The uniform convergence of multivariate natural splines

Aurelian Bejancu *

Department of Applied Mathematics and Theoretical Physics
Silver Street, Cambridge, England CB3 9EW

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Abstract

Let f be a function from $\mathbb{R}^d$ to $\mathbb{R}$ that has square integrable q-th order partial derivatives, where $q > d/2$. Let Ω be an arbitrary bounded set having a $\mathcal{P}_{q-1}(\mathbb{R}^d)$-unisolvent subset and consider a sequence $\{\mathcal{V}_k \ : \ k = 1, 2, 3, \ldots\}$ of sets in $\mathbb{R}^d$, each one containing a finite number of points that tend to "fill up" Ω as $k \to \infty$. We prove that the sequence $\{s_{f,\mathcal{V}_k} \ : \ k = 1, 2, 3, \ldots\}$ of the corresponding multivariate natural spline interpolants to f on $\mathcal{V}_k$ converges to f uniformly on Ω as $k \to \infty$. For $d = q = 2$, this result has been proved by Powell [16]. Here we extend his method of proof by estimating more general power functions. Related results on the uniform error of multivariate natural spline interpolation are also provided.

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1 Introduction

Denote by $BL^q(\mathbb{R}^d)$ the vector space of all distributions for which all the partial derivatives of total order q are square integrable functions over $\mathbb{R}^d$. Throughout the paper, we will assume that $q > d/2$, in which case every element of $BL^q(\mathbb{R}^d)$ is a continuous function on $\mathbb{R}^d$ (cf. [1, 13]). We equip this space with the semi-inner product

$$(u\,,v)_q = \int_{\mathbb{R}^d} \sum_{|\beta|=q} \frac{q!}{\beta!}\, \partial^\beta u \, \partial^\beta v \; dx \tag{1.1}$$

and the induced seminorm $|u|_q = (u\,,u)_q^{1/2}$, where $\beta = (\beta_1, \beta_2, \ldots, \beta_d)$ is a multi-index. Further, $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_d$, $\beta! = \beta_1!\beta_2! \cdots !\beta_d!$ and $\partial^\beta = \frac{\partial^{|\beta|}}{\partial\xi_1^{\beta_1}\partial\xi_2^{\beta_2}\cdots\partial\xi_d^{\beta_d}}$, where

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$x = (\xi_1, \xi_2, \ldots, \xi_d)$. The null-space of the above seminorm is the linear space $\mathcal{P}_{q-1}(\mathbb{R}^d)$ of all polynomials over $\mathbb{R}^d$ of total degree at most $q-1$. We let $N = \binom{d+q-1}{d}$ be the dimension of $\mathcal{P}_{q-1}(\mathbb{R}^d)$.

For $n \geq N$, let $\mathcal{V} = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^d$ be a set of scattered data points that contains a $\mathcal{P}_{q-1}(\mathbb{R}^d)$-*unisolvent* subset $\{v_1, v_2, \ldots, v_N\}$, say. By definition, this is the condition

$$p \in \mathcal{P}_{q-1}(\mathbb{R}^d) \text{ and } p(v_j) = 0,\ j = 1, 2, \ldots, N,\ \Longrightarrow\ p \equiv 0\ . \tag{1.2}$$

Given $f \in BL^q(\mathbb{R}^d)$, the *multivariate natural spline* (MNS) interpolant to f on $\mathcal{V}$ is the unique function $s_{f,\mathcal{V}}$ that solves the optimal interpolation problem

$$\text{minimize } |u|_q \text{ subject to } u \in BL^q(\mathbb{R}^d) \text{ and } u(v_j) = f(v_j),\ j = 1, 2, \ldots, n\ . \tag{1.3}$$

This problem and more general ones have been introduced and studied by Duchon [1, 2]. He proved that the solution $s_{f,\mathcal{V}}$ has the form

$$s_{f,\mathcal{V}}(x) = \sum_{i=1}^{n} \lambda_i \phi(\|x - v_i\|) + p_{f,\mathcal{V}}(x)\ ,\ x \in \mathbb{R}^d. \tag{1.4}$$

Here $p_{f,\mathcal{V}} \in \mathcal{P}_{q-1}(\mathbb{R}^d)$, $\|.\|$ denotes the Euclidean norm of $\mathbb{R}^d$, ϕ is the function

$$\phi(t) = \begin{cases} t^{2q-d} & \text{if } d \text{ is odd} \\ t^{2q-d} \ln t & \text{if } d \text{ is even} \end{cases} \tag{1.5}$$

and the coefficients $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ satisfy the equations

$$\sum_{i=1}^{n} \lambda_i p(v_i) = 0\ ,\ \forall\ p \in \mathcal{P}_{q-1}(\mathbb{R}^d)\ . \tag{1.6}$$

Hence $s_{f,\mathcal{V}}$ is a radial basis function interpolant augmented by a suitable polynomial.

In this paper we study the uniform norm of the error $f - s_{f,\mathcal{V}}$ on bounded sets. Our main result is:

Theorem 1 *Let $\Omega \in \mathbb{R}^d$ be a bounded set that includes a $\mathcal{P}_{q-1}(\mathbb{R}^d)$-unisolvent set and, for any choice of the interpolation points, let $h = h_{\mathcal{V}}$ be the number*

$$h = \sup_{x \in \Omega} \min\{\|x - v_i\|\ :\ i = 1, 2, \ldots, n\}\ . \tag{1.7}$$

Then, for sufficiently small h, the error of the MNS interpolant to any $f \in BL^q(\mathbb{R}^d)$ on $\mathcal{V}$ is bounded by the inequality

$$|f(x) - s_{f,\mathcal{V}}(x)| \leq c\,|f|_q \begin{cases} h^{1/2} & \text{if } 2q - d = 1 \\ h|\ln h|^{1/2} & \text{if } 2q - d = 2 \\ h & \text{if } 2q - d \geq 3 \end{cases} ,\ \forall\ x \in \Omega\ , \tag{1.8}$$

where c is a positive constant that depends on d, q and Ω, but not on $\mathcal{V}$, f or x.

Theorem 1 demonstrates the basic property that the MNS interpolants $s_{f,\mathcal{V}}$ are uniformly convergent to f on Ω as $h \to 0$. When $d = q = 2$, it recovers Powell's result [16, page 110], which has provided the incentive for the present extension. As in [16], our error bounds are derived from a reproducing kernel approach combined with estimates of the so-called power function. The method originates in the paper by Golomb and Weinberger [4] and it has been used in the context of problem (1.3) by Duchon [1, 2] and Meinguet [13, 14]. More recently, many authors have applied it to estimate the error in radial basis interpolation to scattered data: Madych and Nelson [10, 11], Wu and Schaback [19], Schaback [17], Powell [16], Light [5], Light and Wayne [6, 7, 8].

For each fixed $x \in \Omega$, the proof of Theorem 1 requires the selection of a suitable $\mathcal{P}_{q-1}(\mathbb{R}^d)$-unisolvent subset $\{v_1, v_2, \ldots, v_N\}$, say, of the interpolation set $\mathcal{V}$. Next, we define the unique coefficients $\{\alpha_1, \alpha_2, \ldots, \alpha_N\}$ that satisfy

$$p(x) = \sum_{i=1}^{N} \alpha_i p(v_i) \ , \ \forall \ p \in \mathcal{P}_{q-1}(\mathbb{R}^d) \tag{1.9}$$

and, as in Meinguet [13, 14], we consider the Hilbert space of continuous functions from $BL^q(\mathbb{R}^d)$ that vanish at all the points v_i, $i = 1, 2, \ldots, N$. Then, if K_x is the reproducing kernel that gives $u(x) = (K_x \, , u)_q$ for all functions u from this Hilbert space, we obtain

$$|f(x) - s_{f,\mathcal{V}}(x)| = |(K_x \, , f - s_{f,\mathcal{V}})_q| \leq |K_x|_q |f - s_{f,\mathcal{V}}|_q \leq |K_x|_q |f|_q \ . \tag{1.10}$$

Indeed, relations (1.10) depend on the interpolation conditions, on the Cauchy–Schwarz inequality and on the variational calculation, respectively. Further, we calculate the power function $|K_x|_q^2$ using

$$|K_x|_q^2 = (K_x \, , K_x)_q = K_x(x) \ , \tag{1.11}$$

and the outcome is

$$|K_x|_q^2 = c_{q,d} \left\{ \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \phi(\|v_i - v_j\|) - 2 \sum_{i=1}^{N} \alpha_i \phi(\|x - v_i\|) \right\} \ , \tag{1.12}$$

where $c_{q,d}$ is a constant. This approach is described in Section 2. In Section 3 we provide the estimates of the right hand side of (1.12) that are used to deduce (1.8).

We note that the power function (1.12) uses information given only by N of the interpolation points. In Section 2 we also describe the extension of (1.10) that makes use of all the elements of $\mathcal{V}$ and we formulate the problem of finding the best error bound for a given $\mathcal{V}$ as a best approximation problem in the sense of Sard. Section 4 is devoted to some examples of computing precise constants in error bounds, when the interpolation points are the vertices of a grid.

Accuracy orders on bounded domains for the uniform error of the interpolation procedure (1.3) were provided first by Duchon [3]. Specifically, assuming that the domain Ω is sufficiently regular, he obtained the property

$$\|f - s_{f,\mathcal{V}}\|_{L^\infty(\Omega)} = o(h^{q-\frac{d}{2}}) \ , \tag{1.13}$$

where the notations are those of Theorem 1. Various generalizations to other types of problems and a rich bibliography are given in the recent work of Matveev [12]. Both these papers are not using reproducing kernel arguments to derive the error orders.

We note that the bounds (1.8) are less strong than (1.13), the reason being that we do not include a regularity assumption on Ω. Indeed, the remark at the end of Section 3 shows that, for all admissible d and q, there exist a discrete set Ω and a function $f \in BL^q(\mathbb{R}^d)$ for which the actual error $f - s_{f,\mathcal{V}}$ calculated at one point of Ω is of magnitude h as $h \to 0$. Therefore the order of convergence given by (1.8) cannot be improved when $2q - d \geq 3$ under the general hypotheses on Ω stated in Theorem 1.

We remark that the techniques used in this paper to estimate power functions may also be applied to the more general settings described in the papers of Madych and Nelson [10, 11], Wu and Schaback [19] and Light and Wayne [8].

2 General error estimates

This section describes the reproducing kernel method for obtaining pointwise error bounds. Let $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ be a set of scattered points and let the corresponding number $h = h_{\mathcal{V}}$ be defined by (1.7). Assume that $\widetilde{\mathcal{V}} = \{v_1, v_2, \ldots, v_N\}$ is a $\mathcal{P}_{q-1}(\mathbb{R}^d)$ - unisolvent subset of $\mathcal{V}$. Then for each $x \in \mathbb{R}^d$ there exist unique values of the parameters $\{\alpha_1, \alpha_2, \ldots, \alpha_N\}$, depending on x, that are defined by the identities (1.9). Specifically, we consider the monomial basis, $\{P_1, P_2, \ldots, P_N\}$ say, of $\mathcal{P}_{q-1}(\mathbb{R}^d)$. For any set $\{x_1, x_2, \ldots, x_N\}$ of vectors from $\mathbb{R}^d$, we introduce the notation

$$D(x_1, x_2, x_3, \ldots, x_N) = \det(P_j(x_i)) \ . \tag{2.1}$$

Hence the $\mathcal{P}_{q-1}(\mathbb{R}^d)$ - unisolvence of $\widetilde{\mathcal{V}}$ is equivalent to the condition

$$D(v_1, v_2, v_3, \ldots, v_N) \neq 0 \ . \tag{2.2}$$

Since (1.9) holds if and only if

$$P_j(x) = \sum_{i=1}^{N} \alpha_i P_j(v_i) \ , \ j = 1, 2, \ldots, N \ , \tag{2.3}$$

(2.2) implies that the unique coefficients $\{\alpha_1, \alpha_2, \ldots, \alpha_N\}$ satisfying (1.9) have the values

$$\alpha_i = l_i(x) = \frac{D(v_1, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_N)}{D(v_1, v_2, v_3, \ldots, v_N)} \ , \ i = 1, 2, \ldots, N \ . \tag{2.4}$$

Therefore each coefficient α_i , $i = 1, 2, \ldots, N$, is the value at x of the Lagrange interpolation polynomial $l_i \in \mathcal{P}_{q-1}(\mathbb{R}^d)$ defined by

$$l_i(v_j) = \delta_{ij} \ , \ j = 1, 2, \ldots, N \ . \tag{2.5}$$

Our error analysis is based on the following result.

Proposition 1 *Let $f \in BL^q(\mathbb{R}^d)$ and suppose that $\mathcal{V}$ contains a $\mathcal{P}_{q-1}(\mathbb{R}^d)$-unisolvent subset $\widetilde{\mathcal{V}} = \{v_1, v_2, \ldots, v_N\}$. Let x be any fixed point of $\mathbb{R}^d$ and let the coefficients*

$\{\alpha_1, \alpha_2, \ldots, \alpha_N\}$ *be defined by* (1.9). *Then the error of the MNS interpolant to* $f(x)$ *is bounded by*

$$|f(x) - s_{f,\mathcal{V}}(x)| \le [\Phi(x, \widetilde{\mathcal{V}})]^{1/2} |f|_q \ , \tag{2.6}$$

where

$$\Phi(x, \widetilde{\mathcal{V}}) = c_{q,d} \left\{ \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \phi(\|v_i - v_j\|) - 2 \sum_{i=1}^{N} \alpha_i \phi(\|x - v_i\|) \right\} \tag{2.7}$$

and the constant $c_{q,d}$ *is defined in* (2.12) *below.*

Proof. Our arguments use the exposition of Meinguet [13, 14].

For any $u \in BL^q(\mathbb{R}^d)$, consider the $\mathcal{P}_{q-1}(\mathbb{R}^d)$-interpolant Πu of u on $\widetilde{\mathcal{V}}$, which is defined uniquely by the Lagrange-type formula

$$\Pi u = \sum_{j=1}^{N} u(v_j) l_j \ , \tag{2.8}$$

where $l_1, l_2, \ldots, l_N$ are given by (2.5). Then $\Pi : BL^q(\mathbb{R}^d) \longrightarrow BL^q(\mathbb{R}^d)$ is a linear projector with range $\mathcal{P}_{q-1}(\mathbb{R}^d)$ and kernel

$$\ker \Pi = \{u \in BL^q(\mathbb{R}^d) \ : \ u(v_j) = 0 \ , \ j = 1, 2, \ldots, N\} \ . \tag{2.9}$$

It follows that the semi-inner product (1.1) induces a Hilbert structure on the subspace $\ker \Pi$ of $BL^q(\mathbb{R}^d)$. Moreover, as shown in [13], for each fixed $x \in \mathbb{R}^d$, the evaluation linear functional δ_x is continuous with respect to the Hilbert space topology of $\ker \Pi$. Therefore the Riesz representation theorem provides the formula

$$u(x) = \delta_x(u) = (K_x\, , u)_q \ , \ \forall\ u \in \ker \Pi \ , \ \forall\ x \in \mathbb{R}^d \ , \tag{2.10}$$

where $K_x \in \ker \Pi$ is the Riesz representer of δ_x. In other words, the function $\{K(y, z) = K_y(z) \ : \ y, z \in \mathbb{R}^d\}$ is the reproducing kernel function of the Hilbert space $\ker \Pi$. Further, Meinguet [14] finds that K_x is the function

$$\begin{aligned} K_x(y) \ = \ c_{q,d} \Bigg\{ & \phi(\|y - x\|) - \sum_{j=1}^{N} l_j(x) \phi(\|y - v_j\|) - \sum_{i=1}^{N} l_i(y) \phi(\|v_i - x\|) \\ & + \sum_{i=1}^{N} \sum_{j=1}^{N} l_i(y) l_j(x) \phi(\|v_i - v_j\|) \Bigg\} \ , \ y \in \mathbb{R}^d \ , \end{aligned} \tag{2.11}$$

where

$$c_{q,d} = \begin{cases} \dfrac{(-1)^{q-\frac{d-1}{2}} (q - \frac{d+1}{2})!}{2^d \pi^{\frac{d-1}{2}} (q-1)! (2q-d)!} & \text{if } d \text{ is odd} \\[2ex] \dfrac{(-1)^{q-\frac{d}{2}+1}}{2^{2q-1} \pi^{d/2} (q-1)! (q - \frac{d}{2})!} & \text{if } d \text{ is even} \end{cases} \ . \tag{2.12}$$

We deduce that (1.10) holds and we use (1.11), (2.4), (2.11) and the fact that $\phi(0) = 0$ to obtain (1.12).

Letting $\Phi(x, \widetilde{\mathcal{V}}) = |K_x|_q^2$ and combining (1.10) and (1.12) gives the required result. $\square$

The estimate (2.6) makes use of information given only by the subset $\widetilde{\mathcal{V}}$ of $\mathcal{V}$. An improved error bound can be obtained by letting the power function depend on more interpolation points from $\mathcal{V}$. Indeed, finding the best error bound for a given $\mathcal{V}$ can be phrased as a best approximation problem in the sense of Sard, as shown below.

To address this problem in the setting of Proposition 1, we fix a point $x \in \mathbb{R}^d$ and we choose the real coefficients $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ so that

$$p(x) = \sum_{i=1}^{n} \gamma_i p(v_i) \ , \ \forall \ p \in \mathcal{P}_{q-1}(\mathbb{R}^d) \ . \tag{2.13}$$

Since $\widetilde{\mathcal{V}}$ is a $\mathcal{P}_{q-1}(\mathbb{R}^d)$-unisolvent subset of $\mathcal{V}$, (1.9) provides at least one such choice, namely $\gamma_i = \alpha_i$, $i = 1, 2, \ldots, N$ and $\gamma_i = 0$, $i = N+1, N+2, \ldots, n$. Next we consider the linear functional $L_{x,\gamma} = \delta_x - \sum_{i=1}^{n} \gamma_i \delta_{v_i}$, i.e.

$$L_{x,\gamma}(u) = u(x) - \sum_{i=1}^{n} \gamma_i u(v_i) \ , \tag{2.14}$$

for any continuous function u. Condition (2.13) states that $L_{x,\gamma}$ annihilates $\mathcal{P}_{q-1}(\mathbb{R}^d)$. By Theorem 1.1 of Meinguet [14], $L_{x,\gamma}$ is a continuous linear functional on $BL^q(\mathbb{R}^d)$, so its restriction to the subspace $\ker \Pi$ defined in (2.9) is also a continuous linear functional and

$$L_{x,\gamma}(u) = (l_{x,\gamma} \ , u)_q \ , \ \forall \ u \in \ker \Pi \ , \tag{2.15}$$

where $l_{x,\gamma} \in \ker \Pi$ is the Riesz representer of $L_{x,\gamma}$.

The *best approximation to δ_x in the sense of Sard* is the problem of choosing those multipliers $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ in (2.13) and (2.14) which minimize the seminorm $|l_{x,\gamma}|_q$. Schoenberg [18] has characterized the solution to this problem in the context of univariate natural splines by using cardinal functions.

Let $\{S_1, S_2, \ldots, S_n\}$ be the cardinal functions of the interpolation procedure (1.3), so they satisfy

$$S_i(v_j) = \delta_{ij} \ , \ \forall \ i, j \in \{1, 2, \ldots, n\} \ . \tag{2.16}$$

Using our previous notation, for any $f \in BL^q(\mathbb{R}^d)$ we then have

$$s_{f,\mathcal{V}}(x) = \sum_{i=1}^{n} f(v_i) S_i(x) \ , \ x \in \mathbb{R}^d. \tag{2.17}$$

Since the optimal interpolation problem (1.3) has the reproducing property $s_{p,\mathcal{V}} \equiv p$, $\forall \ p \in \mathcal{P}_{q-1}(\mathbb{R}^d)$, (2.17) shows that $L_{x,\tilde{\gamma}}$ annihilates $\mathcal{P}_{q-1}(\mathbb{R}^d)$ if we let $\tilde{\gamma}_i = S_i(x)$, $i = 1, 2, \ldots, n$. Hence (2.15) holds for γ replaced by $\tilde{\gamma}$. Remarkably, the following Schoenberg type result proves that the best approximation to δ_x in the sense of Sard is given by letting the multipliers of (2.13) and (2.14) have the values $\gamma_i = \tilde{\gamma}_i$, $i = 1, 2, \ldots, n$.

Proposition 2 *Let x be any point of $\mathbb{R}^d$, let the real coefficients $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ satisfy (2.13) and define the set $\{\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n\}$ as in the last paragraph. Then the latter coefficients achieve the optimality property*

$$|l_{x,\tilde{\gamma}}|_q \leq |l_{x,\gamma}|_q \ . \tag{2.18}$$

Proof. Using the Cauchy–Schwarz inequality in (2.15), we find

$$|u(x) - \sum_{i=1}^{n} \gamma_i u(v_i)| \leq |l_{x,\gamma}|_q |u|_q \ , \ \forall \ u \in \ker \Pi \ . \tag{2.19}$$

Replacing u in (2.19) by $f - s_{f,\mathcal{V}}$ for any $f \in BL^q(\mathbb{R}^d)$ and taking into account the interpolation conditions $f(v_j) = s_{f,\mathcal{V}}(v_j)$, $j = 1, 2, \ldots, n$, and the variational calculation, we derive

$$|f(x) - s_{f,\mathcal{V}}(x)| \leq |l_{x,\gamma}|_q |f - s_{f,\mathcal{V}}|_q \leq |l_{x,\gamma}|_q |f|_q \ , \ \forall \ f \in BL^q(\mathbb{R}^d) \ . \tag{2.20}$$

Hence (2.17) and $\tilde{\gamma}_i = S_i(x)$, $i = 1, 2, \ldots, n$, imply

$$|f(x) - \sum_{i=1}^{n} \tilde{\gamma}_i f(v_i)| \leq |l_{x,\gamma}|_q |f|_q \ , \ \forall \ f \in BL^q(\mathbb{R}^d) \ . \tag{2.21}$$

On the other hand, for γ replaced by $\tilde{\gamma}$, (2.15) yields

$$u(x) - \sum_{i=1}^{n} \tilde{\gamma}_i u(v_i) = (l_{x,\tilde{\gamma}} \, , u)_q \ , \ \forall \ u \in \ker \Pi \ . \tag{2.22}$$

Therefore, letting $f = u = l_{x,\tilde{\gamma}}$ in (2.21) and (2.22), we obtain

$$|l_{x,\tilde{\gamma}}|_q^2 \leq |l_{x,\gamma}|_q |l_{x,\tilde{\gamma}}|_q \ , \tag{2.23}$$

which shows that (2.18) is true. □

Our next aim is to compute the $BL^q(\mathbb{R}^d)$-seminorm of $l_{x,\gamma}$. By (2.14) and (2.15),

$$|l_{x,\gamma}|_q^2 = L_{x,\gamma}(l_{x,\gamma}) = l_{x,\gamma}(x) - \sum_{i=1}^{n} \gamma_i l_{x,\gamma}(v_i) \ . \tag{2.24}$$

Since $L_{x,\gamma}$ annihilates $\mathcal{P}_{q-1}(\mathbb{R}^d)$, we only need to know the form of $l_{x,\gamma}$ modulo $\mathcal{P}_{q-1}(\mathbb{R}^d)$. Each of the functions l_j, $j = 1, 2, \ldots, N$, is a polynomial. Therefore, using (2.10), (2.11), (2.14) and (2.15), we have the following relations for an arbitrary $y \in \mathbb{R}^d$:

$$\begin{aligned}
l_{x,\gamma}(y) &= (K_y \, , l_{x,\gamma})_q \;=\; L_{x,\gamma}(K_y) \\
&= c_{q,d} L_{x,\gamma} \left\{ \phi(\|y - .\|) - \sum_{j=1}^{N} l_j(y) \phi(\|v_j - .\|) \right\} \\
&= c_{q,d} L_{x,\gamma}(\phi(\|y - .\|)) \pmod{\mathcal{P}_{q-1}(\mathbb{R}^d)} \\
&= c_{q,d} \left\{ \phi(\|y - x\|) - \sum_{i=1}^{n} \gamma_i \phi(\|y - v_i\|) \right\} \pmod{\mathcal{P}_{q-1}(\mathbb{R}^d)} \ .
\end{aligned} \tag{2.25}$$

Hence substituting (2.25) into (2.24) for $y = x, v_1, v_2, \ldots, v_n$, we get

$$|l_{x,\gamma}|_q^2 = c_{q,d} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_i \gamma_j \phi(\|v_i - v_j\|) - 2 \sum_{i=1}^{n} \gamma_i \phi(\|x - v_i\|) \right\} . \tag{2.26}$$

It follows that (2.20) gives the required extension of the error bound (2.6), the power function being provided by the right hand side of (2.26). We remark that Proposition 2 is a reformulation of the result which forms the basis for the approach of Wu and Schaback [19, page 15].

To apply (2.6) or (2.20), one needs estimates of the power function. The next lemma will simplify our arguments.

Lemma 1 *Let all the vectors in $\mathbb{R}^d$ be scaled by the positive factor σ, so any parameters $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ that satisfy* (2.13) *are not changed. Then the power function defined by the right hand side of* (2.26) *is scaled by the factor σ^{2q-d}.*

Proof. From (2.26) and the definition (1.5), this result is clear if d is odd. When d is even, let Φ_{old} and Φ_{new} be the values of the power function (2.26) before and after the scaling, respectively. We deduce the formula

$$\begin{aligned} \Phi_{new} &= \sigma^{2q-d} \Phi_{old} + c_{q,d} \sigma^{2q-d} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_i \gamma_j \|v_i - v_j\|^{2q-d} \right. \\ &\qquad \left. - 2 \sum_{i=1}^{n} \gamma_i \|x - v_i\|^{2q-d} \right\} \log \sigma . \end{aligned} \tag{2.27}$$

Set $\hat{n} = n+1$, $\hat{\gamma}_i = \gamma_i$ and $\hat{v}_i = v_i$ for $i = 1, 2, \ldots, n$, $\hat{\gamma}_{n+1} = -1$ and $\hat{v}_{n+1} = x$. Then the term inside the braces has the value

$$\sum_{i=1}^{\hat{n}} \sum_{j=1}^{\hat{n}} \hat{\gamma}_i \hat{\gamma}_j \|\hat{v}_i - \hat{v}_j\|^{2q-d} \tag{2.28}$$

and we note that, due to (2.13),

$$\sum_{i=1}^{\hat{n}} \hat{\gamma}_i p(\hat{v}_i) = 0 \ , \ \forall \ p \in \mathcal{P}_{q-1}(\mathbb{R}^d) \ . \tag{2.29}$$

But (2.29) is precisely the condition (3.33) from [15]. Therefore we apply Lemma 3.5 of [15] to find that expression (2.28) is zero, which gives the required result. □

3 The uniform convergence on bounded sets

This section provides the proof of Theorem 1. Since the basic ingredient in our proof is Proposition 1, we claim that, given $x \in \Omega$ and given a $\mathcal{P}_{q-1}(\mathbb{R}^d)$-unisolvent subset $\widetilde{\mathcal{V}}$ of $\mathcal{V}$, there is no loss of generality throughout this section in assuming

$$\Omega \cup \widetilde{\mathcal{V}} \subseteq B(0, 1/2) \ , \tag{3.1}$$

where $B(0,1/2)=\{z\in\mathbb{R}^d \ : \ \|z\|\le 1/2\}$. Indeed, scaling all the vectors in $\mathbb{R}^d$ by a positive factor σ does not alter the values of $\{\alpha_1,\alpha_2,\ldots,\alpha_N\}$ in (2.4), and expression (2.7) is scaled by the factor σ^{2q-d} due to Lemma 1. On the other hand, by the scale invariance properties of our interpolation problem (1.3), the MNS interpolant to the new function $\{\bar f(y)=f(\sigma^{-1}y) \ : \ y\in\mathbb{R}^d\}$ on $\sigma\mathcal{V}$ is $\{s_{\bar f,\sigma\mathcal{V}}(y)=s_{f,\mathcal{V}}(\sigma^{-1}y) \ : \ y\in\mathbb{R}^d\}$, i.e. letting $y=\sigma x$ we have

$$f(x)-s_{f,\mathcal{V}}(x)=\bar f(y)-s_{\bar f,\sigma\mathcal{V}}(y)\ , \ x\in\mathbb{R}^d. \tag{3.2}$$

Since $|\bar f|_q^2=\sigma^{-(2q-d)}|f|_q^2$, these arguments show that our error estimates based on (2.6) are scale-invariant.

Note that, if $\{v_1,v_2,\ldots,v_N\}$ is a fixed $\mathcal{P}_{q-1}(\mathbb{R}^d)$-unisolvent set and if we consider a sequence of values of x such that $\|x-v_1\|\to 0$, then (2.4) implies $\alpha_1\to 1$ and $\alpha_i\to 0$, $i=2,3,\ldots,N$. (Indeed, the function D of definition (2.1) is continuous in each of its arguments.) This observation is the basis for the next Lemma, which gives a useful bound on expression (2.7).

Specifically, we fix $x\in\Omega$ and assume that there exists a $\mathcal{P}_{q-1}(\mathbb{R}^d)$-unisolvent subset $\widetilde{\mathcal{V}}=\{v_1,v_2,\ldots,v_N\}$ of $\mathcal{V}$ satisfying

$$\|x-v_1\|\le h \tag{3.3}$$

and

$$\max\{|\alpha_1-1|,|\alpha_2|,\ldots,|\alpha_N|\}\le \tilde c\, h\ , \tag{3.4}$$

where $\{\alpha_1,\alpha_2,\ldots,\alpha_N\}$ are the coefficients defined by (1.9), while $\tilde c$ is a given positive constant.

Lemma 2 *Let Ω, $\mathcal{V}$ and h be as in* Theorem 1 *and let $h\le e^{-1}$. Fix a point $x\in\Omega$ and then pick a set $\widetilde{\mathcal{V}}$ that satisfies the conditions of the last paragraph. Further, suppose that none of the points $v_2,v_3,\ldots,v_N$ is situated on the closed line segment between x and v_1. Then the corresponding expression* (2.7) *has the property*

$$\Phi(x,\widetilde{\mathcal{V}})\le c^2\begin{cases} h & \text{if } \ 2q-d=1\\ h^2|\ln h| & \text{if } \ 2q-d=2\\ h^2 & \text{if } \ 2q-d\ge 3\end{cases}\ , \tag{3.5}$$

where the value of the constant c is given in (3.23) *below.*

Proof. Expression (2.7) can be written in the form

$$\Phi(x,\widetilde{\mathcal{V}})=2c_{q,d}\left\{T_1+T_2-\alpha_1\phi(\|x-v_1\|)\right\}\ , \tag{3.6}$$

where

$$T_1=\sum_{i=2}^{N-1}\sum_{j=i+1}^{N}\alpha_i\alpha_j\phi(\|v_i-v_j\|)+\sum_{i=2}^{N}(\alpha_1-1)\alpha_i\phi(\|v_1-v_i\|) \tag{3.7}$$

and

$$T_2=\sum_{i=2}^{N}\alpha_i[\phi(\|v_1-v_i\|)-\phi(\|x-v_i\|)]\ . \tag{3.8}$$

We seek bounds on each of the three terms on the right hand side of (3.6).

First we consider the variation of the function $\{|\phi(t)| \ : \ 0 \leq t \leq 1\}$. By (1.5), if d is odd, then $|\phi|$ increases strictly monotonically. Alternatively, when d is even, this function has the derivative

$$-\phi'(t) = -t^{2q-d-1}[1 + (2q-d)\ln t] \ , \tag{3.9}$$

so we find strict increasing monotonicity of $|\phi|$ on $[0, \exp(-1/(2q-d))]$ and strict decreasing monotonicity of $|\phi|$ on $[\exp(-1/(2q-d)), 1]$.

In order to bound the last term in (3.6), we note that (3.4) provides

$$|\alpha_1| \leq 1 + |\alpha_1 - 1| \leq 1 + \tilde{c}\, h \leq 1 + \tilde{c}\, e^{-1} \ . \tag{3.10}$$

Since $h \leq e^{-1} \leq \exp(-1/(2q-d))$, (3.3) and the previous paragraph imply

$$|\phi(\|x - v_1\|)| \leq |\phi(h)| = \begin{cases} h^{2q-d} & \text{if } d \text{ is odd} \\ h^{2q-d}|\ln h| & \text{if } d \text{ is even} \end{cases} \ . \tag{3.11}$$

It follows that

$$|-\alpha_1 \phi(\|x - v_1\|)| \leq (1 + \tilde{c}\, e^{-1}) \begin{cases} h^{2q-d} & \text{if } d \text{ is odd} \\ h^{2q-d}|\ln h| & \text{if } d \text{ is even} \end{cases} \ . \tag{3.12}$$

In order to bound T_1, we note that (3.1) implies

$$\|v_i - v_j\| \leq 1 \ , \ i,j = 1, 2, \ldots, N \ . \tag{3.13}$$

By (3.4), (3.13) and the variation of $|\phi|$ on [0,1], each term that is summed in T_1 is bounded above in modulus by

$$\tilde{c}^2 h^2 \max\left\{1, \frac{1}{(2q-d)e}\right\} = \tilde{c}^2 h^2 \ . \tag{3.14}$$

Since T_1 has $(N-2)(N-1)/2 + N - 1 = N(N-1)/2$ terms, we obtain

$$|T_1| \leq \frac{N(N-1)}{2}\, \tilde{c}^2 h^2 \ . \tag{3.15}$$

Next, we set $\psi_i(\theta) = \phi(\|\theta v_1 + (1-\theta)x - v_i\|)$, $0 \leq \theta \leq 1$, $i = 2, 3, \ldots, N$, so (3.8) becomes

$$T_2 = \sum_{i=2}^{N} \alpha_i [\psi_i(1) - \psi_i(0)] \ . \tag{3.16}$$

Moreover, we have the bound

$$\|(1-\theta)(x - v_1) + v_1 - v_i\| \leq \|x - v_1\| + \|v_1 - v_i\| \leq h + 1 \leq e^{-1} + 1 \ . \tag{3.17}$$

The assumption that none of $v_2, v_3, \ldots, v_N$ is on the straight line segment between x and v_1 ensures that ψ_i has the continuous first derivative

$$\begin{aligned} \psi_i'(\theta) &= \frac{d}{d\theta}\phi(\|\theta(v_1 - x) + x - v_i\|) \\ &= \left(\frac{d}{d\theta}\|\theta(v_1 - x) + x - v_i\|\right) \phi'(\|(1-\theta)(x - v_1) + v_1 - v_i\|) \ . \end{aligned} \tag{3.18}$$

Taking account of (3.17) we obtain

$$|\psi_i'(\theta)| \le \|x - v_1\| \max_{0\le r\le 1+e^{-1}} |\phi'(r)| \ , \ i = 2,3,\ldots,N \ , \ \forall\, \theta \in [0,1] \ . \tag{3.19}$$

Since $|\phi'|$ is a continuous function on $[0,\infty)$ by the definition (1.5), the quantity

$$c_0 = \max_{0\le r\le 1+e^{-1}} |\phi'(r)| \tag{3.20}$$

is a positive constant that depends only on d and q. Making use of (3.4), (3.19) and the elementary condition

$$|\psi_i(1) - \psi_i(0)| \le \max\{|\psi_i'(\theta)| \ : \ 0 \le \theta \le 1\} \ , \tag{3.21}$$

we bound expression (3.16) by

$$|T_2| \le \sum_{i=2}^{N} |\alpha_i||\psi_i(1) - \psi_i(0)| \le (N-1)c_0\tilde{c}\, h^2 \ . \tag{3.22}$$

Finally, we use (3.6), (3.12), (3.15) and (3.22) to deduce (3.5), where

$$c^2 = |c_{q,d}|\,\{N(N-1)\tilde{c}^2 + 2(N-1)c_0\tilde{c} + 2(1+\tilde{c}e^{-1})\} \ . \tag{3.23}$$

Note that the case $q = d = 1$ is simpler, since then $N = 1$, which implies $T_1 = T_2 = 0$. □

The proof of Theorem 1 requires the property of the determinant (2.1) that is given in the following lemma. Further, for every x in the closure $\overline{\Omega}$ of Ω, we let $\Delta(x)$ be the number

$$\Delta(x) = \max\{|D(x, x_2, x_3, \ldots, x_N)| \ : \ x_i \in \overline{\Omega} \ , \ i = 2,3,\ldots,N\} \ . \tag{3.24}$$

Lemma 3 *There exists a constant $M > 0$ that depends only on d and q, such that for any vectors $y_1, y_2, u_2, u_3, \ldots, u_N$ from the ball $B(0,1/2)$ we have:*

$$|D(y_1, u_2, u_3, \ldots, u_N) - D(y_2, u_2, u_3, \ldots, u_N)| \le M\|y_1 - y_2\| \ . \tag{3.25}$$

Consequently, the function $\{\Delta(x) \ : \ x \in \overline{\Omega}\}$ is Lipschitz continuous.

Proof. From the definition (2.1),

$$\begin{aligned} &D(y_1, u_2, u_3, \ldots, u_N) - D(y_2, u_2, u_3, \ldots, u_N) \\ &\quad = \det \begin{bmatrix} P_1(y_1) - P_1(y_2) & P_2(y_1) - P_2(y_2) & \cdots & P_N(y_1) - P_N(y_2) \\ P_1(u_2) & P_2(u_2) & \cdots & P_N(u_2) \\ \vdots & \vdots & & \vdots \\ P_1(u_N) & P_2(u_N) & \cdots & P_N(u_N) \end{bmatrix}, \end{aligned} \tag{3.26}$$

where $\{P_1, P_2, \ldots, P_N\}$ is the monomial basis of $\mathcal{P}_{q-1}(\mathbb{R}^d)$. Consider expanding the above determinant in the cofactors of the first row. Since $y_1, y_2 \in B(0,1/2)$, algebraic arguments show that there exists a constant $C > 0$, depending only on d and q, such that

$$|P_j(y_1) - P_j(y_2)| \leq C\|y_1 - y_2\| \ , \ j = 1, 2, \ldots, N \ . \tag{3.27}$$

Further, each cofactor of the first row is bounded above in modulus by a constant, depending only on d and q, since $u_j \in B(0, 1/2)$, $\forall\ j = 2, 3, \ldots, N$. Consequently, using (3.27) we obtain (3.25) as claimed. The Lipschitz continuity of $\Delta(x)$ now follows because, if $y_1, y_2 \in \overline{\Omega}$ satisfy $\Delta(y_1) \geq \Delta(y_2)$ and if $\Delta(y_1) = |D(y_1, u_2, u_3, \ldots, u_N)|$ say, with $u_i \in \overline{\Omega}$, $i = 2, 3, \ldots, N$, then

$$\begin{aligned} \Delta(y_1) - \Delta(y_2) &\leq |D(y_1, u_2, u_3, \ldots, u_N)| - |D(y_2, u_2, u_3, \ldots, u_N)| \\ &\leq |D(y_1, u_2, u_3, \ldots, u_N) - D(y_2, u_2, u_3, \ldots, u_N)| \\ &\leq M\|y_1 - y_2\| \ , \end{aligned} \tag{3.28}$$

where we have used the assumption (3.1) that $\Omega \subseteq B(0, 1/2)$. □

Proof of Theorem 1. By the argument given at the beginning of this section, we assume without loss of generality that (3.1) holds.

We set

$$\Delta = \inf\{\Delta(x) \ : \ x \in \overline{\Omega}\} \ , \tag{3.29}$$

which is positive, due to the continuity of the function $\Delta(x)$ on $\overline{\Omega}$ (Lemma 3) and to the fact that Ω contains at least one set of $\mathcal{P}_{q-1}(\mathbb{R}^d)$-unisolvent points. Let

$$h_0 = \frac{\Delta}{(N+2)M} \ , \tag{3.30}$$

where M is given in Lemma 3.

Let $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ be any finite set of points in $\mathbb{R}^d$ and let $h = h_{\mathcal{V}}$ be defined by equation (1.7). We will prove the assertion of the theorem when

$$h \leq \min\{h_0, e^{-1}\} \ . \tag{3.31}$$

To this end, let x be a general point of Ω. Most of the proof will establish that, after a re-labelling, there exist points $v_1, v_2, \ldots, v_N$ in $\mathcal{V}$ such that the conditions of Lemma 2 are satisfied, because then the required result will be a consequence of Proposition 1.

From the definition (3.29), we can pick points $x_2, x_3, \ldots, x_N$ in $\overline{\Omega}$ such that

$$|D(x, x_2, x_3, \ldots, x_N)| \geq \Delta \ . \tag{3.32}$$

Taking into account the definition (1.7) and after a re-labelling, we select a subset $\widetilde{\mathcal{V}} = \{v_1, v_2, \ldots, v_N\}$ of $\mathcal{V}$ that satisfies

$$\max\{\|x - v_1\|, \|x_2 - v_2\|, \ldots, \|x_N - v_N\|\} \leq h \ . \tag{3.33}$$

Although (1.7) allows some points of $\mathcal{V}$ to occur more than once in $\widetilde{\mathcal{V}}$, this does not happen. Indeed, the following argument shows that (2.2) holds, which is equivalent to the $\mathcal{P}_{q-1}(\mathbb{R}^d)$-unisolvence of the selected subset $\widetilde{\mathcal{V}}$ of $\mathcal{V}$. We use (3.32), (3.33) and we apply (3.25) N times to deduce

$$\begin{aligned} |D(v_1, v_2, v_3, \ldots, v_N)| \;&\geq\; |D(x, x_2, x_3, \ldots, x_N)| \\ &- \; |D(x, x_2, x_3, \ldots, x_N) \;-\; D(v_1, x_2, x_3, \ldots, x_N)| \\ &- \; |D(v_1, x_2, x_3, \ldots, x_N) \;-\; D(v_1, v_2, x_3, \ldots, x_N)| \\ &- \; \cdots \; - \; |D(v_1, v_2, \ldots, v_{N-1}, x_N) \;-\; D(v_1, v_2, \ldots, v_{N-1}, v_N)| \\ &\geq \; \Delta \; - \; NMh \; \geq \; \Delta \; - \; NMh_0 \; . \end{aligned} \tag{3.34}$$

Since (3.30) gives $\Delta - NMh_0 = 2Mh_0 > 0$, it follows that (2.2) holds true.

The next step is to prove the inequality

$$\|x - x_i\| > 2h \; , \; i = 2, 3, \ldots, N \; , \tag{3.35}$$

because (3.33), (3.35) and the triangle inequality then imply

$$\|x - v_i\| > h \; , \; i = 2, 3, \ldots, N \; , \tag{3.36}$$

so none of the points $v_2, v_3, \ldots, v_N$ is on the line segment joining x and v_1. By contradiction, assume that we have

$$\|x - x_k\| \leq 2h \tag{3.37}$$

for at least an index $k \in \{2, 3, \ldots, N\}$. Since $\Omega \subseteq B(0, 1/2)$, we let $y_1 = x$, $y_2 = x_k$, $u_j = x_j$, $j = 2, 3, \ldots, N$ in (3.25) to obtain

$$|D(x, x_2, x_3, \ldots, x_N)| \leq M\|x - x_k\| \; . \tag{3.38}$$

But (3.30), (3.31), (3.32), (3.37) and (3.38) are contradictory, so (3.35) is true.

Since (3.3) is contained in (3.33), the only hypothesis of Lemma 2 that remains to be verified is (3.4), where we use the definition (2.4) of the coefficients $\{\alpha_1, \alpha_2, \ldots, \alpha_N\}$. For $i = 2, 3, \ldots, N$, note that $\widetilde{\mathcal{V}} \subset B(0, 1/2)$ allows us to estimate the numerator of (2.4) similarly to the derivation of inequality (3.38), so we find

$$|D(v_1, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_N)| \leq M\|x - v_1\| \; . \tag{3.39}$$

Hence (2.4), (3.3), (3.30), (3.34) and (3.39) imply

$$|\alpha_i| \leq \frac{Mh}{\Delta - NMh_0} = \frac{h}{2h_0} \; , \; i = 2, 3, \ldots, N \; . \tag{3.40}$$

Further, letting $y_1 = x$, $y_2 = v_1$, $u_j = x_j$, $j = 2, 3, \ldots, N$ in (3.25) we deduce

$$\begin{aligned} |\alpha_1 - 1| \;&=\; \frac{|D(x, v_2, v_3, \ldots, v_N) - D(v_1, v_2, v_3, \ldots, v_N)|}{|D(v_1, v_2, v_3, \ldots, v_N)|} \\ &\leq\; \frac{M\|x - v_1\|}{\Delta - NMh_0} \;\leq\; \frac{Mh}{\Delta - NMh_0} \;=\; \frac{h}{2h_0} \; . \end{aligned} \tag{3.41}$$

It follows from (3.40) and (3.41) that (3.4) is satisfied by letting $\tilde{c} = 1/(2h_0)$, which is a constant that depends only on d, q and Ω. Further, the modifications of all the above arguments when $d = q = 1$ are straightforward. The proof is complete. □

Remark. As mentioned in the Introduction, we provide here examples to show that the error $f - s_{f,\mathcal{V}}$ can take values of magnitude h as $h \to 0$, for all admissible d and q. Let $\Omega \in \mathbb{R}^d$ be the discrete set $\{0, v_2, v_3, \ldots, v_n\}$ of n distinct points, where n is a fixed natural number, $n \geq N$, and $\|v_i\| \geq 1$, $\forall\ i = 2, 3, \ldots, n$. Further, we require that Ω contains a $\mathcal{P}_{q-1}(\mathbb{R}^d)$-unisolvent subset. For a sequence of values of h that tend to zero, let $v_1 = d^{-1/2}(h, h, \ldots, h)^T \in \mathbb{R}^d$, so $\|v_1\| = h$, and let our set of interpolation points be $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$. We choose $f \in C_0^\infty(\mathbb{R}^d)$ satisfying $|f|_q > 0$, $f(x) = \xi_1 + \xi_2 + \cdots + \xi_d$ for $\|x\| \leq 1/2$, $x = (\xi_1, \xi_2, \ldots, \xi_d)$, and $f(x) = 0$ for $\|x\| \geq 1$. Since $f(v_i) = 0$, $i = 2, 3, \ldots, n$, we have

$$s_{f,\mathcal{V}}(0) = f(v_1) S_1(0) , \tag{3.42}$$

where S_1 is the MNS cardinal function defined by $S_1(v_1) = 1$, $S_1(v_i) = 0$, $i = 2, 3, \ldots, n$. Moreover, if we let $g \in C_0^\infty(\mathbb{R}^d)$ be such that $g(x) = 1$ for $\|x\| \leq 1/2$ and $g(x) = 0$ for $\|x\| \geq 1$, then S_1 is the MNS interpolant to g on $\mathcal{V}$ for sufficiently small h. Since $g(0) = 0$, Theorem 1 gives

$$S_h(0) = 1 + \begin{cases} O(h^{1/2}) & \text{if } 2q - d = 1 \\ O(h|\ln h|^{1/2}) & \text{if } 2q - d = 2 \\ O(h) & \text{if } 2q - d \geq 3 \end{cases} , \text{ as } h \to 0 . \tag{3.43}$$

It follows from $f(0) = 0$ and (3.42) that the actual error $f(0) - s_{f,\mathcal{V}}(0)$ is of magnitude $f(v_1) = h\sqrt{d}$ as $h \to 0$, which gives the required result.

4 More applications of the power function method

So far, we have applied the reproducing kernel technique only to scattered interpolation points, by finding generic constants that bound the power function. On the other hand, when the interpolation points form a special finite structure (such as a triangulation or a regular square grid in two dimensions), one may be able to obtain precise numerical constants in the error estimates. Such results have already been given by Powell [15] for the case $q = d = 2$, i.e. for thin plate splines in two dimensions. Since $d = 1$ and $d = 3$ are the only other possibilities allowed by the condition $q > d/2$ when $q = 2$, we will provide here the corresponding constants for these two cases.

When $d = 1$ and $q = 2$, the radial function is $\phi(t) = t^3$, i.e. we are using natural cubic spline interpolation at the knots $v_1 < v_2 < \cdots < v_n$, say, on the real line.

Proposition 3 *Let $f \in BL^2(\mathbb{R})$ and $\sigma = \max\{v_{i+1} - v_i \ : \ i = 1, 2, \ldots, n-1\}$. Then the error of the cubic spline interpolant $s_{f,\mathcal{V}}$ is bounded by the inequality*

$$|f(x) - s_{f,\mathcal{V}}(x)| \leq \frac{1}{4\sqrt{3}} |f|_2\, \sigma^{3/2} , \ \forall\ x \in [v_1, v_n] . \tag{4.1}$$

In addition, the factor $1/(4\sqrt{3})$ cannot be replaced by a smaller number that is independent of n and f.

Proof. Fix $x \in [v_i, v_{i+1}]$, hence $x = (1-\alpha)v_i + \alpha v_{i+1}$, where $\alpha \in [0,1]$ and $i \in \{1, 2, \ldots, n-1\}$. We work with $N = 2$, which is the dimension of $\mathcal{P}_1(\mathbb{R})$, and we apply Proposition 1 for $\widetilde{\mathcal{V}} = \{v_i, v_{i+1}\}$. Since the definition (2.12) gives $c_{2,1} = 1/12$, we find

$$\begin{aligned} \Phi(x, \widetilde{\mathcal{V}}) &= \frac{1}{12}\{2\alpha(1-\alpha)(v_{i+1}-v_i)^3 - 2[(1-\alpha)(x-v_i)^3 + \alpha(v_{i+1}-x)^3]\} \\ &= \frac{\alpha^2(1-\alpha)^2}{3}(v_{i+1}-v_i)^3 \;\leq\; \frac{1}{48}(v_{i+1}-v_i)^3 \;\leq\; \frac{1}{48}\sigma^3 \,. \end{aligned} \tag{4.2}$$

Therefore (2.6) and (4.2) imply (4.1).

In order to prove the last statement of the proposition, we note that expression (4.2) holds as an equation when $\alpha = 1/2$ and $v_{i+1} - v_i = \sigma$. Hence in this case the right hand side of inequality (4.1) has the value $\Phi(x, \widetilde{\mathcal{V}})^{1/2}|f|_2 = |K_x|_2|f|_2$ for every admissible f, where K_x is the Riesz representer of equation (2.10). Further, we pick $f = K_x$ and $n = 2$. Then $\mathcal{V} = \widetilde{\mathcal{V}} = \{v_1, v_2\}$ and since K_x is zero on $\widetilde{\mathcal{V}}$ by definition (2.9), the cubic spline interpolant to f on $\mathcal{V}$ is $s_{K_x,\mathcal{V}} \equiv 0$. Moreover, equation (1.11) gives the value $f(x) = |K_x|_2^2$. It follows that expression (4.1) is satisfied as an equation for these choices, which gives the required result. □

If $d = 3$ and $q = 2$, then the definition (1.5) provides $\phi(t) = t$, i.e. we work with the linear radial function in three dimensions.

Proposition 4 *Let $f \in BL^2(\mathbb{R}^3)$ and suppose that $x \in \mathbb{R}^3$ is inside or on a face of the nondegenerate tetrahedron whose vertices, v_1, v_2, v_3 and v_4 say, are four of the n points of the interpolation set $\mathcal{V}$. Then the error of the tri-variate natural spline interpolant $s_{f,\mathcal{V}}$ at x is bounded by the inequality*

$$|f(x) - s_{f,\mathcal{V}}(x)| \leq \frac{1}{4\sqrt{\pi}}\,|f|_2\,\sigma^{1/2}\;, \tag{4.3}$$

where σ is the length of the longest edge of the tetrahedron. Moreover, the factor $1/(4\sqrt{\pi})$ cannot be replaced by a smaller number that is independent of n and f.

Proof. The dimension of $\mathcal{P}_1(\mathbb{R}^3)$ is $N = 4$. We apply Proposition 1 for $\widetilde{\mathcal{V}} = \{v_1, v_2, v_3, v_4\}$. The corresponding coefficients $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ defined by equation (1.9) are simply the barycentric coordinates of x with respect to the vertices v_1, v_2, v_3 and v_4, because they satisfy

$$\begin{aligned} x &= \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 \\ 1 &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\;, \quad \alpha_i \geq 0\;,\; i = 1, 2, 3, 4\,. \end{aligned} \tag{4.4}$$

Since the definition (2.12) gives $c_{2,3} = -1/8\pi$, expression (2.7), which we need to bound, is the number

$$\Phi(x, \widetilde{\mathcal{V}}) = \frac{1}{4\pi}\left\{\sum_{i=1}^{4}\alpha_i\|x - v_i\| - \sum_{i=1}^{3}\sum_{j=i+1}^{4}\alpha_i\alpha_j\|v_i - v_j\|\right\}\,. \tag{4.5}$$

As in the proof of Lemma 1, we employ the notation $\hat{\alpha}_i = \alpha_i$, $\hat{v}_i = v_i$ for $i = 1, 2, 3, 4$, $\hat{\alpha}_5 = -1$, $\hat{v}_5 = x$, and we consider the expression

$$\sum_{i=1}^{5}\sum_{j=1}^{5} \hat{\alpha}_i\hat{\alpha}_j \|\hat{v}_i - \hat{v}_j\|^2 = 2\left\{\sum_{i=1}^{3}\sum_{j=i+1}^{4} \alpha_i\alpha_j\|v_i - v_j\|^2 - \sum_{i=1}^{4}\alpha_i\|x - v_i\|^2\right\} . \tag{4.6}$$

Since the equations (4.4) are equivalent to

$$\sum_{i=1}^{5} \hat{\alpha}_i p(\hat{v}_i) = 0 \ , \ p \in \mathcal{P}_1(\mathbb{R}^3) \ , \tag{4.7}$$

it follows from Lemma 3.5 of [15] that the left hand side of (4.6) is zero. Hence we find

$$0 = \sum_{i=1}^{3}\sum_{j=i+1}^{4} \alpha_i\alpha_j\|v_i - v_j\|^2 - \sum_{i=1}^{4}\alpha_i\|x - v_i\|^2 \ . \tag{4.8}$$

By multiplying (4.8) by $(4\pi\sigma)^{-1}$ and adding the product to (4.5), we deduce

$$\begin{aligned}
\Phi(x, \widetilde{\mathcal{V}}) &= \frac{\sigma}{4\pi}\left\{\sum_{i=1}^{4}\alpha_i\frac{\|x - v_i\|}{\sigma}\left(1 - \frac{\|x - v_i\|}{\sigma}\right)\right. \\
&\quad \left. - \sum_{i=1}^{3}\sum_{j=i+1}^{4}\alpha_i\alpha_j\frac{\|v_i - v_j\|}{\sigma}\left(1 - \frac{\|v_i - v_j\|}{\sigma}\right)\right\} \\
&\le \frac{\sigma}{4\pi}\sum_{i=1}^{4}\alpha_i\frac{\|x - v_i\|}{\sigma}\left(1 - \frac{\|x - v_i\|}{\sigma}\right) \\
&\le \frac{\sigma}{16\pi}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) \quad = \quad \frac{\sigma}{16\pi} \ .
\end{aligned} \tag{4.9}$$

The last two inequalities are derived by noticing that the numbers $\sigma^{-1}\|x - v_i\|$, $i = 1, 2, 3, 4$ and $\sigma^{-1}\|v_i - v_j\|$, $i, j = 1, 2, 3, 4$, $i < j$, are all in the interval $[0, 1]$, and by applying the property $0 \le t(1 - t) \le 1/4$, $t \in [0, 1]$. Finally, (2.6) and (4.9) give the required estimate (4.3).

We now argue as in the final part of the proof of Proposition 3. Equality occurs throughout (4.9) if x is the midpoint of the longest edge of the tetrahedron. For this choice of x, we let $n = 4$ and $f = K_x$, where K_x is the Riesz representer from (2.10). Then the corresponding interpolant to f on $\mathcal{V} = \widetilde{\mathcal{V}}$ is $s_{K_x,\mathcal{V}} \equiv 0$ as before. Hence inequality (4.3) can be satisfied as an equation in this case, which implies the last statement of the proposition. □

The following result is relevant to the case when the interpolation points are the vertices of a finite regular cubic grid of mesh size σ, say.

Proposition 5 *Let $f \in BL^2(\mathbb{R}^3)$ and suppose that $x \in \mathbb{R}^3$ is inside or on an $\sigma\times\sigma\times\sigma$ cube whose corners are eight of the n interpolation points. Then the error of the tri-variate natural spline interpolant $s_{f,\mathcal{V}}$ at x is bounded by the inequality*

$$|f(x) - s_{f,\mathcal{V}}(x)| \le \left(\frac{7\sqrt{3} - 3\sqrt{2} - 3}{64\pi}\right)^{1/2} |f|_2\,\sigma^{1/2} \ . \tag{4.10}$$

Moreover, the constant factor in the right hand side of (4.10) *cannot be replaced by a smaller number that is independent of* n *and* f.

Proof. Note first that (4.10) is not a consequence of (4.3). Indeed, one can split an $\sigma \times \sigma \times \sigma$ cube into five tetrahedra such that the length of the longest tetrahedron side is $\sigma\sqrt{2}$. Then Proposition 4 establishes the error bound

$$|f(x) - s_{f,\mathcal{V}}(x)| \leq \frac{2^{1/4}}{4\sqrt{\pi}} \, |f|_2 \, \sigma^{1/2} \; , \tag{4.11}$$

where x is any point that is inside or on the boundary of the cube. Since the numbers $\left(\frac{7\sqrt{3}-3\sqrt{2}-3}{64\pi}\right)^{1/2}$ and $\frac{2^{1/4}}{4\sqrt{\pi}}$ are 0.1558 and 0.1677 respectively, to four decimal places, it follows that (4.10) is sharper. For any splitting of the cube into tetrahedra by drawing diagonals, the length of the longest tetrahedron side is at least $\sigma\sqrt{2}$.

We proceed to prove (4.10). Without loss of generality, let the eight corners of the cube be

$$\begin{array}{ll} v_1 = (0,0,0)^T \; , & v_5 = (\sigma,\sigma,0)^T \; , \\ v_2 = (\sigma,0,0)^T \; , & v_6 = (\sigma,0,\sigma)^T \; , \\ v_3 = (0,\sigma,0)^T \; , & v_7 = (0,\sigma,\sigma)^T \; , \\ v_4 = (0,0,\sigma)^T \; , & v_8 = (\sigma,\sigma,\sigma)^T \; . \end{array} \tag{4.12}$$

Let $x = (\xi_1, \xi_2, \xi_3)^T$, where $\xi_i \in [0,\sigma]$, $i = 1,2,3$ and define $\theta_i = \xi_i/\sigma \in [0,1]$, $i = 1,2,3$. Then we set

$$\begin{array}{ll} \gamma_1 = (1-\theta_1)(1-\theta_2)(1-\theta_3) \; , & \gamma_5 = \theta_1\theta_2(1-\theta_3) \; , \\ \gamma_2 = \theta_1(1-\theta_2)(1-\theta_3) \; , & \gamma_6 = \theta_1(1-\theta_2)\theta_3 \; , \\ \gamma_3 = (1-\theta_1)\theta_2(1-\theta_3) \; , & \gamma_7 = (1-\theta_1)\theta_2\theta_3 \; , \\ \gamma_4 = (1-\theta_1)(1-\theta_2)\theta_3 \; , & \gamma_8 = \theta_1\theta_2\theta_3 \; , \end{array} \tag{4.13}$$

because these values provide

$$\sum_{i=1}^{8} \gamma_i = 1 \quad \text{and} \quad \sum_{i=1}^{8} \gamma_i v_i = x \; . \tag{4.14}$$

We achieve equation (2.13) by letting $\gamma_i = 0$ for any other index i for which the interpolation point v_i is not one of the above eight. Therefore to apply the error bound (2.20), it remains to estimate the power function (2.26). Specifically, since $c_{2,3} = -1/8\pi$, we need to bound the expression

$$\Phi = \frac{1}{4\pi} \left\{ \sum_{i=1}^{8} \gamma_i \|x - v_i\| - \sum_{i=1}^{7} \sum_{j=i+1}^{8} \gamma_i\gamma_j \|v_i - v_j\| \right\} \; . \tag{4.15}$$

We employ again the technique from the proof of Lemma 1. By letting $\hat{\gamma}_i = \gamma_i$, $\hat{v}_i = v_i$ for $i = 1,2,\ldots,8$, $\hat{\gamma}_9 = -1$, $\hat{v}_9 = x$ and by applying Lemma 3.5 of [15], we find that the expression

$$\sum_{i=1}^{9} \sum_{j=1}^{9} \hat{\gamma}_i\hat{\gamma}_j \|\hat{v}_i - \hat{v}_j\|^2 = 2 \left\{ \sum_{i=1}^{7} \sum_{j=i+1}^{8} \gamma_i\gamma_j \|v_i - v_j\|^2 - \sum_{i=1}^{8} \gamma_i \|x - v_i\|^2 \right\} \tag{4.16}$$

is zero. Hence, if we multiply the right hand side of (4.16) by $\frac{\zeta}{8\pi}\sigma^{-1}$ and add the product to (4.15), where $\zeta > 0$ is a parameter, we obtain

$$\begin{aligned}\Phi &= \frac{\sigma}{4\pi}\left\{\sum_{i=1}^{8}\gamma_i\frac{\|x-v_i\|}{\sigma}\left(1-\zeta\frac{\|x-v_i\|}{\sigma}\right)\right.\\ &\quad \left.-\sum_{i=1}^{7}\sum_{j=i+1}^{8}\gamma_i\gamma_j\frac{\|v_i-v_j\|}{\sigma}\left(1-\zeta\frac{\|v_i-v_j\|}{\sigma}\right)\right\}. \qquad (4.17)\end{aligned}$$

The first part of (4.14) and the elementary condition $\{r(1-\zeta r) \leq 1/(4\zeta),\ 0 \leq r \leq 1\}$ provide the bound

$$\sum_{i=1}^{8}\gamma_i\frac{\|x-v_i\|}{\sigma}\left(1-\zeta\frac{\|x-v_i\|}{\sigma}\right) \leq \frac{1}{4\zeta}\ . \qquad (4.18)$$

In order to treat the other sum of (4.17), we set

$$\begin{aligned}G_1 &= \gamma_1\gamma_2+\gamma_1\gamma_3+\gamma_1\gamma_4+\gamma_2\gamma_5+\gamma_3\gamma_5+\gamma_2\gamma_6\\ &\quad +\gamma_3\gamma_7+\gamma_4\gamma_6+\gamma_4\gamma_7+\gamma_7\gamma_8+\gamma_6\gamma_8+\gamma_5\gamma_8\\ G_2 &= \gamma_1\gamma_5+\gamma_1\gamma_6+\gamma_1\gamma_7+\gamma_2\gamma_8+\gamma_3\gamma_8+\gamma_4\gamma_8\\ &\quad +\gamma_2\gamma_3+\gamma_2\gamma_4+\gamma_3\gamma_4+\gamma_5\gamma_6+\gamma_5\gamma_7+\gamma_6\gamma_7\\ G_3 &= \gamma_1\gamma_8+\gamma_2\gamma_7+\gamma_3\gamma_6+\gamma_4\gamma_5\ .\end{aligned} \qquad (4.19)$$

because then the definitions (4.12) imply

$$\sum_{i=1}^{7}\sum_{j=i+1}^{8}\gamma_i\gamma_j\frac{\|v_i-v_j\|}{\sigma}\left(1-\zeta\frac{\|v_i-v_j\|}{\sigma}\right) = (1-\zeta)G_1+(\sqrt{2}-2\zeta)G_2+(\sqrt{3}-3\zeta)G_3\ . \qquad (4.20)$$

Hence (4.17), (4.18) and (4.20) give the estimate

$$\Phi \leq \frac{\sigma}{4\pi}\left\{\frac{1}{4\zeta}+(G_1+2G_2+3G_3)\zeta-(G_1+\sqrt{2}G_2+\sqrt{3}G_3)\right\}\ . \qquad (4.21)$$

As a function of ζ, the right hand side of inequality (4.21) is least when ζ has the value

$$\zeta = \frac{1}{2}\,(G_1+2G_2+3G_3)^{-1/2}\ . \qquad (4.22)$$

Therefore we pick this value of ζ, and then (4.21) takes the form

$$\Phi \leq \frac{\sigma}{4\pi}E\ , \qquad (4.23)$$

where

$$E = (G_1+2G_2+3G_3)^{1/2}-(G_1+\sqrt{2}G_2+\sqrt{3}G_3)\ . \qquad (4.24)$$

To find the greatest value of E, let $\eta_i = \theta_i(1-\theta_i)$, $i = 1,2,3$, hence $\eta_i \in [0,1/4]$, $i = 1,2,3$. This notation and the definitions (4.13) and (4.19) give

$$\begin{aligned} G_1 &= \eta_1(1-2\eta_2)(1-2\eta_3) + \eta_2(1-2\eta_3)(1-2\eta_1) + \eta_3(1-2\eta_1)(1-2\eta_2) \\ G_2 &= 2\eta_1\eta_2(1-2\eta_3) + 2\eta_2\eta_3(1-2\eta_1) + 2\eta_3\eta_1(1-2\eta_2) \\ G_3 &= 4\eta_1\eta_2\eta_3 \ , \end{aligned} \tag{4.25}$$

and consequently

$$G_1 + 2G_2 + 3G_3 = \eta_1 + \eta_2 + \eta_3 \ . \tag{4.26}$$

Note that E is symmetric in η_1, η_2 and η_3. It is shown below that E attains its maximum when $\eta_1 = \eta_2 = \eta_3 = 1/4$, which provides the condition

$$E \leq \frac{7\sqrt{3} - 3\sqrt{2} - 3}{16} \ , \tag{4.27}$$

with equality if and only if $\theta_1 = \theta_2 = \theta_3 = 1/2$, i.e. when x is the centre of the cube. Expressions (4.23) and (4.27) combined with the error bound (2.20) give the required inequality (4.10).

To prove the assertion of the last paragraph, suppose by contradiction that $E = E(\eta_1, \eta_2, \eta_3)$ attains its maximum in the interior of the domain $[0,1/4] \times [0,1/4] \times [0,1/4]$. Since

$$\frac{\partial E}{\partial \eta_1} = \frac{1}{2\sqrt{\eta_1+\eta_2+\eta_3}} - [1 - 2(2-\sqrt{2})(\eta_2+\eta_3) + 4(3 - 3\sqrt{2} + \sqrt{3})\eta_2\eta_3] \ , \tag{4.28}$$

by writing the Fermat conditions for a critical point and by using the symmetry of E, we see that an interior point of maximum for E must satisfy $\eta_1 = \eta_2 = \eta_3 = \eta$ say, where $\eta \in (0,1/4)$. It follows from (4.28) that the conditions for a critical point reduce to

$$\frac{1}{2\sqrt{3}} - \sqrt{\eta}[1 - 4(2-\sqrt{2})\eta + 4(3 - 3\sqrt{2} + \sqrt{3})\eta^2] = 0. \tag{4.29}$$

However, a Matlab plot of the left hand side of (4.29) for $\eta \in (0,1/4)$ reveals that this function is bounded below by a positive constant, which provides the required contradiction. By similar arguments, E cannot attain its maximum in the interior of a face or an edge of the cube $[0,1/4] \times [0,1/4] \times [0,1/4]$. The assertion is now proved by comparing the values of E at the eight corners of this cube.

In order to prove the last statement of the proposition, we let x be the centre of the cube, since this choice gives equality in (4.27) and in (4.21). Further, we pick $n = 8$, so $\mathcal{V} = \{v_1, v_2, \ldots, v_8\}$, where $v_1, v_2, \ldots, v_8$ are the vertices of the cube defined in (4.12). Let $\{S_1, S_2, \ldots, S_8\}$ be the three-dimensional natural spline cardinal functions defined by (2.16). If $i \neq j$, $i,j \in \{1,2,\ldots,8\}$, then due to the invariance of the seminorm $|.|_2$ of $BL^2(\mathbb{R}^3)$ under translations and rotations, S_j is the composition between S_i and the rotation of centre x which moves v_i to v_j while permuting the other vertices of the cube. In particular, the "Schoenberg" coefficients must be equal: $\tilde{\gamma}_i = S_i(x) = S_j(x) = \tilde{\gamma}_j$, hence $\tilde{\gamma}_i = 1/8$, $i = 1,2,\ldots,8$. Since (4.13) provides $\gamma_i = 1/8$, $i = 1,2,\ldots,8$ when x is the centre of the cube, the choices $f = l_{x,\gamma} = l_{x,\tilde{\gamma}}$ in (2.17) and $u = l_{x,\tilde{\gamma}}$ in (2.22) provide equalities in both relations of (2.20). Hence the equality sign holds in (4.10) for these choices, which gives the required assertion. □

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E-mail address: ab223@damtp.cam.ac.uk