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Spurious Chaotic Solutions of Differential Equations

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Spurious Chaotic Solutions of Differential Equations

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In this paper we present results concerning a chaotic behaviour of difference equations. It is proved that there exists a horseshoe for the difference equation

$$y_{n+k+1} = \sum_{i=1}^{k} \alpha_i y_{n+i} + \lambda \sum_{i=l}^{m} \beta_i g(y_{n+i}), \ 1 \le l \le m < k$$

whenever λ is sufficiently large, provided that $\sum_{i=l}^{m} \beta_i \neq 0$, the function g is continuous and changes sign at least twice. This equation is of great importance in numerical analysis, since many well known numerical methods for ODEs, in particular multistep methods, can be rendered in this form. Some results concerning the non-chaotic case are also presented.

1. Introduction

When we solve a differential equation numerically, the accuracy of the numerical solution is our main but not the only concern. We also want the solution to preserve important qualitative features of the initial system: the number of stable and unstable steady states, the global behaviour of the trajectories, etc. This can easily be lost even in the simplest cases, as the following example illustrates.

Let us consider a nonlinear second order equation

$$\begin{aligned} &u'' + \lambda u(1-u) = 0 \\ &u(0) = 0, \ u'(0) = v \end{aligned} \tag{1.1}$$

It is known (**?**) that the solution of this equation can exhibit three different types of behaviour:

1 u is periodic if $|v| < v_\lambda$, with period $T \to \infty$ as $|v| \to v_\lambda$;
2 $u \to 1$ if $|v| = v_\lambda$;
3 $u \to \infty$ if $|v| > v_\lambda$.

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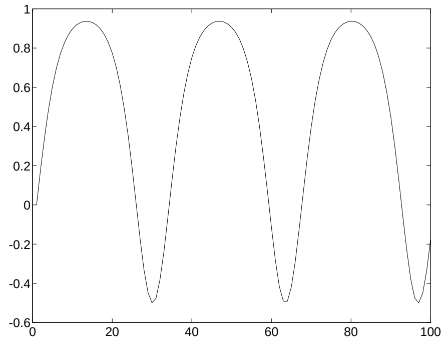

Fig. 1. The solution of the equation 1.2 with $\lambda h^2 = 0.1$ and $v = 1.8$ ($v < v_{\lambda_h}$)

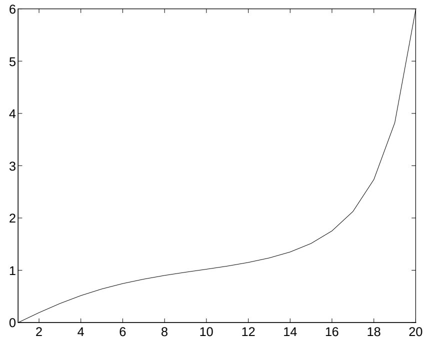

Fig. 2. The solution of the equation 1.2 with $\lambda h^2 = 0.1$ and $v = 1.9$ ($v > v_{\lambda_h}$)

Let us now consider a simple numerical scheme for the equation (??)

$$\frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} + \lambda u_n(1 - u_n) = 0 \tag{1.2}$$
$$u_0 = 0, \ \frac{u_1 - u_0}{h} = v$$

If λh^2 is small ($\lambda h^2 \ll 1$), the solution of the equation (??) has the same structure as the continuous solution, i.e. it is periodic for $v < v_{\lambda_h}$ and unbounded for $v > v_{\lambda_h}$ with a critical value in between (Figures **??**, **??**). However, when $\lambda h^2 \gg 1$, the behaviour of the solution becomes much more complicated. While there still exist periodic solutions (Figure **??**), the set of initial values v for which the solution is periodic does not form an interval as in the continuous case. What is even more interesting, there are bounded solutions which are not periodic and clearly do not approximate any periodic continuous solution (Figure **??**). A closer inspection reveals that most of the solutions sooner or later become unbounded and this happens even if the initial value v is close to 0. In order to investigate this we will prove the following

Lemma 1.1 Let us assume that the following inequality holds

$$u_1 < 1 \tag{1.3}$$
$$u_k > 1 \quad \text{for some } k > 1 \tag{1.4}$$

Then $u_n \to \infty$ as $n \to \infty$

Proof. Let i be the smallest number such that $u_i > 1 \geq u_{i-1}$. It is easy to verify that, under the aforementioned assumptions, $u_{i+1} > u_i$. It follows by induction that u_n monotonically increases if $n \geq i$, hence, there exists a limit u^* of the sequence $\{u_n\}$ which may be finite or infinite. If it is finite then u^* is a solution of the equation

$$(u^* - 2u^* + u^*)/h^2 + \lambda u^*(1 - u^*) = 0$$

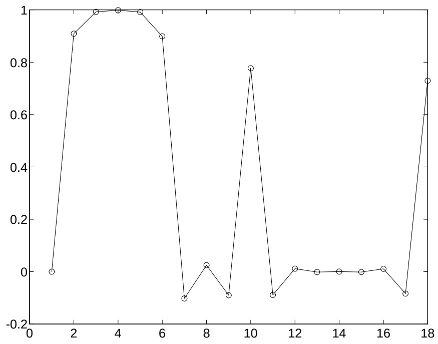

Fig. 3. The solution of the equation 1.2 with $\lambda h^2 = 10$ and $v = 9.0914292244498778$

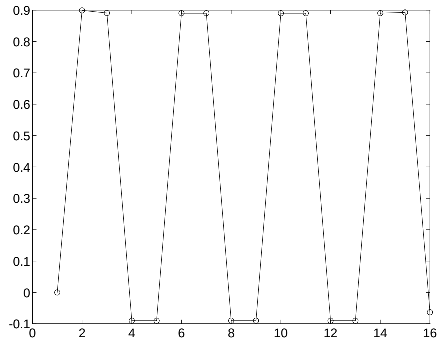

Fig. 4. The solution of the equation 1.2 with $\lambda h^2 = 10$ and $v = 8.99081618987285$

The only solutions for the equation above are $u^* = 0, 1$, which leads to contradiction. Thus, $u^* = \infty$. $\square$

Let D_n denote the set of initial values for which $u_i < 1,\ i = 1, \ldots, n$. Figure **??** displays the sets D_4, D_5 and D_6. Since D_i contracts very fast as $i \to \infty$, for the purpose of a better visualisation we present only a half of the set D_5 at the interval $[-0.12, 0.02]$ and a quarter of the set D_6 at the interval $[-0.002, 0.016]$. The whole set D_5 consists of 8 disjoint intervals and the whole set D_6 consists of 16 disjoint intervals. Numerical experiments suggest that in general D_n consists of 2^{n-2} intervals, so that $D = \cap_{i=1}^{\infty} D_i$ is a Cantor set. Nothing similar to the continuous case, where the set D was a single interval!

The fact that the equation (??) has a different structure of bounded solutions is not the only reason for our studies. Let us consider equation (??) as a boundary value problem with boundary values $u(0) = u(1) = 0$. It is known (**?**) that, if $\pi n^2 < \lambda \leq \pi(n+1)^2$, the aforementioned equation has $2n - 1$ steady solutions. However, if h is not sufficiently small, the discretization (??) has 2^{N-2} different steady solutions where N is the number of discretization points and is independent of λ. Figure **??** shows a bifurcation diagram to the boundary value problem. While it is known (**?**) that in a continuous case all steady solutions bifurcate from a zero solution, one can notice that discretization produces spurious bifurcations. Since the steady states play an important role in a global dynamical behaviour of the nonlinear parabolic equation

$$u_t = u_{xx} + \lambda u(1-u),$$

one may expect a different global behaviour of the solution of the equation above whenever λh^2 is large.

Spurious solutions as in Figure **??** are examples of chaos in difference equations and the equation (??) can be described by what is known in the theory of dynamical systems as a Smale's horseshoe map (**?**). Although very simple in appearance, Smale's horseshoe maps are difficult to investigate. The theory of spurious solutions for ODEs has been discussed

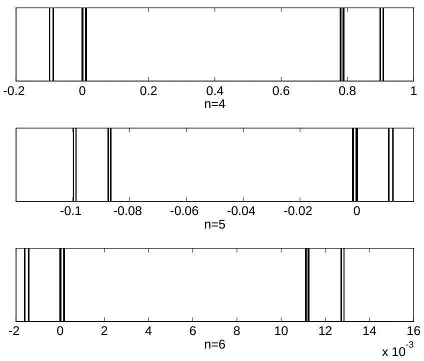

Fig. 5. The sets D_4, D_5 and D_6 for the equation 1.2 with $\lambda = 10$

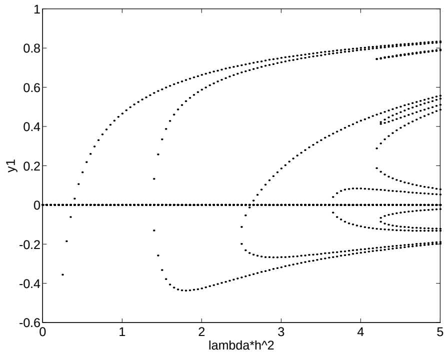

Fig. 6. A bifurcation diagram for the equation 1.2 with $N = 6$

in (?, ?, ?, ?) and chaotic spurious solutions have been found by (?, ?, ?). In our paper we will prove the existence of a horseshoe for a wide class of difference equations which are of great importance in numerical analysis. In the next section we will present a background material from dynamical systems theory and prove the main results of this paper. Section 3 is devoted to some cases of the non-chaotic difference equations.

2. Smale's horseshoe and difference equations

We commence this section by introducing several definitions from the theory of nonlinear dynamical systems.

The original horseshoe map as defined by Stephen Smale (?) was a map of a square, where the image was a figure vaguely reminiscent of a horseshoe (Figure ??). Smale considered the set D_n of the initial values, for which the nth iterate of the map remains in the square and proved that $D = \cap_{i=1}^{\infty} D_i$ is a Cantor set. It is possible to describe this behaviour using symbolic dynamics on binary sequences. It is proved in (?) that D_n consists of 2^n disjoint sets $D_n^{(i)}$, all of them in two vertical strips (Figure ??), which is in fact the set D_1. We can assign a finite binary sequence of length n to each set $D_n^{(i)}$, where the kth element of the sequence is either 0 or 1 depending on whether the kth iterate of the set $D_n^{(i)}$ is in the upper or lower part of the "horseshoe". It is easy to prove (?) that in this case k iterations on a set correspond to the shift by k symbols in the corresponding binary sequence which leads to the following formal definition of the horseshoe map.

Definition 2.1 Let $\mathcal{S}$ be some (finite or infinite) set of symbols with more than one element and let $I_{\mathcal{S}} = \{I_s | s \in \mathcal{S}\}$ be a set of disjoint intervals in $\mathcal{R}$. We say that the map $\phi : I_{\mathcal{S}}^k \mapsto \mathcal{R}$ has a horseshoe if for every sequence of symbols $\{s_i\}_{i=0}^{\infty}$, where $s_i \in \mathcal{S}$, there exists a sequence $\{y_{s_i}\}_{i=0}^{\infty}$, $y_{s_i} \in I_{s_i}$ such that $\phi(y_{s_{i-k+1}}, \ldots, y_{s_i}) = y_{s_{i+1}}$

The simplest case occurs when $\mathcal{S}$ has only two elements, $\mathcal{S} = \{0, 1\}$ say. Our numerical example, as given in the introductory part of this paper, corresponds to this case.

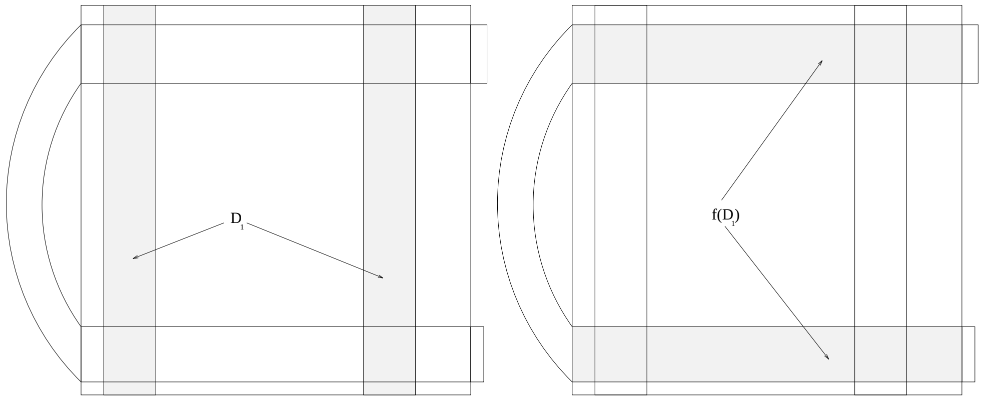

Fig. 7. The horseshoe map, and sets D_1 and $f(D_1)$

Now we are able to formulate our main results

Theorem 2.1 Consider the difference equation

$$y_{n+k+1} = \sum_{i=1}^{k} \alpha_i y_{n+i} + \lambda g(y_{n+j}), \qquad 1 \le j \le k, \tag{2.1}$$

where the function g is continuous and changes sign at least twice. Then for sufficiently large λ there exist two intervals I_0 and I_1 such that for every infinite binary sequence $\{s_i\}_{i=0}^{\infty}$ there exists a sequence $\{y_n\}$ such that $y_n \in I_{s_n}$ and y_n satisfies the equation (??)

Remark. The number of points where function g changes sign corresponds to the number of elements in $\mathcal{S}$ from our definition. This explains why we require the function g to change sign at least twice.

Proof. We prove the theorem by induction. Without loss of generality we assume that g changes sign at the points 0 and 1. Let $I = [-1, 2]$. We define

$$M = \max_{y_{n+i} \in I} \left| \sum_{i=1}^{k} \alpha_i y_{n+i} \right| .$$

Since g is continuous and changes sign at the points 0 and 1, for small enough ϵ there exist intervals I_0 and I_1 such that the image of $I_i,\ i = 0, 1$ under g is $[-\epsilon, \epsilon]$ and I_i does not contain other zeroes of the function g. Taking $\lambda = (M + 2)/\epsilon$ we deduce that the image of intervals I_i under λg is $[-(M + 2), M + 2]$ In other words, for arbitrary values of $y_{n+1}, \ldots, y_{n+j-1}, y_{n+j+1}, \ldots, y_{n+k}$ from interval I, when y_{n+j} spans either of the intervals I_0, I_1, y_{n+k+1} spans both intervals I_0 and I_1.

Now we can formulate our inductive assumption:

A_n *Let S_n be a binary sequence of length n, $S_n = \{s_i\}_{i=1}^{n}$, and $s_i \in \{0, 1\}$. Then there exists a compact set $D_n = D_n(S_n)$ of initial values for equation (??) such that $y_i \in I_{s_i}$*

for $i = 1, \ldots, n+j-1$ *and when* $y_1, \ldots, y_k$ *span* D_n *then* $y_{n+j}, \ldots, y_{n+k}$ *span both* I_0 *and* I_1.

A_1 is obviously true, since $j \leq k$ and we are free to chose initial values for the $y_1, \ldots, y_k$. Suppose that A_n is true. Without loss of generality we may assume that the last element of S_{n+1} is 1. Let D_{n+1} be a set of initial values such that $y_{n+j} \in I_1$. Under the assumption A_n, $D_{n+1} \subset D_n$ is not an empty set, and, from our definition of the interval I_1, when y_{n+j} spans I_1, y_{n+k+1} spans both intervals I_0 and I_1. Since I_1 is compact and g is continuous, D_{n+1} is compact. It follows now that $D = \cap_{i=1}^{\infty} D_i$ is nonempty. $\square$

Theorem 2.2 Consider the following recurrence relation

$$y_{n+k+1} = \sum_{i=1}^{k} \alpha_i y_{n+i} + \lambda \sum_{i=l}^{m} \beta_i g(y_{n+i}), \quad 1 \leq l \leq m < k, \tag{2.2}$$

$$\sum_{i=l}^{m} \beta_i \neq 0, \tag{2.3}$$

where g is continuous and changes sign at least twice. Let

$$K = \frac{|\beta_l|}{\sum_{i=l+1}^{m} |\beta_i|}.$$

Then

1 If

$$K > 1,$$

then for sufficiently large λ there exist two intervals I_0 and I_1 such that for every infinite binary sequence $\{s_i\}_{i=0}^{\infty}$ there exists a sequence $\{y_n\}$ where $y_n \in I_{s_n}$ and y_n satisfies the equation (??) for all $n = 0, 1, 2, \ldots$;

2 If

$$K \leq 1,$$

then for every finite binary sequence $\{s_i\}_{i=0}^{N}$ there exist λ and two intervals I_0 and I_1 such that there exists a sequence $\{y_n\}_{n=0}^{N}$ where $y_n \in I_{s_n}$ and y_n satisfies the equation (??) for all $n = 0, 1, 2, \ldots, N$.

Proof. The proof is similar to the previous one. Since the sum of coefficients β_i is nonzero and our function g changes sign at least twice at the points x_0, x_1, it is easy to see that for sufficiently large λ equation (??) has at least two equilibrium solutions z_0 and z_1:

$$z_\gamma = \sum_{i=1}^{k} \alpha_i z_\gamma + \lambda \sum_{i=l}^{m} \beta_i g(z_\gamma), \qquad \gamma = 0, 1,$$

and $z_\gamma \to x_\gamma$, $\gamma = 0, 1$ as $\lambda \to \infty$. Without loss of generality we may assume that $x_0 = 0$,

$x_1 = 1$. We define

$$f(\lambda, x_1, \ldots, x_k) = \sum_{i=1}^{k} \alpha_i x_i + \lambda \sum_{i=l+1}^{m} \beta_i g(x_i)$$

and rewrite equation (??) as

$$y_{n+k+1} = \lambda \beta_l g(y_{n+l}) + f(\lambda, y_{n+1}, \ldots, y_{n+k}).$$

Since g is continuous and changes sign twice, for small enough ϵ there exist distinct intervals I_γ, $\gamma = 0, 1$ such that

1 Function g maps I_γ to the interval $[-\epsilon, \epsilon]$
2 Intervals I_γ do not contain other zeroes of the function g except for x_0 and x_1

Let us define constants M_1 and M_2

$$\begin{aligned} M_1 &= \max_{x_1,\ldots,x_k \in I_0 \cup I_1} \left| \sum_{i=1}^{k} \alpha_i x_i \right|, \\ M_2 &= \max_{x \in I_0 \cup I_1} |x|. \end{aligned}$$

If $K > 1$, we can take $\lambda > (M_1 + M_2)/(K-1)\epsilon$ which results in the following inequalities:

$$\begin{aligned} \max_{x \in I_0} \lambda \beta_l g(x) &> f(\lambda, x_1, \ldots, x_k) + M_2 \\ -\min_{x \in I_0} \lambda \beta_l g(x) &> f(\lambda, x_1, \ldots, x_k) + M_2 \end{aligned}$$

where $x_1, \ldots, x_k \in I_0 \cup I_1$. Similar inequalities can be obtained for the interval I_1.

This means that when y_{n+l} spans either of the intervals I_0, I_1, y_{n+k+1} spans both intervals I_0 and I_1 for arbitrary values of $y_n, \ldots, y_{n+l-1}, y_{n+l+1}, \ldots, y_{n+k}$ from I_1 or I_0. Hence we may formulate an inductive assumption similar to A_n from Theorem 1:

A'_n *Let S_n be a sequence of n 1s or 0s, $S_n = \{s_i\}_{i=1}^n$, and $s_i \in \{0, 1\}$. Then there exists a compact set $D_n = D_n(S_n)$ of initial values for equation (??) such that $y_i \in I_{s_i}$ for $i = 1, \ldots, n+l$ and when y_j, $j = 1, \ldots, k$ span D_n, then $y_{n+l+1}, \ldots, y_{n+j}$ span both I_0 and I_1*

A'_1 is true since we are free to choose initial values for our equation (??) and (4.4–5) shows that A'_n yields A'_{n+1}. Since $D_n \subset D_{n-1}$ and D_n is compact, $D = \cap D_n \neq \emptyset$.

Consider now the case $K \leq 1$. Let ϵ, I_0, I_1, M_1, M_2 be as in the first part of the theorem We will need the following lemma

Lemma 2.1 Let us fix N. Then, if λ is sufficiently large, there exists a sequence of intervals $I_\gamma^{(n)}$, $\gamma = 0, 1$, $n = 1, \ldots, N-1$, satisfying the following conditions.

$$\begin{aligned} I_\gamma^{(1)} &\subset I_\gamma, \\ I_\gamma^{(n+1)} &\subset I_\gamma^{(n)}, \qquad I_\gamma^{(n+1)} \neq \emptyset \end{aligned}$$

$$\begin{aligned}\max_{x\in I_\gamma^{(n)}} \lambda\beta_l g(x) &> \max|f(\lambda,x_1,\ldots,x_k)|+M_2, \quad x_1,\ldots,x_k\in I_0^{(n+1)}\cup I_1^{(n+1)},\\ \min_{x\in I_\gamma^{(n)}} \lambda\beta_l g(x) &< -\max|f(\lambda,x_1,\ldots,x_k)|-M_2, \quad x_1,\ldots,x_k\in I_0^{(n+1)}\cup I_1^{(n+1)}.\end{aligned}$$

Proof. Let us choose λ such that

$$\lambda = \frac{M_1+M_2}{\epsilon}\sum_{i=0}^{N-1}\frac{1}{K^i}$$

We define c_n recursively as follows

$$\begin{aligned}c_1 &= \lambda\epsilon = (M_1+M_2)\sum_{i=0}^{N-1}\frac{1}{K^i}\\ c_n &= (c_{n-1}-M_1-M_2)K\end{aligned}$$

It is easy to prove by induction that

$$c_n = (M_1+M_2)\sum_{i=0}^{N-n}\frac{1}{K^i},$$

which means that $c_n \geq M_1+M_2$ for all $n\leq N$. Choose $I_\gamma^{(n)}$ so that $\lambda\beta_l g(x)$ maps $I_\gamma^{(n)}$ to the interval $I_{c_n}=(-c_n,c_n)$. The first two conditions are satisfied since $c_{n+1}<c_n$ and $c_1=C_\lambda$. Inequality

$$\max|f(\lambda,x_1,\ldots,x_k)|+M_2 \leq Kc_{n+1}+M_1+M_2 = c_n, \quad x_1,\ldots,x_k\in I_0^{(n+1)}\cup I_1^{(n+1)}$$

yields that the last two conditions are also satisfied. □

Now we can proceed as in the first part of the theorem.

A''_n *Let S_n be a sequence of n 1s or 0s, $S_n=\{s_i\}_{i=1}^n$, and $s_i\in\{0,1\}$. Then there exist a set $D_n=D_n(S_n)$ of initial values for equation (??) such that $y_i\in I_{s_i}^{(i)}$ for $i=1,\ldots,n+l$ and when y_j, $j=1,\ldots,k$ span D_n, then $y_{n+l+1},\ldots,y_{n+j}$ span both $I_0^{(n+l+1)}$ and $I_1^{(n+l+1)}$, where $I_{s_i}^{(i)}\in I_{s_i}$*

A''_1 is true and A''_{n+1} follows from A''_n if we choose λ large enough and define intervals $I_\gamma^{(i)}$ as in Lemma 2.1. □

3. Non-chaotic case

One may ask what happens when the function g has less then two real zeroes. Unfortunately, no general results for difference equations of type (??) or (??) are available. The following two theorems address themselves to a much simpler case. Nevertheless, they are of considerable interest in numerical analysis, because of their relevance to two-step methods for second order nonlinear ODEs.

Theorem 3.1 Let us consider the following difference equation

$$y_{n+1} = 2y_n - y_{n-1} - \lambda g(y_n) \tag{3.1}$$

Suppose that g does not change sign. Then $\hat{y} = \lim_{n\to\infty} y_n$ exists on the extended real line and, unless $g(\hat{y}) = 0$, it is true that $|\hat{y}| = \infty$

Proof. Indeed, from (??) we have by induction

$$y_{n+1} - y_n = y_n - y_{n-1} - \lambda g(y_n) = \cdots = y_1 - y_0 - \lambda \sum_{i=1}^{n} g(y_i) \tag{3.2}$$

Without loss of generality we may assume that the function g is nonpositive. Then the expression on the right hand side of (??) is monotonically increasing as $n \to \infty$ and changes its sign at most once. This means that for sufficiently large n the expression $y_{n+1} - y_n$ has a constant sign, i.e. y_n is monotonic and hence tends to a limit. Let $\hat{y} = \lim_{n\to\infty} y_n$ and suppose that $g(\hat{y}) \neq 0$. Then the expression on the right hand side of (??) has an infinite limit and so does y_n. □

Theorem 3.2 Suppose that g changes sign at the point x_0 and that $g(x) > 0$ for $x < x_0$ and $g(x) < 0$ for $x > x_0$. Then $\hat{y} = \lim_{n\to\infty} y_n$ exists on the extended real line and, unless $g(\hat{y}) = 0$, it is true that $|\hat{y}| = \infty$

Proof. Without loss of generality we may assume that $x_0 = 0$ and that $y_0 < y_1$. Let us recall equation (??). It is easy to see that if $y_1 > 0$, then $y_n \to +\infty$. If $y_1 < 0$ then $y_{i+1} - y_i$ is decreasing and y_i is increasing, so that for some n either $y_n > 0$, $y_n - y_{n-1} > 0$, with following monotonous convergence to $+\infty$, or $y_n < 0$, $y_n - y_{n-1} < 0$ with following monotonous convergence to $-\infty$, or $y_{n+1} - y_n \to 0$, $y_n \to \hat{y}$ as $n \to \infty$. In the latter case we deduce again that the sum on the right hand side of (??) has a finite limit, hence $0 = \lim_{n\to\infty} g(y_n) = g(\hat{y})$. □

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