UNIVERSITY OF CAMBRIDGE

Numerical Analysis Reports

Existence, uniqueness and continuous dependence for neutral equations with state-dependent delays

Yunkang Liu

DAMTP 1997/NA6
March 1997

Department of Applied Mathematics and Theoretical Physics
Silver Street
Cambridge
England CB3 9EW

Existence, uniqueness and continuous dependence for neutral equations with state-dependent delays

Yunkang Liu*

Abstract. The purpose of this work is to consider an initial value problem for neutral functional differential equations with state dependent and vanishing delays. A generalised version of Schauder's fixed point theorem which is based on the ball measure of noncompactness is used to obtain local existence of solutions and a comparison theorem. Results on continuation, uniqueness and continuous dependence of solutions are presented. The necessity of some critical assumptions made on certain Lipschitz constants is discussed.

*Gonville and Caius College, Cambridge University, Cambridge CB2 1TA, United Kingdom

1 Introduction

This paper is concerned with the existence, uniqueness and continuous dependence of solutions of the initial value problem for functional differential equations of neutral type

$$y'(t) = f(t, y(t), y(\alpha(t, y(t))), y'(\beta(t, y(t)))), \qquad t > b, \tag{1.1}$$

$$y(t) = y_0(t), \qquad t \in [a, b], \tag{1.2}$$

where $b > a$, the function $f : [a, \infty) \times \mathbb{R}^3 \mapsto \mathbb{R}$ is continuous, the delay functions $\alpha, \beta : [a, \infty) \times \mathbb{R} \mapsto \mathbb{R}$ are continuous and obey the restriction

$$a \leq \alpha(t, u),\ \beta(t, u) \leq t, \qquad \forall\, t \geq b,\ u \in \mathbb{R},$$

and the initial function $y_0 \in C^1[a, b]$ satisfies the consistent condition

$$y_0'(b) = f(b, y_0(b), y_0(\alpha(b, y_0(b))), y_0'(\beta(b, y_0(b)))). \tag{1.3}$$

The initial value problem (1.1,1.2) and its variations have been studied in Driver [2, 3], Hale and Cruz [6], Grimm [5], Winston [18], Travis [16] and Jackiewicz [11, 12]. Equations with variable but state independent delay have been discussed by Nussbaum [14], Iserles and Terjéki [10], Liu [13], Sengadir [15], Feldstein and Liu [4], and many others. In some of these papers, it is assumed that the delay functions α and β satisfy the restriction

$$b \leq \alpha(t, u),\ \beta(t, u) \leq t, \qquad \forall\, t \geq b,\ u \in \mathbb{R}. \tag{1.4}$$

In this case, the initial condition (1.2) can be replaced by

$$y(b) = y_1, \qquad y'(b) = y_2, \tag{1.5}$$

where y_1 and y_2 satisfy the consistent condition

$$y_2 = f(b, y_1, y_1, y_2). \tag{1.6}$$

Subject to the condition (1.4), the initial value problem (1.1,1.5) can be reduced to (1.1,1.2) by choosing any fixed $a < b$ and letting $y_0(t) = y_1 + (t - b)y_2$ for $t \in [a, b]$.

In this paper, a function y is defined as a solution of the initial value problem (1.1,1.2) if $y \in C^1[a, c]$ for some $c > b$ and that (1.1) and (1.2) are satisfied for $t \in (b, c]$ and $t \in [a, b]$, respectively. It should be noted that in papers such as [2, 3, 6, 16, 18] the consistent condition (1.3) was not assumed and subsequently the solution can not be continuously differentiable. In that case, solutions have to be sought in the space of absolutely continuous functions. In doing so, some restrictions such as $\beta(t, y) < t$ and/or $f(t, u, v, \cdot)$ is linear has to be imposed. Therefore, it is only fair to compare the work to be presented here with papers such as [5, 11, 12], or with [4, 10, 13, 14, 15] where the delays are state-independent.

This paper is structured as follows. In Section 2, fixed point theorems are used to study the existence of local solutions. Let us define for any fixed $c > b$ a continuous mapping $U : C^1[a, c] \times C^1[a, c] \mapsto C^1[a, c]$ by

$$U(y, z)(t) = \begin{cases} y_0(t), & t \in [a, b] \\ y_0(b) + \int_b^t f(\tau, y(\tau), y(\alpha(\tau, y(\tau))), z'(\beta(\tau, y(\tau))))\, \mathrm{d}\tau, & t \in [b, c] \end{cases}$$

and a continuous mapping $F: C^1[a,c] \mapsto C^1[a,c]$ by

$$F(y) = U(y,y). \tag{1.7}$$

It is evident that the existence of a local solution to the initial value problem (1.1,1.2) is equivalent to the existence of a fixed point of F in $C^1[a,c]$ for some $c > b$. In the case $\beta(b, y_0(b)) < b$, it is straightforward to obtain the existence of local solutions by choosing c to be close to b and using Schauder's fixed point theorem as in the standard approach. In the case $\beta(b, y_0(b)) = b$, a generalised version of Schauder's theorem that is based on the measure of noncompactness is used, since the mapping F defined by (1.7) is in general not compact in the function space $C^1[a,c]$. In order to obtain the existence of global solutions we present in Section 3 a continuation theorem and a comparison theorem. The comparison theorem, proved by using the same generalised version of Schauder's theorem, enables us to compare solutions of (1.1,1.2) with solutions of an initial value problem for ordinary differential equation. Results presented in Section 3 relaxed significantly those very restrictive conditions for the existence of global solutions given in [5, 11, 12]. In Section 4, we discuss the uniqueness and continuous dependence of solutions. Except for an example which demonstrates that the usual Lipschitz conditions are not enough to guarantee the uniqueness of solution, all result in Section 4 only improve those in [5, 11, 12] slightly, though sometimes different methods are used to achieve the improvement. From Section 2 to 4, the condition that the Lipschitz constant of the function $f(t,u,v,\cdot)$ is less than 1 (locally or globally) is often assumed. The necessity of this assumption is discussed in Section 5, using mostly ideas due to Nussbaum [14].

2 Existence

This section deals with the existence of local solutions of the initial value problem (1.1,1.2). There are two cases, $\beta(b, y_0(b)) < b$ and $\beta(b, y_0(b)) = b$, are investigated separately.

Theorem 1 *If $\beta(b, y_0(b)) < b$ then the initial value problem (1.1,1.2) has at least one solution in $C^1[a,c]$ for some $c > b$.*

Proof. The continuity of the function β implies that there exist constants $c_0 > b$ and $r_0 > 0$ such that

$$\beta(t,u) \le b \qquad \forall\, t \in [b, c_0],\ |u - y_0(b)| \le r_0.$$

For any fixed $c \in (b, c_0]$ we denote

$$S_c = \left\{ y \in C^1[a,c] \mid y(t) = y_0(t) \text{ for } t \in [a,b],\ \max_{t\in[b,c]} |y(t) - y_0(b)| \le r_0 \right\}.$$

Evidently, S_c is a nonempty, closed, bounded and convex subset of the Banach space C$[a,c]$ equipped with the supremum norm. Noting that for $y \in S_c$ the mapping F defined by (1.7) can be rewritten as

$$F(y)(t) = \begin{cases} y_0(t), & t \in [a,b], \\ y_0(b) + \int_b^t f(\tau, y(\tau), y(\alpha(\tau, y(\tau))), y_0'(\beta(\tau, y(\tau))))\, \mathrm{d}\tau, & t \in [b,c], \end{cases}$$

one can then use Schauder's Theorem (see, e.g., Deimling [1]) to show that F has a fixed point in S_c for some $c \in (b, c_0]$. Hence, (1.1,1.2) has at least one solution in $C^1[a, c]$ for some $c > b$. □

Next, we consider the case $\beta(b, y_0(b)) = b$. In this case, the delay vanishes at $t = b$ at lease and the mapping F defined by (1.7) is in general not compact in the function space $C^1[a, c]$. It is then natural to use a generalised version of Schauder's fixed point Theorem that is based on the ball measure of noncompactness is needed. First, let us introduce some notation. For any bounded set B of a Banach space X the *ball measure of noncompactness* is defined as

$$\gamma(B) = \inf\{r \in \mathbb{R}^+ \mid B \text{ can be covered by finitely many balls of radius } r\}.$$

A continuous mapping $F : \Omega \mapsto X$, where Ω is a bounded subset of X, is called *γ-Lipschitzian* with Lipschitz constant k if $\gamma(F(B)) \leq k\gamma(B)$ whenever $B \subset \Omega$. If $k < 1$ then F is called a strict *γ-contraction*. F is said to be *γ-condensing* if $\gamma(F(B)) < \gamma(B)$ whenever $B \subset \Omega$ and $\gamma(B) > 0$. The following is a generalisation of Schauder's Theorem (see, e.g. Deimling [1])

Theorem 2 *If $S \subset X$ is a nonempty, closed, bounded and convex subset of X, and $F : S \mapsto S$ is a γ-condensing mapping, then F has a fixed point.*

The following lemma will be used to verify whether a mapping, such as the one defined in (1.7), is γ-condensing. It is a slightly improved and generalised version of [17, Therorem 1]. The proof, which is essentially the same as that of [17, Therorem 1], is given here for completeness.

Lemma 3 *Let S be a bounded subset of X, $V : X \times X \mapsto X$ a continuous mapping and $G : S \mapsto S$ a continuous mapping defined by $G(x) = V(x, x)$. If*

1. *there exists a constant k such that for each fixed $x \in S$, $V(x, \cdot) : S \mapsto X$ is Lipschitzian with Lipschitz constant k; and*

2. *for each fixed $x \in S$, $V(\cdot, x)$ maps S to a relatively compact subset of X,*

then G is γ-Lipschitzian with Lipschitz constant k.

Proof. Let D be a subset of S with $\gamma(D) > 0$ and $\epsilon > 0$ an arbitrary number. Then there exist $x_1, \ldots, x_m \in S$, such that $D \subset \bigcup_{i=1}^{m} B(x_i, \gamma(D) + \epsilon)$. For any $1 \leq i \leq m$ there exist $y_{i,1}, \ldots, y_{i,n_i} \in X$ such that $V(D, x_i) \subset \bigcup_{j=1}^{n_i} B(y_{i,j}, \epsilon)$, since $V(\cdot, x)$ maps D to a relatively compact subset of X. For an arbitrary $x \in D$, there exist i and j such that $\|x - x_i\| \leq \gamma(D) + \epsilon$ and $\|V(x, x_i) - y_{i,j}\| \leq \epsilon$. Therefore,

$$\|G(x) - y_{i,j}\| \leq \|V(x, x) - V(x, x_i)\| + \|V(x, x_i) - y_{i,j}\| \leq k\gamma(D) + (1 + k)\epsilon$$

which implies that $G(D) \subset \bigcup_{j=1}^{m} \bigcup_{j=1}^{n_i} B(y_{i,j}, k\gamma(D) + (1 + k)\epsilon)$. Hence, $\gamma(G(D)) \leq k\gamma(D)$ for every $D \subset S$. Equivalently speaking, G is γ-Lipschitzian with Lipschitz constant k. □

Theorem 4 *Suppose that $\beta(b, y_0(b)) = b$ and that there exist constants $c_0 > b$, $r_0 > 0$ and $k_0 < 1$ such that the function $f(t, u, v, \cdot)$ satisfies the Lipschitz condition*

$$\left.\begin{aligned} &|f(t,u,v,w_1) - f(t,u,v,w_2)| \le k_0|w_1 - w_2| \\ &\quad \textit{for all } \ t \in [b, c_0],\ |u - y_0(b)| \le r_0,\ |v - y_0(\alpha(b, y_0(b)))| \le r_0, \\ &\quad \textit{and } \ |w_i - y_0'(b)| \le r_0,\ i = 1, 2. \end{aligned}\right\} \tag{2.1}$$

Then the initial value problem (1.1,1.2) has at least one solution in $C^1[a, c]$ for some $c \in (b, c_0]$.

Proof. For any fixed $c \in (b, c_0]$ we introduce the set

$$S_c = \left\{ y \in C^1[a, c] \mid y(t) = y_0(t) \text{ for } t \in [a, b],\ \max_{t\in[b,c]} |y'(t) - y_0'(b)| \le r_0 \right\}.$$

Evidently, S_c is a nonempty, closed, bounded and convex subset of the Banach space $C^1[a, c]$ equipped with the norm

$$||y|| = |y(b)| + \max_{t\in[a,c]} |y'(t)|.$$

First, let us use Lemma 3 to show that F is γ-condensing on S_c for every $c \in (0, c_0]$. The assumption (2.1) implies that for each fixed $y \in S_c$ the mapping $U(y, \cdot) : S_c \mapsto C^1[a, c]$ is Lipschitzian with Lipschitz constant $k_0 < 1$. For each fixed $z \in S_c$, let us show that $U(\cdot, z)$ maps S_c to a sequentially compact subset of $C^1[a, c]$. For any infinite sequence $\{u_n\}$ in $U(S_c, z)$, there exists an infinite sequence $\{y_n\}$ in S_c such that $u_n = U(y_n, z)$, $n = 1, 2, \ldots$. Using Ascoli-Arzela lemma one can show that S_c is a compact subset of the Banach space $C[a, c]$ equipped with the sup-norm. The equivalence of sequential compactness and compactness implies that there exists a convergent subsequence of $\{y_n\}$ in $C[a, c]$. Denote the limit by y^*. It is obvious that $\{u_n\}$ has a subsequence that converges to $U(y^*, z) \in S_c$. Therefore, $U(\cdot, z)$ maps $C^1[a, c]$ to a sequentially compact (hence compact) subset of $C^1[a, c]$. Invoking lemma 3, we see that F is a strict γ-contraction on S_c for every $c \in (b, c_0]$.

Next, we show that there exists $c \in (b, c_0]$ such that $F : S_c \mapsto S_c$. Since S_c is a family of equicontinuous functions, it is true that the following mappings

$$\begin{aligned} &y \mapsto \alpha(\cdot, y(\cdot)); \\ &y \mapsto \beta(\cdot, y(\cdot)); \\ &y \mapsto y(\alpha(\cdot, y(\cdot))); \\ &y \mapsto f(\cdot, y(\cdot), y(\alpha(\cdot, y(\cdot))), y'(b)); \end{aligned}$$

map S_c to families of equicontinuous functions as well. Therefore, there exists $c \in (b, c_0]$ such that the inequalities

$$\begin{aligned} &|y(t) - y(b)| \le r_0, \\ &|y(\alpha(t, y(t))) - y(\alpha(b, y(b)))| \le r_0, \\ &|y_0'(\beta(t, y(t))) - y_0'(b)| \le r_0 \qquad \text{if } \beta(t, y(t)) < b, \\ &|f(t, y(t), y(\alpha(t, y(t))), y'(b)) - f(b, y(b), y(\alpha(b, y(b))), y'(b))| \le (1 - k_0)r_0, \end{aligned}$$

hold for all $t \in [b,c]$ and all $y \in S_c$. Since $y_0(t)$ satisfies equation (1.1) at $t = b$, $\beta(b, y_0(b)) = b$ and that $y(t) = y_0(t)$, $t \in [a,b]$, for all $y \in S_c$, we can use the assumption (2.1) to obtain the inequality

$$\begin{aligned}
&|(Fy)'(t) - y_0'(b)| \\
\leq\ & |f(t, y(t), y(\alpha(t, y(t))), y'(\beta(t, y(t)))) - f(t, y(t), y(\alpha(t, y(t))), y'(b))| \\
&+ |f(t, y(t), y(\alpha(t, y(t))), y'(b)) - f(b, y(b), y(\alpha(b, y(b))), y'(b))| \\
\leq\ & r_0, \quad t \in [b,c], \qquad \forall y \in S_c.
\end{aligned}$$

Therefore $F : S_c \mapsto S_c$ for some $c \in (0, c_0]$.

Finally, it follows from the generalised Schauder's theorem (Theorem 2) that F has a fixed point in S_c for some $c \in (b, c_0]$. Hence, (1.1,1.2) has at least one solution in $C^1[a,c]$ for some $c > b$. □

3 Continuation and comparison

Suppose that there exist constants $c > b$ and $k_0 < 1$ such that the function $f(t, u, v, \cdot)$ satisfies the Lipschitz condition

$$|f(t, u, v, w_1) - f(t, u, v, w_2)| \leq k_0 |w_1 - w_2| \qquad \forall t \in [b, c),\ u, v, w_1, w_2 \in \mathbb{R}. \tag{3.1}$$

It follows from Theorems 1 and 4 that the initial value problem (1.1,1.2) has local solutions in $C^1[a, c_0]$ for some $c_0 \in (b, c)$. Invoking the two theorems again, we see that every such local solution can be extended to a solution in $C^1[a, c_1]$ for some $c_1 \in (c_0, c)$. By Hausdorff maximality principle, every local solution can be extended to a non-continuable solution, $y \in C^1[a, c_{\max})$, say, where $c_{\max} \in (b, c]$ in the sense that if $c_{\max} < c$ then either $\lim_{t\to c_{\max}-} y(t)$ or $\lim_{t\to c_{\max}-} y'(t)$ does not exist. The following continuation theorem gives an easy to use criterion for determining whether $c_{\max} = c$ or not.

Theorem 5 *Suppose that there exist constants $c > b$ and $k_0 < 1$ such the Lipschitz condition (3.1) is satisfied. If $y \in C^1[a, c_{\max})$ is a non-continuable solution of (1.1,1.2) then $c_{\max} < c$ implies $\limsup_{t\to c_{\max}-} |y(t)| = \infty$.*

Proof. Let us prove it by contradiction. Suppose that $\limsup_{t\to c_{\max}-} |y(t)| < \infty$. From the inequality

$$|y'(t)| \leq |f(t, y(t), y(\alpha(t, y(t))), 0)| + k_0 |y'(\beta(t, y(t)))|, \quad t \in [b, c_{\max}),$$

we obtain the estimate

$$|y'(t)| \leq \max\left\{ \max_{s\in[a,b]} |y_0'(s)|, \frac{1}{1-k_0} \max_{s\in[a,t]} |f(s, y(s), y(\alpha(s, y(s))), 0)| \right\}, \quad t \in [b, c_{\max})$$

which implies that $\limsup_{t\to c_{\max}-} |y'(t)| < \infty$. Therefore, $\lim_{t\to c_{\max}-} y(t)$ exists. If $\lim_{t\to c_{\max}-} \beta(t, y(t)) < c_{\max}$ then it is evident that $\lim_{t\to c_{\max}-} y'(t)$ exists. Suppose

$\lim_{t\to c_{\max}-} \beta(t, y(t)) = c_{\max}$. For any given t_1 and t_2, both of them close to $c_{\max}$ from the left, we obtain by using (3.1) the inequality

$$\begin{aligned} |y'(t_1) - y'(t_2)| \le |f(t_1, y(t_1), y(\alpha(t_1, y(t_1))), y'(\beta(t_1, y(t_1)))) \\ -f(t_2, y(t_2), y(\alpha(t_2, y(t_2))), y'(\beta(t_1, y(t_1))))| \\ +k_0|y'(\beta(t_1, y(t_1))) - y'(\beta(t_2, y(t_2)))|. \end{aligned}$$

It is not difficult to see from the above inequality that

$$\left|\limsup_{t\to c_{\max}-} y'(t) - \liminf_{t\to c_{\max}-} y'(t)\right| \le k_0 \left|\limsup_{t\to c_{\max}-} y'(t) - \liminf_{t\to c_{\max}-} y'(t)\right|$$

which implies that

$$\limsup_{t\to c_{\max}-} y'(t) = \liminf_{t\to c_{\max}-} y'(t).$$

Hence, $\lim_{t\to c_{\max}-} y'(t)$ exists. The fact that both $\lim_{t\to c_{\max}-} y(t)$ and $\lim_{t\to c_{\max}-} y'(t)$ exist contradicts the assumption that y is non-continuable. Hence, $c_{\max} < c$ implies $\limsup_{t\to c_{\max}-} |y(t)| = \infty$. □

We now compare solutions of (1.1,1.2) with that of the initial value problem

$$x' = f_0(t, x), \quad t \in [b, c], \qquad x(b) = x_0, \tag{3.2}$$

where $x_0 = \max_{t\in[a,b]} |y_0(t)|$ and the function $f_0 \in C([b, c] \times \mathbb{R}^+)$ is defined by

$$f_0(t, x) = \max\left\{ \max_{s\in[a,b]} |y_0'(s)|, \frac{1}{1-k_0} \max_{|u|,|v|\le x} |f(t, u, v, 0)|\right\}$$

or, for simplicity, by

$$f_0(t, x) = \max_{s\in[a,b]} |y_0'(s)| + \frac{1}{1-k_0} \max_{|u|,|v|\le x} |f(t, u, v, 0)|.$$

Theorem 6 *Suppose that (3.2) has a solution $x \in C^1[b, c]$ and that there exists a constant $k_0 < 1$ such that*

$$|f(t, u, v, w_1) - f(t, u, v, w_2)| \le k_0|w_1 - w_2| \qquad t \in [b, c]$$

for all $u, v, w_1, w_2 \in \mathbb{R}$ that obey

$$|u|, |v| \le \max_{s\in[b,t]} |x(s)|, \qquad |w_1|, |w_2| \le \max_{s\in[b,t]} |x'(s)|.$$

Then the initial value problem (1.1,1.2) has a solution $y \in C^1[a, c]$ that satisfies the inequality $|y'(t)| \le x'(t)$ for every $t \in [b, c]$.

Proof. It is obvious that

$$S = \left\{y \in C^1[a, c] \mid y(t) = y_0(t) \text{ for } t \in [a, b],\ |y'(t)| \le x'(t) \text{ for } t \in [b, c], \right\}.$$

is a nonempty, closed, bounded and convex subset of the Banach space $C^1[a,c]$ equipped with the norm

$$\|y\| = |y(b)| + \max_{t\in[a,c]} |y'(t)|.$$

Since $f_0(t,\cdot)$ increases monotonically, the inequalities

$$|f(t,y(t),y(\alpha(t,y(t))),0)| \le f_0(t,x(t)), \qquad t\in[b,c]$$

$$|y'(\beta(t,y(t))))| \le x'(t), \qquad t\in[b,c]$$

hold for all $y\in S$. Therefore, the mapping F defined by (1.7) satisfies

$$\begin{aligned}|(Fy)'(t)| &\le |f(t,y(t),y(\alpha(t,y(t))),0)| + k_0|y'(\beta(t,y(t))))| \\ &\le x'(t), \qquad \forall y\in S_c,\ t\in[b,c]\end{aligned}$$

which implies that $F : S_c \mapsto S_c$. Analogously to the proof of Theorem 4, one can show that F is a strict γ-contraction on S_c. According to the generalised Schauder's theorem (Theorem 2), F has a fixed point, say y, in S_c. Evidently, y is a solution of (1.1,1.2) and it obeys $|y'(t)| \le x'(t)$ for every $t\in[b,c]$. □

Using the preceeding two theorems one can obtain the following result on the existence of global solutions.

Corollary 7 *Suppose that there exist constants $c > b$, $k_0 < 1$ and $L > 0$ such that the function $f(t,u,v,\cdot)$ satisfies the Lipschitz condition (3.1) and the inequality*

$$|f(t,u,v,0)| \le L(1+|u|+|v|) \qquad \forall t\in[b,c),\ u,v,w\in\mathbb{R}. \tag{3.3}$$

Then (1.1,1.2) has at least one solution in $C^1[a,c)$ and that the maximal interval of existence of each non-continuable solution is $[a,c)$.

The above result relaxed significantly those rather restrict conditions for the existence of global solutions given in [5, 11, 12].

It should be noted from the proof of Theorem 5 that when $\lim_{t\to c_{\max}-} \beta(t,y(t)) < c_{\max}$ the assumption (3.1) is not needed. Therefore, if the delay function $\beta(t,\cdot)$ is constant, i.e., β depends on t only, both Theorem 5 and Corollary 7 can be modified to the following:

Theorem 8 *If $y\in C^1[a,c_{\max})$ is a non-continuable solution of (1.1,1.2) then $c_{\max} < c$ implies either $\beta(c_{\max}) = c_{\max}$ or $\limsup_{t\to c_{\max}-} |y(t)| = \infty$.*

Corollary 9 *Suppose that there exist constants $c > b$ and $L > 0$ and a continuous function k such that $f(t,u,v,\cdot)$ satisfies the inequality (3.3) and the Lipschitz condition*

$$|f(t,u,v,w_1) - f(t,u,v,w_2)| \le k(t)|w_1 - w_2| \qquad \forall t\in[b,c),\ u,v,w_1,w_2\in\mathbb{R}.$$

If $|k(t)| < 1$ whenever $\beta(t) = t$ then (1.1,1.2) has at least one solution in $C^1[a,c)$ and that the maximal interval of existence of each non-continuable solution is $[a,c)$.

A result that is similar to the preceding corollary but is for a neutral equation with state-independent delays has been obtained by Nussbaum [14].

4 Uniqueness and continuous dependence

The usual requirement for uniqueness and continuous dependence for solutions of initial value problems of ordinary differential equations include various Lipschitz conditions. However, we may need some extra assumptions in the case of neutral differential equations.

Theorem 10 *Suppose that for any given $r > 0$ there exist constants $k_0 < 1$, $B_0 \geq 0$ and $L > 0$ such that f, α and β satisfy*

$$\left.\begin{aligned}
&|f(t,u_1,v_1,w_1) - f(t,u_2,v_2,w_2)| \\
&\qquad\qquad \leq L(|u_1-u_2| + |v_1-v_2|) + k_0|w_1-w_2|, \\
&|\alpha(t,u_1) - \alpha(t,u_2)| \leq L|u_1-u_2|, \\
&|\beta(t,u_1) - \beta(t,u_2)| \leq B_0|u_1-u_2|, \\
&\quad \textit{for all } t \in [b,c],\ |u_1-u_2| \leq r,\ |v_1-v_2| \leq r \\
&\quad \textit{and } |w_1-w_2| \leq r,\ i = 1,2.
\end{aligned}\right\} \tag{4.1}$$

If one of the following two conditions holds:

(i) *$\beta(t,u)$ is independent of u, or equivalently, $B_0 = 0$,*

(ii) *(1.1,1.2) has a solution in $C^{1,1}[a,c]^\dagger$,*

then (1.1,1.2) has no more than one solution.

The proof of the preceding result is omitted here, since it is essentially the same as the one we are going to give on the continuous dependence of solutions. The same result but under slightly stronger Lipschitz conditions were proved by Grimm [5] (subject to the condition (i)) and Jackiewicz [11] (subject to the condition (ii)).

The conditions (i) and (ii) are seemingly unnatural. However, without at least one of them, those Lipschitz conditions listed in (4.1) may not be enough to guarantee the uniqueness of solution. For example, let us consider the initial value problem

$$\begin{cases} y'(t) = k(t)y'(-|y(t)|), & 0 < t < \left(\frac{p+1}{q+1}\right)^{1/(q-p(q+1))}, \\ y(t) = |t|^{p+1}, & t \leq 0, \end{cases} \tag{4.2}$$

where $q > 0$, $0 < p < \frac{q}{q+1}$ and $k(t) = -\frac{q+1}{p+1}t^{q-p(q+1)}$. It is trivial to verify that (4.2) is satisfied by

$$y(t) = \begin{cases} |t|^{p+1}, & t \leq 0, \\ 0, & t \geq 0, \end{cases}$$

as well as by

$$y(t) = \begin{cases} |t|^{p+1}, & t \leq 0, \\ |t|^{q+1}, & t \geq 0. \end{cases}$$

†Here and hereafter the notation $C^{1,1}[a,c]$ denotes the space of differentiable functions whose derivatives are Lipschitz continuous.

Naturally, neither derivative of the two solutions of (4.2) is Lipschitz continuous. On the other hand, we can see from Theorem 10 that if the initial value problem (1.1,1.2) have more than one solutions then it cannot have any solution in the function space $C^{1,1}[a,c]$.

The following result shows that under some extra assumptions a solution of the initial value problem (1.1,1.2) has to be in the function space $C^{1,1}[a,c]$ for some $c > b$,

Theorem 11 *Suppose that $y_0 \in C^{1,1}[a,b]$ and that for any given $r > 0$ there exist constants $k_0 < 1$, $B_0 \in [0, 1/k_0)$ and $L > 0$ such that f, α and β satisfy*

$$\left.\begin{aligned}
&|f(t_1,u_1,v_1,w_1) - f(t_2,u_2,v_2,w_2)| \\
&\qquad\qquad \le L(|t_1-t_2| + |u_1-u_2| + |v_1-v_2|) + k_0|w_1-w_2|, \\
&|\alpha(t_1,u_1) - \alpha(t_2,u_2)| \le L(|t_1-t_2| + |u_1-u_2|), \\
&|\beta(t_1,u_1) - \beta(t_2,u_2)| \le B_0|t_1-t_2| + L|u_1-u_2|, \\
&\qquad \textit{for all } t_1,t_2 \in [b,c],\ |u_1-u_2| \le r,\ |v_1-v_2| \le r \\
&\qquad \textit{and } |w_1-w_2| \le r,\ i = 1,2.
\end{aligned}\right\} \tag{4.3}$$

Then the solution of the initial value problem (1.1,1.2), if exists, must be in the function space $C^{1,1}[a,c]$.

The preceding theorem, together with Corollary 7 and Theorem 10, improves the main result of Jackiewicz [11, 12]. To prove it, the following lemma is needed.

Lemma 12 *Given a function $x \in C[a,b]$, if there exist constants $k \in (0,1)$, $B \in (0, 1/k)$ and $L > 0$ such that for any $t_1, t_2 \in [a,b]$ the inequality*

$$|x(t_1) - x(t_2)| \le L|t_1 - t_2| + k|x(t_1^*) - x(t_2^*)|$$

holds for some $t_1^*, t_2^* \in [a,b]$ obeying

$$|t_1^* - t_2^*| \le B|t_1 - t_2|,$$

then x is Lipschitz continuous with Lipschitz constant $L/(1-kB)$.

Proof. It can be shown by induction that for any $n \in \mathbb{Z}$ there exist $t_1^*, t_2^* \in [a,b]$ obeying

$$|t_1^* - t_2^*| \le B^n|t_1 - t_2|,$$

such that

$$|x(t_1) - x(t_2)| \le L[1 + kB + (kb)^2 + \ldots + (kB)^{n-1}]|t_1 - t_2| + k^n|x(t_1^*) - x(t_2^*)|.$$

Letting n tends to infinity in the preceding inequality yields the desired result. □

Proof of Theorem 11. Let $y \in C^1[a,c]$ be a solution of (1.1,1.2), and let

$$r = 2\max_{t\in[a,c]}\{|y(t)|, |y'(t)|\}.$$

Denote $L_1 = \max_{t\in[a,c]} |y'(t)|$, L_2 the Lipschitz constant of $y_0'(t)$ in the interval $[a,b]$, and

$$\rho = \min\left\{c-b, \frac{1-k_0B_0}{k_0LL_1}\right\}.$$

If $t_1, t_2 \in [a,b]$, we have

$$|y'(t_1) - y'(t_2)| \le L_2|t_1 - t_2|.$$

If $t_1, t_2 \in [b, b+\rho]$, we can use (4.3) to show that there exist $t_1^*, t_2^* \in [a, b+\rho]$ obeying

$$|t_1^* - t_2^*| \le (B_0 + \rho LL_1)|t_1 - t_2|$$

such that

$$|y'(t_1) - y'(t_2)| \le L[1 + L_1[1 + B_0 + LL1(c-b)]\rho]|t_1 - t_2| + k_0|y'(t_1^*) - y'(t_2^*)|.$$

If $t_1 \in [a,b]$ and $t_2 \in [b,\rho]$, we have

$$|y'(t_1) - y'(t_2)| \le |y'(t_1) - y'(b)| + |y'(t_2) - y'(b)|.$$

In any case, the inequality

$$|y'(t_1) - y'(t_2)| \le [L_2 + L(1 + L_1[1 + B_0 + LL1(c-b)]\rho)]|t_1 - t_2| + k_0|y'(t_1^*) - y'(t_2^*)|$$

holds for some $t_1^*, t_2^* \in [a, b+\rho]$ (t_1^* and t_2^* are not necessarily unique) obeying

$$|t_1^* - t_2^*| \le (B_0 + \rho LL_1)|t_1 - t_2|.$$

Applying Lemma 12 yields $y \in C^{1,1}[a, b+\rho]$. Since ρ is constant, it can be shown by using a "method-of-steps" that $y \in C^{1,1}[a,c]$. □

Next, we show that under slightly stronger conditions the solution of (1.1,1.2) depends continuously on the initial function.

Theorem 13 *Suppose that there exist constants $k_0 < 1$, $B_0 \ge 0$ and $L > 0$ such that f, α and β satisfy (4.1) for arbitrary $r > 0$. Let $y, z \in C^1[a,c]$ be solutions of (1.1) that satisfies, respectively, the initial condition (1.2) and*

$$z(t) = z_0(t), \qquad t \in [a,b],$$

where $z_0 \in C^1[a,b]$ satisfies equation (1.1) at $t = b$. If one of the following two conditions holds:

(i) *$\beta(t,u)$ is independent of u, or equivalently, $B_0 = 0$,*

(ii) *the solution $y \in C^{1,1}[a,c]$,*

then the function $w(t) := y(t) - z(t)$ satisfies the inequality

$$|w(t)| + |w'(t)| \le L_1 e^{L_2(t-b)}\left(|w(b)| + \max_{s\in[a,b]} |w'(s)|\right), \qquad t \in [b,c]$$

for some nonnegative constants L_1 and L_2 that are independent of z.

Proof. The existence of y and z follows from Corollary 7, the uniqueness of y follows from Theorem 10. To prove the continuous dependence of y on the initial functions, we note that

$$|w'(t)| \le L(|w(t)| + |w(\alpha(t, z(t)))|) + L|y(\alpha(t, y(t))) - y(\alpha(t, z(t)))|$$
$$+k_0|w'(\beta(t, z(t)))| + k_0|y'(\beta(t, y(t))) - y'(\beta(t, z(t)))| \qquad t \in [b, c].$$

Let $L_3 = \max_{t\in[a,c]} |y'(t)|$. Then

$$|y(\alpha(t, y(t))) - y(\alpha(t, z(t)))| \le LL_3|w(t)|, \qquad t \in [a, c].$$

We claim that the inequality

$$|y'(\beta(t, y(t))) - y'(\beta(t, z(t)))| \le B_0L_4|w(t)|, \qquad t \in [a, c]$$

holds for some constant L_4. More precisely, $L_4 = 0$ if (i) is assumed, L_4 is the Lipschitz constant of y' over the interval $[a, c]$ if (ii) is assumed. Therefore,

$$|w'(t)| \le (2L + LL_3 + k_0B_0L_4) \max_{s\in[a,t]} |w(s)| + k_0 \max_{s\in[a,t]} |w'(s)|, \qquad t \in [b, c].$$

¿From the above inequality we obtain the estimate

$$|w'(t)| \le \max_{s\in[a,b]} |w'(s)| + L_2 \max_{s\in[a,t]} |w(s)|, \qquad t \in [b, c],$$

where $L_2 = (2L + LL_3 + k_0B_0L_4)/(1 - k_0)$. It is quite obvious that w is bounded by the solution of the initial value problem

$$x'(t) = \max_{s\in[a,b]} |w'(s)| + L_2x(t), \ \ t \ge b, \qquad x(b) = |w(b)|.$$

Therefore, there exists a constant L_1 such that the desired estimate of $|w(t)| + |w'(t)|$ holds. □

5 On the condition $k_0 < 1$

The assumption that $k_0 < 1$ appears in most Lipschitz conditions we assumed on the function f in the last three sections. One may wonder how necessary it is. Before discussing this issue we remark that all result in the previous sections can be generalised to equations with multiple delays, to systems of equations, and even to equations with distributional delay, provided that appropriate Lipschitz conditions are assumed. In the sequel we shall make use of some obvious generalised result without given a proof.

First, if all the given functions are smooth then it is possible to obtain existence result under weaker or different assumption. For example, let us assume that f, α, β are all differentiable and that the delay functions α and β satisfy the restriction (1.4). If[‡]

$$|f_4(b, y_1, y_1, y_2)(\beta_1(b, y_1) + \beta_2(b, y_1)| < 1, \tag{5.1}$$

[‡]In the sequel, functions f and β with subscripts are taken to be their partial derivatives.

then the initial value problem (1.1,1.5) has a solution in $C^2[b,c]$ for some $c > b$. This is because in the function space $C^2[b,c]$ equation (1.1) is equivalent to the system of two equations value problem

$$\begin{aligned}
y'(t) &= x(t), \qquad t > b, \\
x'(t) &= f_1(t, y(t), y(\alpha(t, y(t))), x(\beta(t, y(t)))) + \ldots \\
&\quad + f_4(t, y(t), y(\alpha(t, y(t))), x(\beta(t, y(t))) \\
&\quad \times (\beta_1(t, y(t)) + \beta_2(t, y(t))x(t))x'(\beta(t, y(t))), \qquad t > b,
\end{aligned}$$

while the corresponding initial conditions are

$$y(b) = y_1, \; x(b) = y_2.$$

In general, the following result holds.

Theorem 14 *Suppose that α and β satisfy (1.4), f, α, β are C^{n-1} functions for some $n \geq 2$, and that*

$$f_4(b, y_1, y_1, y_2)\left(\beta_1(b, y_1) + \beta_2(b, y_1)\right)^k \neq 1, \qquad k = 0, \ldots, n-2 \tag{5.2}$$

If

$$|f_4(b, y_1, y_1, y_2)(\beta_1(b, y_1) + \beta_2(b, y_1))^{n-1}| < 1$$

then the initial value problem (1.1,1.5) has a solution in $C^n[b,c]$ for some $c > b$.

The condition (5.2) enables us to determine $y^{(k)}(0)$, $k = 2, \ldots, n-1$, uniquely. In general, it is not difficult to see that all the other result obtained in the previous three sections can be generalised in the same manner.

The advantage of investigating existence of solutions in the function space C^n for some $n \geq 2$ was first taking in Nussbaum [14] where a neutral functional differential equation with state independent delays was studied. Similar methods are used for neutral equations with multiple delays [7, 13, 15] and equations with distributional delays [8, 9, 15].

Second, in some fairly general cases, we can still obtain result on existence of solutions without the condition $k_0 < 1$. Let us consider a special case of (1.1) of the form

$$y'(t) = g(t, y(t), y(\alpha(t))) + k(t)y'(\beta(t)), \qquad t > b \tag{5.3}$$

where g,k,α and β are continuous functions, and

$$b \leq \beta(t), \alpha(t) \leq t, \qquad t \geq b.$$

Furthermore, we assume that there exist constants $c_0 > b$ and $L > 0$ such that

$$|g(t, u_1, v_1) - g(t, u_2, v_2)| \leq L(|u_1 - u_2| + |v_1 - v_2|), \;\; t \in [b, c_0], u_1, u_2, v_1, v_2 \in \mathbb{R}.$$

Then the following results hold:

Theorem 15 *If $|k(b)| < 1$ then the initial value problem (5.3,1.5) has a unique solution in $C^1[b,c]$ for some $c \in (b, c_0]$.*

Theorem 16 *Suppose that β is strictly monotonically increasing, that $\beta(t) \leq \alpha(t)$ for $t \in [b, c_0]$, and that $\beta(t) \not\equiv t$ in $[b,c]$ for any positive $c \in (b, c_0)$. If $|k(b)| > 1$ then the initial value problem (5.3,1.5) has infinitely many solutions in $C^1[b, t_0]$ for some $t_0 \in (b, c_0]$.*

Theorem 15 follows directly from our results in the previous three sections. The purpose of the assumption that $\beta(t) \not\equiv t$ in $[b,c]$ for any positive $c \in (b, c_0)$ is obvious. The proof of Theorem 16 is as following. Let ϕ be the inverse function of β. We choose $t_0 > b$ small enough such that $t_1 := \beta(t_0) < t_0$,

$$|k(t)|^{-1} \leq k_1, \qquad t \in [b, t_0]$$

and

$$\max\{2(L+L_1), 3L_1 + |g(t_0, 0, 0)|\}e^{2L_1(t_0-b)}(t_0 - b) \leq 1,$$

where $k_1 = (|k(b)|^{-1} + 1)/2 < 1$ and $L_1 = L/(1-k_1)$. In the interval $[b, t_1]$ equation (5.3) can be rewritten as

$$y'(t) = k(t)^{-1}[y'(\phi(t)) - g(\phi(t), y(\phi(t)), y(\alpha(\phi(t))))], \qquad t \in [b, t_1]. \tag{5.4}$$

Given any function $y_0 \in C^1[t_1, t_0]$ that obeys the consistent condition

$$y_0'(t_0) = g(t_0, y(t_0), y(\alpha(t_0))) + k(t_0)y'(t_1),$$

we claim that equation (5.4), in tandem with the initial condition

$$y(t) = y_0(t), \qquad t \in [t_1, t_0], \tag{5.5}$$

has a unique and continuous dependent solution in $C^1[b, t_0]$. This can be proved by extending Theorems 6, 10 and 13 in a straightforward fashion. In other word, given such a function y_0, there exist constants y_1 and y_2 such that the initial value problem (5.3,1.5) has a solution in $C^1[b, t_0]$ that obeys the condition (5.5). It should be noted that

$$y_2 = \frac{g(b, y_1, y_1)}{1 - k(b)}.$$

We now show that the converse is true. Let $y(t; y^*, t^*)$ be the solution of (5.4,5.5), where the initial function $y_0 \in C^2[b, t_0]$ is chosen in such a way that

$$\begin{aligned}
&y_0(t_0) = y_0(\alpha(t_0)) = y_0(t_1) = y^* \\
&y_0(t) = y^*, \ \ t \in [t^*, t_0] \qquad y_0''(t^*) \neq 0, \\
&y_0'(t_1) = -k(t_1)^{-1} g(t_0, y^*, y^*), \\
&|y_0(t)| \leq |y^*| + 1, \ \ |y_0'(t)| \leq |y_0'(t_1)| + 1, \qquad t \in [t_1, t^*],
\end{aligned}$$

where y^* and $t^* \in ((t_0 + t_1)/2, t_0)$ are parameters. The existence of such an initial function is obvious. It should be noted that change either y^* or t^* yields a different function y_0, hence, a different solution $y(t; y^*, t^*)$. After making simple change of variable (reverse the time variable t) we obtain from Theorem 6 the estimate

$$|y'(t; y^*, t^*)| \le -x'(t), \qquad t \in [b, t_1],$$

where $x \in C^1[b, t_1]$ is the solution of the initial value problem

$$x'(t) = -[M_1 + L_1(1 + 2x(t))], \qquad t \in [b, t_1],$$
$$x(t_1) = M_0,$$

where

$$M_0 = \max_{s \in [b, t_1]} |y_0(s)| \le |y^*| + 1$$

and

$$M_1 = \max_{s \in [b, t_1]} |y_0'(s)| \le |g(t_0, 0, 0)| + 2L|y^*|.$$

Therefore,

$$|y'(t; y^*, t^*)| \le [(2M_0 + 1)L_1 + M_1]\mathrm{e}^{2L_1(t_1 - t)}, \qquad t \in [b, t_1],$$

and hence,

$$|y(b; y^*, t^*) - y^*| \le [(2M_0 + 1)L_1 + M_1]\mathrm{e}^{2L_1(t_0 - b)}(t_0 - b) \le \frac{1}{2}(|y^*| + 1).$$

Since

$$y(b; y^*, t^*) \to \pm\infty \ \text{ as } y^* \to \pm\infty$$

and $y(b; y^*, t^*)$ is continuously dependent on y^*, we conclude that for any $y_1 \in \mathbb{R}$ and $t^* \in ((t_0 + t_1)/2, t_0)$ there exists a $y^* \in \mathbb{R}$ such that

$$y(b; y^*, t^*) = y_1.$$

Hence, (5.3,1.5) has infinitely many solutions in $C^1[b, t_0]$. This completes the proof of Theorem 16.

Nussbaum [14] showed that the initial value problem

$$x'(t) = cx'(mt) + g(t), \;\; 0 \le t \le 1, \qquad x'(0) = 0$$

can have infinitely many solutions in $C^1[0, 1]$ if $|c| > 1$. In the special case where $\alpha \equiv \beta$ and g is linear with respect to its second and third arguments, Theorem 16 has been obtained in Feldstein and Liu [4].

References

[1] K. Deimling. *Nonlinear Functional Analysis.* Springer-Verlag, 1985.

[2] R.D. Driver. Existence and continuous dependence of solutions of a neutral functional-differential equation. *Arch. Rat. Mech. Anal.*, 19:149–166, 1965.

[3] R.D. Driver. A neutral system with state-dependent delay. *J. Diff. Eqs*, 54:73–86, 1984.

[4] A. Feldstein and Y. Liu. On neutral functional differential equations with variable delays. To appear in Math. Proc. Cambridge Phil. Soc. (1998).

[5] L.J. Grimm. Existence and continuous dependence for a class of nonlinear equations. *Proc. Amer. Math. Soc.*, 29:467–473, 1971.

[6] J.K. Hale and M.A. Cruz. Existence, uniqueness, and continuous dependence for hereditary systems. *Ann. Mat. Pura. Appl.*, 85:63–81, 1970.

[7] A. Iserles and Y. Liu. On functional-differential equations with proportional delays. To appear in J. Math. Anal. Appl.

[8] A. Iserles and Y. Liu. On pantograph integro-differential equations. *J. Integral Equations and Appl.*, 6:213–237, 1994.

[9] A. Iserles and Y. Liu. Integro-differential equations and generalized hypergeometric functions. Technical Report 1995/NA6, Department of Applied Mathematics and Theoretical Physics, University of Cambridge, England, 1995. To appear in J. Math. Anal. Appl.

[10] A. Iserles and J. Terjéki. Stability and asymptotic stability of functional-differential equations. *J. London Math. Soc.*, 51:559–572, 1995.

[11] Z. Jackiewicz. Existence and uniqueness of solutions of neutral delay-differential equations with state dependent delays. *Funkcial. Ekvac.*, 30:9–17, 1987.

[12] Z. Jackiewicz. A note on existence and uniqueness of neutral functional-differential equations with state-dependent delays. *Comment. Math. Univ. Carolinae*, 36:15–17, 1995.

[13] Y. Liu. Asymptotic behaviour of functional-differential equations with proportional time delays. *European J. Appl. Maths.*, 7:11–30, 1996.

[14] R. Nussbaum. Existence and uniqueness theorems for some functional differential equations of neutral type. *J. Diff. Eqs*, 11:607–623, 1972.

[15] T. Sengadir. Existence and stability of nonlinear functional differential equations. *J. Math. Anal. Appl.*, 197:890–907, 1996.

[16] S.P. Travis. A one-dimensional two-body problem of classical electrodynamics. *SIAM J. Appl. Math.*, 28:611–632, 1975.

[17] J.R.L. Webb. Fixed point theorems for nonlinear semicontractive operators. *J. London Math. Soc.*, 1:683–688, 1969.

[18] E. Winston. Uniqueness of solutions of state dependent delay differential equations. *J. Math. Anal. Appl.*, 47:620–625, 1974.