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Integro-differential equations and generalized hypergeometric functions

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Abstract This paper is concerned with the integro-differential equations

$$y'(t) = ay(t) + \int_0^1 y(qt)\,\mathrm{d}\mu(q) + \int_0^1 y'(qt)\,\mathrm{d}\nu(q), \qquad t \geq 0$$

and

$$y(t) + \int_0^1 y(qt)\,\mathrm{d}\mu(q) + \int_0^1 y'(qt)\,\mathrm{d}\nu(q) = 0, \qquad t \geq 0,$$

where a is a complex constant, while μ and ν are complex-valued functions of bounded variation on $[0,1]$. The main motivation for the study of these two equations is that for every integers $B+1 \geq A \geq 0$ and real constants $\alpha_1, \cdots, \alpha_A, \beta_1, \cdots, \beta_B$, where $\beta_1, \cdots, \beta_B$ are neither zero nor a negative integer, and $\theta \in [0, 2\pi)$ there exist smooth functions μ and ν such that the generalized hypergeometric function ${}_AF_B(\alpha_1, \cdots, \alpha_A; \beta_1, \cdots, \beta_B; \mathrm{e}^{i\theta}t)$ satisfies the first equation when $A \leq B$ and the second equation when $A = B+1$. The first equation also includes as a special case the well-known pantograph equation and many of its generalizations. The main goals of our study are well-posedness of initial value problems, Dirichlet and Dirichlet–Taylor expansions and asymptotic behaviour of the solutions.

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1 Introduction

In this paper we study the integro-differential equations

$$y'(t) = ay(t) + \int_0^1 y(qt)\,\mathrm{d}\mu(q) + \int_0^1 y'(qt)\,\mathrm{d}\nu(q), \qquad t > 0 \tag{1.1}$$

and

$$y(t) + \int_0^1 y(qt)\,\mathrm{d}\mu(q) + \int_0^1 y'(qt)\,\mathrm{d}\nu(q) = 0, \qquad t > 0, \tag{1.2}$$

where a is a complex constant, $\mu(q)$ and $\nu(q)$ are complex-valued functions of bounded variation on $[0,1]$, and the integrals under consideration are the Riemann–Stieltjes type. We restrict our attention to solutions of (1.1) and (1.2) that satisfy the initial value condition

$$y(0) = y_0. \tag{1.3}$$

It is clear that (1.1) includes many functional differential equations as a special case, for instance, the retarded equation

$$y'(t) = ay(t) + by(pt), \qquad t > 0 \tag{1.4}$$

studied by Fox, Mayers, Ockendon and Tayler [3], Kato and McLeod [10], etc., and the neutral equation

$$y'(t) = ay(t) + \sum_{i=1}^{M} b_i y(q_i t) + \sum_{i=1}^{K} c_i y'(p_i t), \qquad t > 0 \tag{1.5}$$

studied by Feldstein and Jackiewicz [2], Kuang and Feldstein [11], and in more general form, by Derfel [1], Iserles [6] Liu [9] and Iserles and Liu [8]. However, the main motivation for the study of (1.1) and (1.2) is that the generalized hypergeometric function

$${}_AF_B(\alpha_1,\ldots,\alpha_A;\beta_1,\ldots,\beta_B;\mathrm{e}^{i\theta}t) = \sum_{n=0}^{\infty}\left(\prod_{k=0}^{n-1}\frac{\prod_{i=1}^{A}(k+\alpha_i)}{\prod_{i=1}^{B}(k+\beta_i)}\right)\frac{(\mathrm{e}^{i\theta}t)^n}{n!}, \qquad t \geq 0 \tag{1.6}$$

is a solution of (1.1) in the case $A \leq B$ and a solution of (1.2) in the case $A = B+1$ for appropriately chosen smooth functions $\mu(q)$ and $\nu(q)$, where $A, B \in \mathbb{Z}^+$ with $A \leq B+1$, α_i and β_j, $i = 1,\ldots,A$, $j = 1,\ldots,B$ are complex constants, no denominator parameter β_i is allowed to be either zero or a negative integer, and $\theta \in [0, 2\pi)$.

The existence and uniqueness of solutions of the initial value problem (1.1),(1.3) has been studied in Iserles and Liu [7]. For completeness, we cite the result and investigate the well-posedness of (1.2),(1.3) in Section 2 of this paper. The Dirichlet series and the Dirichlet–Taylor series have played an important role in the study of pantograph equation and its generalization [4, 6, 8]. In section 3, we show that under some restrictions on $\mu(q)$ and $\nu(q)$, the analytic solution of (1.1),(1.3) can be expanded into (generalized) Dirichlet or (generalized) Dirichlet–Taylor series. In Section 4, we study the asymptotic behaviour of solutions of (1.1) and (1.2). Our most interesting result is that the asymptotic behaviour of the solution of (1.1) depends mainly on the distribution of the roots of its characteristic equation

$$a + \int_0^1 q^z \,\mathrm{d}\mu(q) = 0 \tag{1.7}$$

when $\mathrm{Re}\, a < 0$, and the asymptotic behaviour of the solution of (1.2) depends mainly on the distribution of roots of (1.7) in the special case $a = 1$. In the last section we demonstrate how to construct smooth functions $\mu(q)$ and $\nu(q)$ such that ${}_AF_B(\alpha_1, \cdots, \alpha_A; \beta_1, \cdots, \beta_B; t)$ is the only analytic solution of (1.1),(1.3) with $y_0 = 1$ if $A \le B$ or the only analytic solution of (1.2),(1.3) with $y_0 = 1$ if $A = B + 1$.

2 Existence and uniqueness of solutions

We begin this section by introducing notation and stating our basic assumptions. Let

$$\mu_s(q) = \int_0^q \tau^s \,\mathrm{d}\mu(\tau), \qquad \nu_s(q) = \int_0^q \tau^s \,\mathrm{d}\nu(\tau)$$

and

$$\mu_s^* = \int_0^1 \tau^s |\,\mathrm{d}\mu(\tau)|, \qquad \nu_s^* = \int_0^1 \tau^s |\,\mathrm{d}\nu(\tau)|$$

for all real constants s for which these integrals exist. Our basic requirement is that

$$\lim_{h\to 0+} \int_{1-h}^1 |\,\mathrm{d}\mu(q)| = \lim_{h\to 0+} \int_{1-h}^1 |\,\mathrm{d}\nu(q)| = 0 \tag{2.1}$$

and this implies

$$\lim_{s\to+\infty} \mu_s^* = \lim_{s\to+\infty} \nu_s^* = 0. \tag{2.2}$$

For equation (1.2), we assume further that $\nu(q) \in C^1[0,1]$, $\nu'(q)$ is of a bounded variation on $[0,1]$ and

$$\lim_{h\to 0+} \int_{1-h}^1 |\,\mathrm{d}\nu'(q)| = 0. \tag{2.3}$$

The following result is implied in the existence and uniqueness theorem proved by Iserles and Liu [7].

Theorem 1 *The equation (1.1) has a unique solution in* $C^\infty[0,\infty)$ *that satisfies the initial condition (1.3) if and only if*

$$\nu_n(1) \ne 1 \qquad \text{for all} \quad n \in \mathbb{Z}^+. \tag{2.4}$$

This solution is of the form

$$y(t) = y_0 \sum_{n=0}^{\infty} \left(\prod_{k=0}^{n-1} \frac{a + \mu_k(1)}{1 - \nu_k(1)} \right) \frac{t^n}{n!} \tag{2.5}$$

and is analytic in $[0,\infty)$.

As for equation (1.2), we have the following result.

Theorem 2 *If $\nu'(1) \neq 0$ then the equation (1.2) has a unique analytic solution on $[0, |\nu'(1)|)$ that satisfies the initial condition (1.3) if and only if*

$$\nu_n(1) \neq 0 \qquad \text{for all} \quad n \in \mathbb{Z}^+. \tag{2.6}$$

This solution is of the form

$$y(t) = y_0 \sum_{n=0}^{\infty} (-1)^n \left(\prod_{k=0}^{n-1} \frac{1 + \mu_k(1)}{\nu_k(1)} \right) \frac{t^n}{n!}, \quad t \in [0, |\nu'(1)|). \tag{2.7}$$

Furthermore, if $\nu'(1) \notin (-\infty, 0]$ and both $\mu'(q)$ and $\nu''(q)$ are Riemann integrable then (1.2) has a unique solution in $C^\infty[0, \infty)$ that is coincident in $[0, |\nu'(1)|)$ with the analytic solution given by (2.7).

Before proving the preceding theorem, we consider the integral operator T defined by

$$Ty(t) := y(t) + \int_0^1 y(qt)\omega(q,t)\,\mathrm{d}q,$$

where $\omega(q,t)$ is Riemann integrable with respect to $q \in [0,1]$ for all $t \geq 0$ and is continuous with respect to $t \in [0, \infty)$ for all $q \in [0,1]$. We have the following result.

Lemma 3 *Suppose that $f \in C[0, \infty)$ and that $y^* \in C[0, r]$, $r > 0$, satisfy the integral equation*

$$Ty(t) = f(t) \tag{2.8}$$

for $t \in [0, r]$. If there exists a constant $h \in (0,1)$ such that

$$\sup_{t \geq 0} \int_{1-h}^1 |\omega(q,t)|\,\mathrm{d}q < 1$$

then there is a unique function $y(t) \in C[0, \infty)$ that satisfies the integral equation (2.8) for $t \in [0, \infty)$ and

$$y(t) \equiv y^*(t), \quad t \in [0, r]. \tag{2.9}$$

Proof. Let us consider the integral equation

$$y(t) + \int_{(1-\delta)r/t}^1 y(qt)\omega(q,t)\,\mathrm{d}q = f(t) - \int_0^{(1-\delta)r/t} y^*(qt)\omega(q,t)\,\mathrm{d}q, \quad t \geq (1-\delta)r, \tag{2.10}$$

where $\delta = h/(2-h) \in (0,1)$. By the contraction mapping theorem, the equation (2.10) has a unique solution, say $y^{**}(t)$, in $C((1-\delta)r, (1+\delta)r]$. Since $y^*(t)$ satisfies (2.10) for $t \in [(1-\delta)r, r]$, the uniqueness of the solution implies $y^{**}(t) = y^*(t)$ for $[(1-\delta)r, r]$. Hence, the function

$$y(t) = \begin{cases} y^*(t), & t \in [0, r] \\ y^{**}(t), & t \in [r, (1+\delta)r] \end{cases}$$

is the only one in $C[0,(1+\delta)r]$ that satisfies (2.8) for $t \in [0,(1+\delta)r]$ and (2.9). By induction, we can prove that for any integer $n \geq 1$ there exists a unique function in $C[0,(1+\delta)^n r]$ that satisfies (2.8) for $t \in [0,(1+\delta)^n r]$ and (2.9). This completes the proof. □

Proof of Theorem 2. By substituting $y(t) = y_0 \sum_{n=0}^{\infty} y_n t^n/n!$ into (1.2), we find that equation (1.2) has a unique analytic solution that satisfies (1.3) if and only if (2.6) holds. Since

$$\lim_{n\to\infty} \frac{1+\mu_n(1)}{(n+1)\nu_n(1)} = \frac{1}{\nu'(1)},$$

the radius of convergence of (2.7) is $|\nu'(1)|$. Suppose that $\nu'(1) \notin (-\infty, 0]$ and that $\mu'(q)$ and $\nu''(q)$ are Riemann integrable. For any fixed $n \in \mathbb{Z}^+$ we let T_n be the linear operators defined by

$$T_n y(t) := y(t) + \int_0^1 y(qt)\omega_n(q,t)\,\mathrm{d}q, \quad t \geq 0,$$

where

$$\omega_n(q,t) = \frac{tq^n\mu'(q) - nq^{n-1}\nu'(q) - q^n\nu''(q)}{t+\nu'(1)}.$$

and let

$$f_n(t) = \begin{cases} \dfrac{\nu'(0)y_0}{t+\nu'(1)}, & n = 0 \\ 0, & n > 0. \end{cases}$$

Let $y_n^*(t) = \mathrm{d}^n y^*(t)/\mathrm{d}t^n$, where $y^*(t)$ is the function analytic on $[0, |\nu'(1)|)$, which has already been defined in (2.7). It is evident that $y_n^*(t)$ satisfies the integral equation

$$T_n y(t) = f_n(t) \tag{2.11}$$

for $t \in [0,r]$ for an arbitrary $r \in (0, |\nu'(1)|)$. According to Lemma 3, there is a unique function, $x_n(t)$, say, in $C[0,\infty)$ that satisfies (2.11) for $t \in [0,\infty)$ and

$$y(t) = y_n^*(t), \quad t \in [0,r].$$

By uniqueness, we have

$$x_0(t) = \sum_{k=0}^{n-1} \frac{y_k^*(0)}{k!} t^k + \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} x_n(\tau)\,\mathrm{d}\tau, \quad t \geq 0$$

for any fixed integer $n > 0$. Hence, $y \equiv x_0 \in C^\infty[0,\infty)$. This completes the proof. □

Corollary 4 *If $\nu'(1) \notin (-\infty, 0]$ and $\mu'(q)$ and $\nu''(q)$ are Riemann integrable then the limit*

$$\lim_{t\to|\nu'(1)|-} \sum_{n=0}^{\infty} (-1)^n \left(\prod_{k=0}^{n-1} \frac{1+\mu_k(1)}{\nu_k(1)} \right) \frac{t^n}{n!}$$

exists and is finite. □

3 Dirichlet and Dirichlet–Taylor series solutions

In this section we discuss the solution of (1.1) under the assumption that $a \neq 0$ and

$$\int_0^1 \frac{1}{1-q} |\,\mathrm{d}\mu(q)|,\ \int_0^1 \frac{1}{1-q} |\,\mathrm{d}\nu(q)| < \infty. \tag{3.1}$$

We show how to expand the solution into (generalized) Dirichlet and (generalized) Dirichlet–Taylor series. First, we define

$$d_{1,0}(q) = \frac{1}{q-1}, \qquad d_{1,1}(q) = \frac{1}{a(q-1)}$$

and recursively for $n \geq 2$

$$\begin{aligned} & d_{n,m}(q_1,\ldots,q_n) \\ = & \begin{cases} \dfrac{q_1\cdots q_{n-1} d_{n-1,0}(q_1,\ldots,q_{n-1})}{q_1\cdots q_n - 1}, & m=0, \\ \dfrac{a q_1\cdots q_{n-1} d_{n-1,m}(q_1,\ldots,q_{n-1}) + d_{n-1,m-1}(q_2,\ldots,q_n)}{a(q_1\cdots q_n - 1)}, & 1 \leq m \leq n-1, \\ \dfrac{d_{n-1,n-1}(q_2,\ldots,q_n)}{a(q_1\cdots q_n - 1)}, & m=n. \end{cases} \end{aligned}$$

Lemma 5 *The integral identity*

$$\begin{aligned} & 1 + \sum_{n=1}^{\infty} z^n \int_0^1 \cdots \int_0^1 \sum_{m=0}^{n} d_{n,m}(q_1,\ldots,q_n)\,\mathrm{d}\mu(q_1)\cdots\mathrm{d}\mu(q_m)\,\mathrm{d}\nu(q_{m+1})\cdots\mathrm{d}\nu(q_n) \\ = & \prod_{n=0}^{\infty} \frac{1-\nu_n(1)z}{1+a^{-1}\mu_n(1)z} \end{aligned} \tag{3.2}$$

holds for $|z| < |a|/\max_{n\geq 0}|\mu_n(1)|$.

Proof. Let

$$\rho = \max\left\{\int_0^1 \frac{1}{1-q}|\,\mathrm{d}\mu(q)|, \int_0^1 \frac{1}{1-q}|\,\mathrm{d}\nu(q)|\right\}$$

and

$$D_n(q_1,\ldots,q_n) = \sum_{m=0}^{n} |d_{n,m}(q_1,\ldots,q_n)|.$$

For $n \geq 2$ and $q_1,\ldots,q_n \in (0,1)$ we have

$$D_n(q_1,\ldots,q_n) \leq \frac{1}{1-q_n} D_{n-1}(q_1,\ldots,q_{n-1}) + \frac{1}{|a|(1-q_1)} D_{n-1}(q_2,\ldots,q_n)$$

which implies that

$$\begin{aligned}
&\max_{0\le m\le n}\int_0^1\cdots\int_0^1 D_n(q_1,\ldots,q_n)|\,\mathrm{d}\mu(q_1)|\cdots|\,\mathrm{d}\mu(q_m)||\,\mathrm{d}\nu(q_{m+1})|\cdots|\,\mathrm{d}\nu(q_n)| \\
\le\ & \frac{(1+|a|)\rho}{|a|}\max_{0\le m\le n-1}\int_0^1\cdots\int_0^1 D_{n-1}(q_1,\ldots,q_{n-1})|\,\mathrm{d}\mu(q_1)|\cdots|\,\mathrm{d}\mu(q_m)| \\
&\times|\,\mathrm{d}\nu(q_{m+1})|\cdots|\,\mathrm{d}\nu(q_{n-1})|.
\end{aligned}$$

Proceeding inductively, we obtain the inequality

$$\begin{aligned}
&\max_{0\le m\le n}\int_0^1\cdots\int_0^1 D_n(q_1,\ldots,q_n)|\,\mathrm{d}\mu(q_1)|\cdots|\,\mathrm{d}\mu(q_m)||\,\mathrm{d}\nu(q_{m+1})|\cdots|\,\mathrm{d}\nu(q_n)| \\
\le\ & \left(\frac{(1+|a|)\rho}{|a|}\right)^n, \qquad n\ge 1.
\end{aligned}$$

Denoting by $\Phi(z;\mu(\,\cdot\,),\nu(\,\cdot\,))$ the series on the left hand side of (3.2), we observe from the preceding estimate that $\Phi(z;\mu(\,\cdot\,),\nu(\,\cdot\,))$ converges absolutely for $|z|<|a|/[(1+|a|)\rho]$. It is obvious that $\Phi(z;\mu_n(\,\cdot\,),\nu_n(\,\cdot\,))$ converges absolutely for an arbitrary $n\in\mathbb{Z}^+$ and $|z|<|a|/[(1+|a|)\rho]$ as well. It is straightforward to verify for any integer $n\ge 0$ the identity

$$(1+a^{-1}\mu_n(1)z)\Phi(z;\mu_n(\,\cdot\,),\nu_n(\,\cdot\,))=(1-\nu_n(1)z)\Phi(z;\mu_{n+1}(\,\cdot\,),\nu_{n+1}(\,\cdot\,)),\ \ |z|\le\frac{|a|}{(1+|a|)\rho},$$

which implies that

$$\begin{aligned}
&\Phi(z;\mu(\,\cdot\,),\nu(\,\cdot\,))\prod_{n=0}^{N}(1+a^{-1}\mu_n(1)z) \\
=\ & \Phi(z;\mu_{N+1}(\,\cdot\,),\nu_{N+1}(\,\cdot\,))\prod_{n=0}^{N}(1-\nu_n(1)z), \qquad |z|\le\frac{|a|}{(1+|a|)\rho}
\end{aligned}$$

holds for any integer $N\ge 0$. Noting that $\lim_{N\to\infty}\Phi(z;\mu_{N+1}(\,\cdot\,),\nu_{N+1}(\,\cdot\,))=1$, we see that (3.2) holds for $|z|\le|a|/[(1+|a|)\rho]$ by letting $N\to\infty$ in the preceding identity. However, the product on the right hand side of (3.2) is analytic for $|z|<|a|/\max_{n\ge 0}|\mu_n(1)|$. Hence, (3.2) holds in the larger disc $|z|<|a|/\max_{n\ge 0}|\mu_n(1)|$. $\square$

Theorem 6 *If* $\max_{n\ge 0}|\mu_n(1)|<|a|$ *then the solution of (1.1),(1.3) can be expanded into a Dirichlet series of the form*

$$\begin{aligned}
y(t) &= \gamma y_0[\mathrm{e}^{at}+\sum_{n=1}^{\infty}\int_0^1\cdots\int_0^1\sum_{m=0}^{n} d_{n,m}(q_1,\ldots,q_n) \\
&\quad\times\mathrm{e}^{q_1\cdots q_n at}\,\mathrm{d}\mu(q_1)\cdots\,\mathrm{d}\mu(q_m)\,\mathrm{d}\nu(q_{m+1})\cdots\,\mathrm{d}\nu(q_n)],
\end{aligned}\tag{3.3}$$

where

$$\gamma=\prod_{n=0}^{\infty}\frac{1+a^{-1}\mu_n(1)}{1-\nu_n(1)}.\tag{3.4}$$

Proof. It is straightforward to verify that the Dirichlet series (3.3) is a solution of (1.1) and, according to Lemma 5, it also satisfies the initial condition (1.3). □

Theorem 7 *If $m := \min\{n \in \mathbb{Z}^+ : \max_{k\geq 0} |\mu_{n+k}(1)| < |a|\} \geq 1$ then the solution of (1.1),(1.3) can be expanded into a Dirichlet–Taylor series of the form*

$$\begin{aligned} y(t) &= \sum_{k=0}^{m-1} \frac{y^{(k)}(0)}{k!} t^k + \gamma y_0 \left\{ \mathrm{e}^{at} + \sum_{n=1}^{\infty} \int_0^1 \cdots \int_0^1 \sum_{l=0}^{n} d_{n,l}(q_1, \ldots, q_n) \right. \\ & \left. \times \left[\mathrm{e}^{q_1 \cdots q_n at} - \sum_{k=0}^{m-1} \frac{(q_1 \cdots q_n at)^l}{l!} \right] \mathrm{d}\mu(q_1) \cdots \mathrm{d}\mu(q_l)\, \mathrm{d}\nu(q_{l+1}) \cdots \mathrm{d}\nu(q_n) \right\}, \end{aligned} \tag{3.5}$$

where

$$y^{(k)}(0) = \prod_{i=0}^{k-1} \frac{a + \mu_i(1)}{1 - \nu_i(1)}, k = 0, 1, \ldots, m-1,$$

and γ is given by (3.4).

Proof. Let $y_*(t)$ be the solution of (1.1),(1.3). It is evident that $y_*^{(m)}(t)$ is the unique analytic solution of the initial value problem

$$y'(t) = ay(t) + \int_0^1 y(qt)\, \mathrm{d}\mu_m(q) + \int_0^1 y'(qt)\, \mathrm{d}\nu_m(q), \quad t > 0, \quad y(0) = y_0 \prod_{k=0}^{m-1} \frac{a + \mu_k(1)}{1 - \nu_k(1)}. \tag{3.6}$$

We expand $y_*^{(m)}(t)$ into a Dirichlet series by the previous theorem and then use the Taylor expansion

$$y(t) = \sum_{k=0}^{m-1} \frac{y^{(k)}(0)}{k!} t^k + \int_0^t \frac{(t-\tau)^{m-1}}{(m-1)!} y^{(m)}(\tau)\, \mathrm{d}\tau \tag{3.7}$$

to obtain the Dirichlet–Taylor series expansion (3.5). □

4 Asymptotic behaviour

In addition to the assumptions made in Section 2 we assume further in the case of equation (1.1), the functions $\mu(q)$ and $\nu(q)$ have the expansions

$$\mu(q) = \int_0^q b(p)\, \mathrm{d}p + \sum_{k=1}^{\infty} b_k S(q - q_k), \tag{4.1}$$

$$\nu(q) = \int_0^q c(p)\, \mathrm{d}p + \sum_{k=1}^{\infty} c_k S(q - q_k), \tag{4.2}$$

while for equation (1.2) we assume that $\mu(q)$ is of the form (4.1), where $b(q)$ and $c(q)$ are Riemann integrable functions on $[0, 1]$, b_k and c_k are complex constants, $q_k \in (0, 1)$ for

$k = 1, \cdots, \infty$, and S is the step function, i.e., $S(q) = 0$ for $q < 0$ and $S(q) = 1$ for $q \geq 0$. There is a loss of generality in these assumptions but they are general enough to include many interesting cases such as (1.4) and (1.5). Given $a \neq 0$, we define

$$\tilde{\alpha}(\mu(\,\cdot\,); a) = \begin{cases} -\infty, & \text{if (1.7) has no solution,} \\ \sup\left\{\operatorname{Re} z : a + \int_0^1 q^z \, \mathrm{d}\mu(q) = 0\right\}, & \text{otherwise.} \end{cases}$$

It is evident that

$$\tilde{\alpha}(\mu_n(\,\cdot\,); a) = \tilde{\alpha}(\mu(\,\cdot\,); a) - n$$

and

$$\tilde{\alpha}(\mu(\,\cdot\,); a) < \inf\{\alpha \in \mathbb{R} : \mu_\alpha^* < |a|\}.$$

Our main result consists of the following two theorems.

Theorem 8 *Let $y(t)$ be the analytic solution of (1.1),(1.3).*

1. *If* $\operatorname{Re} a > 0$ *then*
   - *(a)* $y(t) = \mathcal{O}(\mathrm{e}^{\alpha t})$ *as* $t \to \infty$ *for any constant* $\alpha > \operatorname{Re} a$;
   - *(b) there exists a constant* $\beta \geq 0$ *such that* $y(t) = \mathcal{O}(t^\beta \mathrm{e}^{\operatorname{Re} a t})$ *as* $t \to \infty$, *provided that* $b(q)$ *and* $c(q)$ *are uniformly bounded on* $[0, 1]$ *and that*

   $$\sum_{k=1}^{\infty} \frac{|b_k|}{1 - q_k} < \infty, \qquad \sum_{k=1}^{\infty} \frac{|c_k|}{1 - p_k} < \infty;$$

   - *(c)* $\lim_{t\to\infty} \mathrm{e}^{-at} y(t) = \gamma y_0$, *provided that (3.1) holds, where* γ *is given by (3.4);*
2. *If* $\operatorname{Re} a = 0$ *then*
   - *(a)* $y(t) = \mathcal{O}(\mathrm{e}^{\alpha t})$ *as* $t \to \infty$ *for any constant* $\alpha > 0$;
   - *(b)* $y(t) = \mathcal{O}(t^\alpha)$ *as* $t \to \infty$ *for any constant* $\alpha \in [0, \infty) \cap (\tilde{\alpha}(\mu(\,\cdot\,); a), \infty)$ *provided that* $\operatorname{Im} a \neq 0$ *and (3.1) holds;*
3. *If* $\operatorname{Re} a < 0$ *then*
   - *(a)* $y(t) = \mathcal{O}(t^\alpha)$ *as* $t \to \infty$ *for any* $\alpha \in [0, \infty) \cap (\tilde{\alpha}(\mu(\,\cdot\,); a), \infty)$,
   - *(b) if* $\tilde{\alpha}(\mu(\,\cdot\,); a) < 0$ *then* $y(t) = \mathcal{O}(t^\alpha)$ *as* $t \to \infty$ *for any constant* $\alpha \in (\tilde{\alpha}(\mu(\,\cdot\,); a), 0]$ *that obeys*

   $$\int_0^{1/t} |\,\mathrm{d}\mu(q)| = \mathcal{O}(t^\alpha), \quad \int_0^{1/t} |\,\mathrm{d}\nu(q)| = \mathcal{O}(t^\alpha) \quad \text{as } t \to \infty, \tag{4.3}$$

   or $y(t) = o(t^\alpha)$ *as* $t \to \infty$ *for any constant* $\alpha \in (\tilde{\alpha}(\mu(\,\cdot\,); a), 0]$ *that obeys*

   $$\int_0^{1/t} |\,\mathrm{d}\mu(q)| = o(t^\alpha), \quad \int_0^{1/t} |\,\mathrm{d}\nu(q)| = o(t^\alpha) \quad \text{as } t \to \infty. \tag{4.4}$$

Theorem 9 *Suppose that $\nu'(1) \notin (-\infty, 0]$. Let $y(t) \in C^{\infty}[0,\infty)$ be a solution of (1.1),(1.3).*

1. *If $\tilde{\alpha}(\mu, 1) \geq 0$ then $y(t) = \mathcal{O}(t^{\alpha})$ as $t \to \infty$ for any constant $\alpha > \tilde{\alpha}(\mu(\,\cdot\,); 1)$.*
2. *If $\tilde{\alpha}(\mu, 1) < 0$ then $y(t) = \mathcal{O}(t^{\alpha})$ (or $o(t^{\alpha})$) as $t \to \infty$ for any $\alpha \in (\tilde{\alpha}(\mu, 1), 0]$ that obeys (4.3) (or (4.4)).*

To prove the last two theorems, we first introduce two function spaces and a few lemmas. Let $\mathbb{B}$ be the function space of continuous functions that are uniformly bounded on $[0,\infty)$ and $\mathbb{B}_0$ be the subspace of $\mathbb{B}$ of all functions that tend to zero at infinity, both equipped with the norm

$$\|f(\,\cdot\,)\|_{[t_0,t_1]} = \sup_{t_0 \leq t \leq t_1} |f(t)|.$$

The proof of the following two lemmas are trivial.

Lemma 10 *Suppose that $M_0(t), M_1(t) \in \mathbb{B}$ and $y(t)$ is a function defined on $[0,\infty)$, uniformly bounded on any finite interval and that satisfies the inequality*

$$|y(t)| \leq M_0(t) + M_1(t)\|y(\,\cdot\,)\|_{[0,t]}, \qquad t \geq 0.$$

If $\limsup_{t\to\infty} M_1(t) < 1$ then $y(t) \in \mathbb{B}$. $\square$

Lemma 11 *Suppose that $y(t)$ is a function defined on $[0,\infty)$, and is uniformly bounded and integrable on any finite interval and that satisfies the equation*

$$y(t) + \int_0^1 y(qt)\,\mathrm{d}\omega(q;t) = f(t), \qquad t \geq 0,$$

where $\omega(q;t)$ is a complex function of bounded variation with respect to q on $[0,1]$ for all $t \in [0,\infty)$. If $(t+1)^{\alpha} f(t) \in \mathbb{B}$ and

$$\limsup_{t\to\infty} \int_0^1 q^{\alpha} |\,\mathrm{d}\omega(q;t)| < 1$$

for some real constant α then $(t+1)^{\alpha} y(t) \in \mathbb{B}$. $\square$

Lemma 12 *Let V be either $\mathbb{B}$ or $\mathbb{B}_0$. If $y(t) \in C[0,\infty)$ satisfies the equation*

$$y(t) + \int_0^1 y(qt)\,\mathrm{d}\mu(q) = f(t), \qquad t \geq 0,$$

then $(t+1)^{-\alpha} y(t) \in V$ for any $\alpha > \tilde{\alpha}(\mu(\,\cdot\,); 1)$ that obeys $(t+1)^{-\alpha} \int_0^{1/t} |\,\mathrm{d}\mu(q)| \in V$, $(t+1)^{-\alpha} f(t) \in V$.

Proof. Let $x(t) = y(\mathrm{e}^t)$, $t \in [0, \infty)$. Then

$$x(t) - \int_0^t x(t-s)\,\mathrm{d}\mu(\mathrm{e}^{-s}) = f(\mathrm{e}^t) - \int_0^{\mathrm{e}^{-t}} y(q\mathrm{e}^t)\,\mathrm{d}\mu(q), \qquad t \geq 0.$$

Applying Theorem 4.11 and Corollary 4.7 in [5] to the preceding equation, we deduce that $\mathrm{e}^{-\alpha t}x(t) \in V$, and this implies $(t+1)^{-\alpha}y(t) \in V$. $\square$

Proof of Theorem 8. Let $y_n(t) = y^{(n)}(t)$, $n \in \mathbb{Z}^+$. Then $y_n(t)$ satisfies

$$y_n'(t) = ay_n(t) + \int_0^1 y_n(qt)\,\mathrm{d}\mu_n(q) + \int_0^1 y_n'(qt)\,\mathrm{d}\nu_n(q), \qquad t > 0. \tag{4.5}$$

It is important to observe that if certain properties of $y_{n+1}(t)$ are known then we can use the equation

$$ay_n(t) + \int_0^1 y_n(qt)\,\mathrm{d}\mu_n(q) = f_n(t), \qquad t > 0, \tag{4.6}$$

where

$$f_n(t) = y_{n+1}(t) - \int_0^1 y_{n+1}(qt)\,\mathrm{d}\nu_n(q),$$

to deduce useful information about $y_n(t)$. We separate our proof into special cases.

1(a) and 2(a): According to our assumption (2.2), for any fixed constant $\alpha > \mathrm{Re}\,a$ there exists an integer $m \geq 1$ such that

$$\rho := \frac{\mu_m^* + |a|\nu_{m-1}^*}{\alpha - \mathrm{Re}\,a} + \nu_{m-1}^* < 1.$$

It follows from (4.5) in the case $n = m$ that

$$\begin{aligned} y_m(t) &= \mathrm{e}^{at}(1 - \nu_{m-1}(1))y_m(0) + \int_0^t \mathrm{e}^{a(t-\tau)} \int_0^1 y_m(q\tau)\,\mathrm{d}(\mu_m(q) + a\nu_{m-1}(q))\,\mathrm{d}\tau \\ &\quad + \int_0^1 y_m(qt)\,\mathrm{d}\nu_{m-1}(q), \qquad t \geq 0, \end{aligned} \tag{4.7}$$

Let $x(t) = \mathrm{e}^{-\alpha t}y_m(t)$, $t \geq 0$. We obtain from (4.7) the inequality

$$|x(t)| \leq |(1 - \nu_{m-1}(1))y_m(0)| + \rho\|x(\,\cdot\,)\|_{[0,t]}, \qquad t \geq 0.$$

Applying Lemma 10 to the preceding inequality we deduce that $x(t) = \mathcal{O}(1)$ as $t \to \infty$. Therefore $y_m(t) = \mathcal{O}(\mathrm{e}^{\alpha t})$ as $t \to \infty$ and, by the Taylor expansion (3.7), we have $y(t) = \mathcal{O}(\mathrm{e}^{\alpha t})$ as $t \to \infty$.

1(b): For any constant $\beta > (\|b(\,\cdot\,)\|_{[0,1]} + |a|\|c(\,\cdot\,)\|_{[0,1]})/\mathrm{Re}\,a$ there exists an integer $m \geq 1$ such that

$$\rho := \frac{\|b(\,\cdot\,)\|_{[0,1]} + |a|\|c(\,\cdot\,)\|_{[0,1]}}{\beta\mathrm{Re}\,a} + \sum_{k=1}^{\infty} \frac{|b_k|q_k^m + |c_k|q_k^{m-1}}{1 - q_k} + \nu_{m-1}^* < 1.$$

Let $x(t)=(t+1)^{-\beta}\mathrm{e}^{-at}y_m(t)$, $t\ge 0$. It follows from (4.7) that

$$\begin{aligned} x(t) &= (t+1)^{-\beta}(1-\nu_{m-1}(1))y_m(0) \\ &\quad +(t+1)^{-\beta}\int_0^t\int_0^1 \mathrm{e}^{-(1-q)a\tau}(q\tau+1)^{\beta}x(q\tau)\,\mathrm{d}(\mu_m(q)+a\nu_{m-1}(q))\,\mathrm{d}\tau \\ &\quad +(t+1)^{-\beta}\int_0^1 \mathrm{e}^{-(1-q)at}(qt+1)^{\beta}x(qt)\,\mathrm{d}\nu_{m-1}(q) \end{aligned}$$

which implies the inequality

$$|x(t)|\le |(1-\nu_{m-1}(1))y_m(0)|+M(t)\|x(\,\cdot\,)\|_{[0,t]},\qquad t\ge 0,$$

where

$$\begin{aligned} M(t) &= \frac{\|b(\,\cdot\,)\|_{[0,1]}+|a|\|c(\,\cdot\,)\|_{[0,1]}}{(t+1)^{\beta}}\int_0^t\int_0^1 \mathrm{e}^{-(1-q)\mathrm{Re}\,a\tau}\,\mathrm{d}q\,\mathrm{d}\tau \\ &\quad +\sum_{k=1}^{\infty}\frac{|b_k|q_k^m+|c_k|q_k^{m-1}}{1-q_k}+\nu_{m-1}^*. \end{aligned}$$

By the L'Hopital's rule we have $\lim_{t\to\infty}M(t)=\rho<1$. Again, we obtain by using Lemma 10 that $x(t)=\mathcal{O}(1)$ as $t\to\infty$, i.e., $y_m(t)=\mathcal{O}(t^{\beta}\mathrm{e}^{\mathrm{Re}\,at})$ as $t\to\infty$ for any constant $\alpha>\mathrm{Re}\,a$. Hence, by the Taylor expansion (3.7), we have $y(t)=\mathcal{O}(t^{\beta}\mathrm{e}^{\mathrm{Re}\,at})$ as $t\to\infty$.

1(c): If $\mu_0^*<|a|$ then it follows from Theorem 6 that $\lim_{t\to\infty}\mathrm{e}^{-at}y(t)=\gamma y_0$. If $m:=\min\{n\in\mathbb{Z}^+:\mu_n^*<|a|\}\ge 1$ then, $y_m(t)$ being the solution of (3.6), we have $\lim_{t\to\infty}\mathrm{e}^{-at}y_m(t)=a^{-m}\gamma y_0$. By using the Taylor expansion (3.7) we can prove that $\lim_{t\to\infty}\mathrm{e}^{-at}y(t)=\gamma y_0$.

2(b): If $\mu_0^*<|a|$ then it follows from Theorem 6 that $y(t)=\mathcal{O}(1)$ as $t\to\infty$. If $m:=\min\{n\in\mathbb{Z}^+:\mu_n^*<|a|\}\ge 1$ then we have $y_m(t)=\mathcal{O}(1)$ as $t\to\infty$. Applying Lemma 12 to (4.6) repeatedly from $n=m-1$ to $n=0$ in decreasing order (treating the right hand side of (4.6) as given) yields the desired result.

3(a): For any constant $\alpha>\tilde{\alpha}(\mu(\,\cdot\,);a)\ge 0$ there exists an integer $m\ge 1$ such that

$$\rho:=(\mu_m^*+|a|\nu_{m-1}^*)/|\mathrm{Re}\,a|+\nu_{m-1}^*<1.$$

Let $x(t)=(t+1)^{-\alpha}y_m(t)$, $t\ge 0$. It follows from (4.7) that

$$\begin{aligned} x(t) &= (t+1)^{-\alpha}\mathrm{e}^{at}(1-\nu_{m-1}(1))y_m(0) \\ &\quad +(t+1)^{-\alpha}\int_0^t\int_0^1 \mathrm{e}^{a(t-\tau)}(q\tau+1)^{\alpha}x(q\tau)\,\mathrm{d}(\mu_m(q)+a\nu_{m-1}(q))\,\mathrm{d}\tau \\ &\quad +(t+1)^{-\alpha}\int_0^1 (qt+1)^{\alpha}x(qt)\,\mathrm{d}\nu_{m-1}(q),\qquad t\ge 0 \end{aligned}$$

which implies the inequality

$$|x(t)|\le (t+1)^{-\alpha}\mathrm{e}^{\mathrm{Re}\,at}|(1-\nu_{m-1}(1))y_m(0)|+\rho\|x(\,\cdot\,)\|_{[0,t]},\qquad t\ge 0.$$

Using Lemma 10 to the preceding inequality we see that $x(t) = \mathcal{O}(1)$ as $t \to \infty$, i.e., $y_m(t) = \mathcal{O}(t^\alpha)$ as $t \to \infty$. Applying Lemma 12 to (4.6) repeatedly from $n = m-1$ to $n = 0$ in decreasing order (treating the right hand side of (4.6) as given) yields the desired result.

3(b): For any constant $\alpha \in [0,\infty) \cap (\tilde{\alpha}(\mu(\,\cdot\,);a),\infty)$ that obeys (4.3), there exists an integer $m \geq 3/2 - \alpha$ such that

$$\rho := (\mu^*_{m+\alpha-1/2} + |a|\nu^*_{m+\alpha-3/2})/|\mathrm{Re}\, a| + \nu^*_{m+\alpha-3/2} < 1.$$

Let $x(t) = t^{-\alpha+1/2} y_m(t)$, $t \geq 0$. It follows from (4.5) in the case of $n = m$ that

$$\begin{aligned} x(t) &= t^{-\alpha+1/2}\mathrm{e}^{at}(1-\nu_{m-1}(1))y_m(0) \\ &\quad + t^{-\alpha+1/2}\int_0^t\int_0^1 \mathrm{e}^{a(t-\tau)}(q\tau)^{\alpha-1/2}x(q\tau)\,\mathrm{d}(\mu_m(q)+a\nu_{m-1}(q))\,\mathrm{d}\tau \\ &\quad + t^{-\alpha+1/2}\int_0^1 (qt)^{\alpha-1/2}x(qt)\,\mathrm{d}\nu_{m-1}(q), \quad t \geq 0 \end{aligned}$$

which implies the inequality

$$|x(t)| \leq t^{-\alpha+1/2}\mathrm{e}^{\mathrm{Re}\, at}|(1-\nu_{m-1}(1))y_m(0)| + \rho\|x(\,\cdot\,)\|_{[0,t]}, \qquad t \geq 0.$$

Using Lemma 10 to the preceding inequality we see that $x(t) = \mathcal{O}(1)$ as $t \to \infty$, i.e., $y_m(t) = \mathcal{O}(t^\alpha)$ as $t \to \infty$. Applying Lemma 12 repeatedly to (4.6) from $n = m-1$ to $n = 1$ in decreasing order (treating the right hand side of (4.6) as given) yields the estimate $y_1(t) = \mathcal{O}(t^{\alpha-1/2})$ as $t \to \infty$. Let V be $\mathbb{B}$ if α obeys (4.3) or $\mathbb{B}_0$ if α obeys (4.4). Apply Lemma 12 to (4.6) in the case $n = 0$ yields the desired result. $\square$

Proof of Theorem 9. Let $y_n(t) = y^{(n)}(t)$, $n \in \mathbb{N}$. Then $y_n(t)$ satisfies

$$y_n(t) + \int_0^1 y_n(qt)q^n\,\mathrm{d}\mu(q) = -\int_0^1 y_{n+1}(qt)q^n\,\mathrm{d}\nu(q), \qquad t > 0, \tag{4.8}$$

which is equivalent to

$$y_n(t) + (t+\nu'(1))^{-1}\int_0^1 y_n(qt)(tq^n\,\mathrm{d}\mu(q) - nq^{n-1}\nu'(q)\,\mathrm{d}q - q^n\,\mathrm{d}\nu'(q)) = 0, \quad t > 0. \tag{4.9}$$

Let $m \in \mathbb{N}$ be such that

$$\int_0^1 q^{m-\tilde{\alpha}(\mu(\,\cdot\,);1)}|\,\mathrm{d}\mu(q)| < 1.$$

Applying Lemma 11 to (4.9) with $n = m$ yields $y_m(t) = \mathcal{O}(t^{\tilde{\alpha}(\mu(\,\cdot\,);1)})$ as $t \to \infty$. Application of Lemma 12 repeatedly to (4.8) from $n = m-1$ to $n = 0$ in decreasing order (treating the right hand side of (4.8) as given) yields the desired result, where we let V be $\mathbb{B}$ if α obeys (4.3) or $\mathbb{B}_0$ if α obeys (4.4). $\square$

5 Generalized hypergeometric functions

First, we construct functions μ and ν such that the generalized hypergeometric function is either a solution of (1.1) or a solution of (1.2).

When $A \le B$, we choose

$$\mu(q) = \sum_{l=1}^{B} a_l q^{\gamma_l}, \quad \nu(q) = -\sum_{l=1}^{B} b_l q^{\gamma_l},$$

where γ_l are complex constants that satisfy $\operatorname{Re}\gamma_l > 0$ and $\gamma_l \ne \gamma_m$ if $l \ne m$, while a_l and b_l are complex coefficients to be determined. Accordingly, we have

$$\mu_k(1) = \frac{\sum_{i=1}^{B} a_l\gamma_l \prod_{m=1,m\ne l}^{B}(k+\gamma_m)}{\prod_{m=1}^{B}(k+\gamma_m)}, \qquad \nu_k(1) = -\frac{\sum_{l=1}^{B} b_l\gamma_l \prod_{m=1,m\ne l}^{B}(k+\gamma_m)}{\prod_{m=1}^{B}(k+\gamma_m)}.$$

If $A < B$, we choose

$$a_l = \frac{e^{i\theta}\prod_{m=1}^{A}(\alpha_m-\gamma_l)}{\gamma_l\prod_{m=1,m\ne l}^{B}(\gamma_m-\gamma_l)}, \qquad b_l = \frac{\prod_{m=1}^{B}(\beta_m-\gamma_l)}{\gamma_l\prod_{j=1,j\ne i}^{B}(\gamma_m-\gamma_l)}, \qquad l = 1,2,\ldots,B$$

such that

$$\sum_{l=1}^{B} a_l\gamma_l \prod_{m=1,m\ne l}^{B}(x+\gamma_m) = e^{i\theta}\prod_{l=1}^{A}(x+\alpha_l)$$

and

$$\prod_{l=1}^{B}(x+\gamma_l) + \sum_{l=1}^{B} b_l\gamma_l \prod_{j=1,j\ne i}^{B}(x+\gamma_m) = \prod_{l=1}^{B}(x+\beta_l)$$

hold for all x. Therefore

$$\frac{\mu_k(1)}{1-\nu_k(1)} = \frac{e^{i\theta}\prod_{l=1}^{A}(k+\alpha_l)}{\prod_{l=1}^{B}(k+\beta_l)}.$$

Hence, for $a = 0$ and $y_0 = 1$, (2.5) and (1.6) are identical. In other word, the generalized hypergeometric function (1.6) is the solution of (1.1) with $a = 0$ and $y(0) = 1$.

If $A = B$, we choose

$$a_l = \frac{e^{i\theta}\prod_{m=1}^{B}(\alpha_m-\gamma_l)}{\gamma_l\prod_{m=1,m\ne l}^{B}(\gamma_m-\gamma_l)}, \qquad b_l = \frac{\prod_{m=1}^{B}(\beta_m-\gamma_l)}{\gamma_l\prod_{m=1,m\ne l}^{B}(\gamma_m-\gamma_l)}, \qquad l = 1,2,\ldots,B$$

such that

$$e^{i\theta}\prod_{l=1}^{B}(x+\gamma_l) + \sum_{l=1}^{B} a_l\gamma_l \prod_{m=1,m\ne l}^{B}(x+\gamma_m) = e^{i\theta}\prod_{l=1}^{B}(x+\alpha_l)$$

and

$$\prod_{l=1}^{B}(x+\gamma_l) + \sum_{l=1}^{B} b_l\gamma_l \prod_{m=1,m\ne l}^{B}(x+\gamma_m) = \prod_{l=1}^{B}(x+\beta_l) \tag{5.1}$$

hold for all x. Similarly to the previous case, we find that the generalized hypergeometric function (1.6) is the solution of (1.1) with $a = 1$ and $y(0) = 1$.

When $A = B + 1$, we choose

$$\mu(q) = \sum_{l=1}^{A} a_l q^{\gamma_l}, \quad \nu(q) = -\sum_{l=1}^{A} b_l q^{\gamma_l},$$

where γ_l are complex constants that satisfy $\operatorname{Re}\gamma_l > 0$ and $\gamma_l \neq \gamma_m$ if $l \neq m$, while a_l and b_l are complex coefficients to be determined. Accordingly, we have

$$\mu_k(1) = \frac{\sum_{i=1}^{A} a_l\gamma_l \prod_{m=1,m\neq l}^{A}(k+\gamma_m)}{\prod_{m=1}^{A}(k+\gamma_m)}, \qquad \nu_k(1) = -\frac{\sum_{l=1}^{A} b_l\gamma_l \prod_{m=1,m\neq l}^{A}(k+\gamma_m)}{\prod_{m=1}^{A}(k+\gamma_m)}.$$

Choose

$$a_l = \frac{\prod_{m=1}^{A}(\alpha_m - \gamma_l)}{\gamma_l \prod_{m=1,m\neq l}^{A}(\gamma_m - \gamma_l)}, \qquad b_l = \frac{\mathrm{e}^{-i\theta}\prod_{m=1}^{B}(\beta_m - \gamma_l)}{\gamma_l \prod_{m=1,m\neq l}^{A}(\gamma_m - \gamma_l)}, \qquad l = 1, 2, \ldots, A$$

such that

$$\prod_{l=1}^{A}(x+\gamma_l) + \sum_{l=1}^{A} a_l\gamma_l \prod_{m=1,m\neq l}^{A}(x+\gamma_m) = \prod_{l=1}^{A}(x+\alpha_l) \tag{5.2}$$

and

$$\sum_{l=1}^{A} b_l\gamma_l \prod_{m=1,m\neq l}^{A}(x+\gamma_m) = \mathrm{e}^{-i\theta}\prod_{l=1}^{B}(x+\beta_l)$$

hold for all x. Then the generalized hypergeometric function (1.6) is the solution of (1.2) with $y(0) = 1$.

As we have seen in the previous section, the asymptotic behaviour of (1.1) in the case $\operatorname{Re} a < 0$ and of (1.2) depends mainly on the distribution of roots of their corresponding characteristic equation. According to (5.1) and (5.2) for the cases $A = B$ and $A = B + 1$ respectively, the characteristic equation is

$$\frac{\prod_{l=1}^{A}(z+\alpha_l)}{\prod_{l=1}^{B}(z+\beta_l)} = 0,$$

which implies that the characteristic roots are $-\alpha_1, \ldots, -\alpha_A$. By choosing γ_i such that $Re\gamma_i \geq -\inf_{1\leq i\leq A}\{\operatorname{Re}\alpha_i\}$, $i = 1, \ldots, A$, we obtain from Theorem 8 the following results.

Theorem 13 *For any fixed $\theta \in (\pi/2, 3\pi/2)$,*

$${}_AF_A(\alpha_1, \ldots, \alpha_A; \beta_1, \ldots, \beta_B; \rho\mathrm{e}^{i\theta}) = \mathcal{O}(\rho^{\alpha}) \qquad \text{as} \quad \rho \to \infty$$

for all $\alpha > -\inf_{1\leq i\leq A}\{\operatorname{Re}\alpha_i\}$. □

Theorem 14 *Let $y(z)$ be the analytic extension of ${}_{B+1}F_B(\alpha_1, \ldots, \alpha_{B+1}; \beta_1, \ldots, \beta_B; z)$. Then for any fixed $\theta \in (0, 2\pi)$,*

$$y(\rho\mathrm{e}^{i\theta}) = \mathcal{O}(\rho^{\alpha}) \qquad \text{as} \quad \rho \to \infty$$

for all $\alpha > -\inf_{1\leq i\leq B+1}\{\operatorname{Re}\alpha_i\}$. □

To conclude this section, we list some of the most commonly used generalized hypergeometric functions and their corresponding integral-functional equations.

1. ${}_0F_1(-;\beta;\mathrm{e}^{i\theta}t)$ is the solution of the equation

$$y'(t)+(\beta-1)\int_0^1 y'(qt)\,\mathrm{d}q=\mathrm{e}^{i\theta}\int_0^1 y(qt)\,\mathrm{d}q,\qquad y(0)=1.$$

Recall that the Bessel function $J_\nu(z)$ equals $\frac{1}{\Gamma(\nu+1)}(z/2)^\nu{}_0F_1(-;\nu+1;-z^2/4)$.

2. Confluent hypergeometric function (Kummer's function) ${}_1F_1(\alpha;\beta;\mathrm{e}^{i\theta}t)$ is the solution of the equation

$$y'(t)+\frac{\beta-\gamma}{\gamma}\int_0^1 y'(qt)\,\mathrm{d}q^\gamma=\mathrm{e}^{i\theta}y(t)+\frac{(\alpha-\gamma)\mathrm{e}^{i\theta}}{\gamma}\int_0^1 y(qt)\,\mathrm{d}q^\gamma,\qquad y(0)=1,$$

where γ is an arbitrary constant that obeys $\mathrm{Re}\,\gamma>0$. If $\mathrm{Re}\,\beta>0$, we may choose $\gamma=\beta$, and find that ${}_1F_1(\alpha;\beta;\mathrm{e}^{i\theta}t)$ is the solution of the equation

$$y'(t)=\mathrm{e}^{i\theta}y(t)+(\alpha-\beta)\mathrm{e}^{i\theta}\int_0^1 y(qt)q^{\beta-1}\,\mathrm{d}q,\qquad y(0)=1.$$

Alternatively, we can choose $\gamma=\alpha$ if $\mathrm{Re}\,\alpha>0$.

3. The hypergeometric function (the Gauss function) ${}_2F_1(\alpha_1,\alpha_2;\beta;\mathrm{e}^{i\theta}t)$ is the solution of the equation

$$\mathrm{e}^{-i\theta}\int_0^1 y'(qt)\,\mathrm{d}(b_1q^{\gamma_1}+b_2q^{\gamma_2})=y(t)+\int_0^1 y(qt)\,\mathrm{d}(a_1q^{\gamma_1}+a_2q^{\gamma_2}),\qquad y(0)=1,$$

where γ_1 and γ_2 are arbitrary constants that obey $\mathrm{Re}\,\gamma_1,\mathrm{Re}\,\gamma_2>0$ and $\gamma_1\neq\gamma_2$, and

$$\begin{aligned}a_1&=\frac{(\alpha_1-\gamma_1)(\alpha_2-\gamma_1)}{\gamma_1(\gamma_2-\gamma_1)},\quad a_2=\frac{(\alpha_1-\gamma_2)(\alpha_2-\gamma_2)}{\gamma_2(\gamma_1-\gamma_2)},\\ b_1&=\frac{\beta-\gamma_1}{\gamma_1(\gamma_2-\gamma_1)},\quad b_2=\frac{\beta-\gamma_2}{\gamma_2(\gamma_1-\gamma_2)}.\end{aligned}$$

If $\mathrm{Re}\,\alpha_1,\mathrm{Re}\,\beta>1$, we may choose $\gamma_1=\beta$, $\gamma=\alpha_1$, and find that ${}_2F_1(\alpha_1,\alpha_2;\beta;\mathrm{e}^{i\theta}t)$ is the solution of the equation

$$\mathrm{e}^{-i\theta}\int_0^1 y'(qt)\,\mathrm{d}q^{\alpha_1-1}=y(t)+(\alpha_2-\beta)\int_0^1 y(qt)\,\mathrm{d}q^{\beta-1},\qquad y(0)=1.$$

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