Explicit representations of biorthogonal polynomials

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Abstract Given a parametrised weight function $\omega(x,\mu)$ such that the quotients of its consecutive moments are Möbius maps, it is possible to express the underlying biorthogonal polynomials in a closed form [5]. In the present paper we address ourselves to two related issues. Firstly, we demonstrate that, subject to additional assumptions, every such ω obeys (in x) a linear differential equation whose solution is a generalized hypergeometric function. Secondly, using a generalization of standard divided differences, we present a new explicit representation of the underlying biorthogonal polynomials.

AMS (MOS) *Mathematics Subject Classification.* Primary 42C05; Secondary 33C45.

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1 Introduction

Let us consider a one-parametric family of weight functions $\omega(x,\mu) \geq 0$, $x \geq 0$, $\mu_- < \mu < \mu_+$. We define a *biorthogonal polynomial* $p_n \in P_n \times (\mu_-,\mu_+)^n$, $p_n \not\equiv 0$, by means of the orthogonality relations

$$\int_0^\infty p_n(x;\mu_1,\mu_2,\ldots,\mu_n)\omega(x,\mu_\ell)\,\mathrm{d}x = 0, \qquad \ell = 1,2,\ldots,n, \tag{1.1}$$

where $\mu_1,\mu_2,\ldots,\mu_n$ are distinct. (Here – and elsewhere in this paper – P_n denotes the set of all nth degree polynomials.)

The theory of general biorthogonal polynomials has been described in detail in [4] and further developments have been reported in [5, 6]. It possesses several interesting applications, ranging from numerical methods for ordinary differential equations [3, 8] to approximation theory [10] to location of zeros of polynomial transformations and orthogonal expansions [5, 9, 11]. An exposition of biorthogonal functions and their applications features in Claude Brezinski's monograph [1]. Herewith we revisit these elements of the theory that are germane to the work of the present paper.

Biorthogonal polynomials exist and are unique (up to a nonzero multiplicative constant) if and only if

$$\det \begin{bmatrix} \mathfrak{m}_0(\mu_1) & \mathfrak{m}_1(\mu_1) & \cdots & \mathfrak{m}_{m-1}(\mu_1) \\ \mathfrak{m}_0(\mu_2) & \mathfrak{m}_1(\mu_2) & \cdots & \mathfrak{m}_{m-1}(\mu_2) \\ \vdots & \vdots & & \vdots \\ \mathfrak{m}_0(\mu_m) & \mathfrak{m}_1(\mu_m) & \cdots & \mathfrak{m}_{m-1}(\mu_m) \end{bmatrix} \neq 0$$

for every distinct $\mu_1,\mu_2,\ldots,\mu_n \in (\mu_-,\mu_+)$, where

$$\mathfrak{m}_\ell(\mu) = \int_0^\infty x^\ell \omega(x,\mu)\,\mathrm{d}x$$

is the ℓth moment of ω, $\ell \in \mathbb{Z}^+$ [4]. The zeros of p_n need not, in general, reside in $(0,\infty)$. However, all the zeros are in the support if ω is *strictly sign consistent (SSC)*, i.e. if

$$\det \begin{bmatrix} \omega(x_1,\mu_1) & \omega(x_2,\mu_1) & \cdots & \omega(x_m,\mu_1) \\ \omega(x_1,\mu_2) & \omega(x_2,\mu_2) & \cdots & \omega(x_m,\mu_2) \\ \vdots & \vdots & & \vdots \\ \omega(x_1,\mu_m) & \omega(x_2,\mu_m) & \cdots & \omega(x_m,\mu_m) \end{bmatrix} \neq 0$$

for every $n \geq 1$ and all monotone sequences $0 < x_1 < x_2 < \cdots < x_n$, $\mu_- < \mu_1 < \mu_2 < \cdots < \mu_n < \mu_+$ [4].

Suppose that $p_n(\,\cdot\,;\mu_1,\mu_2,\ldots,\mu_n)$ is explicitly known for every distinct $\mu_1,\ldots,\mu_n \in (\mu_-,\mu_+)$ and that this explicit form can be extended to all $\mu_1,\mu_2,\ldots,\mu_n \in \mathbb{R}$.[3] In other words, given an arbitrary n-tuple $[\mu_1,\mu_2,\ldots,\mu_n] \in \mathbb{R}^d$ we can produce a p_n or, rephrasing again, we explicitly know a transformation $\mathfrak{T} : P_n \to P_n$ such that

$$\mathfrak{T}\left\{\prod_{\ell=1}^n (\,\cdot\, - \mu_\ell)\right\} = p_n(\,\cdot\,;\mu_1,\mu_2,\ldots,\mu_n). \tag{1.2}$$

[3] It is immaterial whether the definition of ω can be continued to all $\mu \in \mathbb{R}$.

If, in addition, we know that ω is SSC – hence that all zeros of p_n reside in $(0,\infty)$ – we deduce that $\mathfrak{T}$ maps polynomials with all their zeros in (μ_-,μ_+) into polynomials with all their zeros in $(0,\infty)$ [5].

To date, the most efficacious mechanism for the generation of an explicit form of p_n has been presented in [5], leading to a long list of interesting transformations of the form (1.2). Thus, having normalized $\mathfrak{m}_0(\mu)\equiv 1$, we asume that there exist real sequences $\{\alpha_n\}_{n\in\mathbb{Z}^+}$, $\{\beta_n\}_{n\in\mathbb{Z}^+}$, $\{\gamma_n\}_{n\in\mathbb{Z}^+}$ and $\{\delta_n\}_{n\in\mathbb{Z}^+}$ such that

$$\alpha_k\delta_\ell-\beta_k\gamma_\ell\neq 0,\qquad k=0,1,\ldots,\ell,\quad \ell\in\mathbb{Z}^+, \tag{1.3}$$

and

$$\mathfrak{m}_n(\mu)=\prod_{\ell=0}^{n-1}\frac{\alpha_\ell+\mu\beta_\ell}{\gamma_\ell+\mu\delta_\ell},\qquad n\in\mathbb{Z}^+.$$

In other words,

$$\mathfrak{m}_{n+1}(\mu)=\frac{\alpha_n+\beta_n\mu}{\gamma_n+\delta_n\mu}\mathfrak{m}_n(\mu),\qquad n\in\mathbb{Z}^+. \tag{1.4}$$

We call weight functions that possess property (1.4) *Möbius quotient functions (MQF).*

It has been proved in [5] that, as long as ω is a Möbius quotient function, the transformation (1.2) assumes the form

$$\mathfrak{T}\left\{\sum_{k=0}^{n} f_k\prod_{i=0}^{k-1}g_i(x)\prod_{j=k}^{n-1}h_j(x)\right\}=\sum_{k=0}^{n}f_kx^k, \tag{1.5}$$

where

$$g_k(\mu)=\alpha_k+\mu\beta_k,\quad h_k(\mu)=\gamma_k+\mu\delta_k,\qquad k\in\mathbb{Z}^+,$$

and where $f_0,f_1,\ldots,f_n$ are arbitrary.

Several MQFs that share strict sign consistency have been presented in [5], for example $\omega_1(x,\mu)=x^\mu$, $\omega_2(x,\mu)=\mu^x$ and $\omega_3(x,\mu)=x^{\log_q\mu}$, $q\in(0,1)$. Moreover, the theme of [8] was the identification of all possible weight functions $\alpha(x)$ such that $\omega_j(x,\mu)\alpha(x)$ is MQF and SSC for some $j\in\{1,2,3\}$. The present paper takes yet another step toward our ultimate goal of identifying all MQFs.

We commence our analysis in Section 2 with the assumption that the coefficients α_n, β_n, γ_n and δ_n are linear in n. This yields a linear ordinary differential equation with polynomial coefficients that must be satisfied by ω and that can be solved explicitly. We demonstrate that, up to a transformation of variables, nothing can be gained beyond a known transformation from [5].

In Section 3 we expand our framework to polynomial coefficients α_n, β_n, γ_n, δ_n. The weight function ω must again obey a linear ordinary differential equation, which we solve explicitly in terms of a generalized hypergeometric function.

Finally, in Section 4 we return to the representation (1.5), in order to derive an alternative form of the transformation and of the underlying biorthogonal polynomial. Let

$$\tilde{q}(x)=\frac{\prod_{k=1}^{n}(x-\mu_k)}{\prod_{k=0}^{n-1}(\gamma_k+x\delta_k)}. \tag{1.6}$$

We prove that

$$p_n(x;\mu_1,\mu_2,\ldots,\mu_n)=\sum_{k=0}^{n}F_k[\tilde q_0,\tilde q_1,\ldots,\tilde q_k]x^k,$$

where $\tilde q_k=\tilde q(-\alpha_k/\beta_k)$, $k=0,1,\ldots,n$, and $F_0,F_1,\ldots$ are a generalization of the familiar divided differences.

2 Linear Möbius quotient functions

Let us suppose that the moment sequence of the weight function ω obeys (1.4) with

$$\alpha_n=a_0-na_1,\quad \beta_n=b_0-nb_1,\quad \gamma_n=c_0-nc_1,\quad \delta_n=d_0-nd_1,\qquad n\in\mathbb{Z}^+. \tag{2.1}$$

We further assume that $\omega(\,\cdot\,,\mu)\in C^1[0,\infty)$ and that $\omega(0,\mu)\equiv 0$ – the last assumption does not lead to severe loss of generality, since it can be always forced by shifting $x\mapsto x+\varepsilon$ for some $\varepsilon>0$, smoothly continuing ω for $x\in[0,\varepsilon)$ so that it vanishes at the origin and ultimately driving ε to zero. We say that ω is a *linear* MQF.[4]

We rewrite (1.4) in the form

$$(\gamma_n+\delta_n\mu)\int_0^\infty x^{n+1}\omega(x,\mu)\,\mathrm{d}x=(\alpha_n+\beta_n\mu)\int_0^\infty x^n\omega(x,\mu)\,\mathrm{d}x,\qquad \mu_-<\mu<\mu_+, \tag{2.2}$$

multiply by $(-t)^n/n!$ and sum for $n=0,1,\ldots$. The outcome is the identity

$$\begin{aligned}&\int_0^\infty\{(a_0-c_0x)+\mu(b_0-d_0x)\}\omega(x,\mu)\mathrm{e}^{-tx}\,\mathrm{d}x \\ =\;&-t\int_0^\infty\{(a_1-c_1x)+\mu(b_1-d_1x)\}x\omega(x,\mu)\mathrm{e}^{-tx}\,\mathrm{d}x.\end{aligned} \tag{2.3}$$

Let

$$\omega_\ell(x,\mu)=x^\ell\{(a_\ell-c_\ell x)+\mu(b_\ell-d_\ell x)\}\omega(x,\mu),\qquad \ell=0,1,\quad \mu\in(\mu_-,\mu_+). \tag{2.4}$$

We can thus rewrite (2.3) as

$$\mathcal{L}\{\omega_0(\,\cdot\,,\mu)\}(t)=-t\mathcal{L}\{\omega_1(\,\cdot\,,\mu)\}(t),$$

where $\mathcal{L}$ is the Laplace transform. However, for every $f\in C^1$,

$$\mathcal{L}\{f'\}(t)=t\mathcal{L}\{f\}(t)-f(0)$$

and we use the smoothness of ω and $\omega_1(0,\mu)\equiv 0$ to argue that

$$\mathcal{L}\left\{\frac{\partial}{\partial x}\omega_1(\,\cdot\,,\mu)\right\}(t)=-\mathcal{L}\{\omega_0(\,\cdot\,,\mu)\}(t).$$

Taking inverse transforms, we thus deduce that

$$\frac{\partial}{\partial x}\omega_1(x,\mu)=-\omega_0(x,\mu),\qquad x\ge 0,\quad \mu\in(\mu_-,\mu_+). \tag{2.5}$$

[4] The phrase ‘linear’ refers, of course, to the coefficients α_n, β_n etc., not to ω itself!

We finally substitute the explicit values of ω_0 and ω_1 from (2.3), and this yields a linear ordinary differential equation for ω,

$$\begin{aligned} & x\{(a_1+\mu b_1)-(c_1+\mu d_1)x\}\frac{\partial}{\partial x}\omega(x,\mu) \\ + \quad & \{[(a_0+\mu b_0)+(a_1+\mu b_1)]-[(c_0+\mu d_0)+2(c_1+\mu d_1)]x\}\,\omega(x,\mu)=0. \end{aligned} \tag{2.6}$$

This can be solved and, in tandem with the initial condition $\omega(0,\mu)\equiv 0$, results in a unique value of ω.

For example, letting

$$\alpha_n=n,\quad \beta_n\equiv 1,\quad \gamma_n=n+1,\quad \delta_n\equiv 1,\qquad n\in\mathbb{Z}^+,$$

we obtain

$$a_0=0,\ a_1=-1,\quad b_0=1,\ b_1=0,\quad c_0=1,\ c_1=-1,\quad d_0=1,\ d_1=0$$

and (2.6) becomes

$$\frac{\partial}{\partial x}\omega(x,\mu)=\frac{\mu-1}{x}\omega(x,\mu),$$

with the solution (up to a multiplicative constant) $\omega(x,\mu)=x^{\mu-1}$ (where $x>0$, $\mu>-1$) – the Jacobi transformation (with $\alpha=0$) from [5].

With greater generality, we observe that, up to a multiplicative constant, the nonzero solution of

$$x(\sigma_0-\sigma_1x)y'+(\rho_0-\rho_1x)y=0,\quad x\ge 0,\qquad y(0)=0,$$

is

$$y(x)=|x|^{-\rho_0/\sigma_0}|\sigma_0-\sigma_1x|^{\rho_0/\sigma_0-\rho_1/\sigma_1}$$

– subject, of course, to the above expression being well defined, i.e. to $\rho_0/\sigma_0<0$. However, if $\rho_0/\sigma_0<\rho_1<\sigma_1$ and the point $\sigma_0\sigma_1>0$ lies within the support of ω then the solution blows up in a finite time, and this is forbidden.

In our case

$$\frac{\rho_0}{\sigma_0}=1+\frac{a_0+\mu b_0}{a_1+\mu b_1},$$

hence $\rho_0/\sigma_0<0$ is equivalent to

$$\frac{a_0+\mu b_0}{a_1+\mu b_1}+1<0.$$

Moreover,

$$\frac{\rho_0}{\sigma_0}-\frac{\rho_1}{\sigma_1}=-1+\frac{a_0+\mu b_0}{a_1+\mu b_1}-\frac{c_0+\mu d_0}{c_1+\mu d_1}.$$

Particular substitutions readily yield an explicit form of ω. For example, we can use

$$\mathbf{a}=\begin{bmatrix}1\\0\end{bmatrix},\quad \mathbf{b}=\begin{bmatrix}0\\-1\end{bmatrix},\quad \mathbf{c}=\begin{bmatrix}1\\0\end{bmatrix},\quad \mathbf{d}=\begin{bmatrix}1\\-1\end{bmatrix},$$

and this results in

$$\omega(x,\mu) = x^{1-\frac{1}{\mu}}, \qquad \mu > 1.$$

More general is a choice of **a**, **b**, **c** and **d** so that $\omega(x,\mu) = cx^{\nu(\mu)}$, $x \in [x_-, x_+] \subset [0,\infty)$ for some $c > 0$ and a function $\nu(\mu) \not\equiv \text{const}$. Of course, we can easily verify that

$$\mathfrak{m}_n(\mu) = \frac{1}{n+\nu(\mu)+1}\left(x_+^{n+\nu(\mu)} - x_-^{n+\nu(\mu)}\right), \qquad n \in \mathbb{Z}^+,$$

and this is consistent with an MQF if and only if $x_- = 0$, $x_+ = 1$ and ν is a Möbius quotient as a function of μ. The following calculation is thus intended merely as an illustration of how our technique can be implemented for specific examples.

Since $\omega(x,\mu) = x^{\nu(\mu)}$, we need $\rho_0\sigma_1 = \rho_1\sigma_0$.

$$a_1c_0 = (a_0 - a_1)c_1, \tag{2.7}$$
$$b_1d_0 = (b_0 - b_1)d_1 \tag{2.8}$$

and

$$(a_0 - a_1)d_1 + (b_0 - b_1)c_1 = a_1d_0 + b_1c_0. \tag{2.9}$$

Let us first assume that $a_1b_1 \neq 0$. It now follows by substituting (2.7) and (2.8) into (2.9) that

$$(a_0b_1 - a_1b_0)(a_1d_1 - b_1c_1) = 0.$$

Suppose first that $a_0b_1 = a_1b_0$. Then

$$\frac{\rho_0}{\sigma_0} \equiv 1 + \frac{a_0}{a_1},$$

hence $\nu \equiv \text{const}$, a contradiction. We thus deduce that $a_1d_1 = b_1c_1$. By letting $c_1 = \kappa a_1$ for some constant $\kappa \in \mathbb{R}$ we deduce the explicit expression

$$d_1 = b_1\kappa, \quad c_0 = (a_0 - a_1)\kappa, \quad d_0 = (b_0 - b_1)\kappa, \tag{2.10}$$

where $\kappa \neq 0$ is arbitrary – as are a_0, a_1, b_0 and b_1. We deduce from (2.10) that

$$\begin{aligned}
\alpha_n &= a_0 - a_1n, \\
\beta_n &= b_0 - b_1n, \\
\gamma_n &= \kappa(a_0 - a_1) - \kappa a_1n, \\
\delta_n &= \kappa(b_0 - b_1) - \kappa b_1n, \qquad n \in \mathbb{Z}^+,
\end{aligned}$$

and

$$\frac{\mathfrak{m}_{n+1}(\mu)}{\mathfrak{m}_n(\mu)} = \kappa^{-1}\frac{(a_0 - a_1n) + (b_0 - b_1n)\mu}{((a_0 - a_1) - a_1n) + ((b_0 - b_1) - b_1n)\mu},$$

$$\nu(\mu) = -1 - \frac{a_0 + \mu b_0}{a_1 + \mu b_1}. \tag{2.11}$$

We conclude that

$$\begin{aligned}
g_n(\mu) &= -(a_1 + b_1\mu)\left(-\frac{a_0 + b_0\mu}{a_1 + b_1\mu} + n\right), \\
h_n(\mu) &= -\kappa(a_1 + b_1\mu)\left(-\frac{a_0 + b_0\mu}{a_1 + b_1\mu} + n + 1\right), \qquad n \in \mathbb{Z}^+.
\end{aligned}$$

Moreover, $a_1b_0 \neq a_0b_1$ readily implies (1.3).

The outstanding case is $a_1b_1 = 0$. Assume first that $a_1 = 0$. If $a_0 \neq 0$ we deduce from (2.7) that $c_1 = 0$. Thus, it follows from (2.9) that $d_1 = b_1c_0/a_0$ and substitution into (2.8) results in $d_0 = (b_0 - b_1)c_0/a_0$. All this is consistent with (2.10) with $\kappa = c_0/a_0$. On the other hand, if both a_0 and a_1 vanish then simple calculation affirms that

$$(b_0 - b_1)c_1 = b_1c_0, \qquad (b_0 - b_1)d_1 = b_1d_1$$

and $b_1 \neq 0$ (otherwise $b_0 \neq 0$, thus $c_0 = d_0 = 0$, and this is ruled out). Hence, after brief manipulation, we derive the explicit form

$$\mathfrak{m}_n(\mu) = \frac{b_0}{b_0 - nb_1}\left(\frac{b_1\mu}{c_1 + d_1\mu}\right)^n \mathfrak{m}_0(\mu), \qquad n \in \mathbb{Z}^+.$$

It is interesting to point out that, as well as being consistent with the stipulated form of ω, this case fits (subject to a trivial change of variable in μ) into the framework of Example E from [4], namely the instance when $\omega(x,\mu)\,\mathrm{d}x = \mathrm{d}\psi(x/\mu)$, in which case $\mathfrak{m}_n(\mu) = c_n\mu^n$, $n \in \mathbb{Z}^+$, where c_n is the nth moment of $\mathrm{d}\psi$.

The explicit form (2.10) is also recovered when $b_1 = 0$ and $b_0 \neq 0$. Since $b_0 \neq 0$, (2.8) yields $d_1 = 0$, (2.9) gives $c_1 = d_0a_1/b_0$ and, finally, (2.7) implies that $c_0 = (a_0 - a_1)d_0/b_0$. Hence $\kappa = d_0/b_0$. Finally, $b_0 = b_1 = 0$ can be similarly reduced to Example E from [4].

The function $\nu(\mu)$ being strictly monotone, we deduce strict sign consistency of ω from strict sign consistency of the function x^μ [12, 13]. Moreover,

$$\prod_{i=0}^{k-1} g_i(x)\prod_{j=k}^{n-1} h_j(x) = (-1)^{n+1}\kappa^{n-k}(a_1 + b_1x)^n \frac{\left(-\frac{a_0+b_0x}{a_1+b_1x}\right)_{n+1}}{k - \frac{a_0+b_0x}{a_1+b_1x}}, \qquad k = 0, 1, \ldots, n,$$

where

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1), \qquad a \in \mathbb{C}, \quad n \in \mathbb{Z}^+,$$

is the familiar *Pocchammer symbol*. Substitution in (1.5) thus yields the transformation

$$\mathfrak{T}\left\{\left(\prod_{\ell=0}^{n}((a_1 + b_1x)\ell - (a_0 + b_0x))\right)\sum_{k=0}^{n}\frac{\kappa^{-k}f_k}{(a_1 + b_1x)k - (a_0 + b_0x)}\right\} = \sum_{k=0}^{n} f_kx^k.$$

This, however, can be converted, by a change of variable, to the Jacobi transformation with $\alpha = 0$ [5], which can be rendered in the form

$$\mathfrak{T}\left\{(x)_{n+1}\sum_{k=0}^{n}\frac{f_k}{k + x}\right\} = \sum_{k=0}^{n} f_kx^k.$$

The case $\rho_0\sigma_1 \neq \rho_1\sigma_0$ can be resolved explicitly by similar techniques.

3 Polynomial Möbius quotient functions

The approach of the last section can be generalized to cater for a more extensive family of MQFs. Thus, suppose that the coefficients can be expanded in the form

$$\begin{aligned}\alpha_n &= \sum_{\ell=0}^{\infty} a_\ell (-n)_\ell,\\ \beta_n &= \sum_{\ell=0}^{\infty} b_\ell (-n)_\ell,\\ \gamma_n &= \sum_{\ell=0}^{\infty} c_\ell (-n)_\ell,\\ \delta_n &= \sum_{\ell=0}^{\infty} d_\ell (-n)_\ell,\end{aligned}$$

and that the functions

$$a(z) = \sum_{\ell=0}^{\infty} a_\ell z^\ell, \quad b(z) = \sum_{\ell=0}^{\infty} b_\ell z^\ell, \quad c(z) = \sum_{\ell=0}^{\infty} c_\ell z^\ell, \quad d(z) = \sum_{\ell=0}^{\infty} d_\ell z^\ell$$

are analytic about the point $z = 0$. We further assume that $\omega(\,\cdot\,,\mu) \in C^\infty[0,\infty)$ and that

$$\frac{\partial^\ell}{\partial x^\ell}\omega(0,\mu) \equiv 0, \qquad \ell = 0, 1, \ldots. \tag{3.1}$$

The last condition, in tandem with infinite differentiability, is restrictive. This can be alleviated along the lines of the last section, by letting

$$\omega_\varepsilon(x,\mu) = \begin{cases} \omega(x-\varepsilon,\mu) & : \varepsilon \le x,\\ \sum_{\ell=0}^{\infty} \theta_\ell \mathrm{e}^{-(\ell+1)x^{-2}} & : 0 < x \le \varepsilon,\end{cases}$$

where $\theta_0, \theta_1, \ldots$ are chosen so as to ensure infinite differentiability at $x = \varepsilon$,[5] and letting $\varepsilon \to 0$ by the end of the procedure. The situation is simplified if – as is the case further in this section – it is assumed that a, b, c and d are polynomials, since then it is enough to insist that $\omega(\,\cdot\,,\mu) \in C^r[0,\infty)$ and that (3.1) holds for $\ell = 0, 1, \ldots, s-1$, where

$$s = \max\{\deg a, \deg b, \deg c, \deg d\}.$$

We again multiply (2.2) by $(-t)^n/n!$ and sum up for all integer nonnegative n. This results in the identity

$$\int_0^\infty x\{c(tx) + \mu d(tx)\}\omega(x,\mu)\mathrm{e}^{-tx}\,\mathrm{d}x = \int_0^\infty \{a(tx) + \mu b(tx)\omega(x,\mu)\mathrm{e}^{-tx}\,\mathrm{d}x. \tag{3.2}$$

Expanding

$$x\{c(tx) + \mu d(tx)\} \qquad \text{and} \qquad \{a(tx) + \mu b(tx)\}$$

[5] The existence of such coefficients follows easily from the Müntz theorem [2].

in powers of t, we deduce from (3.2) that

$$\sum_{\ell=0}^{\infty} t^{\ell} \int_0^{\infty} \omega_\ell(x,\mu) \mathrm{e}^{-tx}\, \mathrm{d}x = 0, \qquad \ell \in \mathbb{Z}^+, \tag{3.3}$$

where ω_ℓ has been given by (2.4). Since (2.4) and (3.1) combine to produce

$$\frac{\partial^j}{\partial x^j}\omega_\ell(0,\mu) = 0, \qquad j = 0,1,\ldots,\ell-1,$$

(indeed, ω_ℓ has a factor x^ℓ) we conclude that

$$t^\ell \int_0^\infty \omega_\ell(x,\mu)\mathrm{e}^{-tx}\,\mathrm{d}x = \mathcal{L}\left\{\frac{\partial^\ell}{\partial x^\ell}\omega_\ell(x,\mu)\right\}(t), \qquad \ell \in \mathbb{Z}^+.$$

Taking inverse transforms, (3.3) becomes

$$\sum_{\ell=0}^{\infty} \frac{\partial^\ell}{\partial x^\ell}\omega_\ell(x,\mu) = 0. \tag{3.4}$$

Let us set

$$P_\ell(\mu) = a_\ell + b_\ell\mu, \quad Q_\ell(\mu) = c_\ell + d_\ell\mu, \qquad \ell \in \mathbb{Z}^+,$$

therefore

$$\omega_\ell(x,\mu) = (P_\ell(\mu) - Q_\ell(\mu)x)x^\ell\omega(x,\mu), \qquad \ell \in \mathbb{Z}^+.$$

Substitution into (3.4) and differentiation with the Leibniz rule result in

$$\begin{aligned}\sum_{j=0}^{\infty}\frac{1}{j!(j+1)!}\Bigg((j+1)\sum_{\ell=0}^{\infty}&\frac{((\ell+j)!)^2P_{\ell+j}(\mu)}{\ell!} \\ &- x\sum_{\ell=0}^{\infty}(\ell+j+1)\frac{((\ell+j)!)^2Q_{\ell+j}(\mu)}{\ell!}\Bigg)x^j\frac{\partial^j}{\partial x^j}\omega(x,\mu) = 0.\end{aligned} \tag{3.5}$$

Note that both

$$\mathfrak{p}_j(\mu) := \sum_{\ell=0}^{\infty}\frac{((\ell+j)!)^2P_{\ell+j}(\mu)}{\ell!} = \sum_{\ell=0}^{\infty}\frac{((\ell+j)!)^2a_{\ell+j}}{\ell!} + \mu\sum_{\ell=0}^{\infty}\frac{((\ell+j)!)^2b_{\ell+j}}{\ell!}$$

and

$$\begin{aligned}\mathfrak{q}_j(\mu) &:= \sum_{\ell=0}^{\infty}(\ell+j+1)\frac{((\ell+j)!)^2Q_{\ell+j}(\mu)}{\ell!} \\ &= \sum_{\ell=0}^{\infty}(\ell+j+1)\frac{((\ell+j)!)^2c_{\ell+j}}{\ell!} + \mu\sum_{\ell=0}^{\infty}(\ell+j+1)\frac{((\ell+j)!)^2d_{\ell+j}}{\ell!}\end{aligned}$$

are linear in μ for all $j \in \mathbb{Z}^+$.

We henceforth assume that $a, b, c, d \in P_s[\mu]$, designating the underlying ω as *polynomial* MQF. Therefore (3.5) is an ordinary differential equation of degree s, equipped with zero initial conditions at the origin.

The equation (3.5) is a special case of the s-degree ODE

$$\sum_{\ell=0}^{s}(p_\ell - q_\ell x)x^\ell y^{(\ell)} = 0. \tag{3.6}$$

We express the solution of (3.6) in the form

$$y(x) = x^\theta \sum_{n=0}^{\infty} y_n x^n.$$

Substitution into (3.6) and elementary manipulation affirm that, for arbitrary y_0, θ must be a zero of

$$\sum_{\ell=0}^{s}(-1)^\ell p_\ell(-\theta)_\ell = 0 \tag{3.7}$$

(thus, provided that $p_s \neq 0$, there are s solutions in $\mathbb{C}$, the right number!), whereas the coefficients y_n are obtained from the recurrence

$$y_n = \frac{\sum_{\ell=0}^{s}(-1)^\ell q_\ell(-n+1-\theta)_\ell}{\sum_{\ell=0}^{s}(-1)^\ell p_\ell(-n-\theta)_\ell} y_{n-1}, \qquad n = 1, 2, \ldots. \tag{3.8}$$

Let

$$P(x) := \sum_{\ell=0}^{s}(-1)^\ell p_\ell(-x-\theta)_\ell, \qquad Q(x) := \sum_{\ell=0}^{s}(-1)^\ell q_\ell(-\theta-x)_\ell.$$

Therefore, (3.7) is equivalent to $P(0) = 0$, whereas (3.8) converts to

$$y_n = \frac{Q(n-1)}{P(n)} y_{n-1} = y_0 \prod_{\ell=1}^{n} \frac{Q(\ell-1)}{P(\ell)}, \qquad n = 1, 2, \ldots.$$

Suppose that

$$P(x) = p_* \prod_{j=1}^{s}(x-\eta_j), \qquad Q(x) = q_* \prod_{j=1}^{s}(x-\zeta_j),$$

where, without loss of generality, $\eta_s = 0$ (recall that $P(0) = 0$). Then

$$\prod_{\ell=1}^{n} P(\ell) = p_*^n \prod_{j=1}^{s}(1-\eta_j)_n, \quad \prod_{\ell=0}^{n-1} Q(\ell) = q_*^n \prod_{j=1}^{s}(-\zeta_j)_n$$

therefore

$$y_n = \left(\frac{q_*}{p_*}\right)^n \frac{\prod_{j=1}^{s}(-\zeta_j)_n}{\prod_{j=1}^{s}(1-\eta_j)_n} y_0, \qquad n \in \mathbb{Z}^+.$$

Letting, without loss of generality, $y_0 = 1$, we deduce that

$$y(x) = x^\theta {}_sF_{s-1}\left[\begin{array}{ll} -\zeta_1, -\zeta_2, \ldots, -\zeta_s; & \\ 1-\eta_1, 1-\eta_2, \ldots, 1-\eta_{s-1}; & \dfrac{q_*}{p_*}x \end{array}\right],$$

a *generalized hypergeometric* function [14].

Of course, we need $\theta \geq s$, otherwise the requisite number of derivatives at the origin fails to vanish – note that we need only $\ell = 0, 1, \ldots, s-1$ in (3.1).

The last result can be somewhat generalized by observing that the degree of either P or Q may be strictly less than s and repeating our analysis without any significant changes.

Theorem 1 Let $\alpha_n, \beta_n, \gamma_n, \delta_n \in P_s[n]$, $n \in \mathbb{Z}^+$, and suppose further that $\omega(\,\cdot\,, \mu) \in C^s[0,\infty)$ and that $\partial^\ell \omega(0,\mu)/\partial x^\ell = 0$, $\ell = 0, 1, \ldots, s-1$. Then there exist nonnegative integers $s_1, s_2 \leq s$ and real numbers θ, $\zeta_1, \zeta_2, \ldots, \zeta_{s_1}$, $\eta_1, \eta_2, \ldots, \eta_{s_2-1}$, ν, all functions of μ, such that

$$\omega(x,\mu) = x^\theta {}_{s_1}F_{s_2-1}\left[\begin{matrix} -\zeta_1, -\zeta_2, \ldots, -\zeta_{s_1}; \\ 1-\eta_1, 1-\eta_2, \ldots, 1-\eta_{s_2-1}; \end{matrix}\ \nu x\right]. \tag{3.9}$$

$\square$

Needless to say, not every ${}_{s_1}F_{s_2-1}$ function will do in (3.9), since we require $\omega(x,\mu) \geq 0$ for all $x \in [0,\infty)$, $\mu_- < \mu < \mu_+$. Even more restrictive requirement is that $\zeta_1, \ldots, \zeta_{s_1}$, $\eta_1, \ldots, \eta_{s_2-1}$, which are functions of μ, are chosen consistently with P and Q being polynomials in μ of requisite degrees. Moreover, the more interesting case is when, in addition, ω is SSC.

As an illustration of our technique, we consider in detail the case when the quotient (1.4) is linear in μ, i.e. $\gamma_n \equiv 1$, $\delta_n \equiv 0$, and assume in addition that $s = 2$. Therefore

$$\alpha_n = a_0 - a_1 n + a_2 n(n-1), \quad \beta_n = b_0 - b_1 n + b_2 n(n-1), \qquad n \in \mathbb{Z}^+,$$

and we have

$$\begin{array}{rclcrcl} P_0(\mu) & = & a_0 + b_0\mu, & \qquad & Q_0(\mu) & \equiv & 1, \\ P_1(\mu) & = & a_1 + b_1\mu, & & Q_1(\mu) & \equiv & 0, \\ P_2(\mu) & = & a_2 + b_2\mu, & & Q_2(\mu) & \equiv & 0. \end{array}$$

In other words,

$$\begin{array}{rclcrcl} \mathfrak{p}_0(\mu) & = & (a_0 + a_1 + 2a_2) + (b_0 + b_1 + 2b_2)\mu, & \qquad & \mathfrak{q}_0(\mu) & \equiv & 1, \\ \mathfrak{p}_1(\mu) & = & (a_1 + 4a_2) + (b_1 + 4b_2)\mu, & & \mathfrak{q}_1(\mu) & \equiv & 0, \\ \mathfrak{p}_2(\mu) & = & 4(a_2 + b_2\mu), & & \mathfrak{q}_2(\mu) & \equiv & 0 \end{array}$$

and the differential equation becomes

$$\begin{array}{rcl} (a_2 + b_2\mu)x^2 \dfrac{\partial^2 \omega(x,\mu)}{\partial x^2} + ((a_1 + 4a_2) + (b_1 + 4b_2)\mu)x \dfrac{\partial \omega(x,\mu)}{\partial x} & & \\ + ((a_0 + a_1 + 2a_2) + (b_0 + b_1 + 2b_2)\mu - x)\,\omega(x,\mu) & = & 0, \end{array}$$

with zero initial conditions at $x = 0$. Since

$$P(x) = \mathfrak{p}_0(\mu) + \mathfrak{p}_1(\mu)(x+\theta) + \tfrac{1}{4}\mathfrak{p}_2(\mu)(x+\theta)(x+\theta-1)$$

and we require $P(0) = 0$, we choose $\theta = \theta(\mu)$ as a root of the quadratic

$$(a_2 + b_2\mu)\theta^2 + ((a_1 + 3a_2) + (b_1 + 3b_2)\mu)\,\theta + (a_0 + a_1 + 2a_2) + (b_0 + b_1 + 2b_2)\mu = 0. \tag{3.10}$$

Therefore

$$\eta_1 = 1 - 2\theta - 4\frac{\mathfrak{p}_1}{\mathfrak{p}_2}$$

and $s_2 = 2$. On the other hand, $Q(x) \equiv 1$, therefore $s_1 = 0$. Moreover, $\nu = 4/\mathfrak{p}_2$ and we deduce from Theorem 1 that

$$\omega(x,\mu) = x^{\theta}{}_0F_1\left[\begin{matrix} -; & \dfrac{4x}{\mathfrak{p}_2} \\ 2\theta + 4\mathfrak{p}_1/\mathfrak{p}_2; & \end{matrix}\right]. \tag{3.11}$$

Recall that a *modified Bessel function* can be represented in the form [14]

$$I_\tau(z) = \frac{\left(\frac{1}{2}z\right)^\tau}{\Gamma(\tau+1)}{}_0F_1\left[\begin{matrix} -; & \frac{1}{4}z^2 \\ \tau+1; & \end{matrix}\right].$$

Let us assume that $\mu_-, a_2, b_2 \ge 0$. Then, dispensing with a multiplicative factor, we have

$$\omega(x,\mu) = x^{\frac{1}{2}-2\mathfrak{p}_1/\mathfrak{p}_2} I_{2\theta+4\mathfrak{p}_1/\mathfrak{p}_2-1}\left(4\left(\frac{x}{\mathfrak{p}_2}\right)^{\frac{1}{2}}\right).$$

As long as θ is real and ≥ 1 we are assured that the initial conditions are satisfied and that $\omega(x,\mu) \ge 0$ for all $x \ge 0$.

Let us consider the special choice

$$a_0 = a_1 = b_0 = b_2 = 0, \qquad a_2 > 0, \qquad b_1 \ne 0.$$

In other words, $\alpha_n = a_2 n(n-1)$ and $\beta_n = -b_1 n$ for $n \in \mathbb{Z}^+$. Letting $\lambda = -b_1\mu/a_2$, (3.10) becomes

$$\theta^2 + (3-\lambda)\theta + (2-\lambda) = 0,$$

with the solutions $\theta = -1$ and $\theta = 2\lambda - 2$. Since we require $\theta \ge 1$, we choose the second solution and impose $\lambda \ge 3$. The outcome is

$$\omega(x,\mu) = x^{\frac{1}{2}(\lambda-3)} I_{\lambda-1}\left(2\left(\frac{x}{a_2}\right)^{\frac{1}{2}}\right),$$

where μ need be restricted consistently with $b_1\mu + 3a_2 \le 0$. This can be simplified, e.g. by considering the variables

$$\tilde{x} = 2\left(\frac{x}{a_2}\right)^{\frac{1}{2}}, \qquad \tilde{\mu} = \lambda - 3,$$

instead of x and μ respectively. This results in

$$\omega(\tilde{x},\tilde{\mu}) = \tilde{x}^{\frac{1}{2}\tilde{\mu}} I_{\tilde{\mu}+1}(\tilde{x}), \qquad \tilde{x} \ge 0, \quad \tilde{\mu} > 0.$$

4 An inversion formula for a biorthogonal representation

The main result underpinning the transformation (1.5) is Lemma 5 of [5], namely that, as long as

$$\prod_{k=1}^{n}(x-\mu_k)=\sum_{k=0}^{n} f_k \prod_{i=0}^{k-1}(\alpha_i+x\beta_i)\prod_{j=k}^{n-1}(\gamma_j+x\delta_j), \tag{4.1}$$

and the moments of ω, namely $\{\mathfrak{m}_s(\mu)\}_{s\in\mathbb{Z}^+}$, are consistent with (1.4), the nth biorthogonal polynomial is

$$p_n(x;\mu_1,\mu_2,\ldots,\mu_n)=\sum_{k=0}^{n} f_k x^k. \tag{4.2}$$

Our purpose in the present section is to express the coefficients $f_0, f_1, \ldots, f_n$ explicitly in terms of the parameters $\mu_1, \mu_2, \ldots, \mu_n$. Let

$$\tilde{q}(x)=\frac{\prod_{k=1}^{n}(x-\mu_k)}{\prod_{k=0}^{n-1}(\gamma_k+x\delta_k)}.$$

Then (4.1) yields

$$\tilde{q}(x)=\sum_{k=0}^{n} f_k \mathfrak{m}_k(x). \tag{4.3}$$

We let $\lambda_j=-\alpha_j/\beta_j$, $j=0,1,\ldots,n$ and observe that, by (1.4), $\mathfrak{m}_j(\lambda_\ell)=0$ for $\ell=0,1,\ldots,j-1$. Therefore, letting $x=\lambda_\ell$ in (4.3), we obtain

$$\tilde{q}_\ell=\sum_{k=0}^{\ell} f_k \mathfrak{m}_k(\lambda_\ell), \qquad \ell=0,1,\ldots,n, \tag{4.4}$$

where $\tilde{q}_\ell=\tilde{q}(\lambda_\ell)$, $\ell=0,1,\ldots,n$. This is a linear system in $f_0, f_1, \ldots, f_n$ with a triangular coefficient matrix and its solution can be easily written explicitly by employing Cramer's rule,

$$f_k=\left(\prod_{j=0}^{k-1}\mathfrak{m}_j(\lambda_j)\right)^{-1}\det\begin{bmatrix}
\mathfrak{m}_0(\lambda_0) & 0 & \cdots & \cdots & 0 & \tilde{q}_0\\
\mathfrak{m}_0(\lambda_1) & \mathfrak{m}_1(\lambda_1) & 0 & & \vdots & \tilde{q}_1\\
\mathfrak{m}_0(\lambda_2) & \mathfrak{m}_1(\lambda_2) & \mathfrak{m}_2(\lambda_2) & \ddots & \vdots & \tilde{q}_2\\
\vdots & \vdots & & \ddots & 0 & \vdots\\
\mathfrak{m}_0(\lambda_{k-1}) & \mathfrak{m}_1(\lambda_{k-1}) & \cdots & \cdots & \mathfrak{m}_{k-1}(\lambda_{k-1}) & \tilde{q}_{k-1}\\
\mathfrak{m}_0(\lambda_k) & \mathfrak{m}_1(\lambda_k) & \cdots & \cdots & \mathfrak{m}_{k-1}(\lambda_k) & \tilde{q}_k
\end{bmatrix}$$

for $k=0,1,\ldots,n$.

An alternative expression can be obtained by defining

$$\mathfrak{v}_j(x):=\frac{\alpha_j+\beta_j x}{\gamma_j+\delta_j x}, \qquad j\in\mathbb{Z}^+.$$

Therefore

$$\mathfrak{m}_k(x)=\prod_{j=0}^{k-1}\mathfrak{v}_j(x), \qquad k\in\mathbb{Z}^+$$

(we assume without loss of generality that $\mathfrak{m}_0(x) \equiv 1$) and (4.4) can be rendered in the form

$$\tilde{q}_\ell = \sum_{k=0}^{\ell} f_k \prod_{j=0}^{k-1} \mathfrak{v}_j(\lambda_\ell), \qquad \ell = 0, 1, \ldots, n. \tag{4.5}$$

The parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ being distinct, regularity implies that $\mathfrak{m}_j(\lambda_\ell) \neq 0$ for $\ell \geq j$. Therefore, $\mathfrak{v}_k(\lambda_\ell) \neq 0$ for $\ell \geq k+1$. Letting $\ell = 0, 1, \ldots$ in (4.4), we obtain

$$\begin{aligned} f_0 &= \tilde{q}_0, \\ f_1 &= \frac{\tilde{q}_1 - \tilde{q}_0}{\mathfrak{v}_0(\lambda_1)}, \\ f_2 &= \frac{\dfrac{\tilde{q}_2 - \tilde{q}_0}{\mathfrak{v}_0(\lambda_2)} - \dfrac{\tilde{q}_1 - \tilde{q}_0}{\mathfrak{v}_0(\lambda_1)}}{\mathfrak{v}_1(\lambda_2)} \end{aligned}$$

etc. To obtain the general form of f_k, we need to generalize the concept of divided differences. Given a sequence $\{r_\ell\}_{\ell \in \mathbb{Z}^+}$, we define

$$\begin{aligned} F_0[r_\ell] &= r_\ell, \\ F_k[r_\ell, r_{\ell+1}, \ldots, r_{\ell+k}] &= \frac{F_{k-1}[r_\ell, \ldots, r_{\ell+k-2}, r_{\ell+k}] - F_{k-1}[r_\ell, \ldots, r_{\ell+k-1}]}{\mathfrak{v}_{k-1}(\lambda_{\ell+k})}, \quad k = 1, 2, \ldots. \end{aligned}$$

Lemma 2 Letting $r_\ell = \tilde{q}_\ell$, $\ell = 0, 1, \ldots, n$, we have $f_\ell = F_\ell[\tilde{q}_0, \tilde{q}_1, \ldots, \tilde{q}_\ell]$, $\ell = 0, 1, \ldots, n$.

Proof. By induction on ℓ. The statement is certainly true for $\ell = 0$ and we assume it for $\ell - 1$. Therefore, it follows from (4.5) that

$$\tilde{q}_\ell = \tilde{q}_0 + \sum_{k=1}^{\ell-1} F_k[\tilde{q}_0, \ldots, \tilde{q}_k] \prod_{j=0}^{k-1} \mathfrak{v}(\lambda_j) + d_\ell \prod_{j=0}^{\ell-1} \mathfrak{v}_j(\lambda_\ell).$$

Therefore,

$$\begin{aligned} f_\ell &= \frac{\tilde{q}_\ell - \tilde{q}_0}{\prod_{j=0}^{\ell-1} \mathfrak{v}_j(\lambda_\ell)} - \sum_{k=1}^{\ell-1} \frac{F_k[\tilde{q}_0 \ldots, \tilde{q}_k]}{\prod_{j=k}^{\ell-1} \mathfrak{v}_j(\lambda_\ell)} \\ &= \frac{F_1[\tilde{q}_0, \tilde{q}_\ell]}{\prod_{j=1}^{\ell-1} \mathfrak{v}_j(\lambda_\ell)} - \sum_{k=1}^{\ell-1} \frac{F_k[\tilde{q}_0 \ldots, \tilde{q}_k]}{\prod_{j=k}^{\ell-1} \mathfrak{v}_j(\lambda_\ell)}. \end{aligned}$$

We will prove by induction on $s = 1, 2, \ldots, \ell - 2$ that

$$f_\ell = \frac{F_s[\tilde{q}_0, \ldots, \tilde{q}_{s-1}, \tilde{q}_\ell]}{\prod_{j=s}^{\ell-1} \mathfrak{v}_j(\lambda_\ell)} - \sum_{k=s}^{\ell-1} \frac{F_k[\tilde{q}_0, \tilde{q}_1, \ldots, \tilde{q}_k]}{\prod_{j=k}^{\ell-1} \mathfrak{v}_j(\lambda_\ell)}. \tag{4.6}$$

This is certainly true for $s = 1$ and, assuming that (4.6) is valid for $s \geq 1$, we have

$$\begin{aligned} f_\ell &= \frac{F_s[\tilde{q}_0, \ldots, \tilde{q}_{s-1}, \tilde{q}_\ell] - F_s[\tilde{q}_0, \ldots, \tilde{q}_{s-1}, \tilde{q}_s]}{\prod_{j=s}^{\ell-1} \mathfrak{v}_j(\lambda_\ell)} - \sum_{k=s+1}^{\ell-1} \frac{F_k[\tilde{q}_0, \tilde{q}_1, \ldots, \tilde{q}_k]}{\prod_{j=k}^{\ell-1} \mathfrak{v}_j(\lambda_j)} \\ &= \frac{F_{s+1}[\tilde{q}_0, \ldots, \tilde{q}_s, \tilde{q}_{s+1}]}{\prod_{j=s+1}^{\ell-1} \mathfrak{v}_j(\lambda_\ell)} - \sum_{k=s+1}^{\ell-1} \frac{F_k[\tilde{q}_0, \tilde{q}_1, \ldots, \tilde{q}_k]}{\prod_{j=k}^{\ell-1} \mathfrak{v}_j(\lambda_j)}. \end{aligned}$$

This proves that (4.6) is true. We now let $s = \ell - 2$, therefore

$$\begin{aligned} f_\ell &= \frac{F_{\ell-1}[\tilde{q}_0, \ldots, \tilde{q}_{\ell-2}, \tilde{q}_\ell] - F_{\ell-1}[\tilde{q}_0, \ldots, \tilde{q}_{\ell-2}, \tilde{q}_{\ell-1}]}{\mathfrak{v}_{\ell-1}(\lambda_\ell)} \\ &= F_\ell[\tilde{q}_0, \tilde{q}_1, \ldots, \tilde{q}_\ell]. \end{aligned}$$

This completes the proof. $\Box$

Theorem 3 Let the moments of ω satisfy (1.4) and assume that $\beta_\ell \neq 0$, $\ell \in \mathbb{Z}^+$, and $\alpha_j \delta_\ell \neq \beta_j \gamma_\ell$ for all $\ell, j \in \mathbb{Z}^+$, $\ell \neq j$. Set

$$\tilde{q}_\ell = \frac{\prod_{k=1}^{n}(\alpha_\ell + \beta_\ell \mu_k)}{\prod_{k=0}^{n-1}(\alpha_\ell \delta_k - \beta_\ell \gamma_k)}, \qquad \ell = 0, 1, \ldots, n.$$

Then

$$p_n(x; \mu_1, \mu_2, \ldots, \mu_n) = \sum_{k=0}^{n} F_k[\tilde{q}_0, \tilde{q}_1, \ldots, \tilde{q}_k] x^k. \tag{4.7}$$

Proof. Follows at once from the lemma, since the above definition of $\{\tilde{q}_\ell\}_{\ell=0}^{n}$ is consistent with $\tilde{q}_\ell = \tilde{q}(\lambda_\ell)$. $\Box$

Acknowledgement

The authors are grateful to André Draux, who has read an earlier version of this paper and whose remarks have greatly improved the exposition.

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