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Abstract The subject matter of this paper are two functional differential equations with complex lag functions. We address ourselves to the existence and uniqueness of solutions and to their asymptotic behaviour.

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1 Introduction

The last few decades have witnessed important advances in our understanding of the behaviour of functional differential equations with rescaling. A pride of place belongs to the *pantograph equation*

$$\boldsymbol{y}'(t) = A\boldsymbol{y}(t) + B\boldsymbol{y}(qt) + C\boldsymbol{y}'(qt), \quad t \geq 0, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0, \tag{1.1}$$

where $\boldsymbol{y} \in \mathbb{C}^d$, A, B, C are $d \times d$ matrices and $q \in (0, \infty) \setminus \{1\}$ and to its generalizations [5, 9, 11, 12, 13]. The interest in (1.1) is motivated by the ubiquity of its applications in a wide range of subject areas, from probability theory to wavelets to applied mathematics [5, 6, 9, 14].

Our knowledge of (1.1) is fairly comprehensive in the case $q \in (0, 1)$ and the results are conveniently expressed in terms of the spectrum σ and the spectral radius ρ of the corresponding matrices. Specifically, unless $q^{-j} \in \sigma(C)$ for some $j \in \mathbb{Z}$, the solution of (1.1) exists and is unique subject to $\rho(C) < 1$ [9]. Moreover, provided that $\max \operatorname{Re} \sigma(A) < 0$, $\rho(A^{-1}B) < 1$ and $\rho(C) < 1$, the trivial solution of (1.1) is asymptotically stable [9]. The behaviour along the stability boundaries $\max \operatorname{Re} \sigma(A) = 0$ and $\rho(A^{-1}B) = 1$ is more complicated and we refer the reader to [9] and [10] respectively.

The equation

$$y^{(n)}(t) = \sum_{j=0}^{\ell} \sum_{k=0}^{m-1} a_{j,k} y^{(k)}(\alpha_j t), \quad t \geq 0, \tag{1.2}$$

where $a_{j,k} \in \mathbb{C}$ and $\alpha_j > 1$ for all $j = 0, 1, \ldots, \ell$ may be called a (generalized) *advanced pantograph equation.* Unlike in the retarded case $\max_j \alpha_j < 1$, the Cauchy problem for such equations is ill-posed. In general, there exists a family of solutions depending on an arbitrary function. However, uniqueness can be guaranteed in a class of quickly decreasing functions. Let $\alpha := \min_j \alpha_j$ and $A := \max_j \alpha_j$. Every solution of (1.2) which for some $c > 0$ and $\gamma > n^2 \ln A/(2 \ln^2 \alpha)$ satisfies the inequality

$$|y(t)| \leq c \exp\left[-\gamma \ln^2(1+t)\right] \tag{1.3}$$

is necessarily trivial [4, 5]. Moreover, this result cannot be improved in a substantive manner since the equation (1.2) has a nontrivial solution that obeys (1.3) for some $c > 0$ and an arbitrary $\gamma < n/(2 \ln A)$ [5].

The equation

$$y^{(n)}(t) = \sum_{j=0}^{\ell} \sum_{k=0}^{m-1} a_{j,k} y^{(k)}(\alpha_j t + \beta_j), \qquad t \geq 0, \tag{1.4}$$

is a generalization of (1.2). A special feature of (1.4) is the existence of compactly supported solutions. This phenomenon was studied in [4] and has direct applications to approximation theory and to wavelets [6].

In this paper we address ourselves to two generalizations of the pantograph equation to the complex plane, namely the *pantograph equation with involution*

$$y'(z) = \sum_{k=0}^{m-1} a_k y(\omega^k z) + \sum_{k=0}^{m-1} b_k y(r\omega^k z) + \sum_{k=0}^{m-1} c_k y'(r\omega^k z), \qquad z \in \mathbb{C}, \tag{1.5}$$

where $a_k, b_k, c_k \in \mathbb{C}$, $k = 0, 1, \ldots, m-1$ are given, $r \in (0,1)$ and ω is the primitive mth root of unity, and to the *pantograph equation of the second type,*

$$y(z) = \sum_{j=0}^{\ell} \sum_{k=1}^{n} a_{j,k} y^{(k)}(\omega_j z), \qquad z \in \mathbb{C}, \tag{1.6}$$

where $a_{j,k}, \omega_j \in \mathbb{C}$, supplemented by appropriate initial conditions at the origin. Our main results are

Theorem 1 *The solution of the linear equation with involution* (1.5) *cannot be uniformly bounded for $\boldsymbol{y}_0 \neq \mathbf{0}$ in the case $m \geq 3$, while in the case $m \geq 2$ the solution cannot be asymptotically stable.* □

and

Theorem 4 *Given that*

$$\max_{j=0,1,\ldots,\ell} |\omega_j| > 1,$$

the following assertions are true:

1. *Every equation* (1.6) *has nontrivial analytic solution in some neighbourhood of the origin.*

2. *The above analytic solution can be continued to the whole complex plane as an entire function of order zero. Hence it is unbounded along any ray approaching infinity.*

We refer to [7] for an early work on functional differential equations with complex delay.

2 The pantograph equation with involution

Let us consider the equation

$$y'(z) = \sum_{k=0}^{m-1} a_k y(\omega^k z) + \sum_{k=0}^{m-1} b_k y(r\omega^k z) + \sum_{k=0}^{m-1} c_k y'(r\omega^k z), \qquad z \in \mathbb{C}, \tag{2.1}$$

where the constants $a_k, b_k, c_k \in \mathbb{C}$, $k = 0, 1, \ldots, m-1$, are given, $r \in (0,1)$ and ω is the mth primitive root of unity,

$$\omega = \exp \frac{2\pi \mathrm{i}}{m}.$$

The equation (2.1) is accompanied by the initial condition $y(0) = 1$. Following the terminology in [15], we term (2.1) a pantograph equation *with involution.*

Let us define

$$y_\ell(z) := y(\omega^\ell z), \qquad \ell = 0, 1, \ldots, m-1.$$

By replacing z with $\omega^\ell z$ in (2.1) we readily obtain

$$\begin{aligned} y'_\ell(z) \;=\; \omega^\ell \Bigg[& \sum_{k=0}^{m-1} a_{(k+\ell) \bmod m} y_{k+\ell}(z) + \sum_{k=0}^{m-1} b_{(k+\ell) \bmod m} y_{k+\ell}(rz) \\ & + \sum_{k=0}^{m-1} c_{(k+\ell) \bmod m} y'_{k+\ell}(z) \Bigg], \qquad \ell = 0, 1, \ldots, m-1. \end{aligned}$$

In a matrix notation this becomes the standard *pantograph equation*

$$\boldsymbol{y}'(z) = A\boldsymbol{y}(z) + B\boldsymbol{y}(rz) + C\boldsymbol{y}'(rz), \quad z \in \mathbb{C}, \qquad \boldsymbol{y}(0) = \mathbf{1}. \tag{2.2}$$

The matrices A, B, C are defined as follows. Given a sequence $\boldsymbol{\alpha} = \{\alpha_k\}_{k=0}^{m-1} \in \mathbb{C}^m$ we let

$$H_m(\boldsymbol{\alpha}, \omega) = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{m-1} \\ \omega\alpha_{m-1} & \omega\alpha_0 & \omega\alpha_1 & \cdots & \omega\alpha_{m-2} \\ \omega^2\alpha_{m-2} & \omega^2\alpha_{m-1} & \omega^2\alpha_0 & \cdots & \omega^2\alpha_{m-3} \\ \vdots & \vdots & \vdots & & \vdots \\ \omega^{m-1}\alpha_1 & \omega^{m-1}\alpha_2 & \omega^{m-1}\alpha_3 & \cdots & \omega^{m-1}\alpha_0 \end{bmatrix}.$$

Then $A = H(\boldsymbol{a}, \omega)$, $B = H(\boldsymbol{b}, \omega)$ and $C = H(\boldsymbol{c}, \omega)$. Although the matrix H might have already appeared somewhere in the literature, we give it the provisional name of a *ω-circulant.*

The components of the vector $\boldsymbol{y}$ obey the identity $y_\ell(z) = y_k(\omega^{\ell-k}z)$, $k, \ell = 0, 1, \ldots, m-1$. This symmetry means that it is enough to restrict our attention to real values of z. In other words, in place of (2.2) we consider

$$\boldsymbol{y}'(t) = A\boldsymbol{y}(t) + B\boldsymbol{y}(rt) + C\boldsymbol{y}'(rt), \quad t \in \mathbb{R}^+, \qquad \boldsymbol{y}(0) = \mathbf{1}. \tag{2.3}$$

We recall from [9] that

1. The solution of (2.3) exists if

$$r^{-p} \notin \sigma(C), \qquad p \in \mathbb{Z}^+.$$

   and it is unique if $\rho(C) < 1$.

2. The solution of (2.3) is uniformly bounded if either

$$\max \operatorname{Re} \sigma(A) < 0, \qquad 0 \notin \sigma(A), \qquad \rho(A^{-1}B) \le 1$$

   or

$$\max \operatorname{Re} \sigma(A) = 0, \qquad 0 \notin \sigma(A), \qquad \rho(A^{-1}B) < 1.$$

3. The trivial solution of (2.3) is asymptotically stable if

$$\max \operatorname{Re} \sigma(A) < 0, \qquad \rho(A^{-1}B) < 1.$$

Therefore, the location of the eigenvalues of $H_m(\,\cdot\,\omega)$ is critical to our discussion.

3 The eigenvalues of ω-circulants

There are several alternative ways of deriving $\sigma(H_m(\boldsymbol{\alpha}, \omega))$ but the following one is probably the neatest. We emphasize the dependence of the spectrum upon the sequence $\boldsymbol{\alpha}$ by denoting

$$\tilde{\sigma}(\boldsymbol{\alpha}) = \sigma(H_m(\boldsymbol{\alpha}, \omega)).$$

Since $\omega^m = 1$, we have

$$\begin{aligned} H_m(\boldsymbol{\alpha}, \omega) &= \begin{bmatrix} \omega^m \alpha_0 & \omega^m \alpha_1 & \cdots & \omega^m \alpha_{m-1} \\ \omega \alpha_{m-1} & \omega \alpha_0 & \cdots & \omega \alpha_{m-2} \\ \vdots & \vdots & & \vdots \\ \omega^{m-1} \alpha_1 & \omega^{m-1} \alpha_2 & \cdots & \omega^{m-1} \alpha_0 \end{bmatrix} \\ &= \begin{bmatrix} \omega^{m-1} \beta_0 & \omega^{m-1} \beta_1 & \cdots & \omega^{m-1} \beta_{m-1} \\ \beta_{m-1} & \beta_0 & \cdots & \beta_{m-2} \\ \vdots & \vdots & & \vdots \\ \omega^{m-2} \beta_1 & \omega^{m-2} \beta_2 & \cdots & \omega^{m-2} \beta_0 \end{bmatrix}, \end{aligned}$$

where $\beta_\ell = \omega \alpha_\ell$, $\ell = 0, 1, \ldots, m-1$. We next permute $H(\boldsymbol{\alpha}, \omega)$ by bringing the top row to the bottom and the leftmost column to the right. This is a similarity transformation (since permutation matrices are orthogonal), hence eigenvalues stay intact – but the end result is $H_m(\boldsymbol{\beta},\omega)$! Therefore we deduce that

$$\tilde{\sigma}(\boldsymbol{\alpha}) = \tilde{\sigma}(\omega\boldsymbol{\alpha}). \tag{3.1}$$

Let us suppose that $\lambda \in \tilde{\sigma}(\boldsymbol{\alpha})$. Then $\det[H_m(\boldsymbol{\alpha},\omega) - \lambda I] = 0$. The fact that the determinant vanishes remains true if we multiply each element of a matrix by the same constant, therefore $\det[H_m(\omega\boldsymbol{\alpha},\omega) - \omega\lambda I] = 0$. Together with (3.1), this implies that $\omega\lambda \in \tilde{\sigma}(\boldsymbol{\alpha})$.

The last statement implies by induction that

$$\lambda \in \tilde{\sigma}(\boldsymbol{\alpha}) \qquad \Rightarrow \qquad \omega^\ell \lambda \in \tilde{\sigma}(\boldsymbol{\alpha}), \quad \ell = 0, 1, \ldots, m-1. \tag{3.2}$$

Up to a multiplicative constant, this gives all the eigenvalues of $H_m(\boldsymbol{\alpha},\omega)$: The statement is trivial if $\lambda \neq 0$ and, moreover, it is enough for one eigenvalue to be nonzero to be able to deduce from (3.2) that so are all the rest. Hence the only exception is when all the eigenvalues are zero – and this is also consistent with both (3.2) and complete knowledge of the spectrum.

Finally, we fill in the missing gap in (3.2) by specifying the multiplicative constant. Since the determinant of a matrix equals the product of its eigenvalues, (3.2) yields

$$\omega^{(m-1)m/2}\lambda^m = \det H_m(\boldsymbol{\alpha},\omega).$$

The determinant is a multiplicative functional, therefore

$$\det H_n(\boldsymbol{\alpha},\omega) = \det \operatorname{diag}\{1,\omega,\ldots,\omega^{m-1}\} \times \det C_m(\boldsymbol{\alpha}),$$

where $C_m(\boldsymbol{\alpha})$ is the circulant of the sequence $\boldsymbol{\alpha}$. But, as is trivial to affirm, $\det \operatorname{diag}\{1,\omega,\ldots,\omega^{m-1}\} = \omega^{(m-1)m/2} = (-1)^m$, while

$$\det C_m(\boldsymbol{\alpha}) = \prod_{k=0}^{m-1} \alpha(\omega^k),$$

where

$$\alpha(z) = \sum_{\ell=0}^{m-1} \alpha_\ell z^\ell, \qquad z \in \mathbb{C},$$

is the symbol of $C_m(\boldsymbol{\alpha})$. We therefore conclude that the spectrum of $H_m(\boldsymbol{\alpha},\omega)$ is

$$\left[\prod_{k=0}^{m-1} \alpha(\omega^k)\right]^{1/m} \omega^\ell, \qquad \ell = 0, 1, \ldots, m-1. \tag{3.3}$$

4 Boundedness and asymptotic stability

On the face of it, the previous section proves that for $m \geq 3$ there is always an eigenvalue of A in the open right half-plane, hence no uniform boundedness, while for $m \geq 2$ there is always an eigenvalue of A in the closed right half-plane, consequently no asymptotic stability. This, however, disregards the possibility that the vector $\boldsymbol{y}(0) = \mathbf{1}$ might lie in a subspace which is orthogonal to unstable modes of $\exp(q^\ell tA)$ for all $\ell \in \mathbb{Z}^+$. We now show that this is impossible.

Let

$$\boldsymbol{v}_s := \begin{bmatrix} 1 \\ \omega^s \\ \omega^{2s} \\ \vdots \\ \omega^{(m-1)s} \end{bmatrix}, \qquad s \in \mathbb{Z}^+.$$

It is easy to verify the formula

$$H_m(\boldsymbol{\alpha}, \omega)\boldsymbol{v}_s = \alpha(\omega^s)\boldsymbol{v}_{s+1}, \qquad s \in \mathbb{Z}^+,$$

hence the identity

$$[H_m(\boldsymbol{\alpha}, \omega)]^s \mathbf{1} = \left[\prod_{j=0}^{s-1} \alpha(\omega^j)\right] \boldsymbol{v}_s, \qquad s \in \mathbb{Z}^+. \tag{4.1}$$

Incidentally, a simple consequence of our analysis and the identity $\boldsymbol{v}_s = \boldsymbol{v}_{s \bmod m}$ is that

$$[H_m(\boldsymbol{\alpha}, \omega)]^m \boldsymbol{v}_s = \left[\prod_{j=0}^{m-1} \alpha(\omega^j)\right] \boldsymbol{v}_s, \qquad s = 0, 1, \ldots, m-1.$$

Therefore

$$[H_m(\boldsymbol{\alpha}, \omega)]^m = \hat{\alpha} I \qquad \text{where} \qquad \hat{\alpha} := \prod_{j=0}^{m-1} \alpha(\omega^j).$$

This is consistent with our derivation of the spectrum of H_m in the last section. Moreover, it is 'almost' a proof that the spectrum is indeed as stated.

Given $\xi \in \mathbb{C}$, we next proceed to evaluate $\exp(\xi A)\mathbf{1}$. The purpose of our analysis is to demonstrate that, unless all the eigenvalues of A are zero, for almost all values of ξ this vector has nonzero components in the direction of all the eigenvectors of A.

Expanding the exponential into a Taylor series,

$$\begin{aligned}
\mathrm{e}^{\xi A}\mathbf{1} &= \sum_{s=0}^{\infty}\frac{1}{s!}\xi^s A^s \boldsymbol{v}_0 = \sum_{s=0}^{\infty}\frac{1}{s!}\xi^s \boldsymbol{v}_s \\
&= \sum_{s=0}^{\infty}\frac{1}{s!}\left[\prod_{\ell=0}^{s-1} a(\omega^\ell)\right]\xi^m \boldsymbol{v}_s \\
&= \boldsymbol{v}_0 + \sum_{r=0}^{\infty}\sum_{j=1}^{m}\frac{1}{(rm+j)!}\left[\prod_{\ell=0}^{rm+j-1} a(\omega^\ell)\right]\xi^m \boldsymbol{v}_j.
\end{aligned}$$

However,

$$\prod_{\ell=0}^{rm+j-1} a(\omega^\ell) = \left[\prod_{\ell=0}^{m-1} a(\omega^\ell)\right]^r \prod_{\ell=0}^{j-1} a(\omega^j) = \hat{a}^r \prod_{\ell=0}^{j-1} a(\omega^\ell),$$

therefore

$$\mathrm{e}^{\xi A}\mathbf{1} = \boldsymbol{v}_0 + \sum_{j=1}^{m}\left[\prod_{\ell=0}^{j-1} a(\omega^\ell)\right]\xi^j \left\{\sum_{r=0}^{\infty}\frac{1}{(rm+j)!}\left[\hat{a}\xi^m\right]^r\right\}\boldsymbol{v}_j.$$

Let

$$f_j(z) = \sum_{r=0}^{\infty}\frac{z^r}{(rm+j)!} = \frac{1}{j!}{}_1F_m\left[\begin{matrix}1; \\ \frac{j+1}{m}, \frac{j+2}{m}, \ldots, \frac{j+m}{m};\end{matrix}\ \frac{z}{m^m}\right], \quad j = 1, 2, \ldots, m,$$

where ${}_1F_m$ is a generalized hypergeometric function. Therefore

$$\mathrm{e}^{\xi A}\mathbf{1} = \boldsymbol{v}_0 + \sum_{j=1}^{m}\left[\prod_{\ell=0}^{j-1} a(\omega^\ell)\right]\xi^j f_j(\hat{a}\xi^m)\boldsymbol{v}_j.$$

Note that, unless $\sigma(A) = \{0\}$, all the terms of the form $a(\omega^\ell)$ are nonzero. Therefore $\exp(\xi A)$ can be expressed as a linear combination of $\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{m-1}$ with coefficients that are nonzero for all the values of $\xi \in \mathbb{C}$ except possibly a countable set (because all the f_j are entire functions and none is identically zero). Moreover, the countable set where some f_js might vanish cannot accumulate at the origin (or at any finite point – the reason is, again, analyticity). Hence, we deduce that all vectors $\exp(tr^\ell A)\mathbf{1}$, except possibly for a finite number of values of tr^ℓ, contain nonzero components in the direction of all the eigenvectors of A. This, in tandem with the work in [9], completes the proof that $m \geq 3$ implies no uniform boundedness, while $m \geq 2$ implies no asymptotic stability of (2.1).

We formulate the main result of this section as a theorem.

Theorem 1 *The solution of the linear equation with involution* (2.1) *cannot be uniformly bounded for* $\boldsymbol{y}_0 \neq \mathbf{0}$ *in the case* $m \geq 3$*, while in the case* $m \geq 2$ *the solution cannot be asymptotically stable.* □

5 q-difference equations and difference equations with exponential coefficients

The next major objective of this paper is the the analysis of the equation

$$y(z) = \sum_{j=0}^{\ell} \sum_{k=1}^{n} a_{j,k} y^{(k)}(\omega_j z), \qquad z \in \mathbb{C}, \tag{5.1}$$

where $a_{j,k}, \omega_j \in \mathbb{C}$, accompanied by appropriate initial conditions at $z = 0$. We term it *the pantograph equation of the second type.*

Before we can proceed to examine (5.1), we need to pay attention to certain properties of difference and q-difference equations.

The equation

$$\sum_{i=0}^{n} a_i(p) f(q^{n-i} p) = 0, \qquad p \in \mathbb{C}, \tag{5.2}$$

where $a_0, a_1, \ldots, a_n$ are given complex functions and $q \in \mathbb{C}$, is called a *q-difference equation.* Let us assume without loss of generality that $|q| > 1$. Moreover, we stipulate that $a_0, a_1, \ldots, a_n$ are analytic in a neighbourhood of the origin and admit there an expansion of the form

$$a_i(p) = \sum_{\ell=0}^{\infty} a_{i,\ell} p^{\ell}.$$

The algebraic equation

$$a_{0,0}\rho^n + a_{1,0}\rho^{n-1} + \cdots + a_{n-1,0}\rho + a_{n,0} = 0$$

is termed the *characteristic equation* of (5.2).

The following result is a consequence of [1].

Theorem 2 *1. Suppose that* $a_{0,0}, a_{n,0} \neq 0$*. Then the q-difference equation* (5.2) *possesses a solution of the form*

$$f(p) = p^{\kappa_1} \sum_{\ell=0}^{\infty} c_\ell p^{\ell}, \qquad p \in \mathbb{C},$$

where $\kappa_1 = \log_q \rho_1$ *and* ρ_1 *is a root of the characteristic equation. The power series converges in a neighbourhood of the origin and the coefficient* c_0 *can be chosen arbitrarily.*
2. Suppose that either $a_{0,0} = 0$ *or* $a_{n,0} = 0$ *or both and let* a_{i,j_i} *denote the first nonzero coefficient in* $a_i(p)$*. We plot the pairs* (i, j_i) *in the* (i, j)*-plane. The* Newton diagram *of the point set* $X \in \mathbb{R}^2$ *is the maximal piecewise-linear convex curve whose endpoints belong to* X *and such that all points of this set lie either on or above the line (cf. Figure 1). The Newton diagram of* $\{(i, j_i)\}$ *is called the* characteristic line *of* (5.2)*. Let* μ *be the slope of the extreme left segment of the characteristic line. Then* (5.2) *has a solution of the form*

$$f(p) = q^{\mu(\log_q^2 p - \log_q p)/2} p^{\kappa_2} \sum_{\ell=0}^{\infty} d_\ell p^{\ell/s}, \qquad p \in \mathbb{C}, \tag{5.3}$$

where κ_2 *is a complex number,* s *is an integer and* d_0 *might be chosen arbitrarily. The power series converges in a neighbourhood of the origin.* □

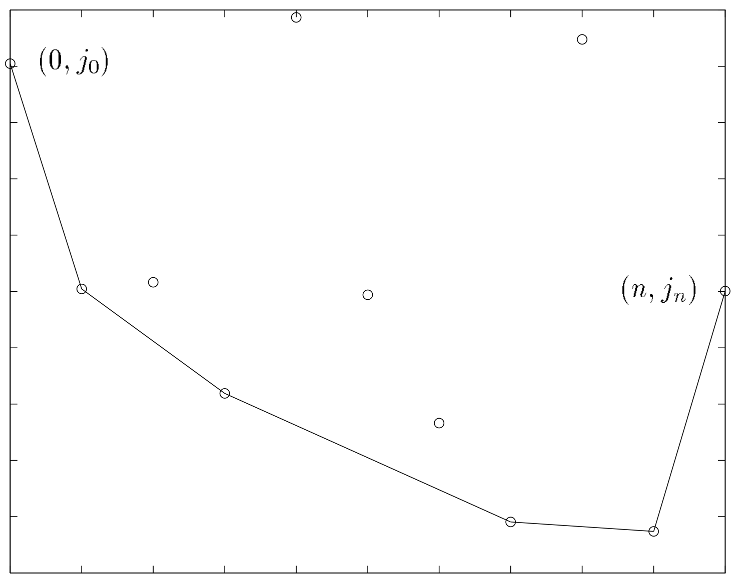

Figure 1 The characteristic line of equation (5.2), which is the Newton diagram of the points (i, j_i).

The following lemma is employed in the proof of Theorem 4, while being of an independent interest.

Lemma 3 *Let us suppose that the sequence* $\{c_k\}_{k=0}^{\infty}$ *satisfies the difference equation*

$$\sum_{j=N_1}^{N_2} \sum_{k=1}^{n} A_{j,k} q^{jr} c_{r+k} + c_r = 0, \qquad r = 0, 1, \ldots, \tag{5.4}$$

where $N_2 \geq 1$*,* $A_{j,k}$ *are given complex coefficients,* $|q| > 1$ *and there exist* $k_1, k_2 \in \{1, 2, \ldots, n\}$ *such that* $A_{N_1,k_1}, A_{N_2,k_2} \neq 0$*. Then for any* $\varepsilon > 0$ *there exists* $D > 0$ *such that* (5.4) *has a nontrivial solution that obeys the estimate*

$$|c_r| \leq D^r \Lambda^{-(1-\varepsilon)r^2/(2n)}, \qquad r = 0, 1, \ldots. \tag{5.5}$$

Proof We commence by dividing (5.4) by the factor $q^{N_2 r}$, and this yields the equivalent equation

$$\sum_{j=N_1}^{N_2} \sum_{k=1}^{n} A_{j,k} \left(\frac{1}{q}\right)^{(N_2-j)r} c_{r+k} + \left(\frac{1}{q}\right)^{N_2 r} c_r = 0, \qquad r = 0, 1, \ldots, \tag{5.6}$$

or, changing the index of summation,

$$\sum_{j=0}^{N_2-N_1} \sum_{k=1}^{n} A_{N_2-j,k} \left(\frac{1}{q}\right)^{jr} c_{r+k} + \left(\frac{1}{q}\right)^{N_2 r} c_r = 0, \qquad r = 0, 1, \ldots. \tag{5.7}$$

Letting

$$a_0(t) := t^{N_2}, \qquad a_k(t) := \sum_{j=0}^{N_2-N_1} A_{N_2-j,k} t^j, \quad k = 1, 2, \ldots, n$$

we may rewrite (5.7) in the form

$$a_0(q^{-r}) c_r + \sum_{k=1}^{n} a_k(q^{-r}) c_{r+k} = 0, \qquad r = 0, 1, \ldots. \tag{5.8}$$

Simultaneously with (5.8), we consider the q-difference equation

$$a_0(t) f(q^n t) + \sum_{k=1}^{n} a_k(t) f(q^{n-k} t) = 0, \qquad t \geq 0. \tag{5.9}$$

Since, according to the assumptions of the lemma, $N_2 \geq 1$, it follows that at least one of the polynomials a_k, $k = 1, 2, \ldots, n$, has a nontrivial constant term. According to Theorem 2 (the Adams theorem, [1]) equation (5.8) possesses a solution f_0 of the form (5.3) in a sufficiently small neighbourhood of the origin, $\mathcal{U}_R = \{t \in \mathbb{C} : 0 < |t| < R\}$, say. In accordance with Figure 1,

$$-\frac{N_2}{n} \leq \mu < 0.$$

The constant d_0 in (5.3) can be chosen arbitrarily, hence we may suppose that $d_0 \neq 0$. Then there exists $R_1 \in (0, R)$ such that $f_0(t) \neq 0$ in $\mathcal{U}_{R_1}$.

Letting $t = q^{-r}$ in (5.9), we obtain

$$a_0(q^{-r}) f_0(q^{n-r}) + \sum_{k=1}^{n} a_k(q^{-r}) f_0(q^{n-(r+k)}) = 0.$$

We denote $c_r = f_0(q^{n-r})$, $r \geq r_0$, where r_0 is the first integer such that $q^{n-r_0} \in \mathcal{U}_{R_1}$. Subsequently, we complete the definition of $c_{r_0-1}, c_{r_0-2}, \cdots, c_0$ by means of the recurrence (5.6). This results in a sequence $\{c_r\}_{r=0}^{\infty}$ which, by virtue of $f_0(t) \neq 0$ in $\mathcal{U}_{R_1}$, is nonzero. We thus deduce that this sequence is a nontrivial solution of (5.8), therefore of (5.6).

To complete the proof of the lemma we need to demonstrate that the inequalities (5.5) hold. We commence by observing that there exists $\theta \in [0, 2\pi)$ such that

$$\arg q^{n-r} \neq \theta, \qquad r = 0, 1, \ldots,$$

and choose the branches of $\log q = \log |q| + \mathrm{i}\psi$ and $\log t = \log |t| + \mathrm{i}\varphi$ so that

$$\theta \leq \psi, \varphi < \theta + 2\pi.$$

We denote by $\mathcal{U}_{R_1}^{\mathrm{c}}$ the neighbourhood $\mathcal{U}_{R_1}$ of the origin, cut along the ray $\arg t = \theta$. Substitution into the expansion of f_0 yields

$$|f_0(t)| \leq D_0 |t|^{\tilde{\kappa}} \mathrm{e}^{\mu \log^2 |t| / \log |q|}, \qquad t \in \mathcal{U}_{R_1}^{\mathrm{c}},$$

where $D_0 > 0$ and $\tilde{\kappa}$ are real numbers and μ has been already restricted to the range $[-N_2/(2n), 0)$. Consequently,

$$\begin{aligned} |c_r| = |f_0(q^{n-r})| &\leq D_0 |q|^{\tilde{\kappa}(n-r)} \mathrm{e}^{\mu(n-r) \log^2 |q| / \log |q|} \\ \leq D_1^r \mathrm{e}^{-(1-\varepsilon) N_2 r^2 \log |q| / (2n)} &= D_1^r \mathrm{e}^{-(1-\varepsilon) r^2 \log |q| / (2n)} = D_1^r \Lambda^{-(1-\varepsilon) r^2 / (2n)} \end{aligned}$$

and (5.5) follows. ☐

6 The pantograph equation of the second type

The present section is devoted to an investigation of the pantograph equation of the second type (5.1). We address ourselves to the question of local existence and analyticity of the solution around $z=0$, to analytic continuation into the whole complex plane and to the asymptotic behaviour of the function y.

We commence by denoting

$$\lambda := \min_{j=0,1,\ldots,\ell} |\omega_j|, \qquad \Lambda := \max_{j=0,1,\ldots,\ell} |\omega_j|$$

and recalling that $\Lambda < 1$ implies that (5.1) has no nontrivial solutions that are analytic at the origin [3]. Therefore, we turn our attention to the case $\Lambda > 1$.

In this case too the Cauchy problem for the equation (5.1), supplemented by the initial data

$$y^{(k)}(0) = c_k, \qquad k = 0, 1, \ldots, n-1, \tag{6.1}$$

has in general no analytic solution at $z=0$ [4].

Example Consider the equation

$$y(z) = -y'(4z) + y''(2z), \qquad z \in \mathbb{C},$$

which, by a trivial change of variable, can be more conveniently written as

$$y''(z) = y'(2z) + y(z/2), \qquad z \in \mathbb{C}. \tag{6.2}$$

We accompany (5.4) with the initial conditions

$$y(0) = 0, \qquad y'(0) = 1. \tag{6.3}$$

Supposing that the solution of (6.2),(6.3) is analytic at $z=0$, we expand it in Taylor series,

$$y(z) = \sum_{r=0}^{\infty} \frac{c_r}{r!} z^r,$$

where $c_r = y^{(r)}(0)$, $r = 0, 1, \ldots$. Differentiating (5.4) r times and letting $z=0$ results in the difference equation

$$c_{r+2} = 2^r c_{r+1} + 2^{-r} c_r, \qquad r = 0, 1, \ldots,$$

where c_0, c_1 are given by (5.5). It is trivial to verify that

$$|c_r| \geq 2^{(r-2)(r-1)/2}, \qquad r = 1, 2, \ldots.$$

Therefore, the radius of convergence of the Taylor series is zero and the solution is not analytic at the origin.

We contend, however, that for each equation of the form (5.1) it is possible to choose initial data (6.1) so that the Cauchy problem (5.1),(6.1) has at least one nontrivial analytic solution at the neighbourhood of the origin.

Suppose that all the ω_j in (5.1) are real. In that case the standard classification of functional differential equations into those of retarded, neutral and advanced type can be applied to (5.1). It is worthwhile to mention that the case $\omega_j > 1$ can correspond in (5.1) to any of these three types.

The central result of this section is a theorem on existence and continuation into the complex plane of solutions to the pantograph equation of the second kind (5.1)

Let us suppose that the numbers ω_j are multiplicatively commensurable, i.e. that

$$\omega_j = q^{\tau_j}, \qquad j = 0, 1, \ldots, \ell,$$

where $|q| > 1$ and $\tau_0, \ldots, \tau_\ell$ are rational numbers.

Theorem 4 *Given that $\Lambda > 1$, the following assertions are true.*

1. *Every equation* (5.1) *has nontrivial analytic solution in some neighbourhood of the origin.*

2. *The above analytic solution can be continued to the whole complex plane as an entire function of order zero. Hence it is unbounded along any ray approaching infinity.*

Proof Recalling that $\omega_j = q^{\tau_j}$, $j = 0, 1, \ldots, \ell$, where the τ_j are rational, we observe that, possibly rescaling q, we may assume without loss of generality that the R_j are integers. Therefore, we may rewrite (5.1) in the form

$$y(z) = \sum_{j=N_1}^{N_2} \sum_{k=1}^{n} A_{j,k} y^{(k)}(q^j z), \qquad z \in \mathbb{C},$$

where $N_1 < N_2$ are integers and there exist $k_1, k_2 \in \{1, 2, \ldots, n\}$ such that $A_{N_1,k_1}, A_{N_2,k_2} \neq 0$. Moreover, $\Lambda > 1$ implies that $N_2 \geq 1$.

We differentiate the equation r times and set $z = 0$. The outcome is

$$y^{(r)}(0) = \sum_{j=N_1}^{N_2} \sum_{k=1}^{n} A_{j,k} q^{jr} y^{(r+k)}(0), \qquad r = 0, 1, \ldots. \tag{6.4}$$

Letting $c_r := y^{(r)}(0)$, $r = 0, 1, \ldots$, we recover the difference equation (6.2). It is easily verifiable that we are within the conditions of Lemma 3, therefore the sequence $\{c_r\}_{r=0}^{\infty}$ obeys the inequalities (6.3). Therefore it readily follows that

$$y(z) = \sum_{r=0}^{\infty} \frac{c_r}{r!} z^r, \qquad z \in \mathbb{C},$$

is an entire function. Its order $\rho(y)$ can be determined easily by the classical formula

$$\rho(y) = \limsup_{r\to\infty} \frac{r\log r}{\log r! - \log|c_r|}$$

[8]. Because of (6.3), it follows at once that $\rho(y) = 0$. According to the Phragmen–Lindelöf principle, every entire function of order zero is unbounded along any ray approaching ∞, and this completes the proof. $\Box$

Corollary Provided that $\Lambda > 1$, the trivial solution $y(z) \equiv 0$ of the pantograph equation of the second kind (5.1) is unstable. $\Box$

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