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# Proof of convergence of an iterative technique for thin plate spline interpolation in two dimensions[1]

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**Abstract:** Thin plate spline methods provide an interpolant to values of a real function and are highly useful in many applications. The need for iterative procedures arises, since hardly any sparsity occurs in the linear system of interpolation equations. This paper considers a generalization of the iterative algorithm developed by Beatson, Goodsell and Powell. A proof of convergence of this method is given. It depends mainly on the remark that all the changes to the thin plate spline coefficients reduce a certain semi-norm of the difference between the required interpolant and the current approximation. The analysis applies also to analogous algorithms for other radial basis functions.

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## 1. Introduction

We seek a function $s^*$ from $\mathcal{R}^2$ to $\mathcal{R}$ that satisfies the interpolation equations

$$s^*(\underline{x}_i) = f_i, \qquad i=1,\ldots,n, \tag{1.1}$$

where $\underline{x}_i \in \mathcal{R}^2$ and $f_i \in \mathcal{R}$, $i=1,\ldots,n$, are data, and where we require the points $\underline{x}_i$, $i=1,\ldots,n$, to be all different and not collinear. Specifically, we let $s^*$ be the thin plate spline solution of the interpolation problem and hence $s^*$ has the form

$$s^*(\underline{x}) = \sum_{j=1}^{n} \lambda_j^* \, \phi\left(\| \, \underline{x} - \underline{x}_j \|_2\right) + p^*(\underline{x}), \qquad \underline{x} \in \mathcal{R}^2, \tag{1.2}$$

where $p^*$ is a polynomial of degree at most one from $\mathcal{R}^2$ to $\mathcal{R}$ and where $\phi$ is the function

$$\phi(r) = r^2 \log r, \quad r>0, \quad \text{and} \quad \phi(0)=0. \tag{1.3}$$

The real coefficients $\lambda_j$, $j=1,\ldots,n$, satisfy the additional conditions

$$\sum_{j=1}^{n} \lambda_j^* = 0 \qquad \text{and} \qquad \sum_{j=1}^{n} \lambda_j^* \, \underline{x}_j = 0, \tag{1.4}$$

which ensure that $s^*$ has square integrable second derivatives. One reason for the success of thin plate spline interpolation is that $s^*$ is the solution of an optimal recovery problem. Specifically, $s^*$ provides the least possible value of $(s^*, s^*)$ subject to the interpolation conditions (1.1), where $(f,g)$ is the scalar product

$$\begin{aligned}(f,g) \;=\; & \frac{1}{8\pi} \int\!\!\int_{\mathcal{R}^2} \left[ \frac{\partial^2 f(x,y)}{\partial x^2} \frac{\partial^2 g(x,y)}{\partial x^2} + 2\, \frac{\partial^2 f(x,y)}{\partial x\, \partial y} \frac{\partial^2 g(x,y)}{\partial x\, \partial y} \right. \\ & \left. + \frac{\partial^2 f(x,y)}{\partial y^2} \frac{\partial^2 g(x,y)}{\partial y^2} \right] dx\, dy, \end{aligned} \tag{1.5}$$

which vanishes on the space of polynomials of degree at most one, $x$ and $y$ being the components of $\underline{x} \in \mathcal{R}^2$.

To calculate $(s^*, s^*)$, we treat the scalar product by integration by parts. There are no contributions from the boundary conditions at infinity, due to the decay of the second and the third derivatives of $s^*(\underline{x})$ as $\|\underline{x}\| \to \infty$, which is a consequence of the conditions (1.4). Thus, following Powell (1992), we deduce the formulae

$$\begin{aligned}(s^*, s^*) &= \frac{1}{8\pi} \int\!\!\int_{\mathcal{R}^2} s^*(\underline{x}) \, \nabla^4 s^*(\underline{x}) \, dx\, dy \\ &= \frac{1}{8\pi} \int\!\!\int_{\mathcal{R}^2} s^*(\underline{x}) \sum_{i=1}^{n} \lambda_i^* \nabla^4 \phi\left(\|\underline{x} - \underline{x}_i\|_2\right) dx\, dy \\ &= \int\!\!\int_{\mathcal{R}^2} s^*(\underline{x}) \sum_{i=1}^{n} \lambda_i^* \delta(\underline{x} - \underline{x}_i) \, dx\, dy \\ &= \sum_{i=1}^{n} \lambda_i^* s^*(\underline{x}_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i^* \lambda_j^* \phi\left(\|\underline{x}_i - \underline{x}_j\|_2\right), \end{aligned} \tag{1.6}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and where $\delta(\underline{x})$, $\underline{x} \in \mathcal{R}^2$, is the Dirac delta function.

Calculating the coefficients $\lambda_j^*$, $j = 1, \ldots, n$, of the required interpolant (1.2) by direct means would require $\mathcal{O}(n^3)$ operations, and thus the work becomes prohibitive for $n > 1000$. Hence the need for iterative algorithms is established. Section 2 presents an iterative method which is a generalization of the algorithm described by Powell (1997). This method generates a sequence of approximations to $s^*(\underline{x})$, $\underline{x} \in \mathcal{R}^2$, of the form

$$s(\underline{x}) = \sum_{j=1}^{n} \lambda_j \, \phi\left(\| \, \underline{x} - \underline{x}_j \|_2\right) + p(\underline{x}), \qquad \underline{x} \in \mathcal{R}^2, \tag{1.7}$$

letting $s \equiv 0$ be the initial choice. Each iteration revises the coefficients $\lambda_j$, $j = 1, \ldots, n$, in a way that is intended to provide convergence to $s^*$. Each $s$ is in the linear space of functions of the form (1.7), whose coefficients satisfy

$$\sum_{j=1}^{n} \lambda_j = 0 \qquad \text{and} \qquad \sum_{j=1}^{n} \lambda_j \, \underline{x}_j = 0. \tag{1.8}$$

We reserve the notation $S$ for this $n$-dimensional linear space.

Our main purpose is to prove the convergence of the method of Section 2. The proof is given in Section 3. It depends on the remark that all the changes to the coefficients $\lambda_j$, $j = 1, \ldots, n$, reduce the scalar product $(s^* - s, s^* - s)$. Furthermore, we find in Section 4 that the interpolation method and our analysis apply to radial basis functions generally and to other functions that are conditionally positive definite. The speed of convergence of the method is shown briefly by some numerical results in Section 5.

## 2. The algorithm

The function (1.7) is the current approximation to $s^*$ throughout the remainder of the paper. Each iteration of the interpolation method is divided into $n - q + 1$ stages, where $q$ is a prescribed integer less than $n$, 30 being a typical value for $q$. For $k = 1, \ldots, n - q$, the $k^{\text{th}}$ stage employs exactly $q$ of the last $n - k + 1$ interpolation equations to provide a new value for $\lambda_k$. This value of $\lambda_k$ is accepted for the remainder of the iteration. The $k^{\text{th}}$ stage does not alter any of the coefficients $\lambda_j$, $j = 1, \ldots, k - 1$ and so the method is analogous to forward substitution. We let $\mathcal{L}_k \subset \{k, \ldots, n\}$ be the set of indices of the interpolation equations that are used in the $k^{\text{th}}$ stage, $|\mathcal{L}_k|$ being $q$. The only other conditions on $\mathcal{L}_k$ are that it contains the index $k$ and that the points $\{\underline{x}_j : j \in \mathcal{L}_k, j \neq k\}$ are not collinear. We let the function

$$\hat{\chi}_k(\underline{x}) = \sum_{j \in \mathcal{L}_k} \hat{\lambda}_{kj} \, \phi\left(\| \, \underline{x} - \underline{x}_j \|_2\right) + \text{linear polynomial}, \qquad \underline{x} \in \mathcal{R}^2, \tag{2.1}$$

be the thin plate spline solution of the Lagrange conditions

$$\hat{\chi}_k(\underline{x}_i) = \delta_{ik}, \qquad i \in \mathcal{L}_k. \tag{2.2}$$

Therefore the coefficients of $\hat{\chi}_k$ satisfy the constraints

$$\sum_{j\in\mathcal{L}_k} \hat{\lambda}_{kj} = 0 \qquad \text{and} \qquad \sum_{j\in\mathcal{L}_k} \hat{\lambda}_{kj}\,\underline{x}_j = 0, \tag{2.3}$$

so $\hat{\chi}_k$ is in $S$ for $k = 1, \ldots, n-q$.

The condition that the points $\{\underline{x}_j : j \in \mathcal{L}_k, j \neq k\}$ are not collinear ensures that $\hat{\chi}_k$ is not a linear polynomial. Therefore the second derivatives of $\hat{\chi}_k$ do not vanish. It follows from definition (1.5) that the scalar product $(\hat{\chi}_k, \hat{\chi}_k)$ is strictly positive. Further, equations (1.6) and (2.1) and the Lagrange conditions provide the value $(\hat{\chi}_k, \hat{\chi}_k) = \hat{\lambda}_{kk} > 0$.

The $k^{\text{th}}$ stage of the algorithm adds the term

$$\frac{\mu_k}{\hat{\lambda}_{kk}} \hat{\chi}_k(\underline{x}), \qquad \underline{x} \in \mathcal{R}^2, \tag{2.4}$$

to $s(\underline{x})$, $\underline{x} \in \mathcal{R}^2$, where $\mu_k$ is the number

$$\mu_k = \sum_{j\in\mathcal{L}_k} \hat{\lambda}_{kj} \left[ f_j - s\left(\underline{x}_j\right) \right]. \tag{2.5}$$

The final stage revises $s(\underline{x})$, $\underline{x} \in \mathcal{R}^2$, by adding to $s$ the thin plate spline solution $\sigma(\underline{x})$, $\underline{x} \in \mathcal{R}^2$, of the interpolation equations

$$\sigma(\underline{x}_i) = f_i - s(\underline{x}_i), \qquad i = n-q+1, \ldots, n. \tag{2.6}$$

We know from a condition on $\mathcal{L}_{n-q}$ that the points $\{\underline{x}_i : i = n-q+1, \ldots, n\}$ are not collinear. Thus $s$ satisfies $s(\underline{x}_i) = f_i = s^*(\underline{x}_i)$, $i = n-q+1, \ldots, n$, at the end of each iteration, and it remains in $S$.

More details are given by Powell (1997). Particular choices of the ordering of the interpolation points, the sets $\mathcal{L}_k$, $k = 1, \ldots, n-q$, and a stopping condition are specified there.

## 3. Convergence analysis

The analysis of convergence for the algorithm was thought to be very hard. Attempts were made to establish suitable properties of the $n \times n$ matrix, $R$ say, such that the vector of residuals $f_i - s(\underline{x}_i)$, $i = 1, \ldots, n$, at the end of an iteration is $R$ times the vector of residuals at the beginning of the iteration. The spectral radius of $R$ is considered briefly by Beatson, Goodsell and Powell (1995), because convergence occurs if it is less than one.

We present a different approach, however, which led to success. The proof of convergence depends mainly on the following lemma.

**Lemma** *Let $s$ be calculated by the algorithm of Section 2. Then, after each stage in every iteration, $(s^*-s, s^*-s)$ has a value less than or equal to its value before that stage. Hence $(s^*-s, s^*-s)$ is monotonically decreasing.*

**Proof of Lemma** For $k=1,\ldots,n-q$, the $k^{\text{th}}$ stage adds the term (2.4) to $s$, which replaces $(s^*-s, s^*-s)$ by $(s^*-s-\theta_k\,\hat{\chi}_k, s^*-s-\theta_k\,\hat{\chi}_k)$, where $\theta_k=\mu_k/\hat{\lambda}_{kk}$. Now it is elementary that the quadratic $(s^*-s-\theta\,\hat{\chi}_k, s^*-s-\theta\,\hat{\chi}_k)$, $\theta\in\mathcal{R}$, is least when $\theta$ has the value

$$\theta=\frac{(s^*-s,\hat{\chi}_k)}{(\hat{\chi}_k,\hat{\chi}_k)}, \tag{3.1}$$

and fortunately this number is just $\theta_k$. Indeed, the identity $(\hat{\chi}_k,\hat{\chi}_k)=\hat{\lambda}_{kk}$ is noted in Section 2. Further, integration by parts as in expression (1.6) and equations (1.1) and (2.5) imply the formula

$$(s^*-s,\hat{\chi}_k)=\sum_{i\in\mathcal{L}_k}\hat{\lambda}_{ki}\,[s^*(\underline{x}_i)-s(\underline{x}_i)]=\mu_k. \tag{3.2}$$

Thus these stages provide the required monotonicity of $(s^*-s, s^*-s)$.

It remains to consider the last stage of an iteration where the thin plate spline solution $\sigma(\underline{x})$, $\underline{x}\in\mathcal{R}^2$, of the equations (2.6) is added to $s$. Now $\sigma$ lies in the subspace $T$ of $S$ which consists of functions of the form

$$\tau(\underline{x})=\sum_{j=n-q+1}^{n}\tau_j\,\phi\left(\|\,\underline{x}-\underline{x}_j\|_2\right)+\text{linear polynomial},\qquad \underline{x}\in\mathcal{R}^2. \tag{3.3}$$

Using integration by parts as before, and the interpolation equations (1.1) and (2.6), we find that $\sigma$ has the property

$$(s^*-s-\sigma,\tau)=\sum_{j=n-q+1}^{n}\tau_j\left[s^*(\underline{x}_j)-s(\underline{x}_j)-\sigma(\underline{x}_j)\right]=0,\qquad \tau\in T. \tag{3.4}$$

Therefore $\sigma$ is the element of $T$ that minimizes $(s^*-s-\tau, s^*-s-\tau)$, $\tau\in T$, so again the change to $s$ by the algorithm does not increase $(s^*-s, s^*-s)$. The proof is complete. $\square$

We now turn to the main theorem. For the remainder of the section, let $s_m$ denote the approximation $s$ at the end of the $m^{\text{th}}$ iteration.

**Theorem** *Let the algorithm specified in Section 2 generate the sequence of functions $s_m$. Then $s_m$ converges to $s^*$ in the linear space $S$ as $m$ tends to infinity.*

**Proof of Theorem** Since $S$ is a finite dimensional space, all norms are equivalent. For our purposes we choose

$$\| s \| = \max\{|s(\underline{x}_i)| : i=1,\ldots,n\}, \qquad s \in S, \tag{3.5}$$

as a norm on $S$. This is well defined, since every thin plate spline is defined uniquely by its values at the interpolation points. Hence, if $\| s^* - s_m \|$ tends to zero as $m \to \infty$, then $s_m$ converges to $s^*$ in $S$. Thus it is sufficient to prove that $s_m(\underline{x}_i)$ tends to $s^*(\underline{x}_i)$ for $i=1,\ldots,n$.

We recall that $s$ is the current approximation. Since $(s^*-s, s^*-s)$ is monotonically decreasing by the lemma, and since it is bounded below by zero, it converges to a limit. Therefore the changes to $(s^*-s, s^*-s)$ tend to zero. Because the method gives $\theta_k$ the value (3.1), it provides the identity

$$(s^* - s - \theta_k \hat{\chi}_k, s^* - s - \theta_k \hat{\chi}_k) = (s^* - s, s^* - s) - \frac{(s^* - s, \hat{\chi}_k)^2}{(\hat{\chi}_k, \hat{\chi}_k)}. \tag{3.6}$$

It follows that $(s^* - s, \hat{\chi}_k)$ converges to zero as the iterations proceed for each integer $k$ in $[1, n-q]$, where the meaning of $s$ is taken from equation (3.1).

At the end of the $m^{\text{th}}$ iteration, $s = s_m$ and thus, due to the $(m+1)^{\text{th}}$ iteration, $(s^* - s_m, \hat{\chi}_1)$ tends to zero as $m$ tends to infinity. At the beginning of the second stage of the $(m+1)^{\text{th}}$ iteration, $s$ is the function $s_m + (\hat{\chi}_1, \hat{\chi}_1)^{-1}(s^* - s_m, \hat{\chi}_1)\,\hat{\chi}_1$, so we find the property

$$\lim_{m\to\infty} (s^* - s_m - \frac{(s^* - s_m, \hat{\chi}_1)\hat{\chi}_1}{(\hat{\chi}_1, \hat{\chi}_1)}, \hat{\chi}_2) = 0. \tag{3.7}$$

It follows from $(s^* - s_m, \hat{\chi}_1) \to 0$ that $(s^* - s_m, \hat{\chi}_2)$ also tends to zero. Further, one can deduce by induction on $k$ that $(s^* - s_m, \hat{\chi}_k)$ converges to zero as $m$ tends to infinity for every integer $k$ in $[1, n-q]$.

Now integration by parts gives the condition

$$(s^* - s_m, \hat{\chi}_k) = \sum_{j \in \mathcal{L}_k} \hat{\lambda}_{kj} \left[ s^*(\underline{x}_j) - s_m(\underline{x}_j) \right], \tag{3.8}$$

so we can write the conclusion of the last paragraph in the form

$$\lim_{m\to\infty} \left\{ \sum_{j=k}^{n} \hat{\lambda}_{kj} \left[ s^*(\underline{x}_j) - s_m(\underline{x}_j) \right] \right\} = 0, \qquad k = 1,\ldots,n-q, \tag{3.9}$$

where $\hat{\lambda}_{kj}$ is defined to be zero for $j \notin \mathcal{L}_k$. We consider the equations (3.9) in reverse order, remembering $s_m(\underline{x}_j) = s^*(\underline{x}_j)$, $j = n-q+1,\ldots,n$, and $\hat{\lambda}_{kk} \neq 0$. It follows by induction on $k$ that $s^*(\underline{x}_k) - s_m(\underline{x}_k)$ tends to zero as $m \to \infty$ for $k = n-q, n-q-1, \ldots, 1$. Thus the required convergence of $s_m$ to $s^*$ in the linear space $S$ is obtained. □

## 4. Extension to other radial basis functions

The iterative algorithm for thin plate splines is very successful, achieving 10 decimals accuracy in fewer than 10 iterations for $n \leq 1000$ when the interpolation points are in general positions (Powell, 1997). Later a similar algorithm was applied to multiquadrics, which also provided excellent efficiency in numerical experiments. We are going to show that the analysis of Section 3 can be applied to the multiquadric case. Further, the algorithm of Section 2 and its convergence theory can be extended to conditionally positive (negative) definite functions, in particular to other radial basis functions and other dimensions $d$, even when an explicit form for a scalar product similar to expression (1.5) is not available.

Let $\Pi_m(\mathcal{R}^d)$ denote the linear space of polynomials of total degree at most $m$. We recall that a function $\Phi : \mathcal{R}^d \to \mathcal{R}$ with $\Phi(-\underline{x}) = \Phi(\underline{x})$, $\underline{x} \in \mathcal{R}^d$, is conditionally positive (or negative) definite of order $m$ on $\mathcal{R}^d$, if, for all sets $X = \{\underline{x}_1, \ldots, \underline{x}_n\} \subset \mathcal{R}^d$ with $n$ distinct points and all vectors $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)^T \in \mathcal{R}^n$ subject to the conditions

$$\sum_{j=1}^{n} \lambda_j \, r(\underline{x}_j) = 0, \qquad r \in \Pi_m(\mathcal{R}^d), \tag{4.1}$$

the quadratic form

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \lambda_j \lambda_k \Phi(\underline{x}_j - \underline{x}_k) \tag{4.2}$$

does not change sign and vanishes only if $\underline{\lambda} = 0$. Micchelli (1986) shows that the radially symmetric functions $\Phi(\underline{x}) = \phi(\| \underline{x}\|_2)$, $\underline{x} \in \mathcal{R}^d$, are conditionally positive (or negative) definite for any $d$, if the derivatives of order greater than $m$ of $\phi(r^{1/2})$, $r \geq 0$, are completely monotonic. This property is achieved by several radial basis functions, including the useful cases

| | |
|---|---|
| thin plate spline | $\phi(r) = r^2 \log r$, |
| Gaussian | $\phi(r) = e^{-r^2}$, |
| inverse multiquadric | $\phi(r) = (r^2 + c^2)^{-\frac{1}{2}}$, |
| multiquadric | $\phi(r) = (r^2 + c^2)^{\frac{1}{2}}$. |

For our purposes, we consider the space $S_\Phi$ which contains functions of the form

$$s(\underline{x}) = \sum_{j=1}^{n} \lambda_j \, \Phi(\underline{x} - \underline{x}_j) + p(\underline{x}), \qquad \underline{x} \in \mathcal{R}^d, \tag{4.3}$$

for a fixed set $X = \{\underline{x}_1, \ldots, \underline{x}_n\} \subset \mathcal{R}^d$, where $p \in \Pi_m(\mathcal{R}^d)$ and where the coefficients $\lambda_j$, $j = 1, \ldots, n$, satisfy equation (4.1). If $\Phi$ is conditionally positive definite, we define on $S_\Phi$ the semi-inner product that is introduced by Schaback

(1993). Letting $t(\underline{x}) = \sum_{i=1}^{n} \nu_i \, \Phi\,(\underline{x} - \underline{x}_i) + r(\underline{x})$, $\underline{x} \in \mathcal{R}^d$, be another function in $S_\Phi$, this semi-inner product between $s$ and $t$ is the bilinear form

$$\begin{aligned} (s,t)_\Phi &= \left( \sum_{j=1}^{n} \lambda_j \, \Phi(\underline{x} - \underline{x}_j) + p(\underline{x}), \sum_{i=1}^{n} \nu_i \, \Phi\,(\underline{x} - \underline{x}_i) + r(\underline{x}) \right)_\Phi \\ &= \sum_{j=1}^{n} \lambda_j \, t(\underline{x}_j) = \sum_{i=1}^{n} \nu_i \, s(\underline{x}_i) = \sum_{j=1}^{n} \sum_{i=1}^{n} \lambda_j \, \nu_i \, \Phi(\underline{x}_j - \underline{x}_i), \end{aligned} \tag{4.4}$$

which vanishes if $s$ or $t$ are in $\Pi_m(\mathcal{R}^d)$. Note the similarity to the last line of equation (1.6). Alternatively, if $\Phi$ is conditionally negative definite, we define $(s,t)_\Phi$ to be minus the right hand side of expression (4.4).

We still require the element $s^* \in S_\Phi$ that satisfies $s^*(\underline{x}_i) = f_i$ for given data $f_i \in \mathcal{R}$ and $\underline{x}_i \in X$, $i = 1, \ldots, n$. Few changes have to be made to the algorithm of Section 2 to generate the sequence of approximations to $s^*$ in $S_\Phi$. We require $q$ to be less than $n$ and greater than the dimension of $\Pi_m(\mathcal{R}^d)$, which has the value $\binom{d+m}{d}$. Further, each set $\mathcal{L}_k$, $k = 1, \ldots, n-q$, must satisfy the condition that the points $\{\underline{x}_j : j \in \mathcal{L}_k, j \neq k\}$ do not lie in the zero set of a nonzero polynomial of degree at most $m$.

For $k = 1, \ldots, n-q$, the functions

$$\hat{\chi}_k(\underline{x}) = \sum_{j \in \mathcal{L}_k} \hat{\lambda}_{kj} \Phi(\underline{x} - \underline{x}_j) + p_k(\underline{x}), \qquad \underline{x} \in \mathcal{R}^d, \tag{4.5}$$

are the unique functions of this form in $S_\Phi$ satisfying (2.2). The choice of $\mathcal{L}_k$ ensures that $\hat{\chi}_k$ is not an element of $\Pi_m(\mathcal{R}^d)$. Thus the definition of the semi-inner product and the conditional positive (or negative) definiteness of $\Phi$ give $(\hat{\chi}_k, \hat{\chi}_k) = \hat{\lambda}_{kk} > 0$.

After letting $s \equiv 0$ initially, the $k^{\text{th}}$ stage of each iteration replaces $s$ by $s + (\mu_k / \hat{\lambda}_{kk}) \hat{\chi}_k$ with $\mu_k$ as in formula (2.5) for $k = 1, \ldots, n-q$. Then the last stage adds $\sigma$ to $s$ as before, where $\sigma$ is the function in $S_\Phi$ with the centres $\underline{x}_i$, $i = n-q+1, \ldots, n$, that is defined by the interpolation conditions (2.6).

The convergence analysis changes little. Instead of $(\cdot,\cdot)$ we use $(\cdot,\cdot)_\Phi$, the space $S$ is replaced by $S_\Phi$, and $T$ becomes the subspace of $S_\Phi$ whose functions have the centres $\underline{x}_i$, $i = n-q+1, \ldots, n$. The norm (3.5) is also well defined in the new set-up, since interpolation by conditionally positive (or negative) definite functions is unique.

Thus the approach presented in this paper can be generalized easily to other conditionally positive (or negative) definite functions. Some numerical experiments with multiquadric functions in two dimensions have been tried already and they were very successful.

| $n$ | $q$ | A | B | C | D |
|---|---|---|---|---|---|
| 400 | 10 | 4; 0.0373 | 28; 0.2474 | 29; 0.2941 | $\star(1)$; 0.7302 |
| | 20 | 3; 0.0009 | 10; 0.0797 | 7; 0.0886 | 294; 0.6896 |
| | 30 | 3; 0.0003 | 8; 0.0651 | 5; 0.0585 | 145; 0.6528 |
| | 50 | 2; 0.0001 | 5; 0.0193 | 4; 0.0149 | 85; 0.6287 |
| 900 | 10 | 5; 0.0377 | 24; 0.2789 | 36; 0.3684 | $\star(2)$; 0.7614 |
| | 20 | 3; 0.0011 | 8; 0.0890 | 8; 0.0941 | $\star(3)$; 0.7134 |
| | 30 | 3; 0.0004 | 6; 0.0716 | 6; 0.0376 | 443; 0.6987 |
| | 50 | 2; 0.0001 | 5; 0.0127 | 5; 0.0144 | 155; 0.6576 |

**Table 1:** Iteration counts and expression (5.1)

## 5. Numerical results and conclusions

If the functions $\hat{\chi}_k$, $k=1,\ldots,n-q$, were orthogonal with respect to the scalar product (1.5), then $s \equiv s^*$ would occur at the end of the first iteration. Therefore we expect the speed of convergence to be linked to the size of the scalar products between the functions $\hat{\chi}_k$, $k=1,\ldots,n-q$. Some numerical experiments were run to test this conjecture. The results are reported below, all the calculations being for thin plate spline interpolation in two dimensions.

Four kinds of distributions of the interpolation points $\underline{x}_i$, $i=1,\ldots,n$, were tried. In Problem A the points are equally spaced on the unit circle $\{\underline{x} : \| \underline{x} \|_2 = 1\}$. The points of Problem B form a finite square grid in $\mathcal{R}^2$ that covers the unit square $[0,1]\times[0,1]$. In Problem C the data points are chosen randomly from the uniform distribution on the unit disc $\{\underline{x} : \| \underline{x} \|_2 \leq 1\}$. Problem D was found by seeking a case where the convergence of the method of Section 2 is very slow. Here the points are equally spaced on one eighth of two concentric circles, half of the points being on each circle, and the radii of the circles being 1 and $1+10^{-5}$.

In all problems, the right hand sides $f_i$, $i=1,\ldots,n$, are independent random numbers from the uniform distribution on $[-1,1]$. The calculation is terminated when all the moduli of the residuals $f_i - s(\underline{x}_i)$, $i=1,\ldots,n$, are less than $10^{-8}$. The values of $n$ and $q$ that are employed are stated in Table 1.

Each pair of numbers in the table gives the number of iterations to achieve the accuracy that has just been mentioned, followed by the quantity

$$\max \left\{ \left| \frac{(\hat{\chi}_i, \hat{\chi}_j)}{\sqrt{\hat{\lambda}_{ii}\,\hat{\lambda}_{jj}}} \right| : 1 \leq i < j \leq n-q \right\}. \tag{5.1}$$

In the cases $\star(1)$, $\star(2)$ and $\star(3)$, the calculations were stopped after 1000 iterations, and then the maximum moduli of the residuals $f_i - s(\underline{x}_i)$, $i=1,\ldots,n$, were $9.0 * 10^{-4}$, $5.2 * 10^{-2}$ and $5.9 * 10^{-7}$, respectively.

As expected, the number of iterations is small when the approximated Lagrange functions $\hat{\chi}_i$, $i=1,\ldots,n-q$, have good orthogonality properties. Indeed, the algorithm performs well when expression (5.1) is less than 0.1, this condition being satisfied for $q \geq 20$ in Problems A, B and C. Furthermore, the results for Problem D show the correlation between slow convergence and lack of orthogonality. The quantity (5.1) is at most one due to the identities $(\hat{\chi}_i, \hat{\chi}_i) = \hat{\lambda}_{ii}$, $i=1,\ldots,n-q$, and the Cauchy-Schwarz inequality.

It is hoped that the importance of the scalar products that has been shown will assist future research on choosing the sets $\mathcal{L}_k$, $k=1,\ldots,n-q$, for the algorithm of Section 2.

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