Local accuracy for radial basis function interpolation on finite uniform grids

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Abstract

We consider interpolation on a finite uniform grid by means of one of the radial basis functions (RBF) $\phi(r) = r^\gamma$ for $\gamma > 0$, $\gamma \notin 2\mathbb{N}$ or $\phi(r) = r^\gamma \ln r$ for $\gamma \in 2\mathbb{N}_+$. For each positive integer N, set $h = N^{-1}$ and let $\{x_i : i = 1, 2, \ldots, (N+1)^d\}$ be the vertices of the uniform grid of mesh-size h on the unit d-dimensional cube $[0,1]^d$. Given $f : [0,1]^d \to \mathbb{R}$, let s_h be its unique RBF interpolant at the grid vertices: $s_h(x_i) = f(x_i)$, $i = 1, 2, \ldots, (N+1)^d$. For $h \to 0$, we show that the uniform norm of the error $f - s_h$ on a compact subset K of the interior of $[0,1]^d$ enjoys the same rate of convergence to zero as the error of RBF interpolation on the infinite uniform grid $h\mathbb{Z}^d$, provided that f is a data function whose partial derivatives in the interior of $[0,1]^d$ up to a certain order can be extended to Lipschitz functions on $[0,1]^d$.

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1 Introduction

$\diamond$ **Motivation.** Several authors (Jackson [7], Powell [11], Schaback [14]) have expressed an interest in the accuracy of RBF interpolation on finite uniform grids. The question suggested by Powell's work [11] is whether these results would reproduce the high orders of accuracy that occur on infinite uniform grids not only for the case of the multiquadric and linear radial functions, but also for other radial basis functions that are currently studied. We provide a positive answer to this question in Theorem 1. In addition, for a specific class of radial basis functions, Theorem 2 shows that one cannot expect higher local rates of accuracy for general smooth data functions.

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$\diamond$ **Notation and problem description.** Let d and N be positive integers and denote $h = N^{-1}$, $D = [0,1]^d$, $\mathcal{J}_N = \mathbb{Z}^d \cap h^{-1}D$, $\mathcal{V}_h = h\mathcal{J}_N$. Hence $\mathcal{V}_h$ contains all the vertices of the uniform grid of mesh size h on the unit d-dimensional cube D. Let $f : D \to \mathbb{R}$ be a data function whose partial derivatives in the interior of D up to a certain order can be extended to Lipschitz functions on D.

For sufficiently large N, we consider interpolation to f on $\mathcal{V}_h$ by means of one of the radial basis functions $\phi(r) = r^\gamma$ for $\gamma > 0$, $\gamma \notin 2\mathbb{N}$ and $\phi(r) = r^\gamma \ln r$ for $\gamma \in 2\mathbb{N}_+$. In any of these cases, let m be the integer part of $\gamma/2$ and note that the $(m+1)$-st derivative of the function $\{\Phi(r) = \phi(r^{1/2}) : 0 \le r < \infty\}$ is strictly completely monotonic, i.e. $(-1)^k \Phi^{(k+m+1)}(r) > 0$, $r > 0$, $k = 0,1,2,\ldots$. Therefore, by the classical theory in [10] and [11], there exists a unique function s_h of the form

$$s_h(x) = \sum_{j \in \mathcal{J}_N} c_j \phi(\|x - hj\|) + p_h(x), \; x \in \mathbb{R}^d, \tag{1.1}$$

that satisfies the interpolation conditions

$$s_h(hj) = f(hj), \; j \in \mathcal{J}_N, \tag{1.2}$$

where $p_h \in \mathcal{P}_m(\mathbb{R}^d)$, $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$, and the coefficients $\{c_j : j \in \mathcal{J}_N\}$ are subject to the constraints

$$\sum_{j \in \mathcal{J}_N} c_j p(j) = 0, \; \forall\, p \in \mathcal{P}_m(\mathbb{R}^d). \tag{1.3}$$

The problem considered in this paper is to investigate the rate of convergence of $\|f - s_h\|_{\mathrm{L}^\infty(K)}$ to zero as $h \to 0$, where K is any fixed compact subset of the interior of D.

$\diamond$ **Method of analysis.** A similar problem has been considered by Powell in Section 9 of [11]. Namely, that work deals with the case when ϕ is the multiquadric or linear radial function, d is odd, the radial translates $\phi(\|x - hj\|)$ in (1.1) are replaced by $\phi(\|h^{-1}x - j\|)$, while $p_h \equiv 0$ (i.e. the polynomial term of s_h is removed and (1.3) is omitted). Powell's method relates the error of interpolation on $\mathcal{V}_h$ to the error of interpolation on the infinite grid $h\mathbb{Z}^d$ for a suitable extension f^* of f to $\mathbb{R}^d$. Specifically, for any fixed x in the interior of D, he writes

$$f(x) - s_h(x) = I_h f^*(x) - s_h(x) + f^*(x) - I_h f^*(x), \tag{1.4}$$

where $I_h f^*$ is the RBF interpolant to f^* on $h\mathbb{Z}^d$ (see next subsection). By a careful analysis of the difference $I_h f^*(x) - s_h(x)$ in the right-hand side of (1.4) and by making use of the well known results concerning the error $f^*(x) - I_h f^*(x)$, Powell deduces that the left-hand side of (1.4) enjoys the same rate of convergence to zero in the interior of D, as the error of interpolation on the infinite grid $h\mathbb{Z}^d$. We present a modification of this method that gives the same conclusion for the problem formulated in the previous subsection.

$\diamond$ **RBF interpolation on the infinite uniform grid.** The two types of basis functions considered in this paper are treated as Examples 5-1 and 5-3 in the fundamental work of Buhmann [3] (see also [4]). He proves the existence of a unique cardinal function

$$\chi(x) = \sum_{j\in\mathbb{Z}^d} \mu_j \phi(\|x-j\|), \quad x \in \mathbb{R}^d, \tag{1.5}$$

where μ_j, $j \in \mathbb{Z}^d$, are real coefficients such that χ is well defined, continuous, and satisfies $\chi(j) = \delta_{0j}$ for all $j \in \mathbb{Z}^d$. Moreover, there exists a maximal integer $\kappa \geq 1$ such that the interpolant

$$Ip(x) = \sum_{j\in\mathbb{Z}^d} p(j)\chi(x-j), \quad x \in \mathbb{R}^d, \tag{1.6}$$

is well defined and $Ip = p$ for all $p \in \mathcal{P}_\kappa(\mathbb{R}^d)$ (i.e. interpolation on $\mathbb{Z}^d$ reproduces polynomials of degree up to κ). Specifically, for any ϕ as above, κ is the least integer greater than or equal to $d+\gamma-1$.

We will also need the rate of decay of χ for large x. For both types of RBFs ϕ and for any dimension d, the bound

$$|\chi(x)| \leq c\|x\|^{-l}, \ \forall\, x \neq 0, \tag{1.7}$$

is satisfied with $l = 2d+\gamma$, where c denotes a generic constant. In addition, if d is odd and $\phi(r) = r^\gamma$, $\gamma \in 2\mathbb{N}+1$, or if d is even and $\phi(r) = r^\gamma \ln r$, $\gamma \in 2\mathbb{N}_+$, then χ enjoys an exponential rate of decay at infinity (cf. Madych and Nelson [8]).

A consequence of these results is that, for functions $f^* : \mathbb{R}^d \to \mathbb{R}$ that have continuous partial derivatives up to order $\kappa+1$, with Lipschitz and/or boundedness assumptions on some of them, the formula

$$I_h f^*(x) = \sum_{j\in\mathbb{Z}^d} f^*(hj)\chi(h^{-1}x - j), \quad x \in \mathbb{R}^d, \tag{1.8}$$

defines an interpolant to f^* on the scaled infinite grid $h\mathbb{Z}^d$. Further, in addition to the interpolation conditions

$$I_h f^*(hj) = f^*(hj), \quad j \in \mathbb{Z}^d, \tag{1.9}$$

$I_h f^*$ has the property

$$\|f^* - I_h f^*\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \leq ch^{d+\gamma} \text{ as } h \to 0, \tag{1.10}$$

where c depends on f^*. The arguments that prove (1.10) are discussed in Section 2.

We remark that the above accuracy orders are known to be sharp, in the sense that there exists a smooth function f^* for which $\|f^* - I_h f^*\|_{\mathrm{L}^\infty(\mathbb{R}^d)}$ does not tend to zero faster than $\mathcal{O}(h^{d+\gamma})$ as $h \to 0$ (cf. [2], [3], [6]).

$\diamond$ **The difference $I_h f^* - s_h$.** An important step in Powell's method of analysis described above is provided by the identity

$$I_h f^*(x) - s_h(x) = I_h(f^* - s_h)x = \sum_{j \in \mathbb{Z}^d} [f^*(hj) - s_h(hj)]\chi(h^{-1}x - j), \ x \in D, \quad (1.11)$$

which will be justified in the next section. Note that, by (1.2) and the fact that f^* is an extension of f, only the indices $j \notin \mathcal{J}_N$ give a nonzero term on the right-hand side of (1.11). Further, f^* will be chosen in such a way that $\mathrm{supp}(f^*)$ is compact, so $|f^*(hj) - s_h(hj)| = |s_h(hj)|$ is satisfied for all but a finite number of indices $j \in \mathbb{Z}^d$. Therefore, in order to exploit the asymptotic decay (1.7) of the cardinal function χ in (1.11), we need an estimate of the growth of $|s_h(x)|$ as $\|x\| \to \infty$, independent of h. Specifically, by using the variational formalism of Wu and Schaback [19], we prove in Proposition 1 of the next section that, whenever ϕ is one of the radial functions considered in this paper, there exists $\tau > 0$ with the property

$$|s_h(x)| \le c_f(1 + \|x\|)^\tau, \ x \in \mathbb{R}^d, \quad (1.12)$$

for all sufficiently small h, where $c_f > 0$ depends on f, but not on h.

$\diamond$ **The results.** A bound of the type (1.12) and the relations between the numbers κ, l, τ are the key ingredients that allow the extension of Powell's approach to the present case.

Before stating the main result, we introduce the Lipschitz spaces $\mathrm{Lip}(k+1, F)$, where k is a nonnegative integer and $F \subset \mathbb{R}^d$ is a closed set (see page 176 of [17] for a general definition). When $F \subset \mathbb{R}^d$ is a closed nonempty set such that $F = \mathrm{cl}(\mathrm{int}(F))$, these spaces are characterized as follows. $\mathrm{Lip}(1, F)$ is the space of bounded and Lipschitz continuous functions on F. For $k \ge 1$, $\mathrm{Lip}(k+1, F)$ is the space of bounded continuous functions f on F with the properties: f is of class C^k on the interior of F, its partial derivatives $\frac{\partial^\alpha f}{\partial x^\alpha}$ extend to bounded continuous functions on F for all multi-indices α such that $|\alpha| \le k$, and the extended functions $\frac{\partial^\alpha f}{\partial x^\alpha}$ on F belong to $\mathrm{Lip}(1, F)$ when $|\alpha| = k$.

The reason for working with Lipschitz spaces is that we use the Whitney theorem to extend the given data function $f : D \to \mathbb{R}$ to $f^* : \mathbb{R}^d \to \mathbb{R}$ and we need some of the partial derivatives of f^* to be Lipschitz. Note also that the boundedness requirement for the partial derivatives of f in the above definition is superfluous when $F = D$.

Theorem 1 *Let ϕ be any one of the basis functions mentioned in the second subsection and let $f \in \mathrm{Lip}(\kappa + 2, D)$. Then, for any compact subset K of the interior of the unit d-cube D, the error of the interpolant* (1.1) *is bounded by the inequality*

$$\|f - s_h\|_{\mathrm{L}^\infty(K)} \le ch^{d+\gamma} \ \text{ as } h \to 0, \quad (1.13)$$

where c is a positive constant that depends on f and K, but not on h.

The remark after equation (1.10) and numerical experiments suggest that the rate (1.13) is sharp. The following result proves this suggestion for the case when d and ϕ are such that the cardinal function (1.5) has an exponential rate of decay at infinity. A proof for the remaining cases is unknown at present.

Theorem 2 *If d is odd and $\phi(r) = r^\gamma$, $\gamma \in 2\mathbb{N}+1$ or if d is even and $\phi(r) = r^\gamma \ln r$, $\gamma \in 2\mathbb{N}_+$, then the order of convergence* (1.13) *is sharp, in the sense that there exist a function $f \in \mathrm{Lip}(\kappa+3, D)$ and a compact set $K \subset \mathrm{int}(D)$ for which the left-hand side of* (1.13) *does not tend to zero faster than $\mathcal{O}(h^{d+\gamma})$ as $h \to 0$.*

The proofs of Theorems 1 and 2 are given in Sections 2 and 3, respectively. They also apply, with minor modifications, to the case when the unit cube D is replaced by any set that is the closure of an open bounded domain in $\mathbb{R}^d$.

$\diamond$ **Comments.** This work was initiated by an interest in the basis function $\{\phi(r) = r^2 \ln r : 0 \le r < \infty\}$, which gives the *thin plate spline* (TPS) interpolant when $d = 2$. In this case, our results imply the rate of convergence $\mathcal{O}(h^4)$ for TPS interpolation inside the unit square.

For the corresponding one dimensional interpolant using the same basis function, Theorem 1 gives $\mathcal{O}(h^3)$ accuracy on compact sets inside [0,1]. This conclusion is confirmed by numerical experiments performed by the author, which also indicate a deterioration in the order of accuracy to $\mathcal{O}(h^{3/2})$ near the endpoints of the interval, for general smooth data functions. Hence, if we want the uniform error behaviour on [0,1] to be closer to the local one for smooth data functions, a modification of the interpolation method is needed. Indeed, if we replace the (two) constraints (1.3) with the condition that s_h interpolates the end derivatives, numerical experiments suggest that the convergence rate is improved to $\mathcal{O}(h^{5/2})$ near the ends of the interval. These features are under investigation, the outcome of which will be reported elsewhere.

Note that, for $d = 1$ and $\phi(r) = r^{2m+1}$, $m \in \mathbb{N}$, the interpolant defined by (1.1)–(1.3) is a *natural* spline of degree $2m + 1$. Our results give the rate of convergence $\mathcal{O}(h^{2m+2})$ in the interior of an interval, which is well known (see [1], for instance).

$\diamond$ **Literature.** The accuracy of interpolation at a *finite* number of points by means of the radial basis functions considered in this paper has been the subject of much research, as shown in [5], [19], [12], [15], [9], [16], [18], for example. Using different techniques, all of these papers allow the data points to be *scattered* and, with the exception of [9], their results apply to the *uniform* error $\|f - s_h\|_{\mathrm{L}^\infty(D)}$, where, for a given set of scattered interpolation points, h denotes the diameter of the largest sphere centred in D that does not contain any interpolation point in its interior. Therefore, due to boundary effects, it is usual for their uniform rates of convergence to be slower than the maximal *local* order $\mathcal{O}(h^{d+\gamma})$ of Theorem 1, which is the main subject of this paper.

2 Proof of Theorem 1

$\diamond$ **The extension f^* of f.** The method of proof that is described in the Introduction requires the construction of a suitable extension f^* of f to $\mathbb{R}^d$, which is done in two steps. Firstly, since $f \in \mathrm{Lip}(\kappa+2, D)$, by the Whitney extension theorem (cf. Chapter VI of [17]), there exists $\tilde{f} \in \mathrm{Lip}(\kappa+2, \mathbb{R}^d)$ such that $\tilde{f}(x) = f(x)$, $x \in D$. Secondly, we let $\eta \in C_0^\infty(\mathbb{R}^d)$ be a cut-off function with $\eta(x) = 1$, $x \in D$, and $\eta(x) = 0$ for sufficiently large $\|x\|$. Hence $f^* = \eta \tilde{f} \in \mathrm{Lip}(\kappa+2, \mathbb{R}^d)$ satisfies

$$f^*(x) = f(x), \ x \in D, \tag{2.1}$$

and $\mathrm{supp}(f^*)$ is compact.

◇ **The order of convergence (1.10).** We recall that κ is the least integer that satisfies $\kappa \geq d + \gamma - 1$. If $f^* \in C^{\kappa+1}(\mathbb{R}^d)$ and f^* has bounded κth and $(\kappa+1)$st-order partial derivatives, the property (1.10) of the error of interpolation to f^* on $h\mathbb{Z}^d$ is proved by Buhmann [3] for all the considered choices of ϕ with the exception of the case when γ is a positive integer and κ is even (see also [4]). In this special case, Buhmann's results imply only that the interpolation error is at most a constant multiple of $h^{d+\gamma}|\ln h|$ as $h \to 0$. However, our conditions on f^* ensure that inequality (1.10) holds for this case too, as shown by the cancellation argument of Powell [11, Proof of Theorem 8.5]. Indeed, since $f^* \in \mathrm{Lip}(\kappa+2, \mathbb{R}^d)$, it follows that f^* satisfies in particular not only Buhmann's assumptions, but also the hypotheses of [11, Theorem 8.5]. That theorem also requires a condition on ϕ. Specifically, for all the choices of ϕ considered by our paper, the distributional Fourier transform of $\psi(x) = \phi(\|x\|)$, $x \in \mathbb{R}^d$, is given by $\widehat{\psi}(t) = c\|t\|^{-(d+\gamma)}$ for $t \neq 0$. Therefore equation (8.34) from [11] is true when $\gamma \in \mathbb{N}_+$ and κ is even. Hence Lemma 8.4 of [11] also holds true, and then the proof of Theorem 8.5 in [11] gives the order of convergence (1.10).

◇ **The justification of (1.11).** By (1.8), it suffices to prove the identity

$$s_h(x) = \sum_{j \in \mathbb{Z}^d} s_h(hj)\chi(h^{-1}x - j), \quad x \in \mathbb{R}^d, \tag{2.2}$$

or, after a change of variables,

$$s_h(hx) = \sum_{j \in \mathbb{Z}^d} s_h(hj)\chi(x - j), \quad x \in \mathbb{R}^d. \tag{2.3}$$

Letting $p = p_h(h\,\cdot)$ in (1.6), where p_h occurs in definition (1.1), and noting that $m < \kappa$, the polynomial reproduction property shows that (2.3) is equivalent to

$$s_h(hx) - p_h(hx) = \sum_{j \in \mathbb{Z}^d} [s_h(hj) - p_h(hj)]\chi(x - j), \quad x \in \mathbb{R}^d. \tag{2.4}$$

By (1.1), the left-hand side of (2.4) is the function

$$\sum_{j \in \mathcal{J}_N} c_j \phi(h\|x - j\|), \ x \in \mathbb{R}^d. \tag{2.5}$$

When $\phi(r) = r^\gamma$, $\gamma > 0$, $\gamma \notin 2\mathbb{N}$, homogeneity implies that (2.5) is a linear combination of terms of the form $\phi(\|x - j\|)$ for $j \in \mathcal{J}_N$. Alternatively, when $\phi(r) = r^\gamma \ln r$, $\gamma \in 2\mathbb{N}_+$, (2.5) becomes

$$\begin{aligned} \sum_{j \in \mathcal{J}_N} c_j \phi(h\|x - j\|) &= \sum_{j \in \mathcal{J}_N} c_j h^\gamma \|x - j\|^\gamma (\ln h + \ln \|x - j\|) \\ &= h^\gamma \sum_{j \in \mathcal{J}_N} c_j \phi(\|x - j\|) + h^\gamma \ln h \sum_{j \in \mathcal{J}_N} c_j \|x - j\|^\gamma, \end{aligned} \tag{2.6}$$

and the last sum is a polynomial of degree less than γ (cf. (1.3)), which is reproduced by interpolation on $\mathbb{Z}^d$, since $\kappa \geq d+\gamma-1 \geq \gamma$. Hence the required equation (2.4) is satisfied if interpolation on the infinite grid $\mathbb{Z}^d$ reproduces the function $\phi(\|x-j\|)$, $x \in \mathbb{R}^d$, for every $j \in \mathcal{J}_N$.

We prove this condition for the above choices of ϕ by applying arguments from [11, Section 7]. Specifically, we consider the linear space $\mathcal{S}$ defined by (7.16), (7.17) of that paper. Due to Theorem 5-7 of [3], the cardinal function (1.5) is in $\mathcal{S}$. Further, it is elementary that any choice of ϕ from Section 1 satisfies inequality (7.18) of [11]. It follows that both the arguments of Lemma 7.5 and of page 169 of [11] are valid for our choices of ϕ. Thus interpolation on the infinite grid $\mathbb{Z}^d$ reproduces the function $\{\phi(\|x-k\|) : x \in \mathbb{R}^d\}$ for every $k \in \mathbb{Z}^d$, which completes the justification of (1.11).

$\diamond$ The uniform growth of s_h.

Proposition 1 *Let ϕ be one of the radial functions considered in this paper and assume that the data function f is in the set* $\mathrm{Lip}(\kappa+2, D)$. *Then the RBF interpolant* (1.1) *satisfies* (1.12), *where c_f is independent of h. Specifically, recalling that m denotes the integer part of γ, we have $\tau = \sigma + (m+\gamma)/2$, where $\sigma = 0$ for $\phi(r) = r^\gamma$, $\gamma > 0$, $\gamma \notin 2\mathbb{N}$, while σ can be any positive constant for $\phi(r) = r^\gamma \ln r$, $\gamma \in 2\mathbb{N}_+$.*

Proof. Let n be the dimension of the space $\mathcal{P}_m(\mathbb{R}^d)$ and denote by $\{P_j : j = 1,2,\ldots,n\}$ the monomial basis of $\mathcal{P}_m(\mathbb{R}^d)$. For any set $\{u_j : j = 1,2,\ldots,n\}$ of vectors from $\mathbb{R}^d$, we define

$$\omega(u_1, u_2, \ldots, u_n) = \det(P_j(u_i)). \tag{2.7}$$

Further, let $\mathcal{V} = \{v_j : j = 1,2,\ldots,n\}$ be a subset of D such that interpolation on $\mathcal{V}$ from the linear space $\mathcal{P}_m(\mathbb{R}^d)$ has a unique solution (for example, $\mathcal{V}$ may be the set of vertices of a suitable regular grid in any simplex that is included in D—cf. [13, page 289]). Therefore we obtain the condition

$$\omega(v_1, v_2, \ldots, v_n) \neq 0, \tag{2.8}$$

and we define $\omega_0 = |\omega(v_1, v_2, \ldots, v_n)| > 0$. By the continuity of the function ω in all its arguments, there exists $\epsilon > 0$ such that, for any subset $\{u_j : j = 1,2,\ldots,n\}$ of $\mathbb{R}^d$ whose elements satisfy $\|u_j - v_j\| < \epsilon$, $j = 1,2,\ldots,n$, we have $|\omega(u_1, u_2, \ldots, u_n)| \geq \omega_0/2$. In particular, for sufficiently small h, there exists a subset $\{x_j : j = 1,2,\ldots,n\}$ say, of $\mathcal{V}_h$, such that

$$\|x_j - v_j\| < \epsilon, \ j = 1,2,\ldots,n, \tag{2.9}$$

and

$$\omega(x_1, x_2, \ldots, x_n) \geq \frac{\omega_0}{2}. \tag{2.10}$$

To simplify the arguments later, denote the elements of the set $\mathcal{V}_h \backslash \{x_j : j = 1,2,\ldots,n\}$ by $\{x_j : j = n+1, n+2, \ldots, (N+1)^d\}$. Moreover, we define the Lagrange interpolation polynomials $l_i \in \mathcal{P}_m(\mathbb{R}^d)$, $i = 1,2,\ldots,n$, which satisfy $l_i(x_j) = \delta_{ij}$, $i,j = 1,2,\ldots,n$, by the expression

$$l_i(x) = \frac{\omega(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n)}{\omega(x_1, x_2, \ldots, x_n)}, \ x \in \mathbb{R}^d, \ i = 1,2,\ldots,n. \tag{2.11}$$

Consequently, we can write the Lagrange representation formula

$$p(x)=\sum_{j=1}^{n} l_j(x)p(x_j),\ p\in\mathcal{P}_m(\mathbb{R}^d),\ x\in\mathbb{R}^d. \tag{2.12}$$

Next, using the extension f^* of f that is constructed at the beginning of this section, we let c_{f^*} be the positive square root of the number

$$c_{f^*}^2=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\widehat{f^*}(t)|^2(\widehat{\psi}(t))^{-1}dt, \tag{2.13}$$

where $\widehat{f^*}$ is the Fourier transform of f^* and already we have let $\widehat{\psi}$ be the distributional Fourier transform of $\{\psi(x)=\phi(\|x\|):x\in\mathbb{R}^d\}$. The finiteness of c_{f^*} can be deduced from the well known property

$$|t^\mu\widehat{f^*}(t)|=|\widehat{D^\mu f^*}(t)|\le\|D^\mu f^*\|_{\mathrm{L}^1(\mathbb{R}^d)},\ t\in\mathbb{R}^d, \tag{2.14}$$

of Fourier transforms, μ being a multi-index. Moreover, for all the considered choices of ϕ, $\widehat{\psi}(t)$ is a multiple of $\|t\|^{-(d+\gamma)}$ for $t\neq 0$. Hence the integrand of (2.13) is in $\mathrm{L}^1(\mathbb{R}^d)$ if $|\widehat{f^*}(t)|^2\|t\|^{d+\gamma}\le c^2\|t\|^{-(d+1)}$ for some constant $c>0$, so that a sufficient condition for c_{f^*} to be finite is $|\widehat{f^*}(t)|\|t\|^{d+(\gamma+1)/2}\le c$ for large $\|t\|$. This condition is true by (2.14), since $d+(\gamma+1)/2\le\kappa+1$, $f^*\in C^{\kappa+1}(\mathbb{R}^d)$ and $\mathrm{supp}(f^*)$ is compact.

Now, because s_h is the interpolant to f^* on $\mathcal{V}_h$, the finiteness of c_{f^*} ensures that Theorem 4 of [19] can be applied. It gives the pointwise bound

$$|s_h(x)-f^*(x)|\le c_{f^*}\kappa_q(x),\ x\in\mathbb{R}^d. \tag{2.15}$$

Here $\{\kappa_q(x):x\in\mathbb{R}^d\}$ is the nonnegative Kriging function whose square has the value

$$\min\ (-1)^{m+1}\left\{\sum_{i=1}^{(N+1)^d}\sum_{j=1}^{(N+1)^d}\beta_i\beta_j\phi(\|x_i-x_j\|)-2\sum_{i=1}^{(N+1)^d}\beta_i\phi(\|x-x_i\|)\right\}, \tag{2.16}$$

where the minimum is taken over the multipliers β_i, $i=1,2,\ldots,(N+1)^d$, that satisfy

$$p(x)=\sum_{i=1}^{(N+1)^d}\beta_i p(x_i),\ p\in\mathcal{P}_m(\mathbb{R}^d), \tag{2.17}$$

for the current value of x. The positivity of expression (2.16) when x is not an interpolation point is a consequence of the fact that $\{(-1)^{m+1}\phi(r^{1/2}):0\le r<\infty\}$ is conditionally strictly positive definite of order $m+1$, for all our choices of ϕ. For each $x\in\mathbb{R}^d$, we choose $\beta_i=l_i(x)$, $i=1,2,\ldots,n$, and $\beta_i=0$, $i=n+1,n+2,\ldots,(N+1)^d$, where l_i, $i=1,2,\ldots,n$ are the Lagrange polynomials (2.11). Hence (2.17) is satisfied due to (2.12), and expression (2.16) implies

$$\kappa_q^2(x)\le\sum_{i=1}^{n}\sum_{j=1}^{n}|l_i(x)||l_j(x)||\phi(\|x_i-x_j\|)|+2\sum_{i=1}^{n}|l_i(x)||\phi(\|x-x_i\|)|. \tag{2.18}$$

The last part of the proof provides an estimate of the right-hand side of (2.18). We use (2.9)–(2.11), the continuity of the determinant function (2.7) in each of its variables, and the fact that each l_i, $i = 1, 2, \ldots, n$, is a polynomial of degree m over $\mathbb{R}^d$, to deduce the bound

$$|l_i(x)| \leq c(1 + \|x\|)^m, \; i = 1, 2, \ldots, n, \; x \in \mathbb{R}^d, \tag{2.19}$$

where c is independent of x and h. Further, the numbers $|\phi(\|x_i - x_j\|)|$, for $i, j = 1, 2, \ldots, n$, are bounded by the constant $\max\{|\phi(\|x - y\|)| : x \in D, \; y \in D\}$, while $x_i \in D$, $i = 1, 2, \ldots, n$, and the growth of ϕ at infinity give

$$|\phi(\|x - x_i\|)| \leq c(1 + \|x\|)^{\gamma + 2\sigma}, \; x \in \mathbb{R}^d, \tag{2.20}$$

where c is still a generic constant. The last inequality holds with $\sigma = 0$ for $\phi(r) = r^\gamma$, $\gamma > 0$, $\gamma \notin 2\mathbb{N}$ and with any $\sigma > 0$ for $\phi(r) = r^\gamma \ln r$, $\gamma \in 2\mathbb{N}_+$. Consequently, (2.15), (2.18)–(2.20) and the compact support of f^* provide the required conclusion (1.12), with the value of τ that is specified in the statement of the proposition. □

◇ **The proof of (1.13).** We now complete the proof of Theorem 1 by deducing a suitable bound on $I_h f^* - s_h$. Since our argument for sums of the type (1.11) is also needed twice in the proof of Theorem 2, it is synthesized in the next result.

Proposition 2 *Let K_1 and K_2 be two compact subsets of $\mathbb{R}^d$, with $K_1 \subset \operatorname{int}(K_2)$, let $\chi : \mathbb{R}^d \to \mathbb{R}$ be a function that satisfies the bound (1.7) for a positive exponent l, let $\mathcal{H}$ be a monotonically decreasing sequence of positive numbers that tends to zero, and let $(g_h)_{h \in \mathcal{H}}$ be a family of real-valued functions, defined on $\mathbb{R}^d$, such that*

$$|g_h(x)| \leq c_0(1 + \|x\|)^\tau, \; x \in \mathbb{R}^d, \; h \in \mathcal{H}, \tag{2.21}$$

where c_0 and τ are positive constants. If τ satisfies the inequality

$$l - \tau > d, \tag{2.22}$$

then the condition

$$\sum_{j \in \mathbb{Z}^d \setminus h^{-1} K_2} |g_h(hj)||\chi(h^{-1}x - j)| = \mathcal{O}(h^{l-d}), \tag{2.23}$$

holds uniformly for $x \in K_1$, as $h \to 0$.

Proof. Let $\Delta = \max\{\|x\| : x \in K_1\}$ and let δ be the distance between K_1 and the boundary of K_2. By (2.21), for any $j \in \mathbb{Z}^d \setminus h^{-1} K_2$ and any $x \in K_1$, we deduce

$$\begin{aligned} |g_h(hj)| &\leq c_0(1 + \|hj\|)^\tau \\ &\leq c_0(1 + \|x\| + \|hj - x\|)^\tau \\ &\leq c_0 \|hj - x\|^\tau \left(1 + \frac{1 + \Delta}{\delta}\right)^\tau. \end{aligned} \tag{2.24}$$

Then we can use (1.7) and (2.24) to obtain

$$\sum_{j\in\mathbb{Z}^d\setminus h^{-1}K_2} |g_h(hj)||\chi(h^{-1}x-j)| \leq c_0 h^\tau \left(1+\frac{1+\Delta}{\delta}\right)^\tau \sum_{j\in\mathbb{Z}^d\setminus h^{-1}K_2} \frac{1}{\|h^{-1}x-j\|^{l-\tau}}. \tag{2.25}$$

Moreover, there exists a constant $c_1 > 0$ such that the sum on the right-hand side of the last expression is bounded above by

$$c_1 \int_{\|t\|\geq h^{-1}\delta} \|t\|^{-l+\tau} dt = c_1\sigma_d \int_{r=h^{-1}\delta}^{\infty} r^{-l+\tau+d-1} dr = \frac{c_1\sigma_d}{l-\tau-d}\left(\frac{h}{\delta}\right)^{l-\tau-d}, \tag{2.26}$$

where σ_d is the surface area of the unit sphere in $\mathbb{R}^d$. The assumption (2.22) ensures that the integral is finite. Combining (2.25) and (2.26) yields (2.23), as required. □

The hypotheses of Proposition 2 are satisfied if we let $K_1 = K$ (the compact set from Theorem 1), $K_2 = D$, $g_h = f^* - s_h$ for all sufficiently small h, and we let χ be the cardinal function (1.5). Indeed, in this case (1.7) holds for $l = 2d + \gamma$, (2.21) is satisfied by $c_0 = c_f + \|f^*\|_{L^\infty(\mathbb{R}^d)}$, the value of τ being given by Proposition 1, and (2.22) is also true when the constant σ of (2.20) is restricted to [0,1]. Therefore (1.11), (1.2), Proposition 2 and $l - d = d + \gamma$ provide

$$\begin{aligned} |I_h f^*(x) - s_h(x)| &= \Big|\sum_{j\in\mathbb{Z}^d\setminus h^{-1}D} (f^*-s_h)(hj)\chi(h^{-1}x-j)\Big| \\ &\leq \sum_{j\in\mathbb{Z}^d\setminus h^{-1}D} |(f^*-s_h)(hj)||\chi(h^{-1}x-j)| \;=\; \mathcal{O}(h^{d+\gamma}),\ \text{as } h\to 0, \end{aligned} \tag{2.27}$$

uniformly for $x \in K$. We now combine (1.4), (1.10) and (2.27) to obtain the desired conclusion (1.13). This finishes the proof of Theorem 1. □

⋄ **Remark.** When the decay of χ is exponential, Proposition 2 implies that the sum (2.23) tends to zero faster than any power of h. This property allows the proof of Theorem 2 that is given in the next section. However, the value $l = 2d + \gamma$ in (1.7) is sufficient to establish Theorem 1 for all the choices of ϕ considered in this paper.

3 Proof of Theorem 2

⋄ **Idea of proof.** We use the method of Jackson [6, Proof of Theorem 5-10] and Buhmann [3, Proof of Theorem 5-13], originally developed for the cases of quasi-interpolation and interpolation, respectively, on infinite regular grids. We note that γ is an integer in Theorem 2, which implies $\kappa = d + \gamma - 1$.

Suppose that the convergence rate

$$\|f - s_h\|_{L^\infty(K)} = o(h^{d+\gamma}),\ h \to 0 \tag{3.1}$$

holds for all functions $f \in \mathrm{Lip}(\kappa+3, D)$ and all compacts $K \subset \mathrm{int}(D)$. Then we deduce the property

$$Ip \equiv p \text{ for all } p \in \mathcal{P}_{\kappa+1}(\mathbb{R}^d), \tag{3.2}$$

where I is the interpolation operator (1.6). Since κ is the maximal degree for reproduction of polynomials on the multi-integer grid [4], however, equation (3.2) provides a contradiction that establishes Theorem 2.

$\diamond$ **The main arguments.** We let p be a homogeneous polynomial of total degree $\kappa+1$ on $\mathbb{R}^d$.

Lemma 1 *Let K be a compact subset of the interior of D that has $\frac{1}{2}u$ as an interior point, where $u = (1, 1, \ldots, 1)^T \in \mathbb{Z}^d$. If the assumption* (3.1) *were true, then the bound*

$$|p(x - \tfrac{1}{2}u) - \sum_{j \in \mathbb{Z}^d} p(hj - \tfrac{1}{2}u)\chi(h^{-1}x - j)| = o(h^{\kappa+1}), \;\; h \to 0, \tag{3.3}$$

would hold uniformly for $x \in K$.

The proof of Lemma 1 is given below. Next we use the following change of variables. We let $y = x - \frac{1}{2}u$, we work with $N = 2\nu$ for increasing positive integers ν, and we recall $h = N^{-1}$. Then , if $j \in \mathbb{Z}^d$ and $k = j - \nu u$, we have $hj - \frac{1}{2}u = hk$ and $h^{-1}y - k = h^{-1}x - j$. Hence (3.3) is equivalent to

$$|p(y) - \sum_{k \in \mathbb{Z}^d} p(hk)\chi(h^{-1}y - k)| = o(h^{\kappa+1}), \;\; h \to 0, \tag{3.4}$$

uniformly for $y \in K - \frac{1}{2}u$.

Now the proof of (3.2) depends on the remark that $K - \frac{1}{2}u$ contains a neighbourhood of the origin in $\mathbb{R}^d$, and on a classical homogeneity argument. Specifically, if $z \in \mathbb{R}^d$ is fixed, then $hz \in K - \frac{1}{2}u$ for any sufficiently small h, so (3.4) implies

$$|p(hz) - \sum_{k \in \mathbb{Z}^d} p(hk)\chi(z - k)| = o(h^{\kappa+1}), \;\; h \to 0. \tag{3.5}$$

Therefore the homogeneity of p yields

$$|p(z) - \sum_{k \in \mathbb{Z}^d} p(k)\chi(z - k)| = o(1), \;\; h \to 0, \tag{3.6}$$

which proves $Ip(z) = p(z)$, because the left-hand side of (3.6) is independent of h. Since z is arbitrary, (3.2) follows.

$\diamond$ **Proof of Lemma 1.** Let $g(x) = p(x - \frac{1}{2}u)$ for all $x \in \mathbb{R}^d$. Given a compact $K \subset \mathrm{int}(D)$, we let M be another compact such that $K \subset \mathrm{int}(M)$ and $M \subset \mathrm{int}(D)$. Let $\rho \in C_0^\infty(\mathbb{R}^d)$ be a cut-off function with the properties: $0 \le \rho(x) \le 1$ for $x \in \mathbb{R}^d$, $\rho(x) = 1$ for $x \in M$, and $\mathrm{supp}(\rho) \subset \mathrm{int}(D)$. Hence the function $\{f(x) = \rho(x)g(x) : x \in$

$\mathbb{R}^d\}$ is infinitely differentiable, with $\mathrm{supp}(f) \subset \mathrm{int}(D)$. Further, we have $f^* = f$, where f^* is the extension of $f\,|_D$ to $\mathbb{R}^d$ that is constructed at the beginning of Section 2.

We replace K by M in assumption (3.1) for the above f. Then $\rho\,|_M \equiv 1$ and $d + \gamma = \kappa + 1$ imply

$$|g(x) - s_h(x)| = o(h^{\kappa+1}),\ \ h \to 0, \tag{3.7}$$

uniformly for $x \in M$, where s_h is the RBF interpolant to $f\,|_D$ on the finite grid $\mathcal{V}_h$. We let $I_h f^*$ and $I_h g$ be the RBF interpolants to $f^* = f$ and g on the infinite grid $h\mathbb{Z}^d$, as given by (1.8). The exponential decay of the cardinal function χ ensures that $I_h g$ is well defined and that $l = d + \kappa + 2$ is allowed in (1.7). For this value of l, we apply Proposition 2 twice.

Firstly, we let $K_1 = M$, $K_2 = D$ and $g_h = s_h$, $h > 0$ in Proposition 2. Then (2.21) holds with $c_0 = c_f$, the value of τ being given by Proposition 1. Further, (2.22) is also satisfied in this case, by keeping the constant σ of (2.20) in [0,1]. Therefore (1.11), the compact support of f^* and Proposition 2 provide

$$\begin{aligned} |s_h(x) - I_h f^*(x)| &= |I_h(s_h - f^*)(x)| = |\sum_{j \in \mathbb{Z}^d \setminus h^{-1}D} s_h(hj)\chi(h^{-1}x - j)| \\ &\le \sum_{j \in \mathbb{Z}^d \setminus h^{-1}D} |s_h(hj)||\chi(h^{-1}x - j)| \ = \ \mathcal{O}(h^{\kappa+2}),\ \text{as } h \to 0, \end{aligned} \tag{3.8}$$

uniformly for $x \in M$.

Secondly, let $K_1 = K$, $K_2 = M$ and $g_h = g$, $h > 0$ (i.e. the family $(g_h)_{h>0}$ reduces to the single element g) in Proposition 2. Since $g \in \mathcal{P}_{\kappa+1}(\mathbb{R}^d)$, it follows that (2.21) holds with $\tau = \kappa + 1$, and condition (2.22) becomes $l - \tau = d + 1 > d$. Hence the properties $\rho\,|_M \equiv 1$, $0 \le \rho \le 1$ and another application of Proposition 2 imply

$$\begin{aligned} |I_h f^*(x) - I_h g(x)| &= |I_h(f^* - g)(x)| = |\sum_{j \in \mathbb{Z}^d \setminus h^{-1}M} [1 - \rho(hj)]g(hj)\chi(h^{-1}x - j)| \\ &\le \sum_{j \in \mathbb{Z}^d \setminus h^{-1}M} |g(hj)||\chi(h^{-1}x - j)| \ = \ \mathcal{O}(h^{\kappa+2}),\ \text{as } h \to 0, \end{aligned} \tag{3.9}$$

uniformly for $x \in K$.

Therefore (3.7)–(3.9) prove (3.3), as required. □

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References

[1] Atkinson, K.E. (1968): On the order of convergence of natural cubic spline interpolation, SIAM J. Numer. Anal. 5, 89–101.

[2] de Boor, C., Ron, A. (1992): Fourier analysis of the approximation power of principal shift-invariant spaces, Constr. Approx. 8, 427–462.

[3] Buhmann, M.D. (1989): Multivariable Interpolation using Radial Basis Functions, PhD. Dissertation, University of Cambridge.

[4] Buhmann, M.D. (1990): Multivariate cardinal interpolation with radial-basis functions, Constr. Approx. 6, 225–255.

[5] Duchon, J. (1978): Sur l'erreur d'interpolation des fonctions de plusieurs variables par les D^m-splines, R.A.I.R.O. An. Num. 12, no. 4, 325–334.

[6] Jackson, I.R.H. (1988): Radial Basis Function Methods for Multivariable Approximation, PhD. Dissertation, University of Cambridge.

[7] Jackson, I.R.H. (1989): Radial basis functions: a survey and new results. In: D.C. Handscomb, ed., The Mathematics of Surfaces III, Oxford Univ. Press, Oxford, pp. 115–133.

[8] Madych, W.R., Nelson, S.A. (1990): Polyharmonic cardinal splines, J. Approx. Theory 60, 141–156.

[9] Matveev, O.V. (1996): Error estimates for splines interpolating scattered data, Conference abstract, Curves and Surfaces, Chamonix 1996.

[10] Micchelli, C.A. (1986): Interpolation of scattered data: distance matrices and conditionally positive definite functions, Constr. Approx. 2, 11–22.

[11] Powell, M.J.D. (1992): The theory of radial basis function approximation in 1990. In: W.A. Light, ed., Advances in Numerical Analysis, Vol. II: Wavelets, Subdivision Algorithms, and Radial Basis Functions, pp. 105–210, Clarendon Press, Oxford.

[12] Powell, M.J.D. (1994): The uniform convergence of thin plate spline interpolation in two dimensions, Numer. Math. 68, 107–128.

[13] Powell, M.J.D. (1997): A review of methods for multivariable interpolation at scattered data points. In: I.S. Duff and G.A. Watson, eds., The State of the Art in Numerical Analysis, Clarendon Press, Oxford, pp. 283–309.

[14] Schaback, R. (1993): Comparison of radial basis function interpolants. In: K. Jetter and F. Utreras, eds., From CAGD to Wavelets, World Scientific, pp. 293–305.

[15] Schaback, R. (1996): Approximation by radial basis functions with finitely many centers, Constr. Approx. 12, 331–340.

[16] Schaback, R. (1997): Improved error bounds for scattered data interpolation by radial basis functions, Preprint, Goettingen (to appear in Math. Comp.).

[17] Stein, E.M. (1970): Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, Princeton.

[18] Wendland, H. (1997): Sobolev-type error estimates for interpolation by radial basis functions. In: A. Le Mehaute, C. Rabut, L.L. Schumaker, eds., Surface Fitting and Multiresolution Methods, Vanderbilt Univ. Press, pp. 337–344.

[19] Wu, Z.-M., Schaback, R. (1993): Local error estimates for radial basis function interpolation of scattered data, I.M.A. J. Numer. Anal. 13, 13–27.

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