Biorthogonal polynomials: Recent developments

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Abstract An important application of biorthogonal polynomials is in the generation of polynomial transformations that map zeros in a predictable way. This requires the knowledge of the explicit form of the underlying biorthogonal polynomials.
The most substantive set of parametrized Borel measures whose biorthogonal polynomials are known explicitly are the *Möbius quotient functions (MQFs),* whose moments are Möbius functions in the parameter. In this paper we describe recent work on the characterization of MQFs, following two distinct approaches. Firstly, by restricting the attention to specific families of Borel measures, of the kind that featured in [4], it is possible sometimes to identify all possible MQFs by identifying a functional relationship between weight functions for different values of the parameter. Secondly, provided that the coefficients in Möbius functions are smooth (in a well defined sense), it is possible to prove that the weight function obeys a differential relationship that, in specific cases, allows an explicit characterization of MQFs. In particular, if all such coefficients are polynomial, the MQFs form a subset of generalized hypergeometric functions.

Dedicated to Syvert P. Nørsett on the occasion of his 50th birthday.

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1 Biorthogonal polynomials – the essential theory

Given points $a < b$ and $c < d$ along the extended real axis, we let $\mathrm{d}\varphi(x, \mu)$ be, as a function of $x \in (a, b)$, a Borel measure for every parameter $\mu \in (c, d)$. Thus, φ is a real, nondecreasing function, right-continuous function of x, with an infinite number of points of increase and such that the moments

$$\mathfrak{m}_k(\mu) = \int_a^b x^k \,\mathrm{d}\varphi(x, \mu)$$

are bounded for all $k \geq 0$ and $\mu \in (c, d)$.

We denote by $\mathbb{P}_n[x]$ the set of all real polynomials of degree $\leq n$. Given any set $\mu_1, \mu_2, \ldots, \mu_n$ of pairwise distinct points in (c, d), an element $p_n \in \mathbb{P}_n[x] \times (c, d)^n$, $p_n \not\equiv 0$, is said to be a *biorthogonal polynomial* if

$$\int_a^b p_n(x; \mu_1, \mu_2, \ldots, \mu_n) \,\mathrm{d}\varphi(x, \mu_\ell) = 0, \qquad \ell = 1, 2, \ldots, n. \tag{1}$$

The reader is referred to [3] for an introduction to biorthogonal polynomials and to [1] for a more general exposition of biorthogonality and its many applications.

Quite clearly, biorthogonal polynomials generalize the standard concept of orthogonality. Thus, letting $\mathrm{d}\varphi(x, \mu) = x^\mu \,\mathrm{d}\psi(x)$ and $\mu_\ell = \ell - 1$, $\ell = 1, 2, \ldots, n$, we recover from (1) polynomials that are orthogonal (in the usual sense) with respect to the Borel measure $\mathrm{d}\psi$. However, the concept of biorthogonal polynomials is substantially more powerful, since much useful information can be derived by considering p_n as a function of its parameters $\mu_1, \mu_2, \ldots, \mu_n$. This approach is central to the work of the present paper.

The most basic question that arises with regard to any mathematical concept is about existence and uniqueness. As far as biorthogonal polynomials are concerned, it can be settled down quite easily.

Theorem 1 (Iserles & Nørsett, [3]) Biorthogonal polynomials $\{p_n\}_{n=1}^\infty$ exist and, up to a nonzero multiplicative constant, are unique if and only if

$$\det \begin{bmatrix} \mathfrak{m}_0(\mu_1) & \mathfrak{m}_1(\mu_1) & \cdots & \mathfrak{m}_{n-1}(\mu_1) \\ \mathfrak{m}_0(\mu_2) & \mathfrak{m}_1(\mu_2) & \cdots & \mathfrak{m}_{n-1}(\mu_2) \\ \vdots & \vdots & & \vdots \\ \mathfrak{m}_0(\mu_n) & \mathfrak{m}_1(\mu_n) & \cdots & \mathfrak{m}_{n-1}(\mu_n) \end{bmatrix} \neq 0$$

for every distinct $\mu_1, \mu_2, \ldots, \mu_n \in (c, d)$ and $n \geq 1$. $\square$

A Borel measure that obeys the conditions of Theorem 1 is said to be *regular.*

The determinant in the statement of the theorem is a generalization of the familiar Hankel determinant from the theory of orthogonal polynomials [2]. This is not an isolated case, since several properties which are familiar in the orthogonal setting are inherited by biorthogonal polynomials: Rodrigues-type formulae [3], Christoffel–Darboux identites [1, 5] etc. However, this is not the case with regard to the location of the zeros: we can be no longer assured that they reside in the support of the underlying measure. The standard proof on the location of zeros of orthogonal polynomials rests upon the trivial observation that $\{1, x, x^2, \ldots, x^{n-1}\}$ is a *Chebyshev set.* Generalizing to biorthogonal polynomials, the Chebyshev set property might be lost, in which case zeros might well stray into $\mathbb{C} \setminus (a, b)$. Consequently, to be assured that the zeros are confined to the support, we need to impose the Chebyshev set property on appropriate functions.

We say that the function $u : (x_-, x_+) \times (y_-, y_+) \to \mathbb{R}$ is *strictly sign consistent (SSC)* if

$$\det \begin{bmatrix} u(x_1, y_1) & u(x_1, y_2) & \cdots & u(x_1, y_n) \\ u(x_2, y_1) & u(x_2, y_2) & \cdots & u(x_2, y_n) \\ \vdots & \vdots & & \vdots \\ u(x_n, y_1) & u(x_n, y_2) & \cdots & u(x_n, y_n) \end{bmatrix} \neq 0$$

for every $x_- < x_1 < x_2 < \cdots < x_n < x_+$, $y_- < y_1 < y_2 < \cdots < y_n < y_+$ and $n \geq 1$ [10].

Theorem 2 (Iserles & Nørsett, [3]) The zeros of $p_n(\,\cdot\,; \mu_1, \mu_2, \ldots, \mu_n)$ reside in (a, b) for every $n \geq 1$ and distinct $\mu_1, \mu_2, \ldots, \mu_n \in (c, d)$ if either φ or its Radon–Nikodym derivative $\mathrm{d}\varphi/\,\mathrm{d}x$ is SSC. $\square$

In the sequel we apply the sobriquet of 'SSC measure' to any $\mathrm{d}\varphi$ that fulfills the conditions of Theorem 2. Many such measures have been listed in [3] and [4]. For example, and provided that $\mathrm{d}\psi$ is a Borel measure, all the following are SSC measures:

1. $\mathrm{d}\varphi(x, \mu) = x^\mu \,\mathrm{d}\psi(x)$, $\quad (a, b) \subseteq (0, \infty)$, $(c, d) \subseteq (-1, \infty)$;
2. $\mathrm{d}\varphi(x, \mu) = \mu^x \,\mathrm{d}\psi(x)$, $\quad (a, b) \subseteq \mathbb{R}$, $(c, d) \subseteq (0, \infty)$;
3. $\mathrm{d}\varphi(x, \mu) = x^{\log_q \mu} \,\mathrm{d}\psi(x)$, $\quad q \in (0, 1)$, $(a, b), (c, d) \subseteq (0, \infty)$;
4. $\mathrm{d}\varphi(x, \mu) = \Gamma(x + \mu) \,\mathrm{d}\psi(x)$, $\quad (a, b), (c, d) \subseteq (0, \infty)$;
5. $\mathrm{d}\varphi(x, \mu) = \prod_{k=0}^{\infty} (1 + q^k \mu x)^{-1} \,\mathrm{d}\psi(x)$, $\quad q \in (0, 1)$, $(a, b), (c, d) \subseteq (0, \infty)$.

2 Zero-mapping transformations of polynomials

Suppose that, given any $n \geq 1$ and pairwise distinct $\mu_1, \mu_2, \ldots, \mu_n \in (c, d)$, we know the underlying polynomial $p_n(\,\cdot\,; \mu_1, \mu_2, \ldots, \mu_n)$ explicitly. In other words, we have a map $\mathfrak{S}$, say, such that

$$\mathfrak{S} : [\; \mu_1 \quad \mu_2 \quad \cdots \quad \mu_n \;] \to p_n(\,\cdot\,; \mu_1, \mu_2, \ldots, \mu_n), \qquad n \geq 1.$$

Suppose further that $\mathfrak{S}$ can be smoothly extended to all $[\; \mu_1 \quad \mu_2 \quad \ldots \quad \mu_n \;] \in \mathbb{C}^n$, $n \geq 1$ (except perhaps for polar singularities) – this, typically, presents no difficulty whatsoever, as long as p_n is normalized by an appropriate multiplicative constant. We employ the same notation to the original map and to its extension, hence $\mathfrak{S} : \mathbb{C}^n \to \mathbb{P}_n[x]$, $n \geq 1$.

Given an arbitrary monic $q \in \mathbb{P}_n[x]$, we denote by $\mathfrak{Z}$ the function that assigns to q the complex n-tuple of its zeros (in any order – the specific ordering of the zeros is of no concern in the sequel but the careful reader might wish to order the zeros, e.g. lexicographically by their real and imaginary parts). Therefore $\mathfrak{Z} : \mathbb{P}_n \to \mathbb{C}^n$, $n \geq 1$, and we may superpose it with $\mathfrak{S}$. We thus define

$$\mathfrak{T} := \mathfrak{S}\mathfrak{Z} : \mathbb{P}_n \to \mathbb{P}_n, \qquad n \geq 1.$$

The following theorem is a straightforward consequence of our construction, in tandem with Theorem 2.

Theorem 3 (Iserles & Nørsett, [4]) Provided that $\mathrm{d}\varphi$ is SSC, the mapping $\mathfrak{T}$ transforms polynomials with all their zeros in (c, d) into polynomials with all their zeros in (a, b). $\square$

Sixteen examples of zero-mapping transformations have been presented in [4]. Here we list just four:

- The *Laguerre* transformation: $(a, b) = (c, d) = (0, \infty)$,

$$\mathfrak{T}\left\{\sum_{k=0}^{n} \alpha_k (x)_k\right\} = \sum_{k=0}^{n} \alpha_k x^k.$$

- The *Charler* transformation: $(a, b), (c, d) = (0, \infty)$,

$$\mathfrak{T}\left\{\sum_{k=0}^{n} \alpha_k x^k\right\} = \sum_{k=0}^{n} (-1)^k \alpha_k (-x)_k.$$

- The *Llaw* transformation: $(a, b) = (1, \infty)$, $(c, d) = (0, \infty)$, $q > 1$,

$$\mathfrak{T}\left\{\sum_{k=0}^{n} \alpha_k (-x; q)_k\right\} = \sum_{k=0}^{n} \alpha_k x^k.$$

- The Γ_1 transformation: $(a, b) = (c, d) = (0, \infty)$, $\lambda > 0$,

$$\mathfrak{T}\left\{\sum_{k=0}^{n} \alpha_k (x)_k\right\} = \sum_{k=0}^{n} (-1)^k \lambda^{n-k} \alpha_k (-x)_k.$$

Here $\alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{R}$, whereas the *Pochhammer* and *Gauß–Heine* symbols are

$$\begin{aligned} (z)_n &:= \prod_{k=0}^{n-1} (z + k), \qquad n \geq 1, \\ (z; q)_n &:= \prod_{k=0}^{n-1} \left(1 - q^k z\right), \qquad n \geq 1 \quad \text{or} \quad n = \infty, \end{aligned}$$

respectively. The names ('Laguerre', 'Charlier' etc.) originate in Borel measures associated with the underlying orthogonal polynomials, while 'Llaw' stems from an inversion of the Wall measure [4].

We mention in passing that another mechanism to generate transformations that map zeros in a predictable manner, likewise based upon biorthogonal polynomials, has been presented by Iserles and Saff in [9]. Yet other transformation can be obtained by employing multiplier sequences or convolution-orthogonal polynomials [8].

As far as the transformations of Theorem 3 are concerned, the one nontrivial requirement is the availability of an explicit form of p_n. This can be accomplished for selected Borel measures by a mixture of tedious manipulation and occasional ingenuity. However, provided that the moments of the underlying measure satisfy a specific functional relationship, the explicit form of p_n can be determined from a general theory [4, 7]. Thus, let $\{\sigma_\ell\}_{\ell=1}^\infty$ be an arbitrary real sequence and define

$$r_k(x) := \prod_{\ell=1}^{k}(x-\sigma_\ell), \qquad k \geq 0.$$

We denote $\mathbf{r} := \{r_k\}_{k=0}^\infty$ and consider the *modified moments*

$$\mathfrak{m}_n(\mu, \mathbf{r}) := \int_a^b r_n(x)\,\mathrm{d}\varphi(x,\mu), \qquad n \geq 1.$$

Note that $\mathfrak{m}_n(\mu)$ can be replaced by $\mathfrak{m}_n(\mu,\mathbf{r})$ in the statement of Theorem 1 without voiding its conclusion.

We say that the Radon–Nikodym derivative (also known as the weight function)

$$\omega(x,\mu) = \frac{\partial}{\partial x}\varphi(x,\mu)$$

is a *Möbius quotient function* if there exist real sequences $\{\alpha_k\}_{k=0}^\infty$, $\{\beta_k\}_{k=0}^\infty$, $\{\gamma_k\}_{k=0}^\infty$ and $\{\delta_k\}_{k=0}^\infty$ such that

$$\alpha_k\delta_\ell \neq \beta_k\gamma_\ell, \qquad k=0,1,\ldots,\ell, \quad \ell \geq 0$$

and

$$\frac{\mathfrak{m}_{n+1}(\mu,\mathbf{r})}{\mathfrak{m}_n(\mu,\mathbf{r})} = \frac{\alpha_n+\beta_n\mu}{\gamma_n+\delta_n\mu}, \qquad n \geq 0. \tag{2}$$

Theorem 4 (Iserles & Nørsett, [4]) Let

$$g_n(\mu) := \alpha_n+\beta_n\mu, \quad h_n(\mu) := \gamma_n+\delta_n\mu, \qquad n \geq 0,$$

and suppose that

$$\prod_{k=1}^{n}(x-\mu_k) = \sum_{\ell=0}^{n}\nu_\ell\prod_{i=0}^{\ell-1}g_i(x)\prod_{j=\ell}^{n-1}h_j(x). \tag{3}$$

Then, up to a nonzero multiplicative constant,

$$p_n(x;\mu_1,\mu_2,\ldots,\mu_n) = \sum_{k=0}^{n}\nu_k r_k(x). \tag{4}$$

□

The representation (3) always exists, provided that the polynomials

$$\left\{ \prod_{i=0}^{\ell-1} g_i(x) \prod_{j=\ell}^{n-1} h_j(x) \right\}_{\ell=0}^{n}$$

are a basis of $\mathbb{P}_n[x]$.

Note that (4) is an 'explicit' representation only in a roundabout way. To some extent, this can be rectified and p_n represented, at least in the case $r_k(x) = x^k$, $k \geq 1$, in a truly explicit manner [7]. However, the form (4) is ideal for the representation of transformations that map zeros of polynomials in a predictable manner.

Corollary (Iserles & Nørsett, [4]) Let $\mathrm{d}\varphi$ be an SSC measure and let ω be a MQF. Then the transformation

$$\mathfrak{T} \left\{ \sum_{\ell=0}^{n} \nu_\ell \prod_{i=0}^{\ell-1} g_i(x) \prod_{j=\ell}^{n-1} h_j(x) \right\} = \sum_{k=0}^{n} \nu_k r_k(x) \tag{5}$$

maps polynomials with all their zeros in (c, d) into polynomials with all their zeros in (a, b). □

The four transformations that have been already mentioned – in fact, all sixteen transformation in [4] – fit into this pattern.

3 Characterization of MQFs by functional equations

In the last section we have made the case for the study of MQFs. As often in mathematical research, this study commenced in a haphazard manner: few obvious weight functions (e.g., x^μ with $(a, b) = (0, 1)$ and $(c, d) = (-1, \infty)$) are MQF. The discovery of other, less trivial, MQFs in [4] was, to a large extent, a hit-and-miss affair. However, with the accumulation of theoretical comprehension comes a point when a more general approach is called for. The sequel of this paper is thus devoted to an attempt to describe all possible MQFs within a given family of candidate weight functions.

In the present section we describe the following framework. Suppose that $\mathrm{d}\varphi(x, \mu) = \rho(x, \mu)\,\mathrm{d}\psi(x)$, with ρ given. What are all possible Borel measures $\mathrm{d}\psi$ (which are independent of μ) such that the underlying weight function is MQF? Note that we have retained not just the freedom to pick $\mathrm{d}\psi$, but also to choose the polynomials r_k, $k \geq 0$.

We present a detailed analysis for the function $\rho(x, \mu) = x^\mu$, $(a, b) \subseteq (0, \infty)$, $(c, d) = (0, \infty)$ (in which case it is well known that the measure is SSC [11]). By the definition of generalized moments,

$$\begin{aligned} \mathfrak{m}_n(\mu, \mathbf{r}) &= \int_a^b x^{\mu-1} \{(x - \sigma_{n+1}) + \sigma_{n+1}\} r_n(x)\,\mathrm{d}\psi(x) \\ &= \mathfrak{m}_{n+1}(\mu - 1, \mathbf{r}) + \sigma_{n+1} \mathfrak{m}_n(\mu - 1, \mathbf{r}), \qquad n \geq 0. \end{aligned} \tag{6}$$

Substitution of (6) into (2) results in

$$\frac{\alpha_n + \beta_n(\mu + 1)}{\gamma_n + \delta_n(\mu + 1)} = \frac{\mathfrak{m}_{n+2}(\mu, \mathbf{r}) + \sigma_{n+2} \mathfrak{m}_{n+1}(\mu, \mathbf{r})}{\mathfrak{m}_{n+1}(\mu, \mathbf{r}) + \sigma_{n+1} \mathfrak{m}_n(\mu, \mathbf{r})}$$

$$= \frac{\frac{\mathfrak{m}_{n+2}(\mu,\mathbf{r})}{\mathfrak{m}_{n+1}(\mu,\mathbf{r})}+\sigma_{n+2}}{1+\sigma_{n+1}\frac{\mathfrak{m}_{n}(\mu,\mathbf{r})}{\mathfrak{m}_{n+1}(\mu,\mathbf{r})}} = \frac{\frac{\alpha_{n+1}+\beta_{n+1}\mu}{\gamma_{n+1}+\delta_{n+1}\mu}+\sigma_{n+2}}{1+\sigma_{n+1}\frac{\gamma_n+\delta_n\mu}{\alpha_n+\beta_n\mu}}.$$

We thus obtain a cubic equation in $\mu > 0$, namely

$$\begin{aligned} & [(\alpha_n+\beta_n)+\beta_n\mu][\gamma_{n+1}+\delta_{n+1}\mu][(\alpha_n+\sigma_{n+1}\gamma_n)+(\beta_n+\sigma_{n+1}\delta_n)\mu] \\ = \;& [\alpha_n+\beta_n\mu][(\gamma_n+\delta_n)+\delta_n\mu][(\alpha_{n+1}+\sigma_{n+2}\gamma_{n+1})+(\beta_{n+1}+\sigma_{n+2}\delta_{n+1})\mu]. \end{aligned}$$

Since each linear factor on the left must match likewise factor on the right, for each and every $n \geq 0$ we end up with the following cases:

Case I: $\exists C \neq 0$ such that $\alpha_n+\beta_n+\beta_n\mu = C(\alpha_n+\beta_n\mu)$. Therefore $C=1$, $\alpha_n = 1$ and $\beta_n = 0$.

Case II: $\exists C \neq 0$ such that $\alpha_n+\beta_n+\beta_n\mu = C(\alpha_{n+1}+\sigma_{n+2}\gamma_{n+1}+(\beta_{n+1}+\sigma_{n+2}\delta_{n+1})\mu)$.

Case III: $\exists C \neq 0$ such that $\alpha_n+\beta_n+\beta_n\mu = C(\gamma_n+\delta_n+\delta_n\mu)$.

Case III is ruled out at once, by virtue of the following trivial consequence of Theorem 1 and of (2).

Proposition 5 If $\exists C \neq 0$ such that $g_n(\mu) = Ch_n(\mu)$ for some $n \geq 0$ then the underlying measure is not regular. $\square$

Hence, for every $n \geq 0$ either case I or Case II holds but, as things stand, there is nothing to prevent different cases to occur for different ns.

Looking closer at Cases I–II and making repeated use of Proposition 5, we can affirm after further manipulation that

$$\gamma_n = \gamma_0 + n\delta_0, \quad \delta_n \equiv \delta_0, \qquad n \geq 0. \tag{7}$$

Moreover, Case I reduces to

$$\left.\begin{aligned} \beta_n &= 0, \\ \alpha_{n+1} &= \alpha_n + (\sigma_{n+1}-\sigma_{n+2})\gamma_n, \\ \beta_{n+1} &= (\sigma_{n+1}-\sigma_{n+2})\delta_n \end{aligned}\right\} \tag{8}$$

whereas Case II is equivalent to

$$\left.\begin{aligned} \sigma_{n+1} &= 0, \\ \alpha_{n+1} &= \alpha_n + \beta_n - \sigma_{n+2}(\gamma_n+\delta_n), \\ \beta_{n+1} &= \beta_n - \sigma_{n+2}\delta_n \end{aligned}\right\} \tag{9}$$

Let us suppose first that $\delta_0 = 0$. Thus, it follows from (7) that, without loss of generality, $\gamma_n \equiv 1$ and $\mathfrak{m}_{n+1}(\mu,\mathbf{r})/\mathfrak{m}_n(\mu,\mathbf{r}) = \alpha_n + \beta_n\mu$. We employ Proposition 5 to argue that $\beta_n \neq 0$. Consequently, (8) is impossible and we deduce that (9) must be valid for all $n \geq 0$. In other words,

$$\sigma_{n+1} = 0, \quad \alpha_n = \alpha_0 + n\beta_0, \quad \beta_n \equiv \beta_0, \qquad n \geq 0,$$

and this is precisely the *Laguerre* transformation from [4].

Unless $\delta_0 = 0$, we assume without loss of generality that $\delta_0 = 1$. Therefore, (7) now reads $\gamma_n = \gamma_0 + n$, $\delta_n \equiv 1$. The situation is now considerably more complicated. Long analysis in [6] affirms that, again, (8) is ruled out and (9) must hold for all $n \geq 0$. An interesting case, not mentioned in [6], is when (8) holds for all $n \geq 0$. In that instance $\sigma_{n+1} \equiv 0$, $\alpha_n \equiv \alpha_0$, $\beta_n \equiv 0$, $\gamma_n = \gamma_0 + n$ and $\delta_n \equiv 1$, $n \geq 0$. By using the technique of [7] (and of the next section of the present paper) it is possible to derive ω explicitly: up to a positive multiplicative constant, if such an ω exists then it is necessarily of the form

$$\omega(x, \mu) = x^{\gamma_0 + \mu - 2} \mathrm{e}^{\alpha_0 / x}.$$

It is, however, an easy exercise to verify that no $(a, b) \subseteq (0, \infty)$ exists so that ω is MQF there. Thus, we integrate by parts

$$\mathfrak{m}_{n+1}(\mu) = \int_a^b x^{\gamma_0 + n - 1 + \mu} \mathrm{e}^{\alpha_0 / x} \, \mathrm{d}x$$

to obtain

$$\mathfrak{m}_{n+1}(\mu) = \frac{1}{\gamma_0 + n + \mu} x^{\gamma_0 + n + \mu} \mathrm{e}^{\alpha_0 / x} \Big|_a^b + \frac{\alpha_0}{\gamma_0 + n + \mu} \mathfrak{m}_n(\mu).$$

Therefore we need

$$\left(\frac{a}{b}\right)^{\gamma_0 + n + \mu} = \exp\left\{\alpha_0 \left(\frac{1}{b} - \frac{1}{a}\right)\right\}$$

to be satisfied for some $0 < a < b \leq \infty$ and μ in a proper interval. Clearly, this is impossible.

Imposing (9) for all $n \geq 0$ results in

$$\sigma_{n+1} = 0, \quad \alpha_n = \alpha_0 + n\beta_0, \quad \beta_n \equiv \beta_0, \qquad n \geq 0.$$

In other words,

$$g_n(\mu) = \alpha_0 + n\beta_0 + \beta_0 \mu, \quad h_n(\mu) = \gamma_0 + n + \mu, \qquad n \geq 0,$$

and we recover the *Jacobi* transformation from [4].

Remarkably, the general approach of this section is not leading to any new transformation. It merely affirms that, as far as $\rho(x, \mu) = x^\mu$ is concerned, the results of [4] are the best possible.

Similar analysis has been extended in [6] to other forms of ρ. Specifically, $\rho(x, \mu) = \mu^x$ leads to a differential equation, instead of the difference equation (6), while for $\rho(x, \mu) = x^{\log_q \mu}$, where $q \in (0, 1)$, (6) is replaced by a functional equation. In the latter case the characterization actually recovers a single case which has been missed in [4], namely the *Wall*$_0$ transformation

$$\mathfrak{T}\left\{\sum_{k=0}^n \nu_k (-q^k x; q)_{n-k}\right\} = \sum_{k=0}^n \nu_k x^k,$$

which maps polynomials with all their zeros in $(c, d) = (0, \infty)$ into polynomials with all their zeros in $(a, b) = (0, 1)$.

4 MQFs and generalized hypergeometric functions

In the present section, which is based upon [7], we continue the theme of characterizing MQFs, while confining our attention to the case $r_k(x) \equiv x^k$, $k \geq 0$. We further assume that the coefficients admit the formal expansion

$$\begin{aligned} \alpha_n &= \sum_{\ell=0}^{\infty} a_\ell (-n)_\ell, \\ \beta_n &= \sum_{\ell=0}^{\infty} b_\ell (-n)_\ell, \\ \gamma_n &= \sum_{\ell=0}^{\infty} c_\ell (-n)_\ell, \\ \delta_n &= \sum_{\ell=0}^{\infty} d_\ell (-n)_\ell \end{aligned}$$

where the functions

$$\begin{aligned} a(z) &:= \sum_{\ell=0}^{\infty} a_\ell z^\ell, & b(z) &:= \sum_{\ell=0}^{\infty} b_\ell z^\ell, \\ c(z) &:= \sum_{\ell=0}^{\infty} c_\ell z^\ell, & d(z) &:= \sum_{\ell=0}^{\infty} d_\ell z^\ell \end{aligned}$$

are analytic about $z = 0$. A particular special case of great interest is when all these functions are polynomial.

We further stipulate that the weight function $\omega(x, \mu)$ is $C^\infty[0, \infty)$ in x and that

$$\frac{\partial^m \omega(0, \mu)}{\partial x^m} = 0, \qquad m \geq 0.$$

The last condition can be relaxed a great deal (and, provided that a, b, c, d are polynomials, need anyway be imposed only for $m = 0, 1, \ldots, \max\{\deg a, \deg b, \deg c, \deg d\}$), but the present form retains the full flavour of the analysis in [7], while being somewhat simpler to investigate.

We write (2) in the form

$$(\gamma_n + \delta_n \mu) \int_0^\infty x^{n+1} \omega(x, \mu) \, \mathrm{d}x = (\alpha_n + \beta_n \mu) \int_0^\infty x^n \omega(x, \mu) \, \mathrm{d}x, \qquad n \geq 0 \tag{10}$$

(note that the $\mathfrak{m}_n$s are now standard moments). Multiplying (10) by $(-1)^n t^n / n!$ and summing up for all $n \geq 0$ produces, after easy manipulation, the identity

$$\int_0^\infty \{c(xt) + \mu d(xt)\} x \mathrm{e}^{-tx} \omega(x, \mu) \, \mathrm{d}x = \int_0^\infty \{a(xt) + \mu b(xt)\} \mathrm{e}^{-tx} \omega(x, \mu) \, \mathrm{d}x.$$

Expanding $\{c(xt) + \mu d(xt)\} x$ and $\{a(xt) + \mu b(xt)\}$ into power series in t, we deduce that

$$\sum_{\ell=0}^{\infty} t^\ell \int_0^\infty \omega_\ell(x, \mu) \mathrm{e}^{-tx} \, \mathrm{d}x = 0, \tag{11}$$

where
$$\omega_\ell(x,\mu) := x^\ell\{(a_\ell - c_\ell x) + \mu(b_\ell - d_\ell x)\}\omega(x,\mu), \qquad \ell \geq 0.$$

The derivatives of ω vanish at $x = 0$, and this allows us to conclude that

$$t^\ell \int_0^\infty \omega_\ell(x,\mu)\mathrm{e}^{-tx}\,\mathrm{d}x = \mathcal{L}\left\{\frac{\partial^\ell \omega_\ell(x,\mu)}{\partial x^\ell}\right\}(t), \qquad \ell \geq 0,$$

where $\mathcal{L}$ is the *Laplace* transform. Thus, taking inverse transforms in (11) leads to the infinite-order differential equation

$$\sum_{\ell=0}^\infty \frac{\partial^\ell \omega_\ell(x,\mu)}{\partial x^\ell} = 0. \tag{12}$$

Let

$$\begin{aligned}
\mathfrak{p}_j(\mu) &:= \sum_{\ell=0}^\infty \frac{((\ell+j)!)^2 a_{\ell+j}}{\ell!} + \mu \sum_{\ell=0}^\infty \frac{((\ell+j)!)^2 b_{\ell+j}}{\ell!},\\
\mathfrak{q}_j(\mu) &:= \sum_{\ell=0}^\infty \frac{(\ell+j)!(\ell+j+1)! c_{\ell+j}}{\ell!} + \mu \sum_{\ell=0}^\infty \frac{(\ell+j)!(\ell+j+1)! d_{\ell+j}}{\ell!}.
\end{aligned}$$

It follows from the definition of ω_ℓ that (12) is identical to

$$\sum_{j=0}^\infty \frac{1}{j!}\left\{\frac{\mathfrak{p}_j(\mu)}{j!} - x\frac{\mathfrak{q}_j(\mu)}{(j+1)!}\right\} x^j \frac{\partial^j}{\partial x^j}\omega(x,\mu) = 0. \tag{13}$$

We assume in the sequel that $\alpha_n, \beta_n, \gamma_n, \delta_n \in \mathbb{P}_s[n]$. Therefore $a, b, c, d \in \mathbb{P}_s[x]$, $\mathfrak{p}_j, \mathfrak{q}_j \equiv 0$ for $j \geq s+1$ and (13) becomes a *bona fide* ordinary differential equation. Its solutions can be determined explicitly. Thus, substituting

$$\omega(x,\mu) = x^\theta \sum_{n=0}^\infty v_n x^n$$

into

$$\sum_{j=0}^s \frac{1}{j!}\left\{\frac{\mathfrak{p}_j(\mu)}{j!} - x\frac{\mathfrak{q}_j(\mu)}{(j+1)!}\right\} x^j \frac{\partial^j}{\partial x^j}\omega(x,\mu) = 0$$

affirms that θ must be a zero of the s-degree polynomial

$$\sum_{\ell=0}^s (-1)^\ell \mathfrak{p}_\ell(\mu)(-\theta)_\ell = 0, \tag{14}$$

while the v_ns obey the recurrence

$$\begin{aligned}
v_0 & \quad \text{arbitrary},\\
v_{n+1} &= \frac{\sum_{\ell=0}^s (-1)^\ell \mathfrak{q}_\ell(\mu)(-n-\theta)_\ell}{\sum_{\ell=0}^s (-1)^\ell \mathfrak{p}_\ell(\mu)(-n-1-\theta)_\ell} v_n, \qquad n \geq 0.
\end{aligned} \tag{15}$$

Note that this construction might occasionally fail to yield all the solutions: zeros in (14) might be complex or, whilst real, too small to ensure that the requisite number of derivatives vanishes at $x = 0$ (specifically, we require $\theta > s$). Moreover, a denominator in (15) might vanish.

Let

$$\begin{aligned} P(x,\mu) &:= \sum_{\ell=0}^{s}(-1)^\ell \mathfrak{p}_\ell(\mu)(-x-\theta)_\ell, \\ Q(x,\mu) &:= \sum_{\ell=0}^{s}(-1)^\ell \mathfrak{q}_\ell(\mu)(-x-\theta)_\ell. \end{aligned}$$

Thus, (14) becomes $P(0,\mu) = 0$, while (15) reads

$$v_{n+1} = \frac{Q(n,\mu)}{P(n+1,\mu)} v_n = \cdots = \left(\prod_{k=0}^{n} \frac{Q(k,\mu)}{P(k+1,\mu)}\right) v_0.$$

The polynomials (in x) P and Q can be factorized,

$$P(x,\mu) = p_* \prod_{j=1}^{s_1}(x-\eta_j), \qquad Q(x,\mu) = q_* \prod_{j=1}^{s_2}(x-\zeta_j),$$

where $s = \max\{s_1, s_2\}$. Obviously, for (14) to hold we need $s_1 \geq 1$, as well as $0 \in \{\eta_1, \eta_2, \ldots, \eta_{s_1}\}$. Without loss of generality, we thus assume that $\eta_{s_1} = 0$. The conclusion is

$$v_n = \left(\frac{p_*}{q_*}\right)^n \frac{\prod_{j=1}^{s_2}(-\zeta_j)_n}{n! \prod_{j=1}^{s_1-1}(1-\eta_j)_n} v_0, \qquad n \geq 0,$$

and we deduce that ω is, up to a multiplication by $x^\theta v_0$, a generalized hypergeometric function.

Theorem 6 (Iserles & Nørsett, [7]) Every MQF ω, within the framework of this section, is, up to a nonzero multiplicative constant, of the form

$$\omega(x,\mu) = x^\theta \, {}_{s_2}F_{s_1-1}\left[\begin{matrix} -\zeta_1, -\zeta_2, \ldots, -\zeta_{s_2} \\ 1-\eta_1, 1-\eta_2, \ldots, 1-\eta_{s_1-1}; \end{matrix} \; cx\right], \tag{16}$$

where $c \in \mathbb{R} \setminus \{0\}$, $\eta_1, \eta_2, \ldots, \eta_{s_1-1}, \zeta_1, \zeta_2, \ldots, \zeta_{s_2}$ are functions of μ and $\theta \in (s, \infty)$. □

It is important to remark that not every function of the form (16) is necessarily an MQF. For this we need in addition that $\omega(x,\mu) \geq 0$ for $x \in (a,b)$ and that the functions

$$\prod_{j=1}^{s_1-1}(x-\eta_j) \qquad \text{and} \qquad \prod_{j=1}^{s_2}(x-\zeta_j)$$

are linear in μ. Moreover, even if all these conditions are obeyed, it does not mean, of course, that every choice of $(a,b) \subseteq (0,\infty)$ leads to a MQF (or even that such a choice exists at all) – the task of determining the correct support remains!

The differential equation can be formally solved also when $s_1 = 0$ – an example with $s_1 = 0$, $s_2 = 1$ features in the previous section.

To illustrate the line of reasoning that led to Theorem 6, we consider the case of $s = 2$ and the quotient (2) being linear in μ. Hence $s_1 = 2$, $s_2 = 0$, (without loss of generality) $\gamma_n \equiv 1$ and $\delta_n \equiv 0$. Therefore

$$\alpha_n = a_0 - a_1 n + a_2 n(n-1), \quad \beta_n = b_0 - b_1 n + b_2 n(n-1), \qquad n \geq 0,$$

and we obtain

$$\begin{array}{rclrcl} \mathfrak{p}_0(\mu) & = & (a_0 + a_1 + 2a_2) + (b_0 + b_1 + 2b_2)\mu, & \mathfrak{q}_0(\mu) & \equiv & 1, \\ \mathfrak{p}_1(\mu) & = & (a_1 + 4a_2) + (b_1 + 4b_2)\mu, & \mathfrak{q}_1(\mu) & \equiv & 0, \\ \mathfrak{p}_2(\mu) & = & 4(a_2 + b_2\mu), & \mathfrak{q}_2(\mu) & \equiv & 0. \end{array}$$

Hence,

$$\begin{array}{rcl} P(x,\mu) & = & (a_0 + a_1 + 2a_2) + (b_0 + b_1 + 2b_2)\mu + (a_1 + 4a_2)(x + \theta) \\ & & \quad + (2_2 + b_2\mu)(x + \theta)(x + \theta - 1), \\ Q(x,\mu) & \equiv & 1 \end{array}$$

and θ is a zero of the quadratic

$$(a_2 + b_2\mu)\theta^2 + \{(a_1 + 3a_2) + (b_1 + 3b_2)\mu\}\theta + \{(a_0 + a_1 + 2a_2) + (b_0 + b_1 + 2b_2)\mu\}.$$

We thus obtain

$$\begin{array}{rcl} \theta & = & \dfrac{1}{2(a_2 + b_2\mu)} \Bigg\{ (a_1 + 3a_2) + (b_1 + 3b_2)\mu \\ & & \quad \pm \Big(((a_1 + a_2) + (b_1 + b_2)\mu)^2 - 4(a_0 + b_0\mu)(a_2 + b_2\mu) \Big)^{\frac{1}{2}} \Bigg\}. \end{array}$$

Since

$$\eta_1 = 1 - 2\theta - 4\frac{\mathfrak{p}_1}{\mathfrak{p}_2},$$

we obtain from Theorem 6 the explicit expression

$$\omega(x,\mu) = x^{\theta} {}_0F_1 \left[\begin{array}{c} -; \\ 2\theta + 4\mathfrak{p}_1/\mathfrak{p}_2; \end{array} \; \frac{4x}{\mathfrak{p}_2} \right],$$

which is MQF as long as $\theta > 2$ for all $\mu \in (c,d)$.

Up to rescaling and a change of variable, the above function is nothing else but the familiar *modified Bessel* function [12],

$$\omega(x,\mu) = x^{\alpha} I_{\beta}\left(4 \left(\frac{x}{\mathfrak{p}_2} \right)^{\frac{1}{2}} \right),$$

where $\alpha = \frac{1}{2} - 2\mathfrak{p}_1/\mathfrak{p}_2$ and $\beta = 2\theta + 4\mathfrak{p}_1/\mathfrak{p}_2 - 1$.

In the special case

$$a_0 = a_1 = b_0 = b_2 = 0, \qquad a_2 > 0, \qquad b_1 \neq 0$$

we obtain the quadratic $\theta^2 + (3-\lambda)\theta + (2-\lambda) = 0$, where $\lambda := -b_1\mu/a_2$. One of its zeros, $\theta = -1$, is too small, whereas the second, $\theta = 2\lambda - 2$ is acceptable as long as $\lambda > 2$. Subject to the latter restriction, and substituting

$$\tilde{x} := 2\left(\frac{x}{a_2}\right)^{\frac{1}{2}}, \qquad \tilde{\mu} := \lambda - 3 > -1,$$

we obtain

$$\omega(x,\mu) = \tilde{x}^{\frac{1}{2}\tilde{\mu}} I_{\tilde{\mu}+1}(\tilde{x}).$$

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