1+\frac{\mu}{\gamma_0}\right)^{\beta_0-k}\\ &= (-1)^{k-s}\frac{(-\beta_0)_k}{(\gamma_0+\mu)^k}\mu^{k-s}\gamma_0^s.\end{aligned}$$

Therefore, we recover (3.12).

The proof of the coincidence of (3.13) and (3.15) is identical. We thus deduce that $\mathrm{d}\varphi$ is of the stipulated form.[3]

In order to identify $\mathrm{d}\varphi$ we note that, to be a Borel measure, the jumps must be nonnegative for all $j\in\mathbb{Z}^+$. There are exactly two possibilities. Firstly, $\beta_0,\gamma_0<0$ imply that

$$(-1)^j\frac{(-\beta_0)_j}{\gamma_0^j}=\frac{(|\beta_0|)_j}{|\gamma_0|^j}>0, \qquad j=0,1,\ldots.$$

In that case $\mu\in(0,|\gamma_0|)$ and we recover the *Meixner* transformation [6]. Secondly, $\beta_0=N$ is a nonnegative integer and γ_0. Thus,

$$(-1)^j\frac{(-\beta_0)_j}{\gamma_0^j}=\begin{cases}\frac{N!}{(N-j)!\gamma_0^j}>0 & : j=0,1,\ldots,N,\\ 0 & : j=N+1,N+2,\ldots.\end{cases}$$

[3] We have not proved that this $\mathrm{d}\varphi$ is the *unique* measure with this property. This is quite straightforward since it is easy to prove that it is determinate (e.g. with the Carleman criterion [4]) but irrelevant since transformations are determined by moment sequences.

This, in the terminology of [6], is the *Krawtchouk* transformation.

Theorem 3 The only transformations consistent with the asserted form of $\mathrm{d}\varphi$ are the Charlier, Meixner and Krawtchouk transformation. □

4 The measure $\mathrm{d}\varphi(x,\mu) = x^{\log_q \mu}\,\mathrm{d}\psi(x)$, $q \in (0,1)$

At first glance, this case is equivalent to $\mathrm{d}\varphi(x,\nu) = x^\nu\,\mathrm{d}\psi(x)$, subject to the transformation $\eta = \log_q \mu$. However, in that case the quotient of I_{k+1} and I_k will cease to be a rational $[1/1]$ function of μ, thereby violating our construction.

We consider $\mathrm{d}\varphi(x,\mu) = x^{\log_q \mu}\,\mathrm{d}\psi(x)$, where $q \in (0,1)$, subject to the condition that $I_k(\mu,\vec{\rho})$ is well-defined and bounded at $\mu = 0$. The last condition sounds strange – after all, $\log_q \mu$ becomes unbounded as $\mu = 0$ – but it is not! Recall that, by our assumption in this paper,

$$I_{k+1}(\mu,\vec{\rho}) = \frac{\alpha_k + \beta_k\mu}{\gamma_k + \delta_k\mu} I_k(\mu,\vec{\rho}), \qquad k = 0,1,\ldots,$$

hence we require that $\gamma_k \neq 0$, $k \in \mathbb{Z}^+$, and that $I_0(0,\vec{\rho})$ is bounded. However,

$$I_0(\mu,\vec{\rho}) = \int_E x^{\log_q \mu}\,\mathrm{d}\psi(x) = \int_E \mu^{\log_q x}\,\mathrm{d}\psi(x).$$

Letting $\mu \downarrow 0$, we obtain

$$I_0(0,\vec{\rho}) = \text{the jump of } \psi \text{ at } x = q,$$

and this is bounded because $\mathrm{d}\psi$ is a (bounded) Borel measure. Therefore, the only requirement is $\gamma_k \neq 0$, $k \in \mathbb{Z}^+$, and we thus lose no generality by requiring $\gamma_k \equiv 1$.

Since $1 + \log_q \mu = \log_q(q\mu)$, we have

$$I_{k+1}(\mu) = \int_E x^{\log_q \mu}(x - \sigma_{k+1})\rho_k(x)\,\mathrm{d}\psi(x) = I_k(q\mu) - \sigma_{k+1} I_k(\mu), \qquad k = 0,1,\ldots. \tag{4.1}$$

Recalling that the ratio of I_{k+1} and I_k is a rational $[1/1]$ function, (4.1) yields the identity

$$I_k(q\mu) = \left(\frac{\alpha_k + \beta_k\mu}{1 + \delta_k\mu} + \sigma_{k+1}\right) I_k(\mu), \qquad k = 0,1,\ldots. \tag{4.2}$$

We distinguish between the following cases:

Case I $\alpha_k + \sigma_{k+1} \neq 0$.

Let

$$A_k = \alpha_k + \sigma_{k+1} \neq 0, \qquad B_k = -\frac{\beta_k + \sigma_{k+1}\delta_k}{\alpha_k + \sigma_{k+1}}.$$

Hence, (4.2) and induction yield

$$I_k(\mu) = A_k^{-\ell} \frac{(-\delta_k\mu;q)_\ell}{(B_k\mu;q)_\ell} I_k(q^\ell\mu), \qquad \ell = 0,1,\ldots. \tag{4.3}$$

We now let $\ell \to \infty$. Since $q \in (0,1)$, $\lim_{\ell\to\infty} I_k(q^\ell \mu) = I_k(0)$. Therefore boundedness of $I_k(\mu)$ requires $A_k \equiv 1$ and we obtain

$$I_k(\mu,\vec{\rho}) = \frac{(-\delta_k\mu;q)_\infty}{(-(\beta_k+(1-\alpha_k)\delta_k)\mu;q)_\infty} I_k(0,\vec{\rho}), \quad \sigma_{k+1} = 1-\alpha_k \qquad k=1,2,\ldots. \tag{4.4}$$

We substitute (4.4) into (4.1), and this gives

$$\begin{aligned} &\frac{(\delta_{k+1}\mu;q)_\infty}{(-(\beta_{k+1}+(1-\alpha_{k+1})\delta_{k+1})\mu;q)_\infty} I_{k+1}(0) \\ = &\left\{ \frac{(-\delta_k q\mu;q)_\infty}{(-(\beta_k+(1-\alpha_k)\delta_k)q\mu;q)_\infty} - (1-\alpha_k)\frac{(-\delta_k\mu;q)_\infty}{(-(\beta_k+(1-\alpha_k)\delta_k)\mu;q)_\infty}\right\} I_k(0). \end{aligned}$$

Moreover, $I_{k+1}(0)/I_k(0) = \alpha_k$. Thus, $\alpha_k \neq 0$, otherwise $I_{k+1}(0) = 0 \Rightarrow I_{k+1} \equiv 0$, in defiance of regularity. We deduce thus that

$$\frac{(-\delta_{k+1}\mu;q)_\infty}{(-(\beta_{k+1}+(1-\alpha_{k+1})\delta_{k+1})\mu;q)_\infty} = \left(1+\frac{\beta_k\mu}{\alpha_k}\right)\frac{(-\delta_k q\mu;q)_\infty}{(-(\beta_k+(1-\alpha_k)\delta_k)\mu;q)_\infty}. \tag{4.5}$$

We distinguish between the following sub-cases, by comparing zeros and poles on both sides of (4.5)

Case I.1 $\alpha_k = 1$.

The expression (4.5) gives

$$\frac{(-\delta_{k+1}\mu;q)_\infty}{(-(\beta_{k+1}+(1-\alpha_{k+1})\delta_{k+1})\mu;q)_\infty} = \frac{(-\delta_k q\mu;q)_\infty}{(-\beta_k q\mu;q)_\infty}.$$

Therefore $\beta_{k+1} = q(\beta_k - (1-\alpha_{k+1})\delta_k$, $\delta_{k+1} = q\delta_k$ and $\sigma_{k+1} = 0$.

Case I.2 $\beta_k = \alpha_k\delta_k$.

This is inconsistent with regularity, since then $I_k(\mu) \equiv \alpha_k$.

Case I.3 $\beta_k = \alpha_k\delta_{k+1}$.

We now have

$$\frac{(-\delta_{k+1}q\mu;q)_\infty}{(-(\beta_{k+1}+(1-\alpha_{k+1})\delta_{k+1})\mu;q)_\infty} = \frac{(-\delta_k q\mu;q)_\infty}{(-(\beta_k+(1-\alpha_k)\delta_k)\mu;q)_\infty}.$$

Therefore $\delta_{k+1} = \delta_k$ and we are back to the case I.2, which is irregular.

Since the above three subcases exhaust all possibilities, we deduce that only I.1 may happen, hence it *must* occur for all $k \in \mathbb{Z}^+$. In other words, we have

$$\left.\begin{array}{rcl} \sigma_{k+1} & \equiv & 0, \\ \alpha_k & \equiv & 1, \\ \beta_k & = & q^k\beta_0, \\ \gamma_k & \equiv & 1, \\ \delta_k & = & q^k\delta_0, \qquad k=0,1,\ldots. \end{array}\right\} \tag{4.6}$$

Moreover, the moments are

$$I_k(\mu) = \frac{(-\beta_0\mu;q)_k}{(-\delta_0\mu;q)_k} I_k(0), \qquad k=0,1,\ldots. \tag{4.7}$$

Let us suppose that $\beta_0 \neq 0$ and that $\mathrm{d}\psi$ is an atomic measure with jumps of

$$\frac{(-\beta_0\mu;q)_\infty}{(-\delta_0\mu;q)_\infty} \times \frac{\left(\frac{\delta_0}{\beta_0};q\right)_\ell}{(q;q)_\ell}(-\beta_0)^\ell$$

at q^ℓ, $\ell \in \mathbb{Z}^+$. Thus, by the Gauß–Heine theorem [14],

$$I_k(\mu) = \frac{(-\beta_0\mu;q)_\infty}{(-\delta_0\mu;q)_\infty} {}_1\phi_0\left[\begin{matrix}\frac{\delta_0}{\beta_0}; \\ -; \end{matrix}\; q, -\beta_0 q^k \mu\right] = \frac{(-\beta_0\mu;q)_k}{(-\delta_0\mu;q)_k}, \qquad k=0,1,\ldots,$$

where ${}_1\phi_0$ is a generalized basic hypergeometric function. Hence, the moments are as required by (4.7).

In order for $\mathrm{d}\psi$ to be a Borel measure we require that the jumps are nonnegative. If $\beta_0 > 0$ then this implies $\delta_0 q^\ell \geq \beta_0$ for all $\ell \in \mathbb{Z}^+$. Therefore $\delta_0 > 0$ and only a finite number of jumps might be nonzero. The latter is possible only if $\delta_0 = q^{-N}\beta_0$ for some $N \in \mathbb{Z}^+$ and we recover the *q-Krawtchouk transformation* [6]. On the other hand, $\beta_0 < 0$ and nonnegative jumps can coexist with any $\delta_0 > \beta_0$ and we obtain the *Wall transformation* [6]. The support of the μs is the interval $D = (0, |\beta_0|^{-1})$, whereas $E = (0,1)$, thus the transformation reads

$$T\left\{\sum_{k=0}^m d_k(-\beta_0 x;q)_k(-\delta_0 q^k x;q)_{m-k}\right\} = \sum_{k=0}^m d_k x^k$$

and it maps polynomials with zeros in D into polynomials with zeros in E – compare with (4.1) in [6].

Finally, we consider the case $\beta_0 = 0$. We derive this as a limiting case of the Wall distribution. Letting $\beta_0 \uparrow 0$, we obtain jumps of

$$\frac{1}{(-\delta_0\mu;q)_\infty}\delta_0^\ell q^{\frac{1}{2}(\ell-1)\ell}$$

at q^ℓ, $\ell \in \mathbb{Z}^+$. Thus, $\delta_0 > 0$ is necessary and sufficient for $\mathrm{d}\psi$ to be a measure. The transformation becomes

$$T\left\{\sum_{k=0}^m d_k(-\delta_0 q^k x;q)_{m-k}\right\} = \sum_{k=0}^m d_k x^k \tag{4.8}$$

and it maps zeros from $[0,\infty)$ to $(0,1)$. We call (4.8) – the only transformation from the three aforementioned choices of ω that has been missed in [6] – the *Wall$_0$ transformation.*

Theorem 4 The only transformations consistent with the asserted form of $\mathrm{d}\varphi$ are the Wall, Wall$_0$ and q-Krawtchouk transformations.

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