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On a functional Riccati equation

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Regular solutions of the Shabat equation

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Abstract. The Shabat equation

$$f'(t)+q^2f'(qt)+f^2(t)-q^2f^2(qt)=\mu, \qquad t\in\mathbb{R},$$

where μ and q are parameters, is the simplest self-similar reduction of the so called dressing chain for constructing and analysing exactly solvable Schrödinger equations. It is also relevant to the study of the q-oscillator algebra in quantum mechanics. The main objective of this paper is to investigate whether the Shabat equation has a nontrivial global solution $f(t)\in \mathrm{C}^1(\mathbb{R})$ that is normalisable in the sense that the corresponding potential $u(t):=f^2(t)-f'(t)-\mu/(1-q^2)$ is in the function space $\mathrm{L}_1(\mathbb{R})$ and whether in the case $\mu/(1-q^2)>0$ it has a global solution $f(t)\in\mathrm{C}^1(\mathbb{R})$ that is regular in the sense that $f(t)=\pm\sqrt{\mu/(1-q^2)}+\mathcal{O}\big(t^{-2}\big)$ and $f'(t)=\mathcal{O}\big(t^{-3}\big)$ as $t\to\pm\infty$.

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1 Introduction

This paper is concerned with the Shabat equation

$$f'(t)+q^2f'(qt)+f^2(t)-q^2f^2(qt)=\mu, \tag{1.1}$$

where q and μ are parameters. The above equation is called a functional Riccati equation in [6]. It is derived in Shabat [26] as the simplest self-similar reduction of the so called dressing chain [15]

$$f_n'(t)+f_{n+1}'(t)+f_n^2(t)-f_{n+1}^2(t)=\mu_n, \qquad n=1,2,\ldots, \tag{1.2}$$

where f_n and μ_n, $n=1,2,\ldots$, are unknown functions and parameters, respectively. If equation (1.1) has a continuous differentiable solution f, say, then the dressing chain (1.2) has the self-similar solution

$$(f_n(t),\mu_n)=(q^nf(q^nt),q^{2n}\mu).$$

The above has been proved in [6, 26, 27, 28, 29, 30, 34] and others to be very useful for formulating and analysing exactly solvable Schrödinger equations of the form

$$-y''(t)+u(t)y(t)=\lambda y(t), \qquad t\in\mathbb{R},$$

where u is the potential function and λ a parameter. Equation (1.1) is also relevant to the study of the q-oscillator (q-Heisenberg or q-Weyl) algebra in quantum mechanics (see, e.g., [3, 23, 31])

$$AA^+ - q^2 A^+ A = \mu, \tag{1.3}$$

where A and A^+ are q-analogues of the annihilation and creation operators, respectively. A well known pair of operators that satisfy (1.3) is (see., e.g. [12])

$$Ay(t) = \frac{y(t) - y(q^2 t)}{(1-q^2)t}, \qquad A^+ y(t) = \mu t y(t).$$

Spiridonov [30] noted that the pair of operators

$$A = T^{-1}[\,\mathrm{d}/\,\mathrm{d}t + f(t)], \qquad A^+ = [-\,\mathrm{d}/\,\mathrm{d}t + f(t)]T \tag{1.4}$$

satisfy (1.3) as well, provided that f is a solution of (1.1) and T is a scaling operator of the form

$$Ty(t) = \sqrt{q}\,y(qt).$$

The well known harmonic oscillator corresponds to the case $q = 1$. We also note that the following equation

$$\hbar[g(t) + \mathrm{e}^{\hbar\eta} g(\mathrm{e}^{\hbar\eta} t)]' + g^2(t) - \mathrm{e}^{2\hbar\eta} g^2(\mathrm{e}^{\hbar\eta} t) = c\left(1 - \mathrm{e}^{2\hbar\eta}\right),$$

derived in [32] from a simple model of classical mechanics obeying a quadratic Poisson symmetry algebra, can be transformed to (1.1) by letting $g(t) = \hbar f(t)$, $q = \mathrm{e}^{\hbar\eta}$ and $\mu = c\left(1 - \mathrm{e}^{2\hbar\eta}\right)/\hbar^2$, where $\hbar$ is Planck's constant, η and c are parameters.

In this paper, we restrict our study of (1.1) to the case where the parameters q and μ are real, though the case of q and μ being complex is also of interest. Some special cases of (1.1) can be solved explicitly. When $q = 1$, (1.1) reduces to the trivial equation $f'(t) = \mu/2$. When $q = -1$, it can be seen that if $\mu = 0$ then every even continuous differentiable function satisfies the equation (1.1) and that the only solution of (1.1) is $f(t) = \mu t/2$ when $\mu \neq 0$. When $q = 0$, (1.1) is simply the Riccati equation

$$f'(t) + f^2(t) = \mu$$

whose solutions can be found explicitly. The case $q \in (1, \infty) \cup (-\infty, -1)$ can be transformed to the case $q \in (-1, 0) \cup (0, 1)$. More specifically, (1.1) is equivalent to the equation

$$g'(t) + p^2 g'(pt) + g^2(t) - p^2 g^2(pt) = -p^2 \mu,$$

where $g(t) = -f(t)$ and $p = 1/q \in (-1, 0) \cup (0, 1)$.

In the sequel, we discuss (1.1) for the cases $q \in (0, 1)$ and $q \in (-1, 0)$ only. When $q \in (0, 1)$ equation (1.1) is a neutral functional differential equation with rescaling (or proportional delay). We shall see later on that (1.1) can be transformed to a system of two equations of neutral type when $q \in (-1, 0)$.

A function $f(t)$ is called a *global solution* of (1.1) if $f(t) \in \mathrm{C}^1(\mathbb{R})$ and it satisfies (1.1) for every $t \in \mathbb{R}$. In the aforementioned applications of (1.1)

$$u(t) := f^2(t) - f'(t) - \mu/(1 - q^2)$$

corresponds to the so called potential function. According to Degasperis and Shabat [6] it is important to know whether $u(t) \in \mathrm{L}_1(\mathbb{R})$, even though it is weaker than the condition

$$\int_{\mathbb{R}} (1+|t|)|u(t)|\,\mathrm{d}t < \infty$$

usually assumed in inverse scattering theory. We call a global solution $f(t)$ of (1.1) *normalisable* if $u(t) \in \mathrm{L}_1(\mathbb{R})$.

An interesting observation is that if $f(t)$ is a normalisable solution of (1.1) then

$$\lim_{t\to\pm\infty} f^2(t) = \mu/(1-q^2), \qquad \lim_{t\to\pm\infty} f'(t) = 0.$$

Hence, (1.1) has no normalisable solution when $\mu < 0$. Actually, we shall conjecture in Section 5 that (1.1) does not even have global solution when $\mu < 0$.

When $\mu > 0$ equation (1.1) has two stationary solutions, namely, $f(t) \equiv f^*$ and $f(t) \equiv -f^*$, where

$$f^* = \sqrt{\mu/(1-q^2)}.$$

Equation (1.1) has only one stationary solution, namely, $f(t) \equiv 0$, if $\mu = 0$, but no stationary solution when $\mu < 0$. Stationary solutions are normalisable but they are not interesting for they correspond to constant potential functions and that the corresponding annihilation operators defined in (1.4) do not have physically acceptable coherent states [29].

In the case $\mu > 0$, a function $f(t)$ is defined by Degasperis and Shabat [6] as a *regular solution* of (1.1) if it is a global solution of (1.1) and it has the following asymptotic expansions

$$f(t) = \pm f^* + \mathcal{O}\big(t^{-2}\big) \;\text{ and }\; f'(t) = \mathcal{O}\big(t^{-3}\big) \qquad \text{as } t \to \pm\infty.$$

We remark in passing that when $q \in (1,\infty) \cup (-\infty,-1)$ and $\mu < 0$ a *regular solution* of (1.1) should be defined as a global solution satisfying $f(t) = \mp f^* + \mathcal{O}\big(t^{-2}\big)$ and $f'(t) = \mathcal{O}\big(t^{-3}\big)$ as $t \to \pm\infty$.

No nontrivial normalisable or regular solution of (1.1) in explicit form are available so far. In this paper, we try to prove the existence of such solutions instead. A natural way of doing this is to investigate the initial value problem of equation (1.1), namely, finding continuous differentiable functions in a neighbourhood of the origin that satisfy

$$f'(t) + q^2 f'(qt) + f^2(t) - q^2 f^2(qt) = \mu, \qquad f(0) = f_0, \tag{1.5}$$

where $f_0 \in \mathbb{R}$ is the initial value. The following questions arise:

1. Is there a (unique) solution to the initial value problem (1.5)?
2. Is the solution global (provided that it exists)?
3. what is the asymptotic behaviour of the global solution as $t \to \pm\infty$?
4. Is the global solution normalisable?
5. Is the global solution regular in the case $\mu > 0$?

Shabat stated in [26] that the estimate

$$0 < qf(qt) < \gamma y(\gamma t) < f(t) < y(t), \qquad t > 0 \tag{1.6}$$

holds if $q \in (0, (2^{1/2} - 1)^{1/2}))$, $\mu = 1 - q^2$ and $f_0 = 0$, where

$$\gamma = \left(\frac{1-q^2}{1+q^2}\right)^{1/2}, \qquad y(t) = \frac{\mathrm{e}^t - \mathrm{e}^{-t}}{\mathrm{e}^t + \mathrm{e}^{-t}}.$$

No proof were given there. Though (1.6) is a nice comparison result, it does not provide enough information about the existence of normalisable or regular solutions. It is obvious that the second inequality in (1.6) does not hold when q is close to 1, while our numerical experiments indicate that the third and the last inequalities hold for every $q \in (0, 1)$. Skorik and Spiridonov [27] proved by using Taylor series that there exists a unique analytic solution to the initial value problem (1.5) in a neighbourhood of the origin when q and μ are complex and $|q| < 1$. They also discussed the case $|q| = 1$. Degasperis and Shabat [6] also used Taylor series to discuss the existence of analytic solution. Adler [2] constructed rational solutions of the equation

$$f'(t) + q^2 f'(qt) + f^2(t) - q^2 f^2(qt) = 1 - q^2$$

for the following initial values

$$f_0 = \frac{q^{k+1/2} + q^{k-1/2}}{1-q}, \qquad k = 0, 1, 2, \ldots,$$

where $q \in (0, 1)$. These rational solutions are not global for they have singularities at some point in $\mathbb{R}$.

The main results of this paper are summarised in the following two theorems.

Theorem 1 *The initial value problem (1.5) has a unique solution in a neighbourhood of the origin and*

1. *the solution is regular if*

$$q \in (0, 1), \qquad \mu > 0, \qquad |f_0| < \frac{(1 - 4q^2 - 7q^4) f^*}{1 + q^4}$$

or

$$q \in (-3^{-1/2}, 0), \qquad \mu > 0, \qquad |f_0| < \frac{[2(1-3q^2)^2(1+q^2)^2 - (1+q^4)^2]^{1/2} f^*}{1+q^4};$$

2. *the solution is not global if*

$$\mu > 0, \qquad |q| < 2^{-1/2}, \qquad |f_0| > f^*$$

or

$$\mu = 0, \qquad |q| < 2^{-1/2}, \qquad f_0 \neq 0$$

or

$$\mu < 0, \qquad |q| < 2^{-1/2}.$$

Theorem 2 *When $\mu > 0$ any global solution $f(t)$ of (1.1) that satisfies $f(t) \to \pm f^*$ as $t \to \pm\infty$ is a regular solution.*

This paper is structured as follows. Section 2 discusses some related topics, including q-difference equations, functional differential equations and Dirichlet series. In Section 3 and Section 4 we give detailed study of equation (1.1) and its initial value problem (1.5) for the cases $q \in (0,1)$ and $q \in (-1,0)$, respectively. Section 5 presents some conjectures on the initial value problem (1.5) when the parameters q, μ and f_0 are not in the regions covered by Theorem 1. These conjectures are supported by extensive numerical experiment.

2 Preliminary

Some notations used in this section are as follows. For a square matrix D, $\sigma(D)$ denotes its *spectrum*, $\rho(D)$ its *spectral radius*, $\alpha[D]$ the maximal real part of its eigenvalues (*spectral abscissa*) and $\|D\|$ denotes the matrix norm induced by a vector norm, likewise denoted by $\|\cdot\|$, which is arbitrary unless otherwise stated.

2.1 On some q-difference equations

First, we consider a q-difference equation of the form

$$y(t) - q^{-\alpha} y(qt) = g(t), \qquad t > 0, \tag{2.1}$$

where α and q are constants, $q \in (0,1)$, and g is a given function that has the asymptotic expansion

$$g(t) = \sum_{n=1}^{N} t^{\beta_n} h_n(\log t) + \mathcal{O}\big(t^{\beta_{N+1}}\big) \qquad \text{as } t \to \infty,$$

with $\beta_1, \ldots, \beta_{N+1}$ being real constants and $h_1, \ldots, h_N$ continuous periodic functions of period $-\log q$. Then the following is true:

Lemma 3 *If $\alpha > \beta_1 > \ldots > \beta_{N+1}$ then any function $y(t) \in \mathrm{C}(0,\infty)$ satisfying (2.1) must have the asymptotic expansion*

$$y(t) = t^{\alpha} h_0(\log t) + \sum_{n=1}^{N} \frac{t^{\beta_n} h_n(\log t)}{1 - q^{\beta_n - \alpha}} + \mathcal{O}\big(t^{\beta_{N+1}}\big) \qquad \text{as } t \to \infty,$$

where h_0 is a continuous periodic function of period $-\log q$.

The case $N = 0$ of the above lemma appeared in the proof of Theorem 3 (iii) of Kato and McLeod [20]. The case $N > 0$ can easily be reduced to the previous case.

Next, we consider a system of q-difference equations of the form

$$\mathbf{y}(t) - C(t)\mathbf{y}(qt) = \mathbf{g}(t), \qquad t \geq t_0 \geq 0 \tag{2.2}$$

where C is a matrix function defined on $[t_0, \infty)$ satisfying $C(t) \to C_0$ as $t \to \infty$ for some constant matrix C_0, and $\mathbf{g}$ is vector function defined on $[t_0, \infty)$.

Lemma 4 *Suppose that $q^\gamma \rho(C_0) < 1$ and $\mathbf{g}(t) = \mathcal{O}(t^\gamma)$ (or $o(t^\gamma)$) as $t \to \infty$ for some constant γ. If $\mathbf{y}(t)$ is uniformly bounded on any bounded closed subinterval of $[qt_0, \infty)$ and satisfies (2.2) for $t \geq t_0$ then $\mathbf{y}(t) = \mathcal{O}(t^\gamma)$ (or $o(t^\gamma)$) as $t \to \infty$.*

The case $\gamma = 0$ can be proved in an elementary way. The case $\gamma \neq 0$ can be transformed to the previous case by letting $\mathbf{y}(t) = (t+1)^\gamma \mathbf{x}(t)$.

We remark in passing that q-difference equations can be transformed to the usual difference equations. For instance, letting $x(t) = y(\mathrm{e}^t)$ then (2.1) becomes

$$x(t) - q^{-\alpha} x(t + \log q) = g(\mathrm{e}^t), \qquad t > -\infty.$$

2.2 A general system of neutral equations

This subsection deals with the initial value problem

$$\mathbf{y}'(t) - C(t)\mathbf{y}'(qt) = \mathbf{g}(t, \mathbf{y}(t), \mathbf{y}(qt)), \quad t \geq 0, \qquad \mathbf{y}(0) = \mathbf{y}_0, \tag{2.3}$$

where C is a $N \times N$ matrix of continuous functions defined on $\mathbb{R}^+$, $\mathbf{g}$ is a N-dimensional vector of continuous functions defined on $\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}$ and $\mathbf{y}_0$ is a given N-dimensional vector. The existence and uniqueness theorem of Nussbaum [24] can be applied to the above initial value problem. However, a slightly better result can be obtained if we use the idea of reducing a neutral equation into a retarded equation as in [18, 33].

Define a sequence of matrices $\{C_n(t)\}_{n=0}^\infty$ by the recurrence relation

$$\begin{aligned} &C_0(t) :\equiv I, \\ &C_n(t) :\equiv C(t)C(qt) \cdots C(q^{n-1}t), \qquad n = 1, 2, \ldots. \end{aligned}$$

Analogous to Lemma 1 of Iserles and Terjéki [18], we have the following result.

Lemma 5 *If $\rho(C(0)) < 1$ then (2.3) is equivalent to the initial value problem*

$$\mathbf{y}'(t) = \sum_{n=0}^\infty C_n(t)\mathbf{g}(q^n t, \mathbf{y}(q^n t), \mathbf{y}(q^{n+1}t)), \quad t \geq 0, \qquad \mathbf{y}(0) = \mathbf{y}_0. \tag{2.4}$$

The assumption $\rho(C(0)) < 1$ implies that $\sum_{n=0}^\infty |C_n(t)|$ converges for every $t \geq 0$. Following the standard approach (see e.g, [13]) on functional differential equations of retarded type we can derive from Lemma 5 the following results.

Lemma 6 *Suppose that $\rho(C(0)) < 1$. Then there exists a constant $T > 0$ such that (2.3) has at least one solution for in the interval $[0, T]$. Moreover,*

1. *if $\mathbf{y}(t)$ is a non-continuable solution, then the maximal interval of existence, $[0, T_{\max})$, is open on the right, and if $T_{\max} < \infty$ then $\limsup_{t \to T_{\max}} \|y(t)\| = \infty$;*

2. *every solution is in $\mathrm{C}^m[0, T]$ if the matrix function $C(t) \in \mathrm{C}^m([0, T])$ and the vector function $\mathbf{g}(t) \in \mathrm{C}^m([0, T] \times \mathbb{R}^{2N})$; and*

3. *the solution of (2.3) is unique if* $\mathbf{g}$ *is Lipschitz continuous; i.e., for any* $t_0 \geq 0$ *there exists a number* $\rho > 0$ *such that*

$$\|\mathbf{g}(t, \mathbf{y}_1, \mathbf{y}_2) - \mathbf{g}(t, \mathbf{x}_1, \mathbf{x}_2)\| \leq M(\|\mathbf{y}_1 - \mathbf{x}_1| + \|\mathbf{y}_2 - \mathbf{x}_2|)$$

for $t_0 \leq t \leq t_0 + \rho$, $\|\mathbf{y}_1 - \mathbf{x}_1\| \leq \rho$, $\|\mathbf{y}_2 - \mathbf{x}_2\| \leq \rho$ *and for some positive constant* M *(depending on* t_0 *and* ρ*).*

Lemma 7 *Suppose that the initial value problem*

$$x'(t) = G(t, x), \quad t \geq 0, \qquad x(0) = x_0$$

has a nonnegative solution $x(t) \in \mathrm{C}^1[0, T]$ *for some* $T > 0$*, where the function* G *is defined as*

$$G(t, x) = \sup_{\|\mathbf{x}_n\| \leq x, n=0,1,\ldots} \left\| \sum_{n=0}^{\infty} C_n(t)\mathbf{g}(q^n t, \mathbf{x}_n, \mathbf{x}_{n+1}) \right\|, \qquad t, x \geq 0.$$

Then for any initial value $\mathbf{y}_0$ *obeying* $\|\mathbf{y}_0\| \leq x_0$ *there exists a solution* $y(t)$ *of (2.3) satisfying* $\|\mathbf{y}(t)\| \leq x(t)$ *for* $t \in [0, T]$.

2.3 A linear system of neutral equations

This subsection deals with a system of linear functional equations of the form

$$[\mathbf{y}(t) - C(t)\mathbf{y}(qt)]' = -A(t)\mathbf{y}(t) + B(t)\mathbf{y}(qt) + \mathbf{g}(t), \qquad t \geq t_0 \geq 0, \tag{2.5}$$

where A, B, C are matrices of continuous functions and $\mathbf{g}$ is a vector of continuous functions, all defined on $[t_0, \infty)$. The vector norm used in this subsection is the l_2 norm induced by the inner product $(\cdot, \cdot)$.

Lemma 8 *Suppose that the matrices* A, B *and* C *have the asymptotic expansions*

$$A(t) \to A_0, \;\; B(t) \to B_0, \;\; C(t) \to C_0 \qquad \text{as } t \to \infty,$$

and that there exists a constant α *such that*

$$\rho(C_0) < q^{-\alpha}, \;\; \lambda_1 > q^{2\alpha-1}|\lambda_2|, \;\; \|B_0 + A_0^T C_0\| < q^{1/2-\alpha}(\lambda_1 + q^{2\alpha-1}\lambda_2),$$

and

$$(t+1)^{-\alpha}\mathbf{g}(t) \in \mathrm{L}_2(t_0, \infty),$$

where λ_1 *and* λ_2 *are the smallest eigenvalues of* $(A_0 + A_0^T)/2$ *and* $(C_0^T B_0 + B_0^T C_0)/2$*, respectively. If* $\mathbf{y}(t) \in \mathrm{C}[qt_0, \infty)$ *such that* $\mathbf{y}(t) - C(t)\mathbf{y}(qt) \in C^1[qt_0, \infty)$ *and (2.5) is satisfied then* $\mathbf{y}(t) = \mathcal{O}(t^\alpha)$ *as* $t \to \infty$.

Proof. We consider the case $\alpha = 0$ only, since the case $\alpha \neq 0$ can be reduced to the case $\alpha = 0$ by letting $\mathbf{y}(t) = (t+1)^\alpha \mathbf{x}(t)$. Define the Liapunov functional

$$V(t) := \frac{1}{2}\|\mathbf{y}(t) - C(t)\mathbf{y}(qt)\|^2 + \beta \int_{qt}^{t} \|\mathbf{y}(\tau)\|^2 \,\mathrm{d}\tau + M \int_{t}^{\infty} \|\mathbf{g}(\tau)\|^2 \,\mathrm{d}\tau, \qquad t \geq t_0,$$

where $\beta = (\lambda_1 - q^{-1}\lambda_2)/2 > 0$ and M is a positive constant to be determined. Using (2.5) we obtain

$$\begin{aligned} V'(t) = &-(\mathbf{y}(t), (A(t) - \beta I)\mathbf{y}(t)) + (\mathbf{y}(t), (B(t) + A(t)^T C(t))\mathbf{y}(qt)) \\ &-(\mathbf{y}(qt), (q\beta + C(t)^T B(t))\mathbf{y}(qt)) + (\mathbf{y}(t) - C(t)\mathbf{y}(qt), \mathbf{g}(t)) \\ &-M\|\mathbf{g}(t)\|^2, \qquad t \geq t_0. \end{aligned} \tag{2.6}$$

Our assumption implies that there exist constants $t_1 \geq t_0$ and $\lambda > 0$ such that

$$\begin{aligned} &(\mathbf{x}_1, (A(t) - \beta I)\mathbf{x}_1) - (\mathbf{x}_1, (B(t) + A(t)^T C(t))\mathbf{x}_2) + (\mathbf{x}_2, (q\beta + C(t)^T B(t))\mathbf{x}_2) \\ &\geq \lambda\left(\|\mathbf{x}_1\|^2 + \|\mathbf{x}_2\|^2\right), \qquad \forall \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^N,\ t \geq t_1. \end{aligned}$$

Using Young's inequality and Cauchy-Schwarz inequality we obtain

$$|(\mathbf{y}(t) - C(t)\mathbf{y}(qt), \mathbf{g}(t))| \leq \lambda(\|\mathbf{y}(t)\|^2 + \|\mathbf{y}(qt)\|^2) + \frac{1 + \|C(t)\|}{4\lambda}\|\mathbf{g}(t)\|^2.$$

By choosing $M = \frac{1}{4\lambda}\left(1 + \sup_{t \geq t_1} |C(t)\|\right)$, we obtain from (2.6) the inequality $V'(t) \leq 0$ for $t \geq t_1$. Therefore, $V(t) \leq V(t_1)$ for $t \geq t_1$. Hence

$$\|\mathbf{y}(t) - C(t)\mathbf{y}(qt)\| \leq 2V(t_1), \qquad t \geq t_1.$$

Applying Lemma 4 to the above inequality yields $\mathbf{y}(t) = \mathcal{O}(1)$ as $t \to \infty$. □

Variations of the preceding lemma can be obtained by using different vector norms, for instance, $\|\mathbf{x}\| = (\mathbf{x}^T D\mathbf{x})^{1/2}$ for some positive definite matrix D.

2.4 Dirichlet series

The Dirichlet series we refer to in this paper are sums of exponential functions. They are useful in the study of linear functional differential equations, especially those with proportional delays. It is found in [11, 9, 5], and other papers, that solutions of

$$y'(t) = ay(t) + by(qt), \qquad t \geq 0,$$

where a, b, q are constant and $q \in (0,1) \cup (1,\infty)$, can be expanded into Dirichlet series of the form $\sum_{n=0}^{\infty} d_n e^{aq^n t}$, where d_n, $n = 0, 1, \ldots$, are constants.

Closely related to the subject of this paper is the generalized pantograph equation

$$\mathbf{y}'(t) = A\mathbf{y}(t) + B\mathbf{y}(qt) + C\mathbf{y}'(qt), \quad t \geq 0, \qquad \mathbf{y}(0) = \mathbf{y}_0, \tag{2.7}$$

where A, B, C are $N \times N$ matrices. Iserles [16] found that its solution can be expanded into Dirichlet series of the form

$$\sum_{n=0}^{\infty} D_n e^{Aq^n t} V\mathbf{y}_0, \tag{2.8}$$

where V is a nonsingular $N \times N$ matrix, $D_0 = I$ and $D_1, D_2 \ldots$ are determined by the recurrence relation

$$AD_n - q^n D_n A = -BD_{n-1} - q^{n-1} CD_{n-1} A, \qquad n = 1, 2, \ldots,$$

subject to the conditions that $\rho(C) < 1$, $\rho(A^{-1}B) < 1$ and

$$\sigma(A) \cap \sigma(q^n A) = \emptyset, \qquad n \in \mathbb{N}. \tag{2.9}$$

The assumption (2.9) can be replaced by the following conditions (see [19])

$$AB = BA, \qquad AC = CA. \tag{2.10}$$

In the scalar case

$$y'(t) = ay(t) + by(qt) + cy'(qt), \quad t \geq 0, \qquad y(0) = y_0, \tag{2.11}$$

one can use the q-binomial theorem [10] to prove that the solution $y(t)$ of (2.11) is of the form [19]

$$y(t) = \frac{y_0(-b/a;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(-ac/b;q)_n}{(q;q)_n} \left(-\frac{b}{a}\right)^n \mathrm{e}^{atq^n} \tag{2.12}$$

subject to the conditions $a \neq 0$, $0 < |b| < |a|$ and $|c| < 1$, where

$$(a;q)_n = \begin{cases} 1, & n = 0 \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & n = 1, 2, \ldots, \infty. \end{cases}$$

It is shown in [16, 19] that the zero solution of (2.7) is asymptotically stable (in the sense that every other solutions tends to zero as t tends to infinity) if $\alpha[A] < 0$, $\rho(A^{-1}B) < 1$ and either (2.9) or (2.10) holds, but unstable if $\alpha[A] > 0$. Similar results and sometimes more general ones can be found in [22], without the assumption (2.9) or (2.10). For instance, if $\alpha[A] < 0$ then every solution $\mathbf{y}(t)$ of (2.7) satisfies the asymptotic estimate

$$\mathbf{y}(t) = \mathcal{O}\left(t^{\frac{\log \rho(A^{-1}B)}{\log q}} (\log t)^{\kappa - 1}\right) \qquad \text{as } t \to \infty,$$

where κ is the maximal geometric multiplicity of those eigenvalues of $A^{-1}B$ with the maximal real part $\alpha(A^{-1}B)$. It will be clear in the next two sections that these results are useful for understanding the behaviour of the solution of (1.5) when the initial value f_0 is close to one of the two stationary solutions in the case $\mu > 0$.

Iserles [17] discussed the Dirichlet series solution for certain nonlinear retarded type functional differential equations with proportional delay. His approach can be generalised to study system of neutral equations with proportional delay. However, it is very difficult to analyse the convergence of the corresponding Dirichlet series, due to the complexity of certain nonlinear recurrence relations involved.

3 The case $q \in (0, 1)$

This section deals with equation (1.1) and its initial value problem (1.5) for the case $q \in (0, 1)$. We note that $f(t) = -g(-t)$, where $g(t)$ is a solution of the equation

$$g'(t) + q^2 g'(qt) + g^2(t) - q^2 g^2(qt) = \mu$$

with the initial condition $g(0) = -f_0$. Therefore, it suffices to study (1.5) for $t \geq 0$.

Theorem 9 *If $q \in (0,1)$ then the initial value problem (1.5) has one and only one solution, say $f(t)$, in a neighbourhood of the origin. Furthermore,*

1. *the maximal interval of existence, say $(T_{\min}, T_{\max})$, is open and*

$$T_{\max}, -T_{\min} \geq \begin{cases} \dfrac{1-q^2}{(1+q^2)|f_0|} & \text{if } \mu = 0, f_0 \neq 0, \\ \left(\dfrac{1-q^2}{|\mu|}\right)^{1/2} \left[\dfrac{\pi}{2} - \arctan \dfrac{(1+q^2)|f_0|}{((1-q^2)|\mu|)^{1/2}}\right] & \text{if } \mu \neq 0; \end{cases}$$

2. *if $T_{\max} < \infty$ then $\limsup_{t \to T_{\max}-0} |f(t)| = \infty$; and*

3. *$f(t) \in \mathrm{C}^\infty(T_{\min}, T_{\max})$.*

Proof. Except the estimate of $T_{\max}$ and the last part, the theorem follows by applying Lemma 6 to (1.5). Applying Lemma 5 to equation (1.5) yields

$$f'(t) = \frac{\mu}{1+q^2} - \sum_{n=0}^{\infty} (-q^2)^n [f^2(q^n t) - q^2 f^2(q^{n+1}t)], \qquad f(0) = f_0. \tag{3.1}$$

It follows from Lemma 7 that the solution of (1.5) is bounded by the solution of the Riccati equation

$$x'(t) = \frac{|\mu|}{1+q^2} + \frac{1+q^2}{1-q^2} x^2(t), \qquad t \geq 0, \qquad x(0) = |f_0|. \tag{3.2}$$

A lower bound for $T_{\max}$ can be established by finding the first singularity point of x. Finally, it can be seen from (3.1) that $f(t) \in \mathrm{C}^\infty[0, T_{\max})$. □

Theorem 10 *Suppose that $q \in (0,1)$ and $f(t) \in \mathrm{C}^1(T_{\min}, T_{\max})$ is the solution of the initial value problem (1.5).*

1. *If $\mu > 0$ and*

$$|f_0 - f^*| < 2f^* \max\left\{\frac{(1-3q^2)(1+q^2)}{1+q^4}, \frac{(1-q^{1/2})(1-q^{3/2})}{(1+q)^2}\right\} \tag{3.3}$$

   then $T_{\max} = \infty$ and $\lim_{t \to \infty} f(t) = f^*$.

2. *If $\mu > 0$, $q < 2^{-1/2}$ and $f_0 < -f^*$ then $T_{\max} < \infty$, $f'(t) < 0$ for every $t \in [0, T_{\max})$ and $\lim_{t \to T_{\max}-0} f(t) = -\infty$.*

3. *If $\mu = 0$, $q < 2^{-1/2}$ and $f_0 < 0$ then $T_{\max} < \infty$, $f'(t) < 0$ for every $t \in [0, T_{\max})$ and $\lim_{t \to T_{\max}-0} f(t) = -\infty$.*

4. *If $\mu < 0$, $q < 2^{-1/2}$ and $f_0 \leq \frac{1}{q}(|\mu|/2)^{1/2}$ then $T_{\max} < \infty$, $f'(t) < 0$ for every $t \in [0, T_{\max})$ and $\lim_{t \to T_{\max}-0} f(t) = -\infty$.*

Before proving Theorem 10 (1,2), we use Dirichlet series to study equations that are linearisation of (1.1) around its stationary solutions f^* and $-f^*$. Let $y(t) = f(t) - f^*$ and $z(t) = f(t) + f^*$. Then $y(t)$ and $z(t)$ satisfy, respectively, the equations

$$y'(t) + q^2 y'(qt) = -ay(t) + aq^2 y(qt) - y^2(t) + q^2 y^2(qt), \qquad t \geq 0 \tag{3.4}$$

and

$$z'(t) + q^2 z'(qt) = az(t) - aq^2 z(qt) - z^2(t) + q^2 z^2(qt), \qquad t \geq 0, \tag{3.5}$$

where $a = 2f^*$. The linearised equation of (3.4) is

$$u'(t) = -au(t) + aq^2 u(qt) - q^2 u'(qt), \qquad t \geq 0. \tag{3.6}$$

It follows from (2.12) that every solution of (3.6) can be expanded into the Dirichlet series

$$u(t) = \frac{u(0)(q^2;q)_\infty}{(-q^2;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1;q)_n}{(q;q)_n} q^{2n} \mathrm{e}^{-atq^n}.$$

Furthermore, $u(t)$ has the asymptotic estimate

$$u(t) = \mathcal{O}\big(t^{-2}\big) \qquad \text{as } t \to \infty.$$

This suggests that the stationary solution f^* of (1.1) may be asymptotically stable in the sense that solutions of (1.5) tend to f^* as $t \to \infty$ if the initial value f_0 is sufficiently close to f^*. The linearised equations of (3.5) is

$$v'(t) = av(t) - aq^2 v(qt) - q^2 v'(qt), \qquad t \geq 0. \tag{3.7}$$

Every solution of (3.7) can be expanded into the Dirichlet series

$$v(t) = \frac{v(0)(q^2;q)_\infty}{(-q^2;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1;q)_n}{(q;q)_n} q^{2n} \mathrm{e}^{atq^n}.$$

The preceding expansion implies that

$$\lim_{t\to\infty} e^{-at} v(t) = v(0) \prod_{k=0}^{\infty} \frac{(q^2;q)_\infty}{(-q^2;q)_\infty},$$

which suggests that the stationary solution $-f^*$ of (1.1) may be unstable.

Proof of Theorem 10 (1). Let $f(t) \in \mathrm{C}^1(T_{\min}, T_{\max})$ be the solution of (1.5). Then the function $y(t) = f(t) - f^*$ satisfies equation (3.4). In the sequel, we exclude the trivial case $f_0 = f^*$.

First, we consider the case

$$|f_0 - f^*| < \frac{2(1-3q^2)(1+q^2)f^*}{1+q^4}. \tag{3.8}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t} y^2(t)|_{t=0} = -\frac{2(1-q^2)}{1+q^2}(f_0 - f^*)^2 (f_0 + f^*) < 0,$$

there exists a real number $t_0 \in (0, T_{\max})$ such that the inequality

$$|y(t)| < |y(0)| \tag{3.9}$$

holds for every $t \in (0, t_0)$.

We claim that $T_{\max} = \infty$ and that (3.9) holds for every $t > 0$. Otherwise there exists a real number $t_1 > t_0$ such that $|y(t)| < |y(0)|$ holds for every $t \in (0, t_1)$ and $|y(t_1)| = |y(0)|$, since $T_{\max} < \infty$ implies that $\limsup_{t \to T_{\max}-0} |f(t)| = \infty$. Applying Lemma 5 to (3.4) yields the equation

$$\begin{aligned} y'(t) = -ay(t) + 2aq^2 \sum_{n=0}^{\infty} (-q^2)^n y(q^{n+1}t) - \left[y^2(t) + 2q^4 \sum_{n=0}^{\infty} q^{4n} y^2(q^{2n+2}t) \right] \\ + 2q^2 \sum_{n=0}^{\infty} q^{4n} y^2(q^{2n+1}t), \qquad t > 0. \end{aligned} \tag{3.10}$$

Define a Liapunov function of the form

$$U(t) = \frac{1}{2} y^2(t).$$

Noting that the last two terms in the right hand side of (3.10) have different sign, we obtain

$$U'(t_1) \le - \left[\frac{(1-3q^2)a}{1-q^2} - \frac{(1+q^4)|y(0)|}{1-q^4} \right] |y(0)|^2 < 0$$

which contradicts our assumption that $y(t_1) = |y(0)| > |y(t)|$ for every $t \in (0, t_1)$. Hence, $T_{\max} = \infty$ and (3.9) holds for every $t > 0$.

Let us prove $\lim_{t\to\infty} y(t) = 0$ by contradiction. Suppose $M_0 := \limsup_{t\to\infty} |y(t)| > 0$. Then there exists a sequence $\{t_k\}_{n=0}^{\infty}$ such that $\lim_{k\to\infty} t_k = \infty$ and

$$\lim_{t\to\infty} |y(t_k)| = M_0.$$

Since $M_0 \le |y(0)|$, the assumption on f_0 implies that there exists a real number $\epsilon > 0$ such that

$$M_1 := a(M_0 - \epsilon)^2 - \frac{2aq^2(M_0^2 + \epsilon)}{1-q^2} - \frac{(1+q^4)M_0^3 + \epsilon}{(1+q^2)(1-q^2)} > 0.$$

Since $\sup_{t\ge 0} |y'(t)| < \infty$, there exists $k_0 \ge 0$ and $\delta > 0$ such that

$$|y(t)| \ge M_0 - \epsilon, \qquad t \in (t_k - \delta, t_k + \delta), \quad k = k_0, k_0 + 1, \ldots.$$

It is also true that there exists $T_0 > 0$ such that

$$\left| y(t) \sum_{n=0}^{\infty} (-q^2)^n y(q^{n+1}t) \right| \le \frac{M_0^2 + \epsilon}{1-q^2}$$

$$\left| y(t) \left[y^2(t) + 2q^2 \sum_{n=0}^{\infty} q^{4n} y^2(q^{2n+2}t) - 2q^2 \sum_{n=0}^{\infty} q^{4n} y^2(q^{2n+1}t) \right] \right| \le \frac{(1+q^4)M_0^3 + \epsilon}{1-q^4}$$

for $t \geq T_0$. Let $k_1 \geq 0$ such that $t_k \geq T_0 + \delta$ for $k \geq k_1$. It follows from (3.10) that

$$U'(t) \leq -M_1, \qquad t \in (t_k - \delta, t_k + \delta), \quad k = k_1, k_1 + 1, \ldots.$$

Therefore

$$U(t_k - \delta) \geq U(t_k) + M_1\delta, \qquad k = k_1, k_1 + 1, \ldots.$$

By letting $k \to \infty$ in the preceding inequality we find that it contradicts the definition of M_0. Therefore, $\lim_{t\to\infty} y(t) = 0$.

Next, we consider the case

$$|f_0 - f^*| < \frac{2(1 - q^{1/2})(1 - q^{3/2})f^*}{(1 + q)^2}. \tag{3.11}$$

Let $x(t) = y(t) + qy(qt)$. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}x^2(t)|_{t=0} = -2(1 + q)(1 - q^2)(f_0 - f^*)^2(f_0 + f^*) < 0,$$

there exists a real number $t_0 \in (0, T_{\max})$ such that

$$|x(t)| < |x(0)| \tag{3.12}$$

holds for every $t \in (0, t_0)$.

We claim that $T_{\max} = \infty$ and that (3.12) holds for every $t > 0$. Otherwise, there exists a real number $t_1 \in [t_0, T_{\max})$ such that (3.12) holds for every $t \in (0, t_1)$ and $|x(t_1)| = |x(0)|$. Define a Liapunov functional of the form

$$V(t) = \frac{1}{2}x^2(t) + a\beta_0 \int_{qt}^{t} y^2(s)\,\mathrm{d}s,$$

where $\beta_0 = \frac{1}{2}(1 + q)q^{1/2}$ such that

$$h(\beta_0) = \max_{\beta>0} h(\beta) = \frac{(1 - q^{1/2})(1 - q^{3/2})}{1 - q}$$

for the function

$$h(\beta) = 1 - \beta - \frac{q(2\beta - 1 - q)^2}{4(1 - q)\beta}, \qquad \beta > 0.$$

Using (3.4), we obtain

$$V'(t) = -a\left(h(\beta_0)x^2(t) + \left\{\frac{2q\beta_0 - q - q^2}{2\left[q(1 - q)\beta_0\right]^{1/2}}x(t) + \left[q(1 - q)\beta_0\right]^{1/2} y(qt)\right\}^2\right)$$
$$-[y(t) - qy(qt)]x^2(t), \qquad t \in [0, T_{\max}).$$

Suppose that $|y(t)|$, $0 \leq t \leq t_1$, attains its maximum at $t_2 \in [0, t_1]$. If follows from (3.12) that

$$|y(t_2)| \leq |x(t_2)| + q|y(qt_2)| \leq |x(0)| + q|y(t_2)|.$$

Therefore

$$\max_{0\le t\le t_1} |y(t)| \le \frac{1+q}{1-q}|f_0 - f^*|,$$

which implies that

$$|y(t) - qy(qt)| \le \frac{(1+q)^2}{1-q}|f_0 - f^*|, \qquad t \in [0, t_1].$$

Therefore, the inequality

$$V'(t) \le -a\left(\gamma x^2(t) + \left\{\frac{2q\beta_0 - q - q^2}{2[q(1-q)\beta]^{1/2}}x(t) + (q(1-q)\beta)^{1/2}y(qt)\right\}^2\right) \tag{3.13}$$

holds for $t \in [0, t_1]$, where

$$\gamma = h(\beta_0) - \frac{(1+q)^2}{a(1-q)}|f_0 - f^*| > 0.$$

Therefore, $V(t)$ is monotonically decreasing in $[0, t_1]$. This, together with the fact that $V'(0) < 0$, implies $V(t_1) < V(0)$, which contradicts the assumption that $|x(t_1)| = |x(0)|$. Hence, $T_{\max} = \infty$ and the inequality (3.12) holds for every $t > 0$.

It follows from the equation

$$y(t) + qy(qt) = x(t), \qquad t \ge 0 \tag{3.14}$$

that

$$\sup_{t\ge 0} |y(t)| \le \frac{|x(0)|}{1-q}.$$

Using the preceding estimate we obtain from (3.4) the estimate

$$\sup_{t\ge 0} |x'(t)| \le \frac{a(1+q^2)|x(0)|}{1-q} + \frac{(1+q^2)x^2(0)}{(1-q)^2}.$$

Let us prove $\lim_{t\to\infty} x(t) = 0$ by contradiction. Suppose $M_0 = \limsup_{t\to\infty} |x(t)| > 0$. Then there exists a sequence $\{t_n\}_{n=0}^{\infty}$ such that

$$t_{n+1} - t_n \ge \frac{M_0}{2M_2}, \quad |x(t_n)| \ge \frac{M_0}{2}, \qquad n = 0, 1, \ldots.$$

where $M_2 = \max\left\{1, \sup_{t\ge 0} |x'(t)|\right\}$. Evidently, there is no intersection between any two of the intervals of I_n, $n = 0, 1, \ldots$, where

$$I_n = \left(t_n - \frac{M_0}{4M_2}, t_n + \frac{M_0}{4M_2}\right), \qquad n = 0, 1, \ldots.$$

Since

$$|x(t)| \ge |x(t_n)| - \frac{M_0}{4M_2} \sup_{\tau\in I_n} |x'(\tau)| \ge \frac{M_0}{4}, \qquad t \in I_n, \quad n = 0, 1, \ldots,$$

and the inequality (3.13) holds for every $t \geq 0$, we deduce that

$$V(t_n + M_0/4M_2) - V(t_n - M_0/4M_2) \leq -\frac{a\gamma M_0^3}{32M_2}, \qquad n = 0, 1, \ldots.$$

It follows from the monotonicity of $V(t)$ and the preceding inequality that for $n \geq 0$

$$V(t_n + M_0/4M_2) - V(t_0 - M_0/4M_2) \leq -\frac{a\gamma M_0^3 n}{32M_2}.$$

By letting $n \to \infty$ in the above inequality we find that it contradicts the nonnegativity of the function V. Therefore, $\lim_{t\to\infty} x(t) = 0$. Applying Lemma 4 to the equation (3.14) yields $\lim_{t\to\infty} y(t) = 0$. Hence, $\lim_{t\to\infty} f(t) = f^*$. □

Theorem 10 (1) implies that the stationary solution f^* of (1.1) is asymptotically stable in the sense that there exists a constant $\delta > 0$ such that the solution $f(t)$ of (1.5) satisfies $f(t) \to f^*$ as $t \to \infty$ whenever $|f_0 - f^*| < \delta$.

We remark that variations of the Razumikhin method and the method of Liapunov functional were used in the proof of Theorem 10 (1). The rest of Theorem 10 is proved below by using a comparison method.

Proof of Theorem 10 (2). Let $z(t) = f(t) + f^*$. Then z satisfies equation (3.5). Since

$$z'(0) = \frac{1-q^2}{1+q^2}(az(0) - z^2(0)) < 0,$$

there exists a real number $t_0 \in (0, T_{\max})$ such that $z'(t) < 0$ for every $t \in [0, t_0]$.

We claim that $z'(t) < 0$ for every $t \in [0, T_{\max})$. Otherwise there exists a real number $t_1 \in (t_0, T_{\max})$ such that $z'(t) < 0$ for every $t \in [0, t_1)$ and $z'(t_1) = 0$. Applying Lemma 5 to (3.5) yields

$$\begin{aligned} z'(t) = a\left\{z(t) - 2q^2 z(qt) + 2\sum_{n=1}^{\infty} q^{4n}\left[z(q^{2n}t) - q^2 z(q^{2n+1}t)\right]\right\} - z^2(t) + 2q^2 z^2(qt) \\ -2\sum_{n=1}^{\infty} q^{4n}\left[z^2(q^{2n}t) - q^2 z^2(q^{2n+1}t)\right], \qquad t \in (0, T_{\max}). \end{aligned} \tag{3.15}$$

Since $q < 2^{-1/2}$, $z(t) < 0$ and $z'(t) < 0$ for $t \in (0, t_1)$, we can see from the above equation that $z'(t_1) < 0$, which leads to a contradiction.

Hence, $z'(t) < 0$ for every $t \in [0, T_{\max})$. It follows from (3.15) that

$$z'(t) < -(1-2q^2)z^2(t), \qquad t \in (0, T_{\max})$$

which implies that

$$z(t) < \frac{1}{(1-2q^2)t + 1/z(0)^{-1}}, \qquad t \in (0, T_{\max}).$$

Therefore, $T_{\max} \leq -((1-2q^2)(f^* + f_0))^{-1}$, $f'(t) < 0$ for every $t \in [0, T_{\max})$ and $\lim_{t\to T_{\max}-} f(t) = -\infty$. □

Proof of Theorem 10 (3,4). Applying Lemma 5 to equation (1.1) yields

$$f'(t) = \frac{\mu}{1+q^2} - f^2(t) + 2\sum_{n=0}^{\infty} q^{4n+2}\left[f^2(q^{2n+1}t) - q^2 f^2(q^{2n+2}t)\right]. \tag{3.16}$$

When $\mu \le 0$ and $f_0 < 0$, the proof is essentially the same as that of Theorem 10 (2). We consider in the sequel the case where $\mu < 0$ and $0 \le f_0 \le \frac{1}{q}(|\mu|/2)^{1/2}$.

We claim that $f'(t) < 0$ for every $t \in [0, T_{\max})$. Otherwise there exists a real number $t_1 \in (0, T_{\max})$ such that $f'(t) < 0$ for every $t \in [0, t_1)$ and $f'(t_1) = 0$. If $f(qt_1) \ge 0$ then

$$0 < f(q^{2n+1}t_1) < f(q^{2n+2}t_1) < f(0), \qquad n = 0, 1, \ldots$$

implies that

$$2\sum_{n=0}^{\infty} q^{4n+2}\left[f^2(q^{2n+1}t_1) - q^2 f^2(q^{2n+2}t_1)\right] < \frac{2q^2 f^2(0)}{1+q^2} \le \frac{|\mu|}{1+q^2},$$

it follows from (3.16) that $f'(t_1) < -f^2(t_1)$, which contradicts the assumption that $f'(t_1) = 0$. If $f(qt_1) < 0$ then there exists an integer $m \ge 1$ such that $f(q^{2n+1}t_1) > 0$ for all $n \ge m$ and $f(q^{2n+1}t_1) \le 0$ for all $0 \le n < m$. It follows from (3.16) that

$$\begin{aligned} f'(t) = {}& \frac{\mu}{1+q^2} - f^2(t) + 2q^2 f^2(qt) - 2\sum_{n=1}^{m-1} q^{4n}\left[f^2(q^{2n}t) - q^2 f^2(q^{2n+1}t)\right] \\ & - 2q^{4m} f^2(q^{2m}t) + 2\sum_{n=m}^{\infty} q^{4n+2}\left[f^2(q^{2n+1}t) - q^2 f^2(q^{2n+2}t)\right]. \end{aligned} \tag{3.17}$$

It can be shown from the preceding equation that $f'(t_1) < -(1-2q^2)f^2(t_1)$, which contradicts the assumption that $f'(t_1) = 0$. Therefore, $f'(t) < 0$ for every $t \in [0, T_{\max})$.

We claim that there exists $t_2 \in (0, T_{\max})$ such that $f(t_2) < 0$. Otherwise, $T_{\max} = \infty$ since $T_{\max} < \infty$ implies that $\lim_{t\to T_{\max}-0} |f(t)| = \infty$. The monotonicity and the boundedness of $f(t)$ imply that $\lim_{t\to\infty} f(t)$ exists and is finite. Since

$$\lim_{t\to\infty} f'(t) = \frac{1}{1+q^2}\left[\mu - (1-q^2)\left(\lim_{t\to\infty} f(t)\right)^2\right] < 0,$$

it is not difficult to show that $f(t) < 0$ when t is large enough. Hence, there exists $t_2 \in (0, T_{\max})$ such that $f(t_2) < 0$.

Similar to the argument used in the second paragraph of this proof we can derived the inequality

$$f'(t) < -(1-2q^2)f^2(t), \qquad t \in (t_2, T_{\max}).$$

Hence,

$$f(t) < \frac{1}{(1-2q^2)(t-t_2) + 1/f(t_2)}, \qquad t \in (t_2, T_{\max})$$

which implies that $T_{\max} \le t_2 - ((1-2q^2)f(t_2))^{-1} < \infty$, $f'(t) < 0$ for every $t \in [0, T_{\max})$ and $\lim_{t\to T_{\max}-} f(t) = -\infty$. □

Before continuing our discussion on the Shabat equation, we introduce two lemmas. The first one is trivial to prove and the second follows easily from the first.

Lemma 11 *Given a function $y(t) \in \mathrm{C}^1(0,\infty)$, if*

$$\lim_{t\to\infty}[y(t) - f_0(t)] = \lim_{t\to\infty}[y'(t) - f_1(t)] = 0$$

for some continuous periodic functions $f_0(t)$ and $f_1(t)$, both of the same period, then $f_0(t) \in \mathrm{C}^1(0,\infty)$ and $f_1(t) \equiv f_0'(t)$.

Lemma 12 *Suppose that $y(t) \in \mathrm{C}^1(0,\infty)$ and that there exist a constant α such that*

$$\lim_{t\to\infty}[t^\alpha y(t) - f_0(\log t)] = \lim_{t\to\infty}\left[t^{\alpha+1}y'(t) - f_1(\log t)\right] = 0$$

for some continuous periodic functions $f_0(t)$ and $f_1(t)$, both of the same period. Then $f_0(t) \in \mathrm{C}^1(0,\infty)$ and $f_1(t) \equiv f_0'(t) - \alpha f_0(t)$.

Theorem 13 *If $q \in (0,1)$, $\mu > 0$ and $f(t) \in \mathrm{C}^1(T_{\min},\infty)$ is a solution of (1.1) that satisfies $f(t) \to f^*$ as $t \to \infty$ then the following asymptotic expansions hold as $t \to \infty$:*

$$f(t) = f^* + t^{-2}h_0(\log t) + t^{-3}h_1(\log t) + \cdots, \tag{3.18}$$

$$f^{(n)}(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}[t^{-2}h_0(\log t)] + \frac{\mathrm{d}^n}{\mathrm{d}t^n}[t^{-3}h_1(\log t)] + \cdots, \quad n = 1,2,\ldots, \tag{3.19}$$

where $h_m \in \mathrm{C}^\infty$, $m = 0,1,\ldots$, are periodic functions of period $-\log q$ that obey the recurrence relation

$$h_m(t) = \frac{1+q^m}{2f^*(1-q^m)}\left[\frac{\mathrm{d}}{\mathrm{d}t} - (m+1)\right]h_{m-1}(t) - \frac{1}{2f^*}\sum_{k=1}^{m-1} h_{k-1}(t)h_{m-k-1}(t) \tag{3.20}$$

for $m = 1,2,\ldots$.

Proof. According to Theorem 9 (3), $f(t) \in \mathrm{C}^\infty[0,\infty)$. Denote $w_n(t) = y^{(n)}(t)$, $n \in \mathbb{Z}^+$, where $y(t) = f(t) - f^*$ satisfies equation (3.4). We separate our proof into ten steps.

Step 1. Equation (3.4) can be rewritten as

$$[y(t) + qy(qt)]' = -[a + a_0(t)]y(t) + [q^2a + a_1(t)]y(qt), \qquad t \ge 0, \tag{3.21}$$

where $a_0(t) = y(t)$ and $b_0(t) = q^2y(qt)$. Applying Lemma 8 to (3.21) yields $y(t) = \mathcal{O}(t^{\alpha_0})$ as $t \to \infty$ for any $\alpha_0 > -1/2$.

Step 2. Differentiating both sides of (3.4) gives an equation of $w_1(t)$ of the form

$$[w_1(t) + q^2w_1(qt)]' = -[a + a_1(t)]w_1(t) + [q^3a + b_1(t)]w_1(qt), \qquad t \ge 0, \tag{3.22}$$

where $a_1(t) = 2y(t)$ and $b_1(t) = 2q^3y(qt)$. Applying Lemma 8 to (3.22) yields $w_1(t) = \mathcal{O}(t^{\alpha_1})$ as $t \to \infty$ for any $\alpha_1 > -3/2$.

Step 3. Differentiating both sides of (3.22) yields an equation of $w_2(t)$ of the form

$$[w_2(t) + q^3w_2(qt)]' = -[a + a_2(t)]w_2(t) + [q^4a + b_2(t)]w_2(qt) + h_2(t), \qquad t \ge 0, \tag{3.23}$$

where $a_2(t) = 2y(t)$, $b_2(t) = 2q^4y(qt)$ and $h_2(t) = -2w_1(t)^2 + 2q^4w_1(qt)^2$. Noting that $h_2(t) = \mathcal{O}\big(t^{2\alpha_1}\big)$ as $t \to \infty$ for any $\alpha_1 > -3/2$, we can obtain by applying Lemma 8 to (3.23) that $w_2(t) = \mathcal{O}(t^{\alpha_2})$ as $t \to \infty$ for any $\alpha_2 > -5/2$.

Step 4. Equation (3.22) can be rewritten as an equation of $w_1(t)$:

$$w_1(t) - \frac{q^3 a + 2q^3 y(qt)}{a + 2y(t)} w_1(qt) = \frac{w_2(t) + q^3 w_2(qt)}{a + 2y(t)}, \qquad t \geq t_0,$$

where $t_0 > 0$ such that $a + 2y(t) > 0$ for every $t \geq t_0$. Applying Lemma 4 to the preceding equation yields $w_1(t) = \mathcal{O}(t^{\alpha_1})$ as $t \to \infty$ for any $\alpha_1 > -5/2$.

Step 5. Equation (3.21) can be rewritten as

$$y(t) - \frac{q^2 a + b_0(t)}{a + a_0(t)} y(qt) = \frac{w_1(t) + q^2 w_1(qt)}{a + a_0(t)}, \qquad t \geq t_1,$$

where $a_0(t) = y(t)$, $b_0(t) = q^2 y(qt)$, and $t_1 > 0$ such that $a + y(t) > 0$ for every $t \geq t_1$. Applying Lemma 4 to the above equation yields $y(t) = \mathcal{O}(t^{\alpha_0})$ as $t \to \infty$ for any $\alpha_1 > -2$.

Step 6. Rewrite equation (3.4) as

$$y(t) - q^2 y(qt) = h(t), \qquad t \geq 0, \tag{3.24}$$

where

$$h(t) = \frac{1}{a}\left[-y^2(t) + q^2 y(qt)^2 - w_1(t) - q^2 w_1(qt)\right] = \mathcal{O}(t^{\alpha}) \qquad \text{as } t \to \infty$$

for any $\alpha > -5/2$. Applying Lemma 3 to (3.24) yields $y(t) = \mathcal{O}\big(t^{-2}\big)$ as $t \to \infty$.

Step 7. Using induction method and repeating the method used in the above 6 steps, we can prove that $w_n(t) = \mathcal{O}\big(t^{-(n+2)}\big)$ as $t \to \infty$ for every $n \in \mathbb{N}$.

Step 8. For any $n \in \mathbb{Z}^+$, differentiating both sides of (3.4) n times yields the equation

$$w_n(t) - q^{n+2} w_n(qt) = h(t), \qquad t \geq 0, \tag{3.25}$$

where

$$\begin{aligned} h(t) = &-\frac{1}{a}\left[w_{n+1}(t) + q^{n+2} w_{n+1}(qt)\right] \\ &-\frac{1}{a}\sum_{m=0}^{n} C_n^m \left[w_m(t) w_{n-m}(t) - q^{n+2} w_m(qt) w_{n-m}(qt)\right]. \end{aligned}$$

Using the estimates proved in the previous two steps, we obtain $h(t) = \mathcal{O}\big(t^{-(n+3)}\big)$ as $t \to \infty$. Applying Lemma 3 to (3.25) yields the asymptotic expansion

$$w_n(t) = t^{-(n+2)} H_{n,0}(\log t) + \mathcal{O}\Big(t^{-(n+3)}\Big), \qquad \text{as } t \to \infty,$$

where $H_{n,0}(t)$ is a continuous periodic function of period $\log q$. It follows from Lemma 12 that $H_{n,0} \in \mathrm{C}^1$ and

$$H'_{n,0}(t) = H_{n+1,0}(t) + (n+2) H_{n,0}(t), \qquad n = 0, 1, \ldots,$$

or equivalently

$$t^{-(n+3)} H_{n+1,0}(\log t) = \frac{\mathrm{d}}{\mathrm{d}t}[t^{-(n+2)} H_{n,0}(\log t)], \qquad n = 0, 1, \ldots.$$

By induction, we see that $H_{n,0} \in \mathrm{C}^\infty$, $n \in \mathbb{Z}^+$, and

$$t^{-(n+2)} H_{n,0}(\log t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}[t^{-2} H_{0,0}(\log t)]$$

for $n \in \mathbb{N}$. Denote $h_0 = H_{0,0}$. We have the following asymptotic expansions

$$f(t) = f^* + t^{-2} h_0(\log t) + \mathcal{O}(t^{-3}),$$
$$f^{(n)}(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}[t^{-2} h_0(\log t)] + \mathcal{O}\Big(t^{-(n+3)}\Big), \qquad n = 1, 2, \ldots$$

Step 9. It can be shown from (3.25) by induction that there are periodic functions $h_m \in \mathrm{C}^\infty$, $m = 1, 2, \ldots$, all of period $\log q$, such that the following asymptotic expansions hold as $t \to \infty$:

$$f(t) = f^* + t^{-2} h_0(\log t) + t^{-3} h_1(\log t) + \ldots + t^{-(m+2)} h_m(\log t) + \mathcal{O}\Big(t^{-(m+3)}\Big),$$
$$f^{(n)}(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}[t^{-2} h_0(\log t)] + \frac{\mathrm{d}^n}{\mathrm{d}t^n}[t^{-3} h_1(\log t)] + \ldots + \frac{\mathrm{d}^n}{\mathrm{d}t^n}[t^{-(m+2)} h_m(\log t)]$$
$$+\mathcal{O}\Big(t^{-(n+m+3)}\Big), \qquad n = 1, 2, \ldots.$$

Therefore, (3.18) and (3.19) hold.

Step 10. Substituting the expansions of $f(t)$ and $f'(t)$ into (1.1) and then comparing the coefficients of $t^{-(m+2)}$, $m \in \mathbb{N}$, yields the recurrence relation (3.20). □

Recall the remark we made at the beginning of this section about the solution $f(t)$, $t \le 0$, of (1.5),it can be seen from the preceding two theorems the following results hold.

Theorem 14 *Suppose that $q \in (0,1)$ and $f(t) \in \mathrm{C}^1(T_{\min}, T_{\max})$ is the solution of the initial value problem (1.5).*

1. *If $\mu > 0$ and*

$$|f_0 + f^*| < 2f^* \max\left\{\frac{(1-3q^2)(1+q^2)}{1+q^4}, \frac{(1-q^{1/2})(1-q^{3/2})}{(1+q)^2}\right\}$$

then $T_{\min} = \infty$ and $\lim_{t\to-\infty} f(t) = -f^*$;

2. *If $\mu > 0$, $q < 2^{-1/2}$ and $f_0 > f^*$ then $T_{\min} > -\infty$, $f'(t) < 0$ for every $t \in (T_{\min}, 0]$ and $\lim_{t\to T_{\min}+0} f(t) = \infty$.*

3. *If $\mu = 0$, $q < 2^{-1/2}$ and $f_0 > 0$ then $T_{\min} > -\infty$, $f'(t) < 0$ for every $t \in (T_{\min}, 0]$ and $\lim_{t\to T_{\min}+0} f(t) = \infty$.*

4. *If $\mu < 0$, $q < 2^{-1/2}$ and $f_0 \ge -\frac{1}{q}(|\mu|/2)^{1/2}$ then $T_{\min} < \infty$, $f'(t) < 0$ for every $t \in (T_{\min}, 0]$ and $\lim_{t\to T_{\min}+0} f(t) = \infty$.*

Theorem 15 *If $q \in (0,1)$, $\mu > 0$ and $f(t) \in \mathrm{C}^1(-\infty, T_{\max})$ is a solution of (1.1) that satisfies $f(t) \to -f^*$ as $t \to -\infty$ then the following asymptotic expansions hold as $t \to \infty$:*

$$-f(-t) = f^* + t^{-2}h_0^*(\log t) + t^{-3}h_1^*(\log t) + \cdots,$$
$$(-1)^{n+1}f^{(n)}(-t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}[t^{-2}h_0^*(\log t)] + \frac{\mathrm{d}^n}{\mathrm{d}t^n}[t^{-3}h_1^*(\log t)] + \cdots, \qquad n = 1, 2, \ldots,$$

where $h_m^* \in \mathrm{C}^\infty$, $m = 0, 1, \ldots$, are periodic functions of period $-\log q$ that obey the recurrence relation

$$h_m^*(t) = \frac{1+q^m}{2f^*(1-q^m)}\left[\frac{\mathrm{d}}{\mathrm{d}t} - (m+1)\right]h_{m-1}^*(t) - \frac{1}{2f^*}\sum_{k=1}^{m-1} h_{k-1}^*(t)h_{m-k-1}^*(t)$$

for $m = 1, 2, \ldots$.

4 The case $q \in (-1, 0)$

This section deals with equation (1.1) and its initial value problem (1.5) for the case $q \in (-1, 0)$. Let $p = -q \in (0, 1)$, $g_1(t) = f(t)$ and $g_2(t) = -f(-t)$ for $t \geq 0$. It can be shown that the initial value problem (1.5) is equivalent to the system

$$\begin{cases} g_1'(t) + p^2 g_2'(pt) + g_1^2(t) - p^2 g_2^2(pt) = \mu, \\ g_2'(t) + p^2 g_1'(pt) + g_2^2(t) - p^2 g_1^2(pt) = \mu, \end{cases} \qquad t \geq 0 \tag{4.1}$$

with the initial condition

$$g_1(0) = f_0, \qquad g_2(0) = -f_0. \tag{4.2}$$

Applying Lemma 5 to (4.1) yields

$$g_1'(t) = \frac{\mu}{1+p^2} - g_1^2(t) + 2p^2 g_2^2(pt) - 2\sum_{n=1}^{\infty} p^{4n}[g_1^2(p^{2n}t) - p^2 g_2^2(p^{2n+1}t)], \tag{4.3}$$

$$g_2'(t) = \frac{\mu}{1+p^2} - g_2^2(t) + 2p^2 g_1^2(pt) - 2\sum_{n=1}^{\infty} p^{4n}[g_2^2(p^{2n}t) - p^2 g_1^2(p^{2n+1}t)]. \tag{4.4}$$

Theorem 16 *If $q \in (-1, 0)$ then the initial value problem (1.5) has one and only one continuously differentiable solution, $f(t)$, say, in a neighbourhood of the origin. Furthermore,*

1. *The maximal interval of existence, say $(T_{\min}, T_{\max})$, say, is open, where $T_{\max} > 0$ and $T_{\min} < 0$;*

2. *Either $T_{\max} < \infty$, $T_{\min} > -\infty$ or $T_{\max} = \infty$, $T_{\min} = -\infty$;*

3. *If $T_{\max} < \infty$ then $\limsup_{t \to T_{\max}-0} |f(t)| = \limsup_{t \to T_{\min}+0} |f(t)| = \infty$; and*

4. $f(t) \in \mathrm{C}^\infty(T_{\min}, T_{\max})$.

Proof. It follows from Lemma 6 that (4.1,4.2) has a unique continuous differentiable solution $(g_1(t), g_2(t))$ in the interval $[0,T)$, where $[0,T)$ is the maximal interval of existence in the sense that $T < \infty$ implies $\lim_{t\to T-0}(g_1^2(t) + g_2^2(t)) = \infty$. This implies that the initial value problem (1.5) has a unique solution

$$f(t) = \begin{cases} g_1(t), & 0 \le t < \infty \\ -g_2(-t), & -\infty < t \le 0 \end{cases}$$

in $C^1(-T,T)$. If $T < \infty$ then $\lim_{t\to T-0} g_1^2(t) = \infty$ and/or $\lim_{t\to T-0} g_2^2(t) = \infty$. Suppose that $\lim_{t\to T-0} g_1^2(t) = \lim_{t\to T-0} g_2^2(t) = \infty$. Then the maximal interval of existence is $(-T,T)$. Suppose that $\lim_{t\to T-0} g_1^2(t) = \infty$. We can use (4.4) to prove that there exists $T_1 \in (T, T/p)$ such that the solution $g_2(t)$ can be extended continuously to the interval $[0,T_1)$. In this case, the maximal interval of existence is $(-T_1,T)$, which is still an open interval. Suppose that $\lim_{t\to T-0} g_1^2(t) = \infty$. We can use (4.3) to prove that there exists $T_2 \in (T, T/p)$ such that the solution $g_1(t)$ can be extended continuously to the interval $[0,T_2)$. In this case, the maximal interval of existence is $(-T,T_2)$, which is an open interval as well. Finally, we can see from (4.3) and (4.4) that $g_1, g_2 \in \mathrm{C}^\infty[0,T_{\max})$. It can be shown by induction from (4.3) and (4.4) that $g_1^{(k)}(0) = (-1)^{k+1} g_2^{(k)}(0)$, $k \in \mathbb{Z}$. Hence, the solution $f(t)$ is in the function space $\mathrm{C}^\infty(T_{\min}, T_{\max})$. □

Theorem 17 *Suppose that $q \in (-1,0)$ and $f(t) \in \mathrm{C}^1(T_{\min}, T_{\max})$ is the solution of the initial value problem (1.5).*

1. *If $\mu > 0$, $q \in (-3^{-1/2}, 0)$ and f_0 satisfy the condition*

$$|f_0| < \frac{[2(1-3q^2)^2(1+q^2)^2 - (1+q^4)^2]^{1/2} f^*}{1+q^4}; \tag{4.5}$$

   then $T_{\max} = \infty$, $\lim_{t\to\pm\infty} f(t) = \pm f^*$.

2. *Suppose that $\mu > 0$ and $q > -2^{-1/2}$. If $f_0 < -f^*$ then $T_{\max} < \infty$, $f'(t) < 0$ for all $t \in [0, T_{\max})$ and $\lim_{t\to T_{\max}-0} f(t) = -\infty$. If $f_0 > f^*$ then $T_{\min} > -\infty$, $f'(t) < 0$ for all $t \in (T_{\min}, 0]$ and $\lim_{t\to T_{\min}+0} f(t) = \infty$.*

3. *Suppose $\mu = 0$ and $q > -2^{-1/2}$. If $f_0 < 0$ then $T_{\max} < \infty$, $f'(t) < 0$ for all $t \in [0, T_{\max})$ and $\lim_{t\to T_{\max}-0} f(t) = -\infty$. If $f_0 > 0$ then $T_{\min} > -\infty$, $f'(t) < 0$ for all $t \in (T_{\min}, 0]$ and $\lim_{t\to T_{\min}+0} f(t) = \infty$.*

4. *Suppose $\mu < 0$ and $q > -2^{-1/2}$. If $f_0 \le 0$ then $T_{\max} < \infty$, $f'(t) < 0$ for every $t \in [0, T_{\max})$ and $\lim_{t\to T_{\max}-0} f(t) = -\infty$. If $f_0 \ge 0$ then $T_{\min} > -\infty$, $f'(t) < 0$ for all $t \in (T_{\min}, 0]$ and $\lim_{t\to T_{\min}+0} f(t) = \infty$.*

Before proving Theorem 17 (1,2), we note that equation (1.1) has two stationary solutions, namely f^* and $-f^*$. This implies that system (4.1) has four stationary solutions, namely,

$$(f^*, f^*)^T,\ (f^*, -f^*)^T,\ (-f^*, f^*)^T \text{ and } (-f^*, -f^*)^T.$$

We can use Dirichlet series as in the previous section to see that the only possibly asymptotically stable stationary solution of (4.1) is $(f^*, f^*)^T$.

Proof of Theorem 17 (1). Let $y_i(t) = g_i(t) - f^*$, $i = 1, 2$. We obtain from (4.3) and (4.4) the following equations

$$\begin{aligned} y_1'(t) &= -a[y_1(t) - 2p^2 y_2(pt)] - 2a \sum_{n=1}^{\infty} p^{4n}[y_1(p^{2n}t) - p^2 y_2(p^{2n+1}t)] \\ &\quad -y_1^1(t) + 2p^2 y_2^2(pt) - 2\sum_{n=1}^{\infty} p^{4n}[y_1^2(p^{2n}t) - p^2 y_2^2(p^{2n+1}t)], \end{aligned} \tag{4.6}$$

$$\begin{aligned} y_2'(t) &= -a[y_2(t) - 2p^2 y_1(pt)] - 2a \sum_{n=1}^{\infty} p^{4n}[y_2(p^{2n}t) - p^2 y_1(p^{2n+1}t)] \\ &\quad -y_2^1(t) + 2p^2 y_1^2(pt) - 2\sum_{n=1}^{\infty} p^{4n}[y_2^2(p^{2n}t) - p^2 y_1^2(p^{2n+1}t)], \end{aligned} \tag{4.7}$$

where $a = 2f^*$. Define a Liapunov function of the form

$$U(t) = \frac{1}{2}[y_1^2(t) + y_2^2(t)].$$

Since

$$y_1'(0) = y_2'(0) = \frac{1-p^2}{1+p^2}[(f^*)^2 - f_0^2],$$

we obtain

$$U'(0) = -\frac{2f^*(1-p^2)}{1+p^2}[(f^*)^2 - f_0^2] < 0.$$

Hence, there exists a real number $t_0 \in (0, T_{\max})$ such that the inequality

$$|U(t)| < |U(0)| \tag{4.8}$$

holds for every $t \in (0, t_0)$. We claim that $T_{\max} = \infty$ and that (4.8) holds for every $t > 0$. Otherwise there exists a real number $t_1 > t_0$ such that $|U(t)| < |U(0)|$ holds for every $t \in (0, t_1)$ and $|U(t_1)| = |U(0)|$. Noting that the last two terms of (4.6) and (4.7) have different signs, we obtain the inequality

$$U'(t_1) \le -\left[\frac{(1-3p^2)a}{1-p^2} - \frac{2^{1/2}(1+p^4)}{1-p^4} U(0)^{1/2}\right] U(0) < 0$$

which leads to a contradiction. The rest of the proof is similar to the second part of the proof of Theorem 10(1). $\square$

Analogous to the second part of the proof of Theorem 10(1), we can use a Liapunov functional of the form

$$V(t) = \frac{1}{2}\left[x_1^2(t) + x_2^2(t)\right] + \beta \int_{qt}^{t} [y_1^2(s) + y_2^2(s)]\,\mathrm{d}s,$$

where

$$x_1(t) = y_1(t) + py_2(pt), \qquad x_2(t) = y_2(t) + py_1(pt),$$

and β is a positive constant. However, we are unable to improve Theorem 17 (1).

The transformation $g(t) = -f(-t)$ does not change equation (1.1). However, the initial condition becomes $g(0) = -f_0$. Therefore, we only need to consider the case $f_0 \leq 0$ in the proof of the rest of Theorem 17.

Proof of Theorem 17 (2). Let $z_1(t) = g_1(t) + f^*$, $z_2(t) = g_2(t) - f^*$. We obtain from (4.3), (4.4) the following equations

$$\begin{aligned} z_1'(t) &= a[z_1(t) + 2p^2 z_2(pt)] + 2a \sum_{n=1}^{\infty} p^{4n}[z_1(p^{2n}t) + p^2 z_2(p^{2n+1}t)] \\ &\quad - z_1^2(t) + 2p^2 z_2^2(pt) - 2\sum_{n=1}^{\infty} p^{4n}[z_1^2(p^{2n}t) - p^2 z_2^2(p^{2n+1}t)], \end{aligned} \tag{4.9}$$

$$\begin{aligned} z_2'(t) &= -a[z_2(t) + 2p^2 z_1(pt)] - 2a \sum_{n=1}^{\infty} p^{4n}[z_2(p^{2n}t) + p^2 z_1(p^{2n+1}t)] \\ &\quad - z_2^2(t) + 2p^2 z_1^2(pt) - 2\sum_{n=1}^{\infty} p^{4n}[z_2^2(p^{2n}t) - p^2 z_1^2(p^{2n+1}t)], \end{aligned} \tag{4.10}$$

where $a = 2f^*$. Since

$$z_1(0) = f_0 + f^* < 0, \qquad z_2(0) = -z_1(0)$$

and

$$z_1'(0) = z_2'(0) = \frac{\mu - (1-p^2)f_0^2}{1+p^2} < 0,$$

there exists $t_0 > 0$ such that

$$z_1(t) + |z_2(t)| < 0 \tag{4.11}$$

for every $t \in (0, t_0)$.

We claim that (4.11) holds for every $t \in (0, T_{\max})$. Otherwise, there exists $t_1 \in [t_0, T_{\max})$ such that (4.11) holds for every $t \in (0, t_1)$ and

$$z_1(t_1) + |z_2(t_1)| = 0. \tag{4.12}$$

By subtracting (4.9) from (4.10) we can see that

$$z_1'(t) - z_2'(t) < 0, \qquad t \in (0, t_1]. \tag{4.13}$$

Adding (4.9) to (4.10) yields

$$\begin{aligned} z_1'(t) + z_2'(t) &= a\{[z_1(t) - z_2(t)] - 2p^2[z_1(pt) - z_2(pt)]\} \\ &\quad + 2a\sum_{n=1}^{\infty} p^{4n}\{[z_1(p^{2n}t) - z_2(p^{2n}t)] - p^2[z_1(p^{2n+1}t) - z_2(p^{2n+1}t)]\} \\ &\quad - [z_1^2(t) + z_2^2(t)] + 2p^2[z_1^2(pt) + z_2^2(pt)] \\ &\quad - 2\sum_{n=1}^{\infty} p^{4n}\{[z_1^2(p^{2n}t) + z_2^2(p^{2n}t)] - p^2[z_1^2(p^{2n+1}t) + z_2^2(p^{2n+1}t)]\}, \end{aligned} \tag{4.14}$$

Let

$$W(t) = z_1^2(t) + z_2^2(t).$$

Since $W'(0) = 0$ and $W''(0) > 0$, there exists $t_2 \in (0, t_1)$ such that

$$W'(t) > 0 \tag{4.15}$$

for $t \in (0, t_2)$. We claim that (4.15) holds for every $t \in (0, t_1)$. Otherwise, there exists $t_3 \in (t_2, t_1)$ such that (4.15) holds for every $t \in (0, t_3)$ and $W'(t_3) = 0$. Noting that $p^2 < 1/2$, we can see from (4.14) that

$$z_1'(t) + z_2'(t) < 0, \qquad t \in (0, t_3].$$

Therefore, $W'(t_3) > 0$, which leads to a contradiction. Hence, (4.15) holds for every $t \in (0, t_1)$. Using (4.13) and (4.15) we derive from (4.14) the inequality

$$z_1'(t) + z_2'(t) < 0, \qquad t \in (0, t_1]. \tag{4.16}$$

Noting that

$$z_1'(t) < 0 \tag{4.17}$$

holds for $t \in (0, t_1)$, we have $z_1(t_1) < 0$. Using (4.13) and (4.16) we obtain

$$(z_1^2(t) - z_2^2(t))'|_{t=t_1} > 0,$$

which contradicts our assumptions that (4.11) holds for every $t \in (0, t_1)$ and that (4.12) holds. Therefore, (4.11) holds for every $t \in (0, T_{\max})$.

It follows from (4.9) that

$$\begin{aligned} z_1'(t) < {} & a[z_1(t) - 2p^2 z_1(pt)] + 2a \sum_{n=1}^{\infty} p^{4n}[z_1(p^{2n}t) - p^2 z_1(p^{2n+1}t)] \\ & - z_1^2(t) + 2p^2 z_1^2(pt) - 2\sum_{n=1}^{\infty} p^{4n}[z_1^2(p^{2n}t) - p^2 z_1^2(p^{2n+1}t)] \end{aligned} \tag{4.18}$$

holds for every $t \in (0, T_{\max})$. Similar to the proof of Theorem 10 (2), it can be shown that (4.17) holds for every $t \in (0, T_{\max})$. Hence, we obtain from (4.18) that

$$z_1'(t) < -(1 - 2p^2) z_1^2(t), \qquad t \in (0, T_{\max}).$$

Since

$$z_1(t) < \frac{1}{(1 - 2p^2)t + 1/z_1(0)}, \qquad t \in (0, T_{\max}),$$

we must have $T_{\max} < -((1 - 2p^2) z_1(0))^{-1}$. □

Proof of Theorem 17 (3,4). First, we consider the case $\mu < 0$, $f_0 = 0$. Since the solution is unique, we must have $g_1(t) \equiv g_2(t) = g(t)$, where $g(t)$ is a solution of the equation

$$g'(t) + p^2 g'(pt) + g^2(t) - p^2 g^2(pt) = \mu.$$

According to Theorem 10 (4), $T_{\max} < \infty$ and $\lim_{t \to T_{\max} - 0} g(t) = -\infty$.

Next, we consider the case $\mu \leq 0,\ f_0 < 0$. Since

$$g_1(0) = f_0 < 0, \qquad g_2(0) = -f_0 > 0$$

and

$$g_1'(0) = g_2'(0) = \frac{\mu - (1-p^2)f_0^2}{1+p^2} < 0,$$

there exists $t_0 > 0$ such that

$$g_1(t) + |g_2(t)| < 0 \tag{4.19}$$

for every $t \in (0, t_0)$.

We claim that (4.19) holds for every $t \in (0, T_{\max})$. Otherwise, there exists $t_1 \in [t_0, T_{\max})$ such that (4.19) holds for every $t \in (0, t_1)$ and

$$g_1(t_1) + |g_2(t_1)| = 0. \tag{4.20}$$

Subtracting (4.3) from (4.4) yields

$$g_1'(t) - g_2'(t) = -[g_1^2(t) - g_2^2(t)] - 2\sum_{n=1}^{\infty} p^{2n}[g_1^2(p^n t) - g_2^2(p^n t)], \tag{4.21}$$

which, together with (4.19), implies that

$$g_1'(t) - g_2'(t) < 0, \qquad t \in (0, t_1]. \tag{4.22}$$

Adding (4.3) to (4.4) yields

$$\begin{aligned} g_1'(t) + g_2'(t) = {} & \frac{2\mu}{1+p^2} - [g_1^2(t) + g_2^2(t)] + 2p^2[g_1^2(pt) + g_2^2(pt)] \\ & -2\sum_{n=1}^{\infty} p^{4n}\left\{g_1^2(p^{2n}t) + g_2^2(p^{2n}t) - p^2\left[g_1^2(p^{2n+1}t) + g_2^2(p^{2n+1}t)\right]\right\}. \end{aligned} \tag{4.23}$$

Let

$$W(t) = g_1^2(t) + g_2^2(t).$$

Since $W'(0) = 0$ and $W''(0) > 0$, there exists $t_2 \in (0, t_1)$ such that

$$W'(t) > 0 \tag{4.24}$$

for $t \in (0, t_2)$. We claim that (4.24) holds for every $t \in (0, t_1)$. Otherwise, there exists $t_3 \in (t_2, t_1)$ such that (4.24) holds for every $t \in (0, t_3)$ and $W'(t_3) = 0$. Noting that $p^2 < 1/2$, we can see from (4.23) that

$$g_1'(t) + g_2'(t) < 0, \qquad t \in (0, t_3].$$

Therefore, $W'(t_3) > 0$, which leads to a contradiction. Hence, (4.24) holds for every $t \in (0, t_1)$ which, together with (4.23), implies that

$$g_1'(t) + g_2'(t) < 0, \qquad t \in (0, t_1). \tag{4.25}$$

Therefore

$$(g_1^2(t) - g_2^2(t))'|_{t=t_1} > 0,$$

which contradicts our assumptions that (4.19) holds for every $t \in (0, t_1)$ and (4.20) holds. Therefore, (4.19) holds for every $t \in (0, T_{\max})$, which implies that

$$g_1'(t) < \frac{\mu}{1+p^2} - g_1^2(t) + 2p^2 g_1^2(pt) - 2\sum_{n=1}^{\infty} p^{4n}[g_1^2(p^{2n}t) - p^2 g_1^2(p^{2n+1}t)] \tag{4.26}$$

holds for every $t \in (0, T_{\max})$ as well. Similar to the proof of Theorem 10 (3,4), we can prove from the above inequality that

$$g_1'(t) < 0, \qquad t \in (0, T_{\max}).$$

Hence,

$$g_1'(t) < -(1-2p^2)g_1^2(t), \qquad t \in (0, T_{\max}).$$

Since

$$g_1(t) < \frac{1}{(1-2p^2)t + 1/g_1(0)} \qquad t \in (0, T_{\max}),$$

we must have $T_{\max} < -((1-2p^2)g_1(0))^{-1}$. □

Theorem 18 *If $q \in (-1, 0))$, $\mu > 0$ and $f(t) \in \mathrm{C}^1[0, \infty)$ is a solution of (1.1) that satisfies $f(t) \to \pm f^*$ as $t \to \pm\infty$ then the asymptotic expansions hold as $t \to \infty$*

$$\begin{aligned}
&f(t) = f^* + t^{-2}h_0(\log t) + t^{-3}h_1(\log t) + \ldots, \\
&f^{(n)}(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}[t^{-2}h_0(\log t)] + \frac{\mathrm{d}^n}{\mathrm{d}t^n}[t^{-3}h_1(\log t)] + \cdots, \quad n = 1, 2, \ldots, \\
&-f(-t) = f^* + t^{-2}h_0^*(\log t) + t^{-3}h_1^*(\log t) + \ldots, \\
&(-1)^{n+1}f^{(n)}(-t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}[t^{-2}h_0^*(\log t)] + \frac{\mathrm{d}^n}{\mathrm{d}t^n}[t^{-3}h_1^*(\log t)] + \cdots, \quad n = 1, 2, \ldots,
\end{aligned}$$

where $h_n, h_n^* \in \mathrm{C}^\infty$, $n = 0, 1, \ldots$, are periodic functions of period $-2\log|q|$ that obey

$$h_0(t) = h_0^*(t + \log|q|)$$

and the recurrence relations

$$\begin{aligned}
h_m(t) = &\frac{1}{2f^*(1-q^{2m})}\left[\frac{\mathrm{d}}{\mathrm{d}t} - (m+1)\right]\left[(1+q^{2m})h_{m-1}(t) + 2|q|^m h_{m-1}^*(t + \log|q|)\right] \\
&-\frac{1}{2f^*}\sum_{k=1}^{m-1} h_{k-1}(t)h_{m-k-1}(t), \quad m = 1, 2, \ldots, \\
h_m^*(t) = &\frac{1}{2f^*(1-q^{2m})}\left[\frac{\mathrm{d}}{\mathrm{d}t} - (m+1)\right]\left[(1+q^{2m})h_{m-1}^*(t) + 2|q|^m h_{m-1}(t + \log|q|)\right] \\
&-\frac{1}{2f^*}\sum_{k=1}^{m-1} h_{k-1}^*(t)h_{m-k-1}^*(t), \quad m = 1, 2, \ldots.
\end{aligned}$$

Proof. Let $p = -q \in (0, 1)$, $g_1(t) = f(t)$ and $g_2(t) = -f(-t)$ for $t \geq 0$. Then g_1 and g_2 satisfies (4.1). Denote $u_n(t) = y_1^{(n)}(t)$ and $v_n(t) = y_2^{(n)}(t)$ for $n \in \mathbb{Z}^+$, where $y_i(t) = g_i(t) - f^*$, $i = 1, 2$, satisfy the system

$$y_1'(t) + p^2 y_2'(pt) + a y_1(t) - p^2 a y_2(pt) + y_1^2(t) - p^2 y_2^2(pt) = 0, \tag{4.27}$$

$$y_2'(t) + p^2 y_1'(pt) + a y_2(t) - p^2 a y_1(pt) + y_2^2(t) - p^2 y_1^2(pt) = 0, \tag{4.28}$$

where $a = 2f^*$. The proof is essentially the same as that of Theorem 13. For instance, similar to the first three steps of the proof of (13), we can obtain the estimates

$$u_n(t), v_n(t) = \mathcal{O}(t^{\alpha_n}) \quad \text{as } t \to \infty$$

for $n = 0, 1, 2$ and for any $\alpha_0 > -1/2$, $\alpha_1 > -3/2$ and $\alpha_2 > -5/2$. Differentiating both sides of (4.27) and (4.28) yield

$$u_1(t) - \frac{p^3[a + 2v(pt)]}{a + 2u(t)} v_1(pt) = \frac{u_2(t) + p^3 v_2(pt)}{a + 2u(t)},$$

$$v_1(t) - \frac{p^3[a + 2u(pt)]}{a + 2v(t)} u_1(pt) = \frac{v_2(t) + p^3 u_2(pt)}{a + 2v(t)}.$$

We obtain from the above equations the following equation

$$u_1(t) - \frac{p^6[a + 2v(pt)][a + 2u(p^2t)]}{[a + 2u(t)][a + 2v(pt)]} u_1(p^2t)$$

$$= \frac{u_2(t) + p^3 v_2(pt)}{a + 2u(t)} + \frac{p^3[a + 2u(p^2t)][v_2(pt) + p^3 u_2(p^2t)]}{[a + 2v(pt)][a + 2v(pt)]}.$$

Therefore, we can obtain the estimate

$$u_1(t), v_1(t) = \mathcal{O}(t^{\alpha_1}) \quad \text{as } t \to \infty$$

for any $\alpha_1 > -5/2$. We omit the rest of the proof. □

5 Some conjectures

In this section we postulate some conjectures on the initial value problem (1.5). Our conjectures are supported by numerical experiment. Some numerical investigation has been carried out in [4, 6]. The method used for our numerical experiment is the so called θ-method for functional differential equations with proportional delay [21]. Since the cases $\mu > 0$ and $\mu < 0$ can be transformed to the cases $\mu = 1$ and $\mu = -1$, respectively, it suffices to do experiment numerically for the cases $\mu = 1, 0$ and -1 only. In the sequel, $f(t)$ denotes the solution of the initial value problem (1.5) with the maximal interval of existence $(T_{\min}, T_{\max})$.

First, we consider the case $q \in (0, 1)$. As explained in Section 3, we only need to solve (1.5) for $t \geq 0$ only. In Figures 1 and 2, solutions of (1.5) with $\mu = 1$, $q = 0.8$ are plotted for various initial values. We choose q close to 1, because most of our result in Section 3 are proved under assumptions that usually restrict the size of q. Figure 1 shows that the solution tends to the stationary solution $f^* \approx 1.6667$ (the dash-dot line) when $f_0 > -f^*$, whereas Figure 2 demonstrates that the solution blows up in finite time if $f_0 < -f^*$. These two figures suggest the following conjecture:

Conjecture 1: Suppose that $q \in (0,1)$ and $\mu > 0$. If $f_0 > f^*$ then $T_{\max} = \infty$, $f^* < f(t) < f_0$ for every $t > 0$, and $\lim_{t\to\infty} f(t) = f^*$; If $-f^* < f_0 < f^*$ then $T_{\max} = \infty$, $f^* < f(t) < f_0$ for every $t > 0$, and $\lim_{t\to\infty} f(t) = f^*$; If $f_0 < -f^*$ then $T_{\max} < \infty$ and $\lim_{t\to T_{\max}-0} f(t) = -\infty$.

Part of Conjecture 1 has been formulated in Degasperis and Shabat [6].

The solutions that are displayed in the previous two figures are all monotonic. However, the monotonicity will be lost when the initial values are close to the unstable stationary solution $-f^*$. This can be seen from Figure 3 ($\mu = 1$, $q = 0.5$), where solutions with initial values -1.13 (the solid line), -1.15 (the dash-dot line) and 1.1547 (the dash line) are displayed.

In Figure 4 ($\mu = 1, q = 0.5$), we plot t against $e^{2t}(f(e^t) - f^*)$ with the initial values $f_0 = 1.3$ and $f_0 = 1$ (from top to bottom in the picture). This picture clearly indicates that $e^{2t}(f(e^t) - f^*)$ tends to a non-trivial periodic function of period $-\log 2 \approx 0.6931$. This suggest that h_0 (hence h_n, $n = 1, 2, \ldots$) in the asymptotic expansions (3.18) can be a genuine periodic function. This contradicts the conjectured asymptotic expansion given in [4].

In the next two figures, solutions of (1.5) with $\mu = 0$ and $\mu = -1$, respectively, and $q = 0.8$ are plotted for various initial values f_0. Figure 5 shows clearly that the solution tends to zero (the only stationary solution of (1.1)) when $f_0 > 0$, but blows up in finite time when $f_0 < 0$. Figure 6 demonstrates that the solution always blows up in finite time, regardless of the value of f_0. These two figures prompt the next two conjectures.

Conjecture 2: Suppose that $q \in (0,1)$ and $\mu = 0$. If $f_0 > 0$ then $T_{\max} = \infty$, $0 < f(t) < f_0$ for every $t > 0$ and $\lim_{t\to\infty} f(t) = 0$. If $f_0 < 0$ then $T_{\max} < \infty$ and $\lim_{t\to T_{\max}-0} f(t) = -\infty$.

Conjecture 3: If $q \in (0,1)$ and $\mu < 0$ then $T_{\max} < \infty$ and $\lim_{t\to T_{\max}-0} f(t) = -\infty$.

We have also made a comprehensive numerical experiment with the case $q \in (-1,0)$. Our numerical result suggests the following conjectures:

Conjecture 4: Suppose that $q \in (-1,0)$ and $\mu > 0$. If $|f_0| < f^*$ then $T_{\max} = \infty$, $T_{\min} > -\infty$, $|f(t)| < f^*$ for every $t \in \mathbb{R}$, and $\lim_{t\to\pm\infty} f(t) = \pm f^*$. If $f_0 < -f^*$ then $T_{\max} < \infty$ and $\lim_{t\to T_{\max}-0} f(t) = -\infty$. If $f_0 > f^*$ then $T_{\min} > -\infty$ and $\lim_{t\to T_{\min}+0} f(t) = \infty$.

Conjecture 5: Suppose that $q \in (-1,0)$ and $\mu = 0$. If $f_0 < 0$ then $T_{\max} < \infty$ and $\lim_{t\to T_{\max}-0} f(t) = -\infty$. If $f_0 > 0$ then $T_{\min} > -\infty$ and $\lim_{t\to T_{\min}+0} f(t) = \infty$.

Conjecture 6: Suppose that $q \in (-1,0)$ and $\mu < 0$. If $f_0 \le 0$ then $T_{\max} < \infty$ and $\lim_{t\to T_{\max}-0} f(t) = -\infty$. $f_0 \ge 0$ then $T_{\min} > -\infty$ and $\lim_{t\to T_{\min}+0} f(t) = \infty$.

Since the case $q \in (-\infty, -1) \cup (1, \infty)$ can be transformed to the case $q \in (-1, 0) \cup (0, 1)$, the above conjectures together with Theorem 2 imply that when $q \neq 1, -1$

1. the solution of (1.5) is regular if
$$(1-q^2)\mu > 0, \qquad |f_0| < f^*;$$
and
2. the solution of (1.5) is not global if
$$(1-q^2)\mu > 0, \qquad |f_0| > f^*$$
or
$$\mu = 0, \qquad f_0 \neq 0$$
or
$$(1-q^2)\mu < 0.$$

We have proved (Theorems 1 and 2) that the above are true when $|q|$ is sufficiently small (or sufficiently large).

Finally, we remark that the technique developed in this paper can be used to certain extent to study the following system of equations

$$\begin{cases} f_n'(t) + f_{n+1}'(t) + f_n^2(t) - f_{n+1}^2(t) = \mu_n, & n = 1, 2, \ldots, N-1, \\ f_N'(t) + q^2 f_1'(qt) + f_N^2(t) - q^2 f_1^2(qt) = \mu_N, \end{cases}$$

where q and μ are parameters. Solutions of the above system give a class of self-similar solutions, or q-periodic closure, of the form (see Spiridonov [28, 30])

$$f_{n+N}(t) = q f_n(qt), \quad \mu_{n+N} = q^2 \mu_n, \qquad n = 1, 2, \ldots,$$

to the dressing chain (1.2). Spiridonov [30] also noted that solutions of the system

$$\begin{cases} f_n'(t) + f_{n+1}'(t) + f_n^2(t) - f_{n+1}^2(t) = \mu_n, & n = 1, 2, \ldots, N-1, \\ f_N'(t) + f_1'(t+1) + f_N^2(t) - f_1^2(t+1) = \mu_N \end{cases} \tag{5.1}$$

provide the dressing chain with another class of self-similar solutions of the form

$$f_{n+N}(t) = f_n(t+1), \quad \mu_{n+N} = \mu_n, \qquad n = 1, 2, \ldots.$$

However, the technique used in this paper does not apply to (5.1).

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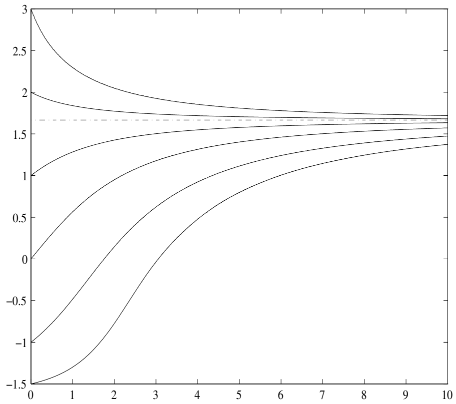

Figure 1. $q = 0.8, \mu = 1$.

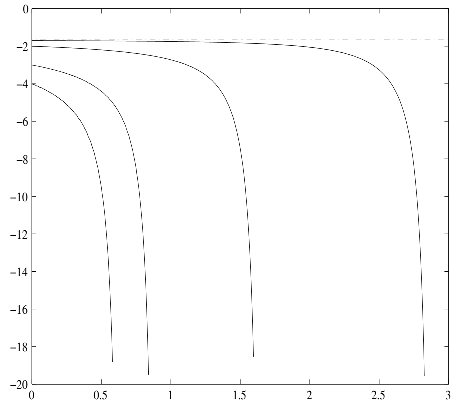

Figure 2. $q = 0.8, \mu = 1$.

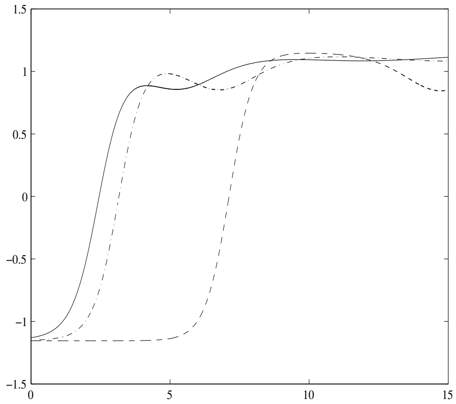

Figure 3. $q = 0.5, \mu = 1$.

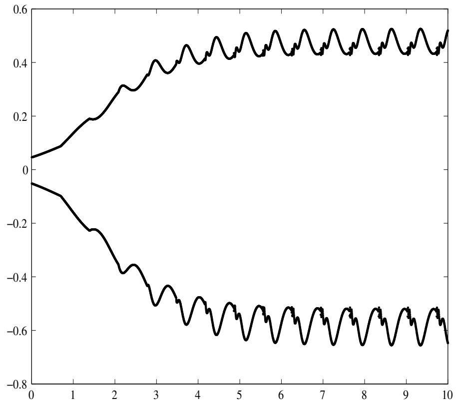

Figure 4. $q = 0.8, \mu = 1$.

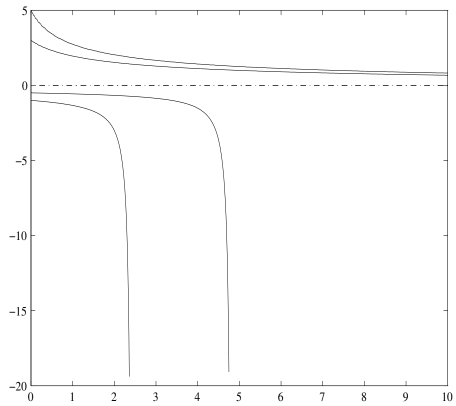

Figure 5. $q = 0.8, \mu = 0$.

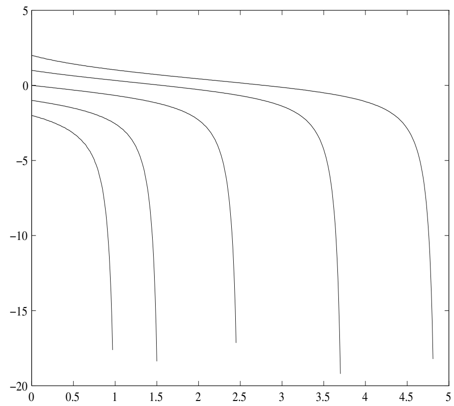

Figure 6. $q = 0.8, \mu = -1$.

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