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Exact and discretized stability of the pantograph equation

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1 Introduction

The *pantograph* equation

$$\boldsymbol{y}'(t) = A\boldsymbol{y}(t) + B\boldsymbol{y}(qt) + C\boldsymbol{y}'(qt), \quad t \geq 0, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0, \tag{1.1}$$

where A, B, C are $d \times d$ complex matrices, $\boldsymbol{y}_0 \in \mathbb{C}^d$ and $q \in (0,1)$, possesses large number of applications in applied mathematics and engineering [14] and is fast becoming the standard model problem in the design of numerical methods for functional-differential equations with unbounded delays [3, 4, 5, 12, 17, 21, 2].

There exists a natural inclination to extrapolate from the known theory of linear functional-differential equations with constant delay,

$$\boldsymbol{y}'(t) = A\boldsymbol{y}(t) + B\boldsymbol{y}(t-\tau) + C\boldsymbol{y}'(t-\tau), \quad t \geq 0, \tag{1.2}$$

where $\tau > 0$ and an initial value is given for all $t \in (-\tau, 0]$, to guess the behaviour of (1.1). This inclination should be resisted, because (1.1) and (1.2) represent diverse mathematical structures and pose completely different mathematical challenges. On the one hand, the solution of (1.2) exhibits a familiar pattern of derivative discontinuities, while the solution of (1.1), when unique, is infinitely smooth. Moroever, the asymptotic stability condition for (1.2), namely that all the zeros of

$$\det[zI - A - \mathrm{e}^{-\tau z}(B + zC)] = 0, \qquad z \in \mathbb{C},$$

reside in the left half-plane, is very different from the equivalent condition for the pantograph (cf. Section 2). Surprisingly enough, the constant-delay asymptotic stability condition is, at least in principle, more difficult to verify, although simplified conditions exist in special cases (cf. for example [13]).

Lest there is an impression that the constant-delay equation (1.2) is somehow more difficult then (1.1), we hasten to recount the other side of the story. While (1.2) is a relatively strightforward interval map, the pantograph (1.1) is altogether more interesting: it is a d-dimensional dynamical system at $t = 0$ but an infinite-dimensional one for every $t > 0$. Its behaviour, being determined by a countable number of modes, is more akin to *nonlinear* differential equations. In particular, the equation displays a formidable richness of interesting phenomena on its stability boundary.

The intrinsic complexity of the pantograph equation is further emphasized under discretization. While a standard finite-difference formula, whether multistep or Runge–Kutta, renders (1.2) into a stationary (albeit, high-degree) difference equation, the pantograph (1.1) yields a *nonstationary* system, whose analysis is significantly more difficult. Indeed, the results to date are incomplete and their derivation calls for new mathematical techniques.

The theme of the paper is the range of phenomena which the pantograph equation – whether the exact or the discretized – displays inside and on its stability boundary. In Section 2 we describe the behaviour of the exact equation, while Section 3 is devoted to its discretization. We hasten to emphasize that, insofar as the numerical analysis of (1.1) is concerned, the results to date are incomplete and open problems abound. Indeed, one of the goals of this brief survey is to single out the many interesting open problems and challenges for the attention of the broader numerical community.

The pantograph equation (1.1) is a fruitful point of departure for generalizations:

1. The case $q > 1$ has attracted significant attention because of its many applications in probability theory, theory of spin glasses and, perhaps most intriguingly, theory of wavelets [9, 10].

2. There is a growing measure of interest in $q \in \mathbb{C}$, because of its conenction with self-similar dressing chains of Schrödinger operators [11, 21, 22].

3. Although only preliminary and tentative results are available with regard to *nonlinear* functional-differential equations with proportional delays, it is quite apparent that even the humble equation

$$y'(t) = \alpha y(t) + y(qt)(1 - y(qt)), \quad t \geq 0, \qquad y(0) = y_0,$$

   displays a whole range of intriguing phenomena [15].

4. It is natural to generalize (1.1) by considering equations with several proportional delays. As a matter of fact, it is perfectly viable to consider a *continuum* of delays – in other words, integro-differential equations of the form

$$y'(t) = ay(t) + \int_0^1 y(qt)\mathrm{d}\mu(q) + \int_0^1 y'(qt)\mathrm{d}\nu(t), \quad t \geq 0, \qquad y(0) = 1,$$

   where both $\mathrm{d}\mu$ and $\mathrm{d}\nu$ are complex-valued Borel measures. Such equations are interesting *inter alia* because they admit generalized hypergeometric solutions [18].

2 The exact equation

Let us suppose that $\sigma(C)$, the spectrum of the matrix C, does not contain points of the form $q^{-\ell}$ for $\ell \in \mathbb{Z}^+$. Then the solution of (1.1) exists [14]. Moreover, if $\rho(C) < 1$ then the solution is unique [21].

It is instructive to examine what happens if the above conditions fail. We first consider the scalar equation

$$y'(t) = ay(t) + by(qt) + y'(qt), \quad t \geq 0, \qquad y(0) = 1, \tag{2.1}$$

which is typical of the general case $q^{-\ell} \in \sigma(C)$. It is easy to verify directly that a power series solution to (2.1) exists if and only if $a + b = 0$ and that in the latter case every function $y(t) = (1 - \eta) + \eta e^{at}$, $\eta \in \mathbb{C}$, is a solution.

The equation

$$y'(t) = ay(t) + by(qt) + cy'(qt), \quad t \geq 0, \qquad y(0) = 1, \tag{2.2}$$

possesses an infinite number of solutions whenever $a, b \neq 0$ and $|c| > 1$. This can be proved by showing that (2.2) admits two different *Dirichlet expansions.* We defer the discussion of this equation until we have introduced this important concept, later in this section.

We next proceed to analyse asymptotic stability. Considering the simple scalar equation (2.2) with $c = 0$, we proceed by the classical technique of direct upper bounds, not because it is effective but since it exposes the shortcoming inherent in this approach. Thus, let

$$\eta(t) := \max_{0 \leq \tau \leq t} |y(\tau)|^2, \qquad t \geq 0.$$

Since

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|y(t)|^2 &= \mathrm{Re}\, \bar{y}(t)y'(t) \\ &= (\mathrm{Re}\, a)|y(t)|^2 + \mathrm{Re}\,[b\bar{y}(t)y(qt)] \\ &\leq (\mathrm{Re}\, a)|y(t)|^2 + |b|\eta(t), \end{aligned}$$

it follows that

$$\frac{1}{2}\eta' \leq (\mathrm{Re}\, a + |b|)\eta.$$

Therefore $\eta(t) \leq \exp[2t(\mathrm{Re}\, a + |b|)]$, $t \geq 0$, and the solution tends to zero if

$$\mathrm{Re}\, a < 0, \qquad |b| < -\mathrm{Re}\, a. \tag{2.3}$$

The main problem with (2.3) is that it constitutes but a small subset of the genuine domain of asymptotic stability.

The asymptotic behaviour of the pantograph equation (1.1) has been analysed, often in a simplified form, by a number of authors [7, 8, 9, 14, 19, 21]. The most powerful tool in this endeavour is an expansion of the solution in *Dirichlet series* of the form

$$\boldsymbol{y}(t) = \sum_{n=0}^{\infty} D_n e^{q^n t A} \boldsymbol{v}, \quad t \geq 0, \tag{2.4}$$

where each D_n is a $d \times d$ complex matrix and $\boldsymbol{v} \in \mathbb{C}^d$. To argue that (2.4) exists we stipulate that A is nonsingular and that $\lambda \neq q^\ell \mu$ for every $\lambda, \mu \in \sigma(A)$, and $\ell \in \mathbb{N}$. Substitution of the expansion (2.4) into (1.1) results in linear recurrences

$$AD_n - q^n D_n A = -BD_{n-1} - q^{n-1} C D_{n-1} A, \qquad n \in \mathbb{N}.$$

Each step of the recursion entails a solution of a linear system and all such *commutator equations* are nonsingular because of the aforementioned condition on the eigenvalues of A.

Setting $D_0 = I$ and letting $D(t) = \sum_{n=0}^{\infty} D_n t^n$, it has been demonstrated that

$$\det D(t) = \prod_{n=0}^{\infty} \frac{\det(I - tq^n C)}{\det(I + tq^n BA^{-1})}, \qquad t \geq 0,$$

therefore $\det D(0) = 1$ and we let $\boldsymbol{v} = D^{-1}(0)\boldsymbol{y}_0$. This proves the existence of the Dirichlet series (2.4) [14].

It is possible to prove that (2.4) converges for any $t \geq 0$ if $\rho(A^{-1}B) < 1$ (otherwise Dirichlet series need be replaced by *Dirichlet–Taylor* series [14]). Moreover, convergence is uniform for $t \in [0, \infty)$ as long as all the eigenvalues of A reside in the complex left half-plane. However, the great virtue of the expansion (2.4) is that it tells the asymptotic behaviour of (1.1) at a glance. In particular, if

$$\operatorname{Re} \sigma(A) < 0, \qquad \rho(A^{-1}B) < 1 \tag{2.5}$$

then $\lim_{t\to\infty} \boldsymbol{y}(t) = \boldsymbol{0}$ and the zero solution is asymptotically stable [14].

Specializing (2.4) to the equation (2.2), we derive for $b \neq 0$ the explicit expansion

$$y(t) = \frac{(-ac/b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(-ac/b;q)_n}{(q;q)_n} \left(-\frac{b}{a}\right)^n \mathrm{e}^{q^n at}, \qquad t \geq 0, \tag{2.6}$$

where the *Gauß–Heine symbols* $(z;q)_n$ are defined by the formula

$$(z;q)_n = \prod_{k=0}^{n-1} (1 - q^k z), \qquad z, q \in \mathbb{C}, \quad n \in \mathbb{Z}^+ \cup \{\infty\}.$$

In particular, the asymptotic stability condition (2.5) becomes

$$\operatorname{Re} a < 0, \qquad |b| < |a|. \tag{2.7}$$

An interesting observation is that (2.7) is independent of c. An even more profound observation, though, is that the set (2.7) is usually *much* greater than the set (2.5). Figure 1 displays all the values of b consistent with (2.5)

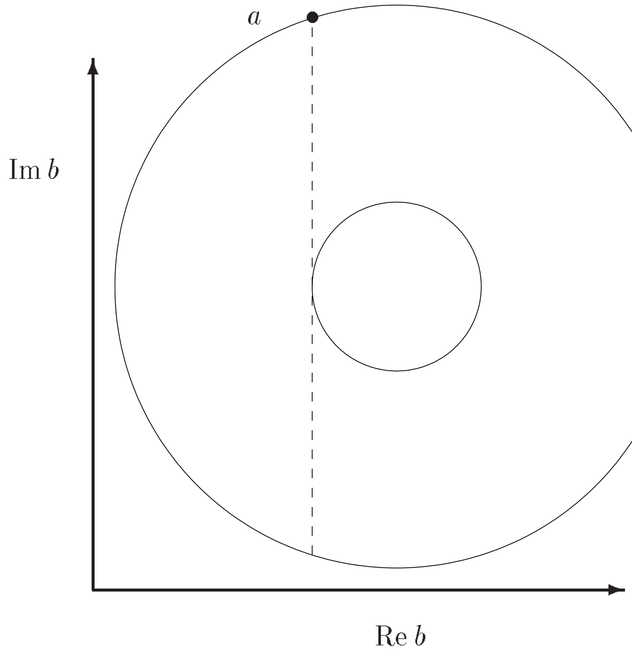


Figure 1: Values of $b \in \mathbb{C}$ consistent with (2.5) (the smaller disc) and (2.7) (the larger disc) for a given value of $a \in \mathbb{C}$. The dotted line denotes $\mathrm{Re}\, a$.

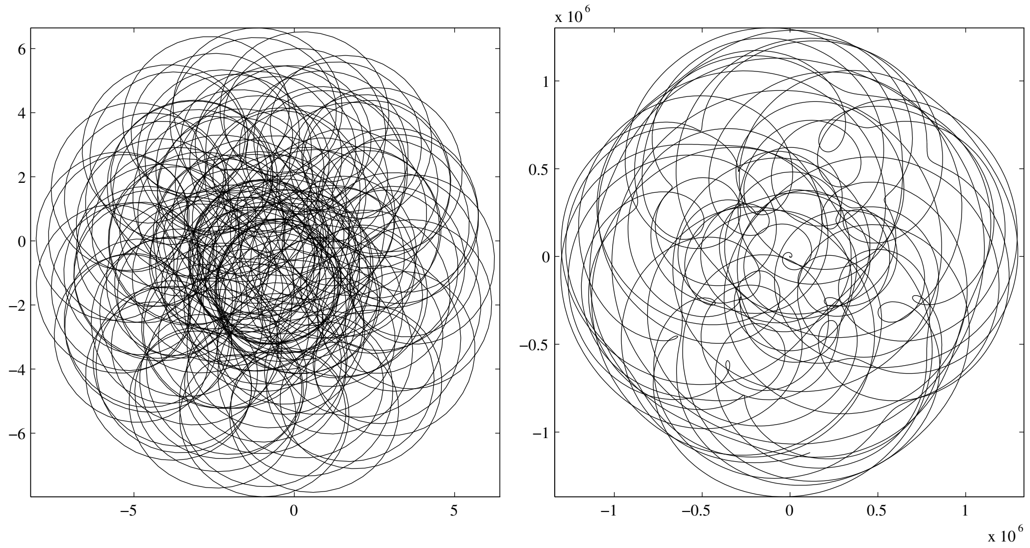
(the inner disc) and with (2.7) (the outer disc) for a given value of $a \in \mathbb{C}$. Note that the nearer a is to the pure imaginary axis, the larger the discrepancy between the two sets.

Before proceeding to examine the stability boundary, we discharge the earlier promise of considering the case $|c| > 1$. The function (2.6) remains a valid solution of (2.2), but it is possible to show that other solutions exist. Specifically, direct substitution and an elementary calculus with *basic hypergeometric functions* affirm that the Dirichlet expansion

$$y(t) = \frac{(ac/b;q)_\infty}{(-qc/b;q)_\infty} \sum_{n=0}^{\infty} \frac{(-ac/b;q)_n}{(q;q)_n} \left(\frac{q}{c}\right)^n \mathrm{e}^{-q^{-n}bt/c}, \qquad t \geq 0, \tag{2.8}$$

is another solution of the same equation. Note that the expansion (2.8) converges for $|c| > q$ but its derivative converges only for $|c| > 1$. Moreover, (2.8) is differentiable only a finite number of times – it can be proved that the only C^∞ solution of (2.2) is the expansion (2.6) [21].

The boundary of the stability domain (2.5) consists of several parts. Firstly, in the case $\mathrm{Re}\,\sigma(A) < 0$, $\rho(A^{-1}B) = 1$, provided that all the eigenvalues of $A^{-1}B$ with unit modulus have the same geometric and algebraic multiplicity, there exists a *periodic* function ϕ and for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such



$y'(t) = \mathrm{i}y(t) + \frac{99}{100}\mathrm{i}y(t/10)$ $\qquad$ $y'(t) = \mathrm{i}y(t) + \frac{99}{100}y(9t/10)$

Figure 2: Almost-periodic solutions of the pantograph equation (2.2) in the $(\mathrm{Re}\, y, \mathrm{Im}\, y)$ plane.

that

$$\|\boldsymbol{y}(t) - \boldsymbol{\phi}(q^{-t}\| < \varepsilon, \qquad t \geq t_\varepsilon.$$

Otherwise the solution of (1.1) is unbounded in $[0, \infty)$ for some intial values.

The situation is, if at all, even more interesting when

$$\max\{\mathrm{Re}\, \lambda \,:\, \lambda \in \sigma(A)\} = 0.$$

If $\det A = 0$ then the solution is unbounded in $[0, \infty)$ for some initial values. Otherwise there exists an *almost-periodic* function $\boldsymbol{\psi}$ and for every $\varepsilon > 0$ there exists $s_\varepsilon > 0$ such that

$$\|\boldsymbol{y}(t) - \boldsymbol{\psi}(t)\| < \varepsilon, \qquad t \geq s_\varepsilon.$$

We remind the reader that a function $\boldsymbol{\psi} \in L_\infty[0, \infty)$ is almost-periodic if for every δ there exists $v_\delta > 0$ such that

$$\|\boldsymbol{\psi}(t + v_\delta) - \boldsymbol{\psi}(t)\| < \delta, \qquad t \geq 0,$$

and $\liminf_{\delta \downarrow 0} v_\delta > 0$.

If *all* the eigenvalues of A reside on $\mathrm{i}\mathbb{R}$ then the solution $\boldsymbol{y}$ itself is almost-periodic. This is the case, for example, for the equation (2.2) when $a \in \mathrm{i}\mathbb{R} \setminus \{0\}$. The solutions of two such equations are displayed in Figure 2 in the plane

$(\operatorname{Re} y, \operatorname{Im} y)$. Both are suggestive of a great deal of structure. Indeed, the little we know to date on the dynamics of almost-periodic functions of this form is certainly quite interesting. If $q = k/\ell$, where $k, \ell \in \mathbb{N}$ are relatively prime, $1 \le k \le \ell - 2$, then y possesses an almost-symmetry (a concept that parallels almost-periodicity) under rotations by integer multiples of $2\pi/(\ell - k - 1)$ [14]. This can be seen quite vividly in Figure 3, which displays four solutions for different rational values of q.

Further features of the solution can be discerned in numerical experimentation but, as things stand, they lack complete verification. Thus, again for $q = k/\ell$, where k and ℓ are relatively prime, each plot is composed of ℓ 'strands'. Each such 'strand' becomes itself, upon zooming, a braid of ℓ 'strands' and so on. If q is irrational the solution seems to be space-filling in an annulus [14].

3 The numerical solution

It is relatively easy to extend standard ODE methods to the pantograph equation (1.1). The only extra ingredient is the provision of continuous output but this can be always ensured by interpolation or by using natural Runge–Kutta methods. In a sense, it is easier to design time-stepping algorithms for (1.1) than for (1.2), because the latter equation exhibits derivative discontinuities. Having said this, stability analysis of the discretized pantograph presents a formidable challenge.

We mention in passing that perhaps the most efficient means of discretizing the pantograph equation is provided by a Dirichlet expansion. Yet, as long as we consider (1.1) as a model equation and wish to infer from its analysis to the broader issue of the numerical analysis of functional differential equations, we are not allowed to exploit features which are characteristic of the pantograph.

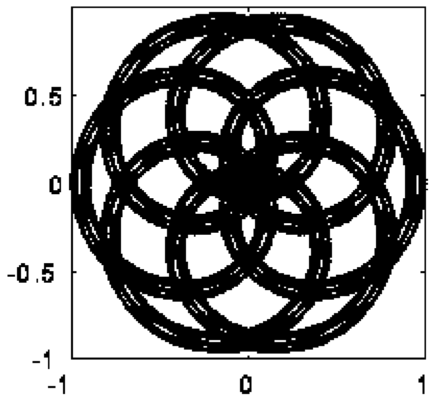
The familiar trapezoidal rule is a good starting point for the analysis of standard numerical methods, although Runge–Kutta schemes present an added complexity [6]. Interpolating $\boldsymbol{y}(qt)$ by a piecewise-linear function on an equispaced grid results in the nonstationary recurrence relation

$$\begin{aligned} y_{n+1} \;=\; & \frac{1+\frac{1}{2}ha}{1-\frac{1}{2}ha} y_n + \frac{hb}{1-\frac{1}{2}ha}[(1-\alpha_n) y_{\lfloor qn \rfloor} + \alpha_n y_{\lfloor qn \rfloor + 1}] \\ & + \frac{c}{1-\frac{1}{2}ha}(y_{\lfloor qn \rfloor + 1} - y_{\lfloor qn \rfloor}), \end{aligned} \qquad n \in \mathbb{Z}^+, \tag{3.1}$$

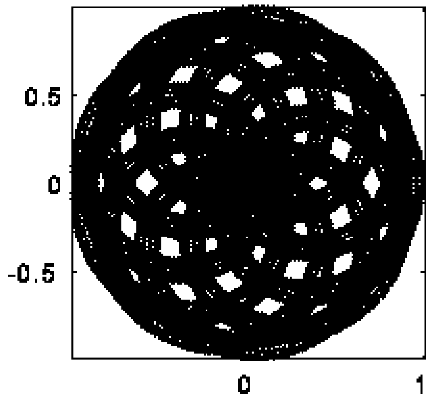
where

$$\alpha_n = q\left(n + \tfrac{1}{2}\right) - \lfloor qn \rfloor, \qquad n \in \mathbb{Z}^+,$$

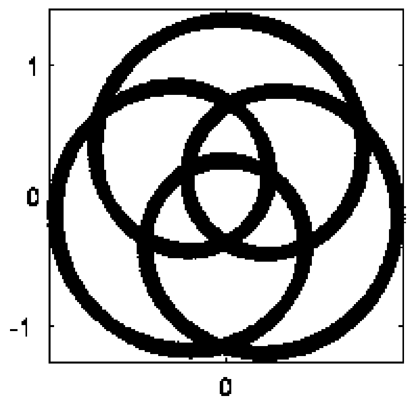
which approximates (2.2) with a local error of $\mathcal{O}(h^3)$.



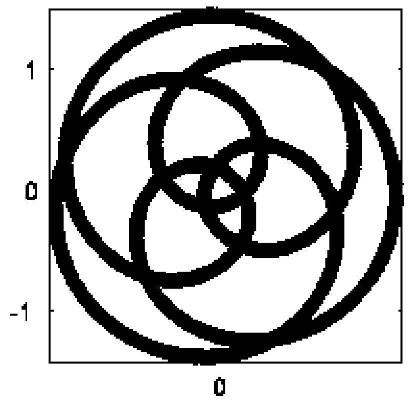
$y'(t) = \mathrm{i}y(t) - \frac{3}{4}y'(t/7)$



$y'(t) = \mathrm{i}y(t) - \frac{3}{4}y'(2t/11)$



$y'(t) = \mathrm{i}y(t) - \frac{1}{4}y(t/4) - \frac{3}{4}\mathrm{i}y'(t/4)$



$y'(t) = \mathrm{i}y(t) - \frac{1}{4}y(2t/5) - \frac{2}{3}\mathrm{i}y'(2t/5)$

Figure 3: Rotational almost-symmetry in the exact solution of the pantograph equation (2.2). Note the dependence on the value of $\ell - k - 1$, where $q = k/\ell$.

Asymptotic behaviour of the method (3.1) can be analysed by classical methods, but this approach suffers all the shortcomings of similarly naive estimates for the exact solution of (2.2) that have been already debated in Section 2 and that have led to the set (2.3) [5]. In a special case when q is a reciprocal of an integer, though, different analysis has been presented by Martin Buhmann and the author and it leads to a result consistent with both numerical experiment and intuition [4]. Cf. [20] for an extension of this analysis to θ-methods.

In the case $q = \frac{1}{2}$ the recurrence (3.1) reads

$$\begin{aligned} y_{2n+1} &= \mathfrak{R} y_{2n} + \tfrac{1}{4}\mathfrak{S}(3y_n + y_{n+1}) + \mathfrak{T}(y_{n+1} - y_n), \\ y_{2n+2} &= \mathfrak{R} y_{2n+1} + \tfrac{1}{4}\mathfrak{S}(y_n + 3y_{n+1}) + \mathfrak{T}(y_{n+1} - y_n), \end{aligned} \qquad n \in \mathbb{Z}^+, \tag{3.2}$$

where

$$\mathfrak{R} = \frac{1 + \frac{1}{2}ha}{1 - \frac{1}{2}ha}, \qquad \mathfrak{S} = \frac{hb}{1 - \frac{1}{2}ha}, \qquad \mathfrak{T} = \frac{c}{1 - \frac{1}{2}ha}.$$

Although (3.2) can be analysed in the above form, we turn our attention to a somewhat simplified model,

$$\begin{aligned} y_{2n+1} &= \lambda y_{2n} + 2\mu y_n \\ y_{2n+2} &= \lambda y_{2n+1} + \mu(y_n + y_{n+1}), \end{aligned} \qquad n \in \mathbb{Z}^+, \tag{3.3}$$

where $\lambda, \mu \in \mathbb{C}$, because (as will be apparent in the sequel) it sheds more light on the behaviour on the stability boundary. Intuitively speaking, λ should be identified with $\mathfrak{R}$ and μ with $\frac{1}{2}\mathfrak{S}$.

Letting $x_n := y_{n+1} - y_n$, we obtain from (3.3) the recurrence

$$x_0 = \lambda + 2\mu - 1,$$

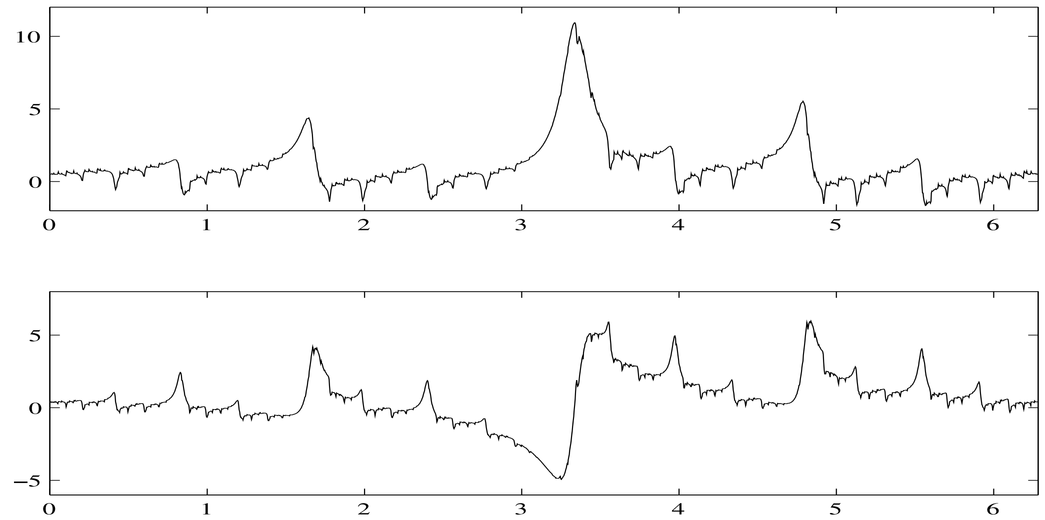
$$\left.\begin{aligned} x_{2n+1} &= \lambda x_{2n} + \mu x_n \\ x_{2n+2} &= \lambda x_{2n+1} + \mu x_n \end{aligned}\right\} \qquad n \in \mathbb{Z}^+.$$

Let us denote by $\hat{x}(\theta)$, $\theta \in [0, 2\pi]$, the *Fourier transform* of the sequence $\{x_n\}_{n\in\mathbb{Z}^+}$. It is possible to prove that $\hat{x}$ obeys the functional equation

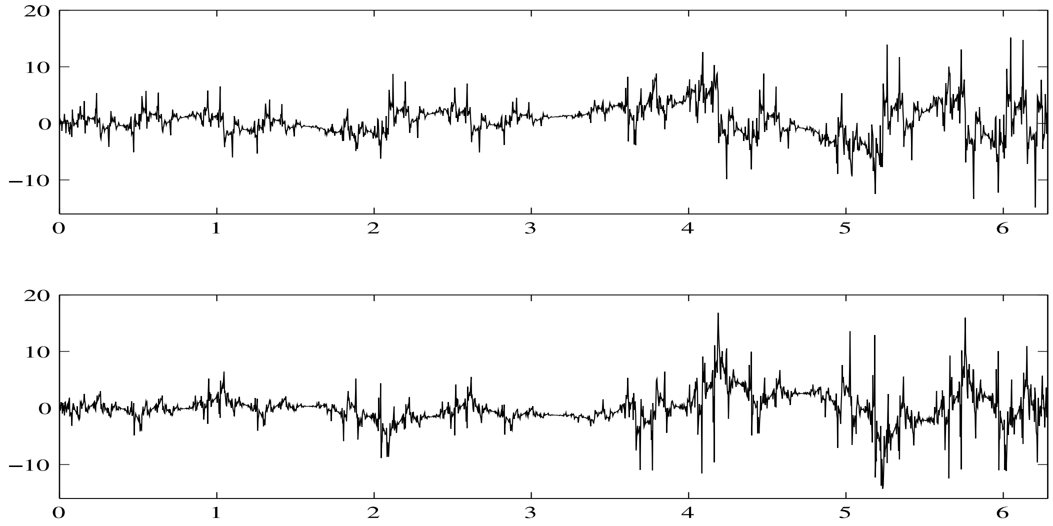
$$\hat{x}(\theta) = \frac{1}{1 - \lambda e^{i\theta}}[x_0 + \mu(1 + e^{i\theta})\hat{x}(2\theta)] \pmod{2\pi}, \qquad \theta \in [0, 2\pi]. \tag{3.4}$$

The function $\hat{x}$ is exceedingly complicated even for 'nice' values of λ and μ, as should be evident from Figure 4. The situation is even worse for $|\lambda| = 1$, since then the function is unbounded. Yet, it is not outside the realm of analysis! Applying (3.4) repeatedly, we expand the Fourier transform into infinite series,

$$\hat{x}(\theta) = (\lambda + 2\mu - 1)\sum_{m=0}^{\infty} \frac{\prod_{k=0}^{m-1}(1 + e^{2^k i\theta})}{\prod_{k=0}^{m}(1 - \lambda e^{2^k i\theta})}\mu^m, \qquad \theta \in [0, 2\pi],$$



$\lambda = \frac{9}{10}e^{\frac{30}{31}\pi i}$, $\mu = \frac{1}{2}e^{\frac{11}{40}\pi i}$



$\lambda = -\frac{4}{5}$, $\mu = \frac{3}{4}e^{\frac{4}{5}\pi i}$

Figure 4: Real and imaginary parts of the Fourier transform $\hat{x}$ corresponding to different choices of λ and μ.

and use the mean ergodic theorem to deduce that $\hat{x}$ is an $L_2[0,2\pi]$ function for all $|\lambda|<1$ and $2|\mu|<|1-\lambda|$. From this there is but a short step to argue that also the Fourier transform of $\{y_n\}_{n\in\mathbb{Z}^+}$ is $L_2[0,2\pi]$, whence $y_n\to 0$ and the recurrence (3.3) is asymptotically stable. Similar analysis, albeit more technically convoluted, applies to the numerical method (3.2), whereby the asymptotic stability condition is $|\mathfrak{R}|<1$, $|\mathfrak{S}|<|1-\mathfrak{R}|$, which translates *exactly* into the condition (2.5). Hence, at least when q is a reciprocal of an integer, the trapezoidal rule recovers the correct asymptotic stability domain of the pantograph equation [4]. (Note that in the neutral case $0<|c|<1$ we need to restrict the step-size so that $|c\pm\frac{1}{4}hb|<|1-\frac{1}{2}ha|$.)

The behaviour of (3.3) on the stability boundary $|\lambda|=1$ is considerably more complicated. However, a number of results are available and they indicate that the attractor of the sequence $\{y_n\}_{n\in\mathbb{Z}^+}$ can be an exceedingly complicated object, as can be seen in Figure 5. Unfortunately, virtually all (bar numerical experiments) that is known in precise mathematical terms with regard to the attractor refers to the special case $\lambda=-1$, which does not correspond to a realistic numerical method (since $\mathfrak{R}=1+ha+\mathcal{O}(h^2)$).

We expand each y_n in powers of μ,

$$y_n=(-1)^n+\sum_{\ell=1}^{\infty}a_{n,\ell}\mu^\ell,\qquad n\in\mathbb{Z}^+.$$

Let us commence by examining the first few values of y_n,

$$\begin{aligned}
y_1 &= -1+2\mu,\\
y_2 &= +1-2\mu+2\mu^2,\\
y_3 &= -1 \qquad\quad +2\mu^2,\\
y_4 &= +1 \qquad\quad -2\mu^2+2\mu^3,\\
y_5 &= -1+2\mu-2\mu^2+2\mu^3,\\
y_6 &= +1-2\mu \qquad\quad +2\mu^3,\\
y_7 &= -1 \qquad\qquad\qquad +2\mu^3,\\
y_8 &= +1 \qquad\qquad\qquad -2\mu^3+2\mu^4,\\
y_9 &= -1+2\mu \qquad\quad -2\mu^3+2\mu^4,\\
y_{10} &= +1-2\mu+2\mu^2-2\mu^3+2\mu^4,\\
y_{11} &= -1 \qquad\quad +2\mu^2-2\mu^3+2\mu^4,\\
y_{12} &= +1 \qquad\quad -2\mu^2 \qquad\quad +2\mu^4.
\end{aligned}$$

Obviously, we can discern a pattern! Firstly, it appears that $a_{n,\ell}\in\{0,+2,-2\}$ and, secondly, there is clearly a binary structure to the sequence. It is not

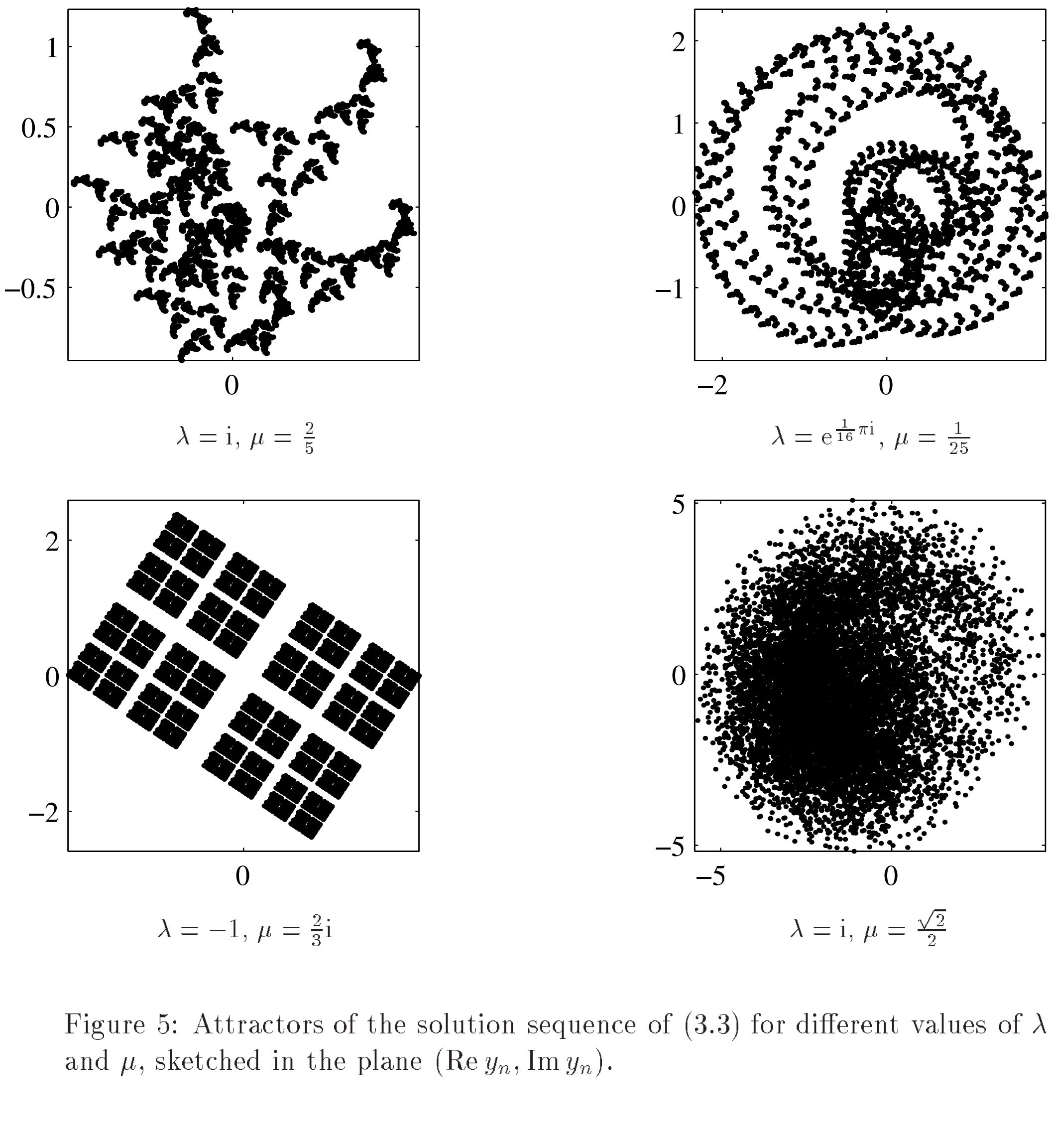


Figure 5: Attractors of the solution sequence of (3.3) for different values of λ and μ, sketched in the plane $(\operatorname{Re} y_n, \operatorname{Im} y_n)$.

difficult to show that this is indeed the case and that the explicit form of the coefficients is

$$a_{2^{\ell+1}m+j,\ell} = \begin{cases} 0, & j = 0, 1, \ldots, 2^{\ell-1} - 1, \\ 2, & j = 2^{\ell-1}, \ldots, 2^{\ell} - 1, \\ -2, & j = 2^{\ell}, \ldots, 3 \times 2^{\ell-1} - 1, \\ 0, & j = 3 \times 2^{\ell-1}, \ldots, 2^{\ell+1} - 1 \end{cases}$$

for all $\ell \in \mathbb{N}$ and $m \in \mathbb{Z}^+$ [16].

Let $\mathcal{A}_\mu$ be the *ω-set* of the sequence $\{y_n\}_{n\in\mathbb{Z}^+}$. In other words, $\mathcal{A}_\mu$ is the closure of all the accumulation points of the sequence. The explicit form of the coefficients allows to evaluate specific members of $\mathcal{A}_\mu$. For example, as long as $|\mu| < 1$ it is true that

$$\lim_{r\to\infty} y_{(4^r-1)/3} = -\frac{1-\mu}{1+\mu}$$

and

$$\lim_{r\to\infty} y_{2(4^r-1)/3} = \frac{1-\mu}{1+\mu}.$$

We present next an explicit evaluation of y_{2^r-1}. Since for all $\ell = 1, 2, \ldots, r-1$ it is true that

$$2^r - 1 = 2^{\ell+1}m + j \qquad \text{where} \qquad m = 2^{r-\ell-1} - 1,\ j = 2^{\ell+1} - 1,$$

we deduce that $a_{2^r-1,\ell} = 0$ in that range. Moreover, for $\ell = r$ we have $m = 0$, $j = 2^r - 1$, whence $a_{2^r-1,r} = 2$, and trivially $a_{2^r-1,\ell} = 0$ for $\ell \ge r+1$. Therefore $y_{2^r-1} = -1 + 2\mu^r$ and

$$\lim_{r\to\infty} y_{2^r-1} = -1.$$

It is possible to prove that, in general, each $\mathcal{A}_\mu$ can be inscribed into a complex disc of radius $|1-\mu|/(1-|\mu|)$ [16]. It is known that $\mathcal{A}_0 = \{-1, +1\}$ and $\mathcal{A}_{1/2} = [-1, 1]$. Perhaps more surprising result is that every element of the sequence $\{y_n\}_{n\in\mathbb{Z}^+}$ is in $\mathcal{A}_\mu$. In order to prove this we pick an arbitrary $n \in \mathbb{Z}^+$ and let $m_k := n + 2^{k+1}$, $k \in \mathbb{Z}^+$. Obviously, the m_ks form a pairwise-distinct sequence and, provided that k is large enough, binary expansions of n and m_k differ only in the kth digit. It is easy to prove that $a_{m_k,\ell} = a_{n,\ell}$, $\ell = 0, 1, \ldots, k-1$, whence

$$|y_n - y_{m_k}| \le 2\sum_{\ell=k}^{\infty} |\mu|^\ell = \frac{2|\mu|^k}{1-|\mu|} \stackrel{k\to\infty}{\longrightarrow} 0$$

for all $|\mu| < 1$ [16]. As a matter of fact, unless $\mu = 0$, the set $\mathcal{A}_\mu$ is of an infinite cardinality. Moreover, it follows at once from the definition that there exists a

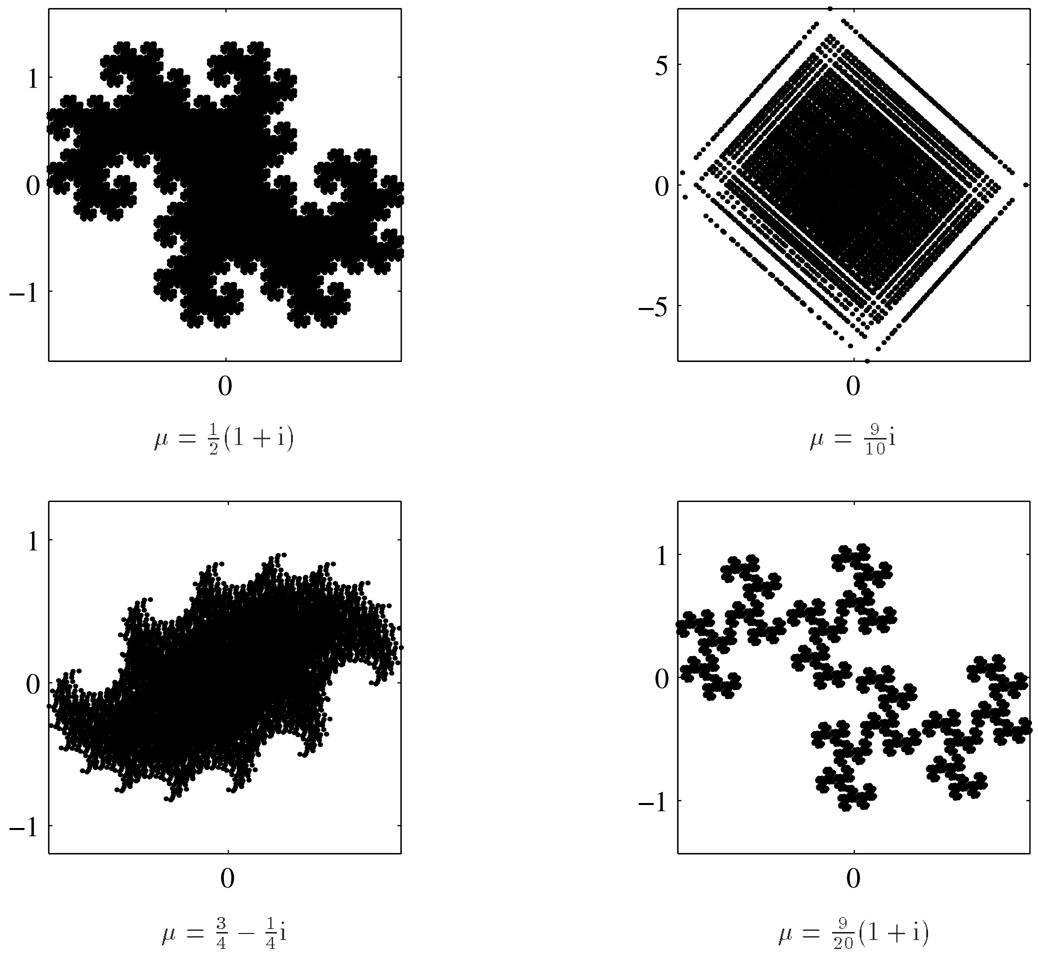


Figure 6: ω-sets $\mathcal{A}_\mu$ for different values of $\mu \in \mathbb{C}$, displayed in the plane $(\mathrm{Re}\,\mathcal{A}_\mu, \mathrm{Im}\,\mathcal{A}_\mu)$.

countable number of components of $\mathcal{A}_\mu$ in every nontrivial neighbourhood of every element of this set.

Figure 6 displays the sets $\mathcal{A}_\mu$ for four choices of $\mu \in \mathbb{C}$. It is indicative of fractal sets, as is the above analysis. We proceed next to identify $\mathcal{A}_\mu$, $\mu \neq 0$, with a different mathematical construct, thereby proving that our suspicion is well-founded: the attractors indeed possess a fractional Hausdorff dimension.

Letting $\lambda = -1$ in (3.3), we eliminate μy_n and μy_{n+1} from these expressions. The first equation can be rewritten in the form

$$\mu y_n = \tfrac{1}{2}\mu(y_{2n} + y_{2n+1})$$

and substitution in the second equation yields

$$y_{2n+1} - y_{2n} = y_{2n-1} - y_{2n-2}.$$

Consequently, by induction,

$$y_{2n+1} - y_{2n} = y_1 - y_0 = -2(1-\mu).$$

Adding and subtracting the first equation results in

$$\begin{aligned} y_{2n} &= \mu y_n + (1-\mu), \qquad n \in \mathbb{N}, \\ y_{2n+1} &= \mu y_n - (1-\mu), \qquad n \in \mathbb{Z}^+, \end{aligned} \tag{3.5}$$

respectively. Letting

$$f_1(z) := \mu z + (1-\mu), \quad f_2(z) := \mu z - (1-\mu), \qquad z \in \mathbb{C},$$

the recurrence (3.5) becomes

$$y_{2n} = f_1(y_n), \qquad y_{2n+1} = f_2(y_n).$$

This, in turn, has the same attractor as a dynamical system which is obtained when for each $n \in \mathbb{Z}^+$ we obtain y_{n+1} from y_n by iterating either f_1 or f_2 with equal probability [1]. The attractor $\mathcal{A}_\mu$ is thus identified with a *probabilistic mixture of Julia sets,* whose properties have been a subject to a thoroughgoing mathematical analysis in the last decade.

To state that the present body of knowledge on the discretized pantograph equation is uncomplete constitutes an understatement. Insofar as asymptotic stability is concerned, the method of proof extends just to q which is a reciprocal of an integer and on the stability boundary we can explain only the nonrealistic case $\lambda = -1$. Yet, computer experimentation indicates that our results represent general patterns. It is fair to formulate the conjecture that, subject to further step-size restrictions in the neutral case, the discretization of

(2.2) by an arbitrary multistep method is stable whenever $|b| < |a|$ and ha belongs to the linear stability domain of the underlying ODE scheme. Moreover, computer experiments seem to indicate that values of ha on the boundary of the linear stability domain lead to fractal attractors. Only limitations of our current methods of proof and of our understanding stand between us and a verification (or refutation) of these conjectures.

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