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Abstract This paper deals with an initial value problem of a neutral functional differential equation with variable time delay. The effect of the neutral term on the existence and uniqueness of solutions is investigated. A comparison theorem is given and is applied to the Shabat equation. The asymptotic behaviour of solutions of a linear perturbation of the generalised pantograph equation is studied.

1 Introduction

Consider the initial value problem

$$y'(t) - c(t)y'(\phi(t)) = f(t, y(t), y(\phi(t))), \qquad t \geq 0, \qquad y(0) = y_0, \tag{1.1}$$

where c and f are complex continuous functions on $\mathbb{R}^+$ and $\mathbb{R}^+ \times \mathbb{C} \times \mathbb{C}$, respectively, ϕ is a real continuous function on $\mathbb{R}^+$ with $0 \leq \phi(t) \leq t$ for all $t \in \mathbb{R}^+$, and $y_0 \in \mathbb{C}$. The restriction on ϕ implies that the time lag $t - \phi(t)$ vanishes at least at $t = 0$. Special cases of equation (1.1) include the generalised pantograph equation [11, 27]

$$y'(t) - c_0 y'(qt) = a_0 y(t) + b_0 y(qt), \qquad t \geq 0, \tag{1.2}$$

where $a_0, b_0, c_0 \in \mathbb{C}$, $q \in (0, 1)$, the Shabat equation [25, 27]

$$y'(t) + q^2 y'(qt) = \mu - y^2(t) + q^2 y^2(qt), \qquad t \geq 0, \tag{1.3}$$

where $\mu \in \mathbb{C}$, $q \in (0, 1)$, and the linear case studied in [15]. Various special cases of (1.1) have been used as test equations in the study of numerical stability [1, 3, 20]. One can also use special cases of (1.1) to construct small solutions of neutral equations by following the idea in [4].

This paper is organised as follows: Section 2 considers the well-posedness of the initial value problem (1.1). A large part of that section is devoted to the study of the effect of the coefficient $c(t)$ on the existence and uniqueness of solutions. Examples

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that illustrate the influence of $c(t)$ on the convergence of traditional numerical methods can be found in [20]. In Section 3, some comparison results are presented. Under some restrictions, it is shown that solutions of (1.1) can be compared with solutions of some ordinary differential equations. In Section 4, several examples, including the Shabat equation (1.3), demonstrate how the comparison theorem can be used to study the existence of solutions and to give sharper estimates of the solutions. Section 5 studies the equation

$$y'(t) - c(t)y'(\phi(t)) = a(t)y(t) + b(t)y(\phi(t)), \qquad t \geq 0, \tag{1.4}$$

as a perturbation of the generalised pantograph equation (1.2). Under certain conditions on $a(t), b(t), c(t)$ and $\phi(t)$, it is proved that solutions of (1.4) have asymptotic behaviour similar to that of solutions of equation (1.2)

2 Existence and uniqueness of solutions

As far as existence and uniqueness of solutions are concerned, the well-known "method of steps" does not apply to (1.1), since $\phi(t) = t$ at least at $t = 0$. The general theory of Nussbaum [23] does apply to (1.1). However, a slightly better result can be obtained if the idea of reducing the neutral equation into a retarded equation (see, e.g., [15, 28]) is used. Define ϕ_n recursively as follows

$$\phi_0(t) = t, \qquad \phi_{n+1}(t) = \phi(\phi_n(t)), \qquad n = 0, 1, \ldots,$$

and let

$$c_0(t) = 1, \qquad c_{n+1}(t) = \prod_{k=0}^{n} c(\phi_k(t)), \qquad n = 0, 1, \ldots.$$

It is important to note for any fixed $t > 0$ that the sequence $\{\phi_n(t)\}_{n=0}^{\infty}$ is monotonically decreasing and that $\lim_{n\to\infty} \phi_n(t) = t^*$, where $t^* = \sup\{\tau \in [0, t] \mid \phi(\tau) = \tau\}$. Clearly, t^* is the closest fixed point of ϕ on the left hand side of t.

Theorem 1 *Suppose that $|c(t)| < 1$ whenever $\phi(t) = t$. Then there exists a number $T > 0$ such that (1.1) has a solution on $[0, T]$. Moreover, if $y(t)$ is a non-continuable solution of (1.1), then the maximal interval of existence, $[0, T_{\max})$, is open on the right, and if $T_{\max} < \infty$, then $\limsup_{t\to T_{\max}} |y(t)| = \infty$. Furthermore, the solution is unique if f is locally Lipschitz continuous; i.e., for any $t_0 \geq 0$ there exists a number $\rho > 0$ such that*

$$|f(t, y_1, y_2) - f(t, x_1, x_2)| \leq M(|y_1 - x_1| + |y_2 - x_2|)$$

for $t_0 \leq t \leq t_0 + \rho$, $|y_1 - x_1| \leq \rho$, $|y_2 - x_2| \leq \rho$ and for some positive constant M (depending on t_0 and ρ).

Proof. Since $|c(t)| < 1$ whenever $\phi(t) = t$, it can be shown by using iteration that (1.1) is equivalent to the initial value problem

$$y'(t) = \sum_{n=0}^{\infty} c_n(t) f(t, y(\phi_n(t)), y(\phi_{n+1}(t))), \qquad t \geq 0, \qquad y(0) = y_0. \tag{2.5}$$

The rest of the proof follows similarly to those in Chapter 12 of Gripenberg, Londen and Staffans [9]. □

One natural question to ask is what happens when $|c(t)| \geq 1$ at some points t where $\phi(t) = t$. To this end, it is worthwhile to cite the following paragraph from Hale [10, p. 26]:

> *As the theory evolves, one cannot keep from observing that retarded equations are very similar to ordinary differential equations and parabolic partial differential equations and that neutral equations resemble difference equations and hyperbolic partial differential equations.*

In the rest of this section, we consider the following linear case of the initial value problem (1.1):

$$y'(t) - c(t)y'(\phi(t)) = f(t) + a(t)y(t) + b(t)y(\phi(t)), \qquad t \geq 0, \qquad y(0) = y_0, \quad (2.6)$$

where $a, b, c, \phi, f \in C[0, t_0]$, $|c(t)| > 1$ and $\phi(t)$ is strictly monotonically increasing on $[0, t_0]$ for some $t_0 > 0$. Obviously, the functional equation that (2.6) resembles should be

$$x(t) - c(t)x(\phi(t)) = f(t), \qquad t \geq 0. \quad (2.7)$$

Denote

$$U = \{u(t) \in C(I_0) \mid u(t_0) - c(t_0)u(\phi(t_0)) = f(t_0)\}.$$

Analogous to Theorem 1 of Liu [20], the following holds:

Lemma 2 *Given any function $u(t) \in U$, equation (2.7) has one and only one solution $x(t) \in C[0, t_0]$ such that $x(t) = u(t)$ on I_0.*

The above lemma yields the following result.

Theorem 3 *Suppose that $a(t) \equiv b(t) \equiv 0$. Given any function $u(t) \in U$, the initial value problem (2.6) has one and only one solution $y(t) \in C^1[0, t_0]$ such that $y'(t) = u(t)$ on I_0.*

The above result can be viewed as an extention of the example given in Remark 4 in Nussbaum [23] except that Theorem 3 characterises all solutions of (2.6) in the case $a(t) \equiv b(t) \equiv 0$. Without the restriction that $a(t) \equiv b(t) \equiv 0$, the following (slightly weaker) result holds:

Theorem 4 *The initial value problem (2.6) has infinitely many linearly independent solutions in $[0, t_0]$ for any given initial value y_0.*

Proof. Denote the function space

$$V = \{v(t) \in C^1(I_0) \mid v'(t_0) - c(t_0)v'(\phi(t_0)) = f(t_0) + a(t_0)v(t_0) + b(t_0)v(\phi(t_0))\}$$

and let

$$t_n = \phi_n(t_0), \quad \text{and} \quad I_n = [t_{n+1}, t_n] \qquad \text{for } n \in \mathbb{Z}^+.$$

The assumptions on the function ϕ imply that

$$\lim_{n\to\infty} t_n = 0. \tag{2.8}$$

It can be shown that given any function $v(t) \in V$, the equation

$$y'(t) - c(t)y'(\phi(t)) = f(t) + a(t)y(t) + b(t)y(\phi(t)), \qquad t \in [0, t_0], \tag{2.9}$$

has one and only one solution $y(t) \in C^1[0, t_0]$ that obeys $y(t) = v(t)$ for $t \in I_0$. Indeed, such a solution $y(t)$ can be formulated in the following way:

1. $y(t) = v(t)$ for $t \in I_0$; and
2. When $t \in I_n$ for $n \in \mathbb{Z}^+$, $y(t)$ is the unique solution of the initial value problem

$$\begin{cases} y'(t) + \frac{b(\zeta(t))}{c(\zeta(t))} y(t) = \frac{1}{c(\zeta(t))} \left[y'(\zeta(t)) - f(\zeta(t)) - a(\zeta(t))y(\zeta(t)) \right], \\ y(t_n) = y(t_n + 0). \end{cases} \tag{2.10}$$

where $\zeta(t)$ is the inverse function of $\phi(t)$. Note that $y(t)$ is constructed recursively from (2.10), where $y(\zeta(t))$ and $y'(\zeta(t))$ are treated as given functions because $\zeta : I_n \mapsto I_{n-1}$ for $n \geq 1$. Obviously, $y(t) \in C^1(0, t_1]$. Therefore, it suffices to prove that $y(t)$ can be extended continuously to $t = 0$ from the right. It follows from (2.10) that

$$|y'(t)| \leq \rho \left[M_0 + M_1 |y(\zeta(t))| + M_2 |y(t)| + |y'(\zeta(t))| \right], \qquad t \in (0, t_1],$$

where $\rho = \|c^{-1}|_{[0,t_0]}\|_\infty$, $M_0 = \|f|_{[0,t_0]}\|_\infty$, $M_1 = \|a|_{[0,t_0]}\|_\infty$ and $M_2 = \|b|_{[0,t_0]}\|_\infty$. The preceding inequality yields the following estimate:

$$|y'(t)| \leq \|v'|_{I_0}\|_\infty + \beta \|y|_{[t,t_0]}\|_\infty, \qquad t \in (0, t_0], \tag{2.11}$$

where $\beta = \rho(M_1 + M_2)/(1 - \rho)$. Hence,

$$|y(t)| \leq |v(t_0)| + \frac{\rho M_0}{1 - \rho} + \int_t^{t_0} |y'(s)| \, \mathrm{d}s \leq \alpha + \beta \int_t^{t_0} \|y|_{[t,t_0]}\|_\infty \, \mathrm{d}s, \qquad t \in (0, t_0],$$

where $\alpha = |v(t_0)| + t_0[\|v'|_{I_0}\|_\infty + \rho M_0/(1 - \rho)]$. Analogous to Gronwall's inequality, the following estimate holds:

$$|y(t)| \leq \alpha \mathrm{e}^{\beta(t_0 - t)}, \qquad t \in (0, t_0]. \tag{2.12}$$

The inequalities (2.11) and (2.12) imply that $y'(t)$ is uniformly bounded on $(0, t_0]$. Therefore, $\lim_{t\to 0+} y(t)$ exists so $y(t)$ can be extended continuously to $t = 0$ from the right.

Next, it is easy to see that the homogeneous initial value problem

$$y'(t) - c(t)y'(\phi(t)) = a(t)y(t) + b(t)y'(\phi(t)), \qquad t \in [0, t_0], \qquad y(0) = 0 \tag{2.13}$$

has infinitely many linearly independent solutions. This is because V is an infinite dimensional function space, whether $f \equiv 0$ or not.

Finally, it can be shown that given any function $f \in C[0, t_0]$, the equation

$$y'(t) - c(t)y'(\phi(t)) = f(t) + a(t)y(t) + b(t)y(\phi(t)), \qquad t \in [0, t_0] \tag{2.14}$$

has solutions that obey $y(0) \neq 0$. It follows from (2.8) that there exists an integer $n^* \geq 0$ such that

$$\left(M_3 + \left\{1 + t_{n^*}\left[M_3 + \rho M_0/(1-\rho)\mathrm{e}^{\beta t_{n^*}}\right]\right\}\right) t_{n^*+1} < 1,$$

where $M_3 = 3|f(t_{n^*}) + a(t_{n^*}) + b(t_{n^*})|$. Then, one such solution to (2.14) can be obtained by solving the problem

$$\begin{cases} y'(t) - a(t)y(t) = f(t) + b(t)y(\phi(t)) + c(t)y'(\phi(t)), & t \notin I_{n^*}, \\ y(t) = y_0(t), \qquad t \in I_{n^*}, \end{cases}$$

where

$$y_0(t) = 1 + \frac{f(t_{n^*}) + a(t_{n^*}) + b(t_{n^*})}{(t_{n^*} - t_{n^*+1})^2}(t - t_{n^*})(t - t_{n^*+1})^2, \qquad t \in I_{n^*}.$$

The proof that $y(0) \neq 0$ proceeds as follows. Analogous to (2.11) and (2.12), the following estimates

$$|y'(t)| \leq M_3 + \beta \|y|_{[t, t_{n^*}]}\|_\infty$$

$$|y(t)| \leq \{1 + t_{n^*}[M_3 + \rho M_0/(1-\rho)]\}\, \mathrm{e}^{\beta(t_{n^*} - t)}$$

holds for $t \in [0, t_{n^*}]$. Therefore,

$$|y'(t)| \leq M_3 + \beta\{1 + t_{n^*}[M_3 + \rho M_0/(1-\rho)]\}\, \mathrm{e}^{\beta t_{n^*}}, \qquad t \in [0, t_{n^*}]$$

and

$$|y(0) - 1| \leq \int_0^{t_{n^*+1}} |y'(\tau)|\, \mathrm{d}\tau < 1.$$

Hence, $y(0) \neq 0$. Since (2.14) is linear, Theorem 4 follows from the last two parts of the proof. □

3 Comparison results

Consider the neutral functional differential equation

$$x'(t) - d(t)x'(\psi(t)) = g(t, x(t), x(\psi(t))), \qquad t \geq 0, \tag{3.15}$$

where d and g are real continuous functions on $\mathbb{R}^+$ and $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$, respectively, and ψ is a real continuous function in $\mathbb{R}^+$ with $0 \leq \psi(t) \leq t$ for all $t \in \mathbb{R}^+$. Assume that $|d(t)| < 1$ whenever $\psi(t) = t$, and that

$$|c(t)| \leq d(t), \;\; \phi(t) \leq \psi(t), \;\; |f(t, y_1, y_2)| \leq g(t, |y_1|, |y_2|), \quad \forall t \geq 0, \; (y_1, y_2) \in \mathbb{C}^2.$$

Then the following comparison theorem holds:

Theorem 5 *Suppose that (3.15) has a nonnegative solution $x(t) \in C^1[0,T]$ for some $T > 0$. If $\phi \equiv \psi$, then, for any initial value y_0 obeying $|y_0| \le x(0)$, there exists a solution $y(t)$ of (1.1) satisfying $|y(t)| \le x(t)$ for all $t \in [0,T]$. If $\phi \not\equiv \psi$, then the same result holds provided that d, ψ and g are monotonically increasing.*

Proof. Define

$$\psi_0(t) = t, \qquad \psi_{n+1}(t) = \psi(\psi_n(t)), \qquad n = 0, 1, \ldots,$$
$$d_0(t) = 1, \qquad d_{n+1}(t) = \prod_{k=0}^{n} d(\psi_k(t)), \qquad n = 0, 1, \ldots.$$

It is not difficult to verify by iteration from (3.15) that $x(t)$ satisfies the equation

$$x'(t) = \sum_{n=0}^{\infty} d_n(t) g(\psi_n(t), x(\psi_n(t)), x(\psi_{n+1}(t))), \qquad t \in [0,T]$$

which implies that $x(t)$ is monotonically increasing and that

$$x(t) = x(0) + \int_0^t \sum_{n=0}^{\infty} d_n(s) g(\psi_n(s), x(\psi_n(s)), x(\psi_{n+1}(s)))\,\mathrm{d}s, \qquad t \in [0,T]. \quad (3.16)$$

Let

$$V = \{v \in C[0,T] \mid |v(t)| \le x(t),\ \forall x \in [0,T]\}.$$

Based on the equation (2.5), an operator Q: $C[0,T] \mapsto C[0,T]$ is defined as follows:

$$Q(v)(t) = y_0 + \int_0^t \sum_{n=0}^{\infty} c_n(s) f(\phi_n(s), v(\phi_n(s)), v(\phi_{n+1}(s)))\,\mathrm{d}s, \qquad t \in [0,T].$$

If $\phi \equiv \psi$, then (3.16) implies that

$$|Q(v)(t)| \le x(t), \qquad t \in [0,T], \qquad \forall v \in V. \quad (3.17)$$

If $\phi \not\equiv \psi$ and d, ψ and g are monotonically increasing, the inequality (3.17) still holds. Therefore, Q maps the function space V into itself. Since V is a bounded, closed and convex subspace of the Banach space $C[0,T]$ (equipped with the maximum norm), and Q is a compact map of V into V, it follows from Schauder's fixed point theorem that there exists a function $y \in V$ such that $y(t) = Q(y)(t)$ for all $t \in [0,T]$. It is obvious that this function $y(t)$ is a solution of (1.1). □

Let

$$d(t) = \max_{0 \le s \le t} |c(t)|, \qquad g(t, y_1, y_2) = \max_{0 \le s \le t, |x_1| \le y_1, |x_2| \le y_2} |f(t, x_1, x_2)|.$$

Then Theorem 5 yields the following result:

Corollary 6 *If $|c(t)| < 1$ for $t \in [0, T]$ and the equation*

$$x'(t) = \frac{1}{1-d(t)} g(t, x(t), x(t)), \qquad t \geq 0$$

has a nonnegative solution $x(t) \in C^1[0, T]$ for some $T > 0$, then for any initial value y_0 obeying $|y_0| \leq x(0)$ there exists a solution $y(t)$ of (1.1) satisfying $|y(t)| \leq x(t)$ for all $x \in [0, T]$.

4 Applications

Theorems 1, 5 and Corollary 6 yield interesting information about the existence and uniqueness of solutions of some special cases of (1.1). Corollary 7, which follows from Theorem 1 and Corollary 6, is a much stronger result than Corollary 1 in Iserles and Terjéki [15].

Corollary 7 *Suppose that $|c(t)| < 1$ for all $t \in \mathbb{R}^+$ and that there exist continuous functions $M_i(t)$, $i = 0, 1, 2$, such that $|f(t, y_1, y_2)| \leq M_0(t) + M_1(t)|y_1| + M_2(t)|y_2|$ for all $t \in \mathbb{R}^+$ and $(y_1, y_2) \in \mathbb{C}^2$. Then (1.1) has solutions in $[0, \infty)$ for any initial value y_0. Furthermore, if $f(t, y_1, y_2)$ is locally Lipschitz continuous, then the solution is unique.*

Though Corollary 6 is easier to use, Theorem 5 can give a sharper estimate about the growth of the solution as shown in the next example.

Example 1. Consider the initial value problem

$$y'(t) - c(t)y'(\phi(t)) = a(t)y(t) + b(t)y(\phi(t)), \qquad t \geq 0, \qquad y(0) = y_0, \tag{4.18}$$

where a, b, c, ϕ are continuous functions that satisfy the following assumptions:

$$|a(t)| \leq a_0, \ |b(t)| \leq b_0, \ |c(t)| \leq c_0 < 1, \ \phi(t)/t \leq q \in (0, 1), \ \text{ for all } t > 0.$$

Since the solution $y(t)$ of (4.18) is unique in this case, Theorem 5 implies that $|y(t)| \leq x(t)$, where $x(t)$ is the solution of the generalised pantograph equation

$$x'(t) = a_0 x(t) + b_0 x(qt) + c_0 x'(qt), \quad t \geq 0, \qquad x(0) = |y_0|. \tag{4.19}$$

If $a_0 > 0$, then Theorem 6 of Iserles and Liu [13] implies that $x(t) = \mathcal{O}(\mathrm{e}^{a_0 t})$ as $t \to \infty$. Therefore, $y(t) = \mathcal{O}(\mathrm{e}^{a_0 t})$ as $t \to \infty$. If $a_0 = 0$, then

$$x(t) = |y_0| \sum_{n=0}^{\infty} \left(\prod_{k=0}^{n-1} \frac{q^k b_0}{1 - c_0 q^k} \right) \frac{t^n}{n!} \leq \frac{|y_0|}{\prod_{k=0}^{\infty}(1 - c_0 q^k)} z(b_0 t),$$

where $z(t)$ is the solution of

$$z'(t) = z(qt), \qquad z(0) = 1.$$

According to a result of de Bruijn [2] (see also Kato and McLeod [16]), $z(t) = \mathcal{O}(h(t))$ as $t \to \infty$, where

$$h(t) = t^{\alpha}(\log t)^{\beta} \exp\left(-\frac{1}{2\gamma}(\log t - \log\log t)\right)$$

with

$$\gamma = \log q, \quad \alpha = \frac{1}{2} - \frac{1}{\gamma} - \frac{1}{\gamma}\log(-\gamma), \quad \beta = 1 + \frac{1}{\gamma}\log(-\gamma).$$

Therefore, $y(t) = \mathcal{O}(h(b_0 t))$ as $t \to \infty$. In contrast, Corollary 6 yields the following weaker estimate

$$y(t) = \mathcal{O}\left(\exp\left(\frac{a_0 + b_0}{1 - c_0} t\right)\right) \qquad \text{as } t \to \infty.$$

This example is motivated by a recent work of Derfel and Vogl [5], though our comparison theorem is fundamentally different from theirs.

However, for some special cases, one can use the basic comparison idea to obtain better results as shown in the next example.

Example 2. Consider the Shabat equation (1.3). It follows from Theorem 1 that (1.3) has a unique non-continuable solution in the interval $[0, T_{\max})$ for some $T_{\max} \le \infty$ which obeys the initial value condition

$$y(0) = y_0 \neq 0.$$

Denote this solution by $y(t)$. According to Corollary 6, $y(t)$ is bounded by the solution of the Riccati's equation

$$x'(t) = \frac{|\mu|}{1 - q^2} + \frac{1 + q^2}{1 - q^2} x^2(t), \qquad t \ge 0, \qquad x(0) = |y_0|. \tag{4.20}$$

However, on noticing that (1.3) is equivalent to the equation

$$y'(t) = \frac{\mu}{1 + q^2} - \sum_{n=0}^{\infty} (-1)^n q^{2n} (y^2(q^n t) - q^2 y^2(q^{n+1} t)),$$

one can shown similarly that $y(t)$ can be bounded by the the Riccati's equation

$$x'(t) = \frac{|\mu|}{1 + q^2} + \frac{1 + q^2}{1 - q^2} x^2(t), \qquad t \ge 0, \qquad x(0) = |y_0|. \tag{4.21}$$

When $\mu = 0$, the solution of (4.21) is

$$x(t) = \left(\frac{1}{|y_0|} - \frac{1 + q^2}{1 - q^2} t\right)^{-1}.$$

It follows from Corollary 6 that

$$T_{\max} \ge T_0 := \frac{1 - q^2}{(1 + q^2)|y_0|} \qquad \text{and} \qquad |y(t)| \le x(t), \;\; t \in [0, T_0).$$

When $\mu \neq 0$, the solution of (4.21) is

$$x(t) = \frac{1}{\alpha} \tan\left(\arctan(\alpha|y_0|) + \frac{\alpha|\mu|}{1+q^2} t\right),$$

where $\alpha = (1+q^2)/((1-q^2)|\mu|)^{1/2}$. Therefore,

$$T_{\max} \geq T_1 := \left(\frac{1-q^2}{|\mu|}\right)^{1/2} \left[\frac{\pi}{2} - \arctan(\alpha|y_0|)\right]$$

and

$$|y(t)| \leq x(t), \qquad t \in [0, T_1).$$

It is not difficult to see that the preceding lower bound of $T_{\max}$ is better than the one that can be obtained by comparing the solution $y(t)$ with that of (4.20).

5 Asymptotic behaviour

For the generalised pantograph equation (1.2), it is proved in Liu [21] that when $\operatorname{Re} a_0 < 0$, any solution $y(t) \in C^\infty[0, \infty)$ of (1.2) must satisfy the following estimate

$$y(t) = \mathcal{O}(t^{\tilde{\alpha}}) \qquad \text{as } t \to \infty, \tag{5.22}$$

where $\tilde{\alpha} := \frac{1}{\ln q} \ln|a_0/b_0|$ if $b_0 \neq 0$, otherwise $\tilde{\alpha}$ can be any arbitrarily small constant. It can be seen from results of Kato and McLeod [16] and Kato [17] that the preceding estimate cannot be improved in the case of $c_0 = 0$.

This section shows that the estimate (5.22) still holds for the equation (1.4), provided that the functions $a, b, c, \phi \in C^\infty[0, \infty)$ are perturbation of a_0, b_0, c_0 and qt, respectively, in the sense that

$$a(t) = a_0 + a_1(t), \quad b(t) = b_0 + b_1(t), \quad \phi(t) = qt + \phi_1(t)$$

and as $t \to \infty$ the following hold:

$$\begin{aligned}
&a_1(t), b_1(t) = o(1), \\
&b_0 a_1(t) - a_0 b_1(t) = \mathcal{O}(t^{-\beta}) \qquad \text{for some constant } \beta > 0, \\
&a_1^{(n)}(t), b_1^{(n)}(t) = \mathcal{O}(t^{-n}), \qquad n = 1, 2, \ldots, \\
&c(t), c'(t) = \mathcal{O}(1), \ \ \phi_1(t) = \mathcal{O}(1), \ \ \phi_1'(t) = o(1), \\
&c^{(n)}(t), \ \phi_1^{(n)}(t) = \mathcal{O}(t^{-n+1}), \qquad n = 2, 3, \ldots.
\end{aligned}$$

Under the above conditions, equation (1.4) can be viewed as a linear perturbation of (1.2).

Theorem 8 *If* $\operatorname{Re} a_0 < 0$*, then the estimate (5.22) holds for any solution* $y(t) \in C^\infty[0, \infty)$ *of equation (1.4).*

Three lemmas are needed to prove the preceding result.

Lemma 9 *Suppose that $x(t) \in C[0,\infty)$ is a nonnegative function that satisfies the inequality*

$$x(t) \le g(t) + C(t)x(\psi(t)) + \int_{t_0}^{t} D(t,s)x(\psi(s))\,\mathrm{d}s, \qquad t \ge t_0 \ge 0,$$

where $g, C, \psi \in C[t_0,\infty)$ are nonnegative functions with $0 \le \psi(t) \le t$ for all $t \ge t_0$, and $D(t,s)$ is a nonnegative continuous function for all $t_0 \le s \le t$. If $g(t) = \mathcal{O}(1)$ as $t \to \infty$ and

$$\limsup_{T\to\infty} \max_{t\ge T} \left[|C(t)| + \int_T^t D(t,s)\,\mathrm{d}s \right] < 1,$$

then $x(t) = \mathcal{O}(1)$ as $t \to \infty$.

Lemma 10 *Suppose that $y(t) \in C[qt_0,\infty)$ is a complex function that satisfies the equation*

$$y(t) + q^{-\mu} y(qt) = g(t), \qquad t \ge t_0,$$

where q and μ are real constants with $q \in (0,1)$. If $g(t) = \mathcal{O}(t^{\nu})$ as $t \to \infty$ with $\nu < \mu$, then $y(t) = \mathcal{O}(t^{\mu})$ as $t \to \infty$.

Lemma 11 *Suppose that $x(t) \in C^1[0,\infty)$ is a complex function that satisfies the equation*

$$(x(t) - C(t)x(\psi(t)))' = A(t)x(t) + B(t)x(\psi(t)) + g(t), \qquad t \ge t_0 \ge 0, \tag{5.23}$$

where $A, B, C \in C[t_0,\infty)$ are complex functions and $\psi(t) \in C[t_0,\infty)$ is a real function with $\operatorname{Re} A(t) < 0$ and $0 \le \psi(t) \le t$ for all $t \ge t_0$. If $g(t)/\operatorname{Re} A(t) = \mathcal{O}(1)$ as $t \to \infty$ and

$$\limsup_{T\to\infty} \max_{t\ge T} \left[|C(t)| + \max_{T\le s\le t} \frac{|B(s) + A(s)C(s)|}{|\operatorname{Re} A(s)|} \right] < 1,$$

then $x(t) = \mathcal{O}(1)$ as $t \to \infty$.

Lemma 9 is probably not new and its proof is trivial. Lemma 10 has been proved in [21]. To prove Lemma 11, one can use the variation-of-constants formula to obtain from (5.23) the equation

$$x(t) = [x(t_0) - C(t_0)x(\psi(t_0))]\,\mathrm{e}^{\int_{t_0}^{t} A(s)\,\mathrm{d}s} + \int_{t_0}^{t} \mathrm{e}^{\int_s^t A(\tau)\,\mathrm{d}\tau} g(s)\,\mathrm{d}s$$
$$+ C(t)x(\psi(t)) + \int_{t_0}^{t} D(t,s)x(\psi(s))\,\mathrm{d}s, \qquad t \ge t_0,$$

where

$$D(t,s) = \mathrm{e}^{\int_s^t A(\tau)\,\mathrm{d}\tau}[B(s) + A(s)C(s)].$$

Since

$$\left| \int_{t_0}^{t} \mathrm{e}^{\int_s^t A(\tau)\,\mathrm{d}\tau} g(s)\,\mathrm{d}s \right| \le \sup_{t\ge t_0} \left| \frac{g(t)}{\operatorname{Re} A(t)} \right|, \qquad t \ge t_0$$

and

$$\int_T^t |D(t,s)|\,\mathrm{d}s \le \max_{T\le s\le t} \frac{|B(s)+A(s)C(s)|}{|\mathrm{Re}\, A(s)|},$$

it follows from Lemma 9 that $x(t) = \mathcal{O}(1)$ as $t \to \infty$.

Proof of Theorem 8. Consider only the case $b_0 \neq 0$. The case $b_0 = 0$ can be dealt with similarly. Let

$$\alpha := 2 - \frac{1}{\ln q} \ln\left(c_0 + \frac{q|b_0| + c_1 + |a_0|c_0}{|\mathrm{Re}\, a_0|}\right),$$

where $c_k = \limsup_{t\to\infty} |c^{(k)}(t)|$, $k = 0, 1$. It will be shown by induction on n that

$$y^{(n)}(t) = \mathcal{O}\big(t^{\alpha-n}\big), \qquad \text{as } t \to \infty \tag{5.24}$$

holds for any $n \in \mathbb{Z}^+$. First, consider the case $n = 0$. By assumption, there exists $t_0 > 0$ such that

$$\mathrm{Re}\, a(t) - \frac{\alpha}{t+1} < 0, \qquad \phi'(t) > 0, \qquad \text{for all } t \ge t_0.$$

Let $x(t) = (t+1)^{-\alpha} y(t)$, then $x(t)$ satisfies (5.23) with

$$\begin{aligned}
A(t) &= a(t) - \frac{\alpha}{t+1},\\
B(t) &= \left[\frac{\phi(t)+1}{t+1}\right]^{\alpha} \left[b(t) + \frac{c'(t)}{\phi'(t)} - \frac{\alpha c(t)}{(t+1)\phi'(t)} - \frac{c(t)\phi''(t)}{\phi'(t)^2}\right],\\
C(t) &= \left[\frac{\phi(t)+1}{t+1}\right]^{\alpha} \frac{c(t)}{\phi'(t)},\\
\psi(t) &\equiv \phi(t), \qquad g(t) \equiv 0.
\end{aligned}$$

Since

$$\limsup_{T\to\infty} \max_{t\ge T} \left[|C(t)| + \max_{T\le s\le t} \frac{|B(s)+A(s)C(s)|}{|\mathrm{Re}\, A(s)|}\right] \le q < 1,$$

it follows from Lemma 11 that $x(t) = \mathcal{O}(1)$ as $t \to \infty$. Hence (5.24) holds for $n = 0$. Suppose that there exists an integer $m \ge 1$ such that (5.24) holds for all $n < m$. Denote $y_n(t) = y^{(n)}(t)$, $n = 0, 1, \ldots$. Differentiating both sides of (1.4) m times yields the equation

$$\begin{aligned}
&y_m'(t) - c(t)\phi'(t)^m y_m'(\phi(t))\\
&= a(t)y_m(t) + \left[b(t)\phi'(t)^m + mc'(t)\phi'(t)^{m-1}\right] y_m(\phi(t)) + f_m(t), \quad t \ge 0,
\end{aligned} \tag{5.25}$$

where

$$\begin{aligned}
f_m(t) = {} & [a(t)y(t) + b(t)y(\phi(t)) + c(t)y'(\phi(t))]^{(m)} - a(t)y_m(t)\\
& - \left[b(t)\phi'(t)^m + mc'(t)\phi'(t)^{m-1}\right] y_m(\phi(t)) - c(t)\phi'(t)^m y_m'(\phi(t)).
\end{aligned}$$

The assumption that (5.24) holds for all $n < m$ implies that

$$f_m(t) = \mathcal{O}\big(t^{\alpha-m}\big) \qquad \text{as } t \to \infty.$$

Analogous to the case $n = 0$, the estimate (5.24) holds for $n = m$. Therefore, (5.24) holds for all $n \geq 0$. This completes that induction proof.

Next, it will be shown that the estimate

$$y_m(t) = \mathcal{O}\big(t^{\alpha-n}\big) \qquad \text{as } t \to \infty, \qquad 0 \leq m \leq n \tag{5.26}$$

holds for any nonnegative integer $n \leq n^*$, where $n^* = [\alpha - \tilde{\alpha}] \geq 1$. The proof is also by induction on n. According to (5.24), the estimate (5.26) holds for $n = 0$. Suppose for some positive integer $k \leq n^*$ that (5.26) holds for $n < k$. It will be shown for $0 \leq j \leq k$ that the estimate

$$y_j(t) = \mathcal{O}\big(t^{\alpha-k}\big) \qquad \text{as } t \to \infty \tag{5.27}$$

holds. The estimate (5.24) implies that (5.27) holds for $j = k$. For $j < k$, it follows from (5.25) that

$$y_j(t) + \frac{b(t)\phi'(t)^j + jc'(t)\phi'(t)^{j-1}}{a(t)} y_j(\phi(t)) = g_j(t), \qquad t \geq 0, \tag{5.28}$$

where

$$g_j(t) = \frac{1}{a(t)}\left[y_{j+1}(t) - c(t)\phi'(t)^j y_{j+1}(\phi(t)) - f_j(t)\right].$$

Let $x_j(t) = (t+1)^{k-\alpha}|y_j(t)|$. It follows from (5.28) that

$$x_j(t) \leq (t+1)^{k-\alpha}|g_j(t)| + C_j(t)x_j(\phi(t)), \qquad t \geq 0, \tag{5.29}$$

where

$$C_j(t) = \frac{|b(t)||\phi'(t)^j(t+1)^{k-\alpha} + j|c'(t)||\phi'(t)^{j-1}}{|a(t)|(\phi(t)+1)^{k-\alpha}}.$$

The assumption that (5.26) holds for $n = k-1$ implies that $f_j(t) = \mathcal{O}\big(t^{\alpha-k}\big)$ as $t \to \infty$ for $0 \leq j \leq k-1$. Since $k \leq n^*$ implies that

$$\limsup_{t\to\infty} |C_j(t)| = q^{\alpha-\tilde{\alpha}-k+j} < 1, \qquad 0 \leq j \leq k-1,$$

it follows by applying Lemma 9 to (5.29) repeatedly from $j = k-1$ to 0 in descending order that $x_j(t) = \mathcal{O}(1)$ as $t \to \infty$. This implies that (5.27) holds for $j = k-1, k-2, \ldots, 0$. Therefore, the estimate (5.26) holds for $1 \leq n \leq n^*$. This completes the other induction proof.

Similarly, it can be proved by induction on m from $m = n^* + 1$ to 1 in descending order that

$$y_m(t) = \mathcal{O}\Big(t^{\alpha-n^*-1}\Big) \qquad \text{as } t \to \infty, \qquad 1 \leq m \leq n^* + 1.$$

This implies that

$$g_0(t) = \frac{1}{a(t)}\{y_1(t) - c(t)y_1(\phi(t))\} = \mathcal{O}\Big(t^{\alpha-n^*-1}\Big) \qquad \text{as } t \to \infty.$$

Since

$$y(t) + \frac{b(t)}{a(t)} y(\phi(t)) = g_0(t), \qquad t \geq 0, \tag{5.30}$$

Lemma 9 can be applied to prove that

$$y(t) = \mathcal{O}\left(t^{\tilde{\alpha}+\beta/2}\right) \qquad \text{as } t \to \infty.$$

Rewrite (5.30) as

$$y(t) + \frac{b_0}{a_0} y(qt) = h(t), \qquad t \geq 0, \tag{5.31}$$

where

$$h(t) = g_0(t) + \frac{b_0}{a_0} y(qt) - \frac{b(t)}{a(t)} y(\phi(t)).$$

Since

$$y(qt) - y(\phi(t)) = \int_{qt}^{\phi(t)} y'(\tau)\, \mathrm{d}\tau = \mathcal{O}\left(t^{\alpha-n^*-1}\right) \qquad \text{as } t \to \infty$$

and

$$\left[\frac{b_0}{a_0} - \frac{b(t)}{a(t)}\right] y(qt) = \mathcal{O}\left(t^{\tilde{\alpha}-\beta/2}\right) \qquad \text{as } t \to \infty,$$

applying Lemma 10 to (5.31) yields the estimate

$$y(t) = \mathcal{O}\left(t^{\tilde{\alpha}}\right) \qquad \text{as } t \to \infty.$$

This concludes the proof of Theorem 8. □

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