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Insight, not just numbers

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ABSTRACT

We survey briefly recent developments in geometric integration and numerical methods on manifolds. The underlying philosophy is that numerical methods should, whenever practicable, recover correctly known qualitative behaviour of the underlying differential system. Classical numerical methods retain invariants very poorly indeed, and this justifies the recent introduction of a new breed of discretization schemes.

QUANTITY OR QUALITY?

Few mantras have been repeated more often by lecturers of numerical analysis and scientific computing than the motto of R.W. Hamming's book [6], "The purpose of computing is insight, not numbers". Like motherhood and apple pie, the quest for insight is readily acknowledged, yet not many have ever asked what are its *practical* implications: how should we change the way we compute (a procedure which, after all, produces numbers and not much else) to seek insight?

The dichotomy of insight and numbers is specific to numerical analysis. A scientist, an engineer, a pure mathematician, are all natural insight-seekers in their respective spheres of research. On a formal level, this is manifested by the construction and analysis of mathematical models. Typical scientific investigation commences with data-gathering and continues through the formulation of a model, employing the language of mathematics. The next stage is mathematical analysis, the deduction of qualitative features from the formal description of the model. As a matter of course, mathematical analysis leads to a wealth of important results, but seldom does it provide a complete and comprehensive solution to the underlying problem. To flesh out the numbers and the shapes we need to compute! Yet, not for nothing is the last stage colloquially called "throwing a problem into a computer". More often than not, computation means that the mathematical model is discretized in a form suitable for number-crunching and subsequently subjected to algorithms – fast Fourier transform, multigrid, wavelets, …– which, although often exhibiting impressive intellectual dexterity, have nothing to do with the original problem *per se.* The aforementioned qualitative results, achievements of 'classical' analysis, are frequently casualties of computation: when the computers speak, pure mathematics often falls silent.

There is, however, an alternative and it is being embraced by a growing number of practitioners in the last few years. Analysis and computation, the quest for quality and the quest for quantity, need not be enemies or, indeed, separate activities! At their best, they should be complementary, undertaken *simultaneously* and in a synergistic manner. Thus, computation should not wait until analysis has run out of steam: it should be undertaken from the very beginning, to help us in testing theoretical hypotheses and setting down conjectures. On the other hand, the role of analysis is not over at the outset of 'number crunching', because qualitative features, the great achievement of the analytic approach, should not be abandoned under discretization! *We should, whenever possible and* *practical, employ computational algorithms that reflect known qualitative features of the underlying system.*

A scientist or an engineer is not interested in 'analysing' or 'computing' a differential equation, say, but in *understanding* it (an activity which, naturally, is context-dependent). The goal of applied mathematics should be neither to analyse nor to compute but to *understand.* Insight, not numbers.

What are the practical consequences of this new paradigm of computing? The traditional approach of numerical analysts is to design, analyse and implement algorithms that cater for the widest-possible range of problems. The underlying message has much validity: the user, typically, is neither a skilled numerical analyst nor a mathematician. Hence, the purpose of a well-written computational package is to relieve the user from decisions that require deep numerical or mathematical knowledge.

However, the broad-front approach is, by its very nature, inimical to insight. The 'package' does not know the mathematics (or the physics) of the underlying problem. It does not realise that one variable stands for pressure (which cannot become negative), another for temperature (which cannot dip under $0°$K) or that the solution should preserve energy or volume, say. A tool that has been designed for a wide range of tasks is, by its very nature, blunt. As soon as we wish to combine computation with analysis, incorporate qualitative information into number-crunching, a wide-purpose methodology must be replaced by a variety of special, bespoke techniques, purpose-designed for specific (and necessarily restrictive) sets of problems.

It is important to emphasize that there is room for both the sledgehammer of wide-purpose methods and the scalpel of 'qualitative' techniques. Different applications require different courses of action and the best guarantee for the correct answer rests in asking the right question.

GEOMETRIC INTEGRATION

Qualitative information comes in all shapes and forms. Yet, arguably, the two most important qualitative aspects of an evolving dynamical system are *asymptotic behaviour* and *invariants.* The retention of correct asymptotic behaviour has been recognised as a major requirement of good discretization algorithms since the dawn of the computer era. The last decade has witnessed the emergence of a powerful approach, based to a large extent on the theory of nonlinear dynamical systems, which explains the long-term behaviour of discretization methods for initial-values ODEs and PDEs [18].

It is fair to observe that the numerical community has, by and large, neglected the entire issue of invariants until the last few years. The importance of invariants in a numerical setting is, however, a subject to a growing interest and it is the theme of this brief survey. Adopting an increasingly-popular terminology, we reserve the name of *geometric integrators* to methods that are designed not just to minimize, in a classical sense, the numerical error but also to render correctly the invariants (i.e., the geometry) of the solution [17].

A vivid example of the importance of invariants and of their sensitivity to discretization is the *Kepler problem*

$$\begin{aligned} p' &= -\frac{1}{q^2}+\frac{l^2}{q^3}, \\ q' &= p, \end{aligned} \tag{1}$$

which models (after proper normalization and transition to polar coordinates) the rotation of a material body with the radial momentum p and the radial position q about a fixed body. The constant l is the angular momentum [7]. The Kepler problem possesses a number of invariants, but perhaps the most interesting are the *symplectic form* $\mathrm{d}p \wedge \mathrm{d}q$ and the *Hamiltonian energy*

$$H(p,q)=\frac{p^2}{2}-\frac{1}{q}+\frac{l^2}{2q^2}.$$

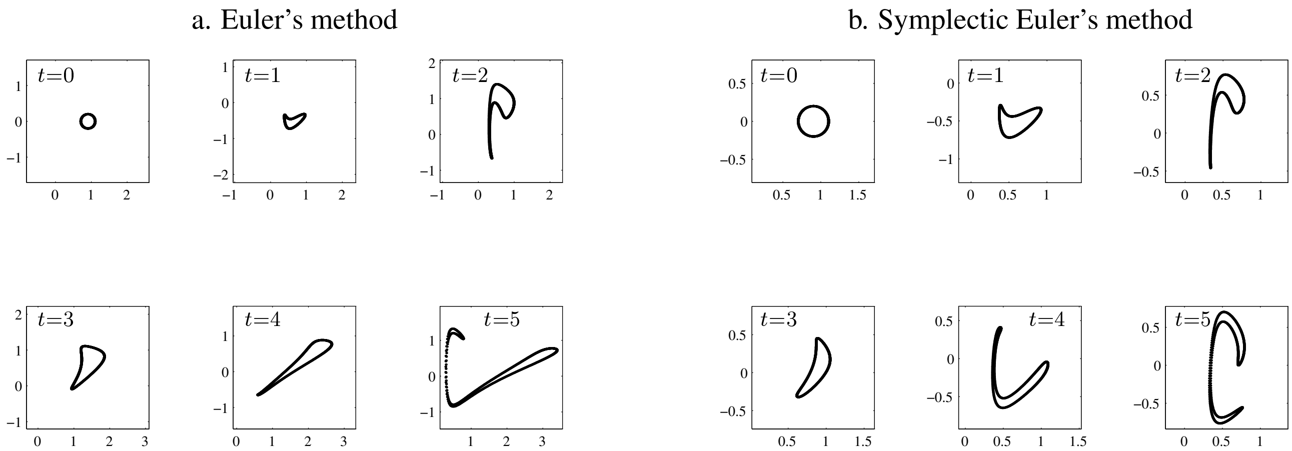

Figure 1: The evolution of a circular domain under a discretized flow, displayed at times $t = 0, 1, \ldots, 5$. The step sizes are $h = \frac{1}{20}$ for Euler and $h = \frac{1}{10}$ for symplectic Euler.

Conservation of symplecticity for a Hamiltonian system with a single degree of freedom is equivalent to the statement that, for every $t > 0$, the map $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$\varphi_t(p(0), q(0)) = (p(t), q(t)),$$

is area-preserving in the phase plane (cf. Figure 1b for an example of such a map). This property is important, not least since it guarantees the existence of periodic solutions [7,17].

We discretize the system (1) with two explicit, first-order time-stepping methods: the *Euler* scheme

$$\begin{aligned} p_{n+1} &= p_n + h\left(-\frac{1}{q_n^2} + \frac{l^2}{q_n^3}\right), \\ q_{n+1} &= q_n + hp_n, \end{aligned} \tag{2}$$

where $p_n \approx p(nh)$ and $q_n \approx q(nh)$, $n \in \mathbb{Z}^+$, and the *symplectic Euler* algorithm,

$$\begin{aligned} p_{n+1} &= p_n + h\left(-\frac{1}{q_n^2} + \frac{l^2}{q_n^3}\right), \\ q_{n+1} &= q_n + hp_{n+1}. \end{aligned} \tag{3}$$

On the face of it, the difference between (2) and (3) is very minor indeed. However, let us examine the numerical retention of symplecticity in Figures 1a and 1b. Note that, while in Figure 1a the area evidently grows, this is not the case with 1b, where the numerical map is area-conserving.

The importance of a symplectic discretization is underscored by the theorem that a symplectic solution sequence is always within an exceedingly small distance from the exact solution of a nearby Hamiltonian problem [17]. Therefore, it shares with the latter a range of important dynamic features, not least the presence of invariant tori. Yet, as seen in Figure 1 and affirmed by an extensive theory, only some methods can be guaranteed to retain symplecticity.

We have already mentioned that the Kepler problem (1), in tandem with other Hamiltonian equations, has another invariant: the Hamiltonian energy. As can be seen from Figure 2, neither method keeps Hamiltonian energy constant, yet there is a crucial difference. While, in the case of Euler's method, $H(p_n, q_n)$ grows literally in leaps and bounds, the variation of Hamiltonian energy for the method (3) is oscillatory [10]. This is typical of symplectic methods.

The Kepler problem (1) is just one instance of *Hamiltonian equations,* which share fascinating mathematical structure (symplecticity, conservation of Hamiltonian energy, and much more!) with a very

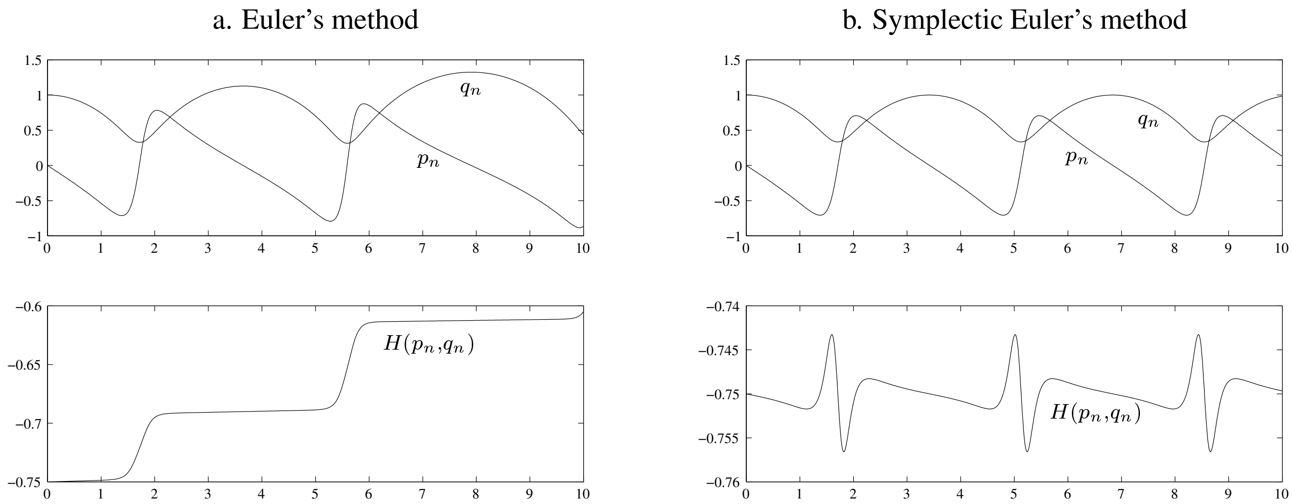

Figure 2: The solution components and Hamiltonian energy for Euler's and symplectic Euler's methods, with $h = \frac{1}{20}$ and $h = \frac{1}{10}$ respectively.

broad range of applications. Many important problems in celestial mechanics, quantum chemistry, molecular dynamics and mathematical biology can be formulated in a Hamiltonian form and their discretization (not least when long-term integration is at issue) should pay heed to the Hamiltonian structure [11,13,17].

Our next example of a differential system with an important invariant is the *Toda lattice flow*

$$y_j'' = \varphi'(y_{j+1} - y_j) - \varphi'(y_j - y_{j-1}), \qquad j = 1, 2, \ldots, m, \tag{4}$$

where φ is a given function and we assume that $y_0, y_{m+1} \equiv 0$. Although the system (4) has many applications, perhaps the most well-known is as a model to a system of unit-mass particles with a nearest-neighbour interaction (e.g., particles linked by springs), whereby y_j is the displacement of the jth particle and φ is the interaction potential [19]. Letting $\varphi(r) = \mathrm{e}^r + r$, we can rewrite the Toda flow in a Hamiltonian form,

$$\begin{aligned} p_j &= \mathrm{e}^{q_{j-1}-q_j} - \mathrm{e}^{q_j - q_{j+1}}, \\ q_j' &= p_j, \end{aligned} \qquad j = 1, 2, \ldots, m.$$

Next, we let

$$a_j = \frac{1}{2}\mathrm{e}^{(q_j - q_{j+1})/2}, \qquad b_j = \frac{1}{2}p_j$$

and

$$A = \begin{bmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & a_{m-2} & b_{m-1} & a_{m-1} \\ 0 & \cdots & 0 & a_{m-1} & b_m \end{bmatrix}, \qquad B(A) = \begin{bmatrix} 0 & a_1 & 0 & \cdots & 0 \\ -a_1 & 0 & a_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -a_{m-2} & 0 & a_{m-1} \\ 0 & \cdots & 0 & -a_{m-1} & 0 \end{bmatrix}.$$

It is easy to reformulate the Toda lattice equations in a matrix form, namely

$$A' = B(A)A - AB(A), \quad t \geq 0, \qquad A(0) = A_0. \tag{5}$$

Little more effort is required to demonstrate that the integrable system (5) possesses m invariants: the eigenvalues of the matrix $A(t)$ [19]. This is true, as a matter of fact, for an arbitrary matrix function $B(A)$, motivating the name *isospectral flow*.

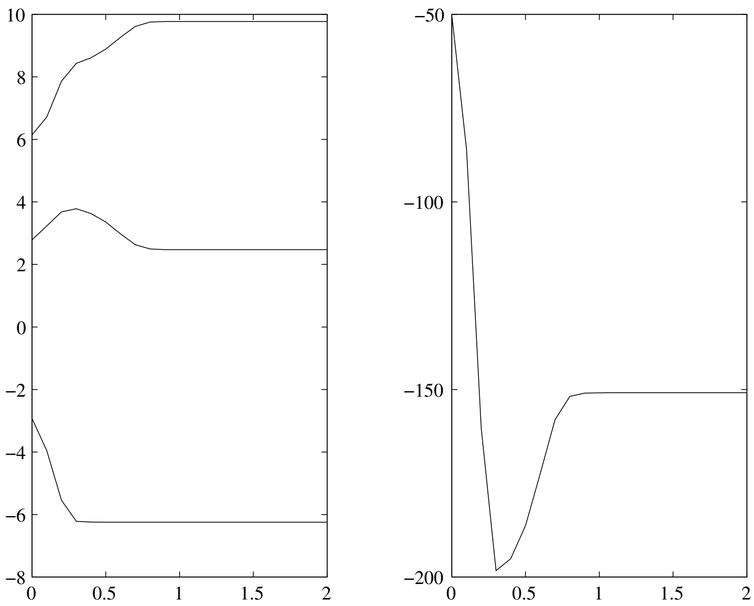

Figure 3: Breakdown in isospectrality in the Euler method with $h = \frac{1}{10}$. The left figure displays the individual eigenvalues and the right displays the determinant.

Figure 3 displays the solution of (5) by means of the Euler method,

$$A_{n+1} = A_n + h[B(A_n)A_n - A_n B(A_n)], \qquad n \in \mathbb{Z}^+, \tag{6}$$

with $m = 3$ and the initial value

$$A_0 = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 1 & 4 \\ 0 & 4 & 2 \end{bmatrix}.$$

It is obvious that the method produces completely wrong eigenvalues. This is vividly demonstrated by observing the huge error in the determinant of A_n – observe that, being the product of the eigenvalues, also the determinant is an invariant of the system (5).

The failure of the Euler method is typical and it can be proved that *all* classical methods, e.g. Runge–Kutta and multistep, render eigenvalues wrongly for $m \geq 3$ [1,8]. Retention of isospectrality can be accomplished, though, by exploiting analytic insight. Consider the matrix ODE

$$U' = B(UA_0U^{-1})U, \quad t \geq 0, \qquad U(0) = I. \tag{7}$$

It is possible to prove that the solution of (5) is $A(t) = U(t)A_0U^{-1}(t)$ [19]. This, incidentally, demonstrates that $A(t)$ is similar to A_0, thereby having the same eigenvalues. However, from the numerical point of view, the main importance of (7) is as a mechanism for the generation of isospectral discretization methods. For example, applying Euler's method to this system results in the isospectral scheme

$$A_{n+1} = (I + hB(A_n))A_n(I + hB(A_n))^{-1}, \qquad n \in \mathbb{Z}^+. \tag{8}$$

Note that both (6) and (8) are first-order methods and, at least before the onset of asymptotic behaviour, they are likely to display similar errors in solution components. Yet, while (6) distorts the eigenvalues, the isospectral Euler method (8) renders them faithfully.

It is possible to apply higher-order discretization schemes to (7). This may lead to loss of symmetry in the important special case when $B(A)$ is skew-symmetric (and U is orthogonal) but this can be ameliorated by a technique from [2].

An important source of isospectral equations is numerical analysis itself, since many computational problems in linear algebra can be formulated in this form [4].

GENERAL INVARIANTS

Let us assume that the solution of the differential system

$$\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y}), \quad t \geq 0, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0 \in \mathbb{R}^d, \tag{9}$$

possesses an invariant of the form $\rho(\boldsymbol{y}(t)) \equiv \rho(\boldsymbol{y}_0)$, $t \geq 0$. Assuming for simplicity's sake that ρ is a C^∞ function, we say that (9) is $\mathcal{M}$*-invariant,* where

$$\mathcal{M} = \{\boldsymbol{x} \in \mathbb{R}^d \,:\, \rho(\boldsymbol{x}) = \rho(\boldsymbol{y}_0)\}$$

is a smooth manifold, embedded in $\mathbb{R}^d$. Likewise, a time-stepping method is said to be $\mathcal{M}$*-invariant* if it generates a sequence $\boldsymbol{y}_n \in \mathcal{M}$, $n \in \mathbb{Z}^+$, when applied to an arbitrary $\mathcal{M}$-invariant ODE (9).

It is possible to classify classical $\mathcal{M}$-invariant numerical methods for smooth manifolds $\mathcal{M}$ and the results are exceedingly disappointing. Multistep methods are $\mathcal{M}$-invariant if and only if ρ is a linear function [8]. The situation is somewhat better with regard to Runge–Kutta methods. As proved by Calvo, Iserles and Zanna [3], a necessary condition for $\mathcal{M}$-invariance is that every two-dimensional slice through the manifold $\mathcal{M}$ is a level set of some function σ which obeys the nonlinear degenerate parabolic PDE

$$\sigma_y^3\sigma_{xxx} - 3\sigma_x\sigma_y^2\sigma_{xxy} + 3\sigma_x^2\sigma_y\sigma_{xyy} - \sigma_x^3\sigma_{yyy} = 0. \tag{10}$$

An important case when equation (10) holds along each slice is when ρ is quadratic in all its components: special cases are symplectic manifolds, flow on a sphere, orthogonal flows and flows on Stiefel and Grassmann manifolds [3]. In that case it is possible to single out a class of $\mathcal{M}$-invariant Runge–Kutta methods. Specifically, the method

$$\begin{aligned}
\boldsymbol{k}_l &= \boldsymbol{f}(t_{n+c_l}, \boldsymbol{y}_n + h\sum_{j=1}^{\nu} a_{l,j}\boldsymbol{k}_j), \qquad l = 1, 2, \ldots, \nu, \\
\boldsymbol{y}_{n+1} &= \boldsymbol{y}_n + h\sum_{l=1}^{\nu} b_l\boldsymbol{k}_l,
\end{aligned}$$

where $t_\alpha = \alpha h$ and $c_l = \sum_{j=1}^{\nu} a_{l,j}$, $l = 1, 2, \ldots, \nu$, is $\mathcal{M}$-invariant on every quadratic manifold if and only if

$$b_l a_{l,j} + b_j a_{j,l} = b_l b_j, \qquad l, j = 1, 2, \ldots, \nu.$$

Invariance on general manifolds requires new techniques, outside the familiar menagerie of multistep and Runge–Kutta schemes. An obvious approach is *projection:* use an arbitrary method to time-step and project the outcome after every step back to the requisite manifold. This is, essentially, equivalent to reformulating the underlying problem as a *differential-algebraic system.* There is no denying that projections are often of real use, but they suffer from a number of shortcomings. Firstly, the manifold in question is often formulated in a form which is not (easily) amenable to projection: both symplectic and isospectral manifolds are of this kind. Secondly, computational experience seems to indicate that projection methods might be prone to instabilities. Thirdly, and most importantly, projection forces a solution to an arbitrary manifold even if it should not be there! In principle, we might be mistaken, whether phenomenologically or in our computer programme. It might well be that the solution should not reside on a manifold at all and a good solution method should be able to recognise this situation.

A number of general $\mathcal{M}$-invariant techniques have been introduced in the last few years. A promising approach, due to Crouch and Grossman [5], rewrites the $\mathcal{M}$-invariant ODE (9) in the form

$$\boldsymbol{y}' = \sum_{k=1}^{m} f_k(t, \boldsymbol{y})\boldsymbol{v}_k,$$

where the *rigid frame* $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ lies in the tangent bundle $T\mathcal{M}$. It is possible to extend Runge–Kutta methods within this formalism and render them $\mathcal{M}$-invariant although, in principle, this requires the knowledge of the tangent space (often either unavailable or difficult to compute) and is fraught with conceptual and technical difficulties.

LIE GROUP METHODS

Retention of invariants under discretization is considerably better understood within the framework of Lie groups. A manifold G is a *Lie group* if it is endowed with a continuous group structure. Familiar examples are the *special orthogonal group* $\mathrm{SO}_d[\mathbb{R}]$ (the set of $d \times d$ real orthogonal matrices with unit determinant), the *special linear group* $\mathrm{SL}_d[\mathbb{R}]$ (the set of all real $d \times d$ matrices with unit determinant) and the *real projective space* $\mathbb{RP}^{d-1}$ (the manifold of lines through the origin in $\mathbb{R}^d$) [16].

Invariance on a Lie group is ubiquitous in many applications, e.g. in mechanics and in relativity theory. Moreover, many other invariant manifolds are *homogeneous spaces:* they are invariant under a transitive action of a Lie group. If we know how to retain Lie-group invariance under discretization, we can also keep to a homogeneous space [15]. Therefore, Lie groups play a fundamental role in numerical theory of invariance.

Because of its group structure, it is enough to know the tangent space of G at identity. This space $\mathfrak{g}$, called the *Lie algebra* of G, has a number of features that make it amenable to analysis and numerics alike. Thus, $\mathfrak{g}$ is a linear space, equipped with an antisymmetric binary operation $[\,\cdot\,,\,\cdot\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and with a natural mapping $\exp : \mathfrak{g} \to G$. In the important special case when G is a subset of $d \times d$ matrices these operations are the familiar commutation and matrix exponentiation, respectively. We henceforth restrict ourselves to *matricial* Lie groups.

A differential equation which is invariant on the Lie group G can be always rendered in the form

$$y' = a(t, y)y, \quad t \ge 0, \qquad y(0) = y_0 \in G, \tag{11}$$

where $a : \mathbb{R} \times G \to \mathfrak{g}$. It is further possible to prove that $y(t) = \mathrm{e}^{\sigma(t)} y_0$, $t \ge 0$, where the function σ lies in $\mathfrak{g}$ and obeys there the differential equation

$$\sigma' = a(t, y) + \frac{1}{2}[a(t, y), \sigma] + \frac{1}{12}[[a(t, y), \sigma], \sigma] - \frac{1}{720}[[[[a(t, y), \sigma], \sigma], \sigma], \sigma] + \cdots, \qquad t \ge 0, \tag{12}$$

where $\sigma(0) = 0$ [12]. A common thread in a number of recent discretization algorithms is that they approximate the solution of (12) in the Lie algebra $\mathfrak{g}$, subsequently mapping it to the Lie group G by exponentiation. The outcome is a method that evolves a numerical solution of (11) on the Lie group.

A very promising example of such methods is RKMK, a modification of Runge–Kutta due to Munthe-Kaas [14]. Limitations of space preclude a discussion of RKMK methods in their full generality. Instead, we present on the left a familiar third-order Runge–Kutta and, on the right, its Lie-group modification *à la* Munthe-Kaas,

$$\begin{aligned} k_1 &= a(t_n, y_n)y_n, \\ k_2 &= a(t_{n+1/2}, y_n + \tfrac{1}{2}hk_1)(y_n + \tfrac{1}{2}hk_1), \\ k_3 &= a(t_{n+1}, y_n - hk_1 + 2hk_2)(y_n - hk_1 + 2hk_2), \\ u &= h(\tfrac{1}{6}k_1 + \tfrac{2}{3}k_2 + \tfrac{1}{6}k_3), \\ y_{n+1} &= y_n + u. \end{aligned} \qquad \begin{aligned} l_1 &= a(t_n, y_n), \\ l_2 &= a(t_{n+1/2}, \mathrm{e}^{\frac{1}{2}hl_1} y_n), \\ l_3 &= a(t_{n+1}, \mathrm{e}^{-hl_1 + 2hl_2} y_n), \\ v &= h(\tfrac{1}{6}l_1 + \tfrac{2}{3}l_2 + \tfrac{1}{6}l_3), \\ y_{n+1} &= \mathrm{e}^{v + \frac{1}{6}h[v, l_1]} y_n. \end{aligned}$$

Similar construction extends to all Runge–Kutta methods. As a rule, the derivative evolves in $\mathfrak{g}$ and is subjected solely to Lie-algebraic operations, inclusive of commutation, and, whenever we wish to return to the Lie group, exponentiation.

We conclude this brief survey by a very recent development, a Lie-group method for *linear* equations (hence, with $a = a(t)$) based on a Lie-algebraic expansion of the solution of (12). It is possible to show that the function σ can be written explicitly in the form

$$\begin{aligned}\sigma(t) \;=\; &\int_0^t a(\tau_1)\mathrm{d}\tau_1 + \frac{1}{2}\int_0^t\int_0^{\tau_1}[a(\tau_1),a(\tau_2)]\mathrm{d}\tau_2\mathrm{d}\tau_1 + \frac{1}{4}\int_0^t\int_0^{\tau_1}\int_0^{\tau_2}[a(\tau_1),[a(\tau_2),a(\tau_3)]]\mathrm{d}\tau_3\mathrm{d}\tau_2\mathrm{d}\tau_1\\ &+\frac{1}{12}\int_0^t\int_0^{\tau_1}\int_0^{\tau_1}[[a(\tau_1),a(\tau_2)],a(\tau_3)]\mathrm{d}\tau_3\mathrm{d}\tau_2\mathrm{d}\tau_1+\cdots.\end{aligned}$$

The first few terms of the above expansion have been derived by Magnus in 1954 [12] but its generation by recursive rules has been discovered only recently by Iserles and Nørsett [9], who have also presented convergence analysis and discussed numerical implementation. They have termed it the *Magnus expansion,* in honour of Wilhelm Magnus's pioneering contribution.

Perhaps the most surprising aspect of the Magnus expansion is that, although its evaluation entails extensive multivariate integration, a notoriously labour-intensive procedure, the special nature of the integrands lends itself to relatively affordable quadratures. The cost in function evaluations, regardless of the number of integrals, is just the same as of univariate Gaussian quadrature. Moreover, the cost of numerical linear algebra can be reduced as well, implementing techniques from the theory of Lie algebras, graph theory and combinatorics [9]. For example, herewith a fourth-order Lie-group method for the solution of $y' = a(t)y$,

$$\begin{aligned}&c_1=\frac{1}{2}-\frac{\sqrt{3}}{6},\quad c_2=\frac{1}{2}+\frac{\sqrt{3}}{6}, \qquad\qquad a_k=a(t_{n+c_k}),\quad k=1,2\\ &\tilde{\sigma}=\frac{1}{2}h(a_1+a_2)+\frac{\sqrt{3}}{12}h^2[a_2,a_1]+\frac{1}{80}h^3[a_2-a_1,[a_2,a_1]],\qquad y_{n+1}=\mathrm{e}^{\tilde{\sigma}}y_n.\end{aligned}$$

Numerical experience indicates that the above method produces remarkably accurate solution even when the conservation of a Lie-group structure is not at issue [9].

This concludes a brief survey of the new breed of discretization methods, whose purpose is not just to deliver the right numbers but also to render faithfully the qualitative behaviour of the underlying differential system. These are early days and, no sooner a survey paper is complete, it is already out of date. Yet, hopefully, we have managed to share with the reader the excitement of recent developments, to emphasize their potential as a practical tool in scientific computing and to make a case for a structured numerical quest for insight.

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