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# Modifications in cellular models

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**Abstract:** Continuity of modification is seen as a key to the incremental updating of the explicit representation of parametric models. This note describes an exploration of the issues carried out by the authors during the summer of 1998. We identify that geometrically continuous modification may need to be topologically discontinuous, and explore what happens at such discontinuities.

This report represents a snapshot of the authors' thinking, taken at the end of the visit to Cambridge of Prof. Shapiro during July and August 1998. We are well aware that certain aspects need considerably more thought, but feel that sufficient progress was made during that visit to warrant making the results to date accessible.

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# 1 Preliminaries

## 1.1 Motivation

1) the design process usually includes modification of a candidate design as an essential tool.

Many major CAD systems provide for parametric design whereby a representation in terms of design variables is automatically mapped to a more explicit representation. (Systems with constraints also ensure certain validity conditions apply between these design variables.)

Systems based on holding both a parametric representation and a boundary representation tend to have problems maintaining auxiliary information saved with the boundary elements when the Brep is recomputed. (This is termed the *persistent naming* problem.)

We have explored the possibility of replacing the recomputation of the Brep by incremental updating, so that naming information is not lost.

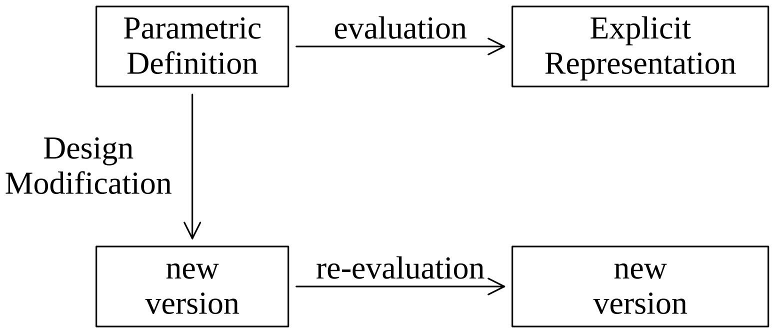


Figure 1: Conventional Parametric Modelling.

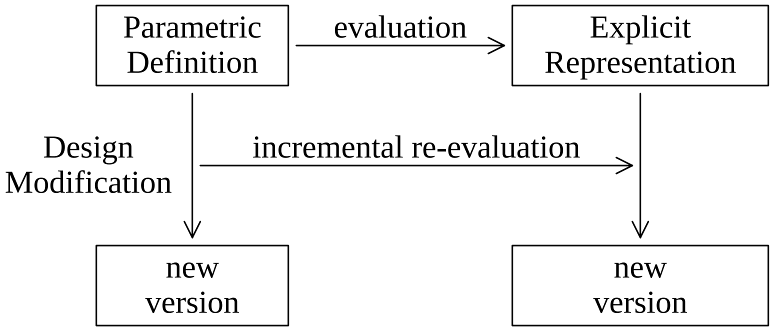


Figure 2: Incremental Parametric Modelling.

2) cellular models are required for some applications, and they include very naturally the non-manifold and multi-dimensional aspects which manifold solid modellers find hard.

The questions we set out to address are:-

*Is incremental parametric modelling as shown in Figure 2 feasible in the cellular context ?*

*Future modellers will be cellular from the outset. What impact will the need to support modification have on their architecture ?*

## 1.2 Notation

$d$ – the manifold dimension of a cell
$\bar{d}$ – modelling dimension, typically equal to 3.
$K$ – a cellular complex
$A$, $B$, $C$, $\sigma$ – cells
$P$ – a point
$t$ – a scalar
$T$ – a transform
$M$ – Moveset
$B$ – Borderset
$S$ – Staticset
$m$, $b$, $s$ – cells in $M$, $B$, $S$
$E$ – Expandset
$E_n$ – Euclidean space of dimension $n$
$R$ – space of real numbers
$R^n$ – space of $n$-tuples of real numbers
**Clo**() – closure (of pointsets and of subsets of cell complexes)
**Int**() – interior (of pointsets and of subsets of cell complexes)
**Bdy**() – boundary (of pointsets and of subsets of cell complexes)
**Front**() – frontier (of pointsets and of subsets of cell complexes)
**Star**() – star (of pointsets and of subsets of cell complexes)
**Strat**() – stratification of a partition of space to give a complex

# 2 Cellular Models

## 2.1 Cellular Topology

### 2.1.1 Cells

A cell is an abstract entity which at the very minimum posseses a name (label), a set of points in $E_{\bar{d}}$ (the underlying modelling space), and an orientation, evidenced by the sense of the tangent to a 1-cell and the normal on a 2-cell[1] , which vectors are required to vary coherently with position in the cell.

Cells[2] are required to satisfy the following requirements:

- Every cell must be homogeneous in manifold dimension.

- Every cell must be open with respect to that dimension (It does not contain its frontier).

But we do not require cells to be simply-connected or smooth. In fact, cells may even be disconnected, though maintaining cellular models with disconnected cells significantly complicates the cellular set operations and transformations described below.

Nor do we insist that cells be finite in extent.

Our results will still apply when conditions of connectedness, smoothness, finiteness of extent or simple-connectedness are imposed.

### 2.1.2 Cellular Complex

Cellular modelling uses the concept of a **cellular complex**, which is a finite collection of cells such that

- Every cell must be disjoint from every other.

[1] 0- and 3-cells are deemed to have a positive scalar orientation, so that we can make unambiguous statements about relative orientation of 0-cells with respect to adjacent 1-cells, and of 3-cells with respect to adjacent 2-cells.

[2] When there is no ambiguity we may abuse the language and refer to the pointset as a "cell". In particular we use the notations $P \in C$ as a shorthand for $P \in$ pointset of $C$ and $A \subset B$ as a shorthand for pointset of $A \subset$ pointset of $B$.

- The frontier of every cell must consist of the union of other cells.

If every cell $\sigma_i$ is connected, the above requirements have a simple intuitive interpretation. It says that it is possible to choose a small enough radius $\epsilon$, so that the three-dimensional neighborhoods of all points of $\sigma_i$ meet the same cells in the cell complex.

When cell $A$ is in the frontier of cell $B$ (or vice versa), we say that $A$ and $B$ are incident on each other. The relative orientation of two incident cells is positive if their individual orientations are consistent and is negative otherwise.

### 2.1.3 Closure and Star

Cellularity imposes a finite (and therefore computable) piecewise structure on pointsets and facilitates construction and management of logical and combinatorial relations between them. In particular, the requirement that a frontier of every cell is a union of other cells allows cellular definitions of topological operations.

- The **Closure** of cell $\sigma$ is a cell complex consisting of $\sigma$ and all cells in the frontier of $\sigma$;

$$\mathbf{Clo}(\sigma \in K) \equiv \sigma \cup \bigcup \{A | A \in \mathbf{Front}(\sigma)\}$$

- The **Star** of cell $\sigma$ is a set of cells that includes $\sigma$ and all higher-dimensional cells incident on $\sigma$.

$$\mathbf{Star}(\sigma \in K) \equiv \sigma \cup \bigcup \{A | \sigma \in \mathbf{Front}(A)\}$$

These definitions extend naturally to a collection of cells by taking the union of the closures (stars) of individual cells in the collection.

### 2.1.4 Stratification

The process of constructing a valid cell complex for a partitioning of space is called **stratification**. Just as the CSG (Constructive Solid Geometry) approach to homogeneous solid modelling discovered the need for 'regularisation' to avoid unexpected effects, the equivalent for cellular models is 'stratification'.

The operation denoted by **Strat**(.) takes as argument a partitioning of space into 'proto-cells', (disjoint equivalence classes of points), and carries out further division of those proto-cells which do not satisfy all the above until they form a valid complex again. This is performed as the last act in every operation, to ensure that the model is ready and clean for further operations.

We can also apply **Strat**(.) to a pointset, which is interpreted as a partitioning of space into IN and OUT. Pointsets may be split into pieces homogeneous in dimension. Also, open pointsets acquire bounding cells: closed ones are split into boundary and interior. Thus stratification may be thought of as a subdivision of the closure of a pointset.

Our definitions of cell complex and stratification differ from others (Whitney stratification, SGCs, and the classical cell complexes) in that we do not insist that every cell should be smooth; the nature of stratification is essentially articulated in terms of neighbourhoods, and introducing a geometric condition of differentiability seems artificial and unnecessary.

We also use open cells, in order to be able to map uniquely from points to the cells in which they lie.

## 2.2 Cellular Geometry

Every cell in the complex is a subset of a **carrier**, which may or may not be defined outside of the closure of the cell. The carrier of a $d$-cell is an orientable manifold connected closed pointset of manifold dimension $d$, defined by some point classification rule. When $d$ is the dimension $\bar{d}$ of the modelling space (typically 3), the carrier of every $\bar{d}$-cell is $E_{\bar{d}}$.

The geometric constraints implied by the cell complex can be summarized by two conditions:

(F1) Every cell is a subset of the *intersection* of all carriers in its star; and

(F2) Every cell is an *interpolation* of all cells in its frontier.

These conditions are typically taken to apply to the carriers of the cells as well, i.e. the carrier of every cell is a subset of the intersection of the carriers of all the cells in star (cell), and the carrier of every cell is an interpolation of the carriers of all the cells in the cell's frontier. This extension is a stronger condition which is not necessary, but may be convenient.

Notice that this says nothing about how the model is constructed or modified. In particular, the model may be overconstrained, e.g. with more than three surfaces intersecting at a point or more than two in an edge. This also implies a break in the duality symmetry, in the sense that a face can interpolate any number of boundaries, which only need to meet in their common vertices, whereas faces which intersect in a single vertex are strongly constrained.

## 2.3 Cellular Operations

### 2.3.1 Complex as point classification

To maintain the closure of the modelling space under modelling operations, all operations must be defined to preserve the properties of the cellular models. To achieve this we note that each cell complex may be regarded as a map from position in $R^3$ to a unique cell. This map replaces the usual point membership classification: instead of classifying points with respect to a set as IN, ON, OUT, the points are classified into cells[3].

$$Map : P \in \{point\} \mapsto C \in \{cell\} \text{ if } P \in pointset(C)$$

### 2.3.2 Binary operations

In order to define binary set-operations over cellular complexes, we can define a composite structure which is the map from position to ordered pair of cells, one in each operand.

This map defines a classification of points into equivalence classes, all those points in an equivalence class lying in the intersection of the same ordered pair of cells. These classes do *not* necessarily form a cell complex, but a cell complex may be made by stratification, and this is called **cellular union**.

Other operations are defined in terms of the selection of cells from this complex to form a subcomplex.

[3] OUT is considered a cell for this purpose, although it may not satisfy all the requirements placed above on cells.

### 2.3.3 Cellular Union

- Cellular union of two cellular models $A = \{a\}$, $B = \{b\}$ is constructed by considering all pairs of elements $a \in A$ and $b \in B$:

$$A \cup^c B \equiv \bigcup_{a \in A, b \in B} \{\mathbf{Strat}(a - b)\} \cup \{\mathbf{Strat}(b - a)\} \cup \{\mathbf{Strat}(a \cap b)\},$$

where the stratification ensures the cellularity of the composite structure.

### 2.3.4 Partitioning of one complex by another

The cellular union of $A$ and $B$ symmetrically imprints the complex $A$ with the cellular structure of complex $B$ and vice versa. Sometimes it may be more convenient to only refine the cellular structure of the first operand using the cellular structure of the second operand, making them compatible. To this end, we define a cellular operation $A|B$, **partitioning of A by B** whose result is a new cellular structure of its first operand.

- Partitioning $A|B$

$$A|B \equiv \{\mathbf{Strat}(a - b)\} \cup \{\mathbf{Strat}(a \cap b)\} = \{\sigma \in (A \cup^c B) | \sigma \subset A\}$$

### 2.3.5 Cellular Intersection

- Another subset of cellular union is the cellular intersection

$$A \cap^c B \equiv \{\mathbf{Strat}(a \cap b)\} = \{\sigma \in (A \cup^c B) | \sigma \subset A \wedge \sigma \subset B\}$$

### 2.3.6 Cellular Difference

- Definition of cellular difference is a little more complicated, because the difference of two closed sets need not be closed. Thus we modify the operation to return the closure of the difference [4]

$$A -^c B \equiv \mathbf{Clo}\{\mathbf{Strat}(a - b)\}$$

It is easy to see that $A -^c B$ is a subcomplex of $A|B$.

[4] Closure of the set complement is sometimes termed a *pseudo-complement*.

### 2.3.7 Implementation

In order that the relationship of the cells of the result to the cells of the operands should be explicit, a relation, keyed on result cell, can be returned with columns:-

[Result cell, A-cell, relative orientation wrt A, B-cell, relative orientation wrt B]

The relative orientations are relevant when the result cell has the same dimension as its parent operand cell, and indicates whether the orientation (as evidenced by the first derivative (tangent or normal vector)) is the same as, or opposite to, that of its parent.

# 3 Transforms

We define a transform to be a continuous map $T(t)$ from a scalar range $[0 \ldots 1]$ to a 1:1 continuous map from $E_{\bar{d}}$ to $E_{\bar{d}}$.

$$T(t) : [0 \ldots 1] \mapsto \{\mathcal{R}^{\bar{d}} \mapsto \mathcal{R}^{\bar{d}}\}$$

with the conditions that

- $T(0)$ must be the identity mapping.
- $T(t)$ is one to one and continuous.
- $T$ is continuous in $t$.

Translations, Rotations and Scaling by any positive factor are all acceptable.

Mirroring is not acceptable, because there is no continuous way of getting to a mirroring from the identity without passing through a degenerate map[5] where at least one point is the image of two or more source points.

However, the maps need not be linear: for example, a twist where rotation in the $xy$-plane is proportional to $z$ satisfies the requirements of 1:1 and continuity.

The purpose of the constraint on the space mappings is to ensure that all topological statements made about neighbourhoods of cells remain invariant under application of the map.

The purpose of the continuity constraint on the map from the scalar is to ensure that modifications made by applying the map represent a continuous deformation of space.

[5]or going to a higher dimensional space

# 4 Sweeps

The sweep of a point-set $G$ under a transform $T(t)$ is the continued union of images of $G$ under the maps $T(t)$, $t \in [0 \ldots 1]$.

The closed interval $[0 \ldots 1]$ is regarded as cellular, with three cells - the value 0, the value 1 and the open interval $(0 \ldots 1)$.

The sweep of a cellular complex $C$ is a complex whose pointset is the sweep of the pointset of $C$. Each point $P$ in the sweep of $C$ can have assigned to it a set of all pairs

$$\{(Q \in C, t \in [0 \ldots 1]) \mid P = T(t)(Q)\}$$

.

Each such pair can be classified by the cell of $C$ in which $Q$ lies and the 'cell' of $T$ in which $t$ lies. Provided that the number of cells in $C$ is finite, there are a finite number of pairs of source cells, and therefore a finite number of sets of pairs of all source cells.

The cellular structure of the sweep of $C$ is the stratification of a partitioning in which two points are in the same cell if and only if they have the same set of all source cell pairs.

(The case where the continued union is such that each point can be assigned to a unique point in $C$ and a unique $t$ value is termed **factorisable**. It is obviously a special case of the above where each set of all source cell pairs has only one member. The definition above is not limited to this special case.)

# 5 Tweaks

## 5.1 Basic Concepts

A tweak is the modification of a model by moving some of its cells relative to others.

It is a concept derived from Brep modelling, where a topological entity is moved, and the adjacent entities move to maintain the necessary incidence relationships.

CSG modelling has a similar capability, where one of the primitives used in a set-theoretic expression is changed, thus giving a new shape.

In the cellular context we wish to define a tweak of a complex $C$ by identifying a set $M$ (the **move set**) of cells in $C$ which are to be moved and also a transformation $T(t)$ to be applied to them.

We observe that because the movement is relative to others, there must be some cells which do not move. We call the set of these the **static set**, $S$.

Finally, there will in general need to be cells which provide the junction between these two sets. We call the set of these the **border set**, $B$.

The conditions F1 and F2 under Cellular Geometry above apply to all complexes. They therefore apply dynamically to the carriers of all cells during the course of the tweak. When the carrier of a cell is moved, those of cells in the closure and star must have their carriers changed too, to maintain those conditions.

Depending on whether the cells in the border set are of higher or lower dimension than those with which they must remain compatible, we require either interpolation or intersection to be used to determine their new carriers. We expect the border cells to move continuously (though not necessarily differentiably) with the transform parameter $t$.

### 5.1.1 Symmetries

There are certain symmetries which we would like to respect.

1) The effect of moving the move set should be equivalent to that of moving the whole object by the same transformation and then moving the static set by the

inverse.

2) There should be a duality symmetry so that in 3-space, vertices are dual to solid cells, edges to faces.

There can also be a duality over the boundary of a manifold solid, so that vertices are dual to faces, edges to edges.

### 5.1.2 Other desirable properties

- The Border set ought to contain as few cells as possible
- The Border set ought to be determinable from either the move or the static set.

## 5.2 Kinds of Tweaks

If the original complex $C$ is 'in general position' and the modification defined by $T$ is sufficiently small, the geometric changes during the tweak do not imply any changes of incidence.

Such a tweak is termed a **Small Tweak**.

### 5.2.1 Events

For any given value of $t$ during a tweak there is a corresponding configuration of the model. There can be values of $t$ at which this configuration changes, and we call these values the **events**.

Events may be **one-sided** or **two-sided**. In the case of a two-sided event the configuration at the event's value of $t$ is different from those at both larger and smaller values: in the case of a one-sided event the configuration is the same as that on one side but different from that on the other.

The mechanism for a one-sided event is that carriers which need to be intersected intersect only for configurations at, and on one side of, the event, but not on the other.

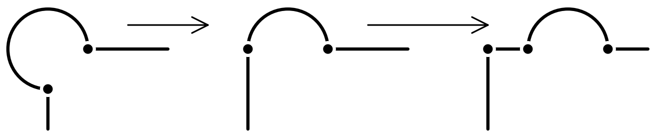

Figure 3: A one-sided event.

### 5.2.2 Induced Cellularity in $t$

Events partition the range $[0 \ldots 1]$ into a cellular structure containing the events themselves and the open intervals[6] between them.

A **small tweak** is one which lies entirely within a single open interval.

If there are events, then topological changes happen, potentially both as an event is reached and as it is left.

As a two-sided event is reached (due, for example, to the moving part of the model colliding with the static part), cells have to collapse.

This process of achieving the right-hand limit of the open interval is termed a **Collapsing Tweak**

As a two-sided event is left, new cells have to be created. The process of entering an open interval from its left-hand limit is termed an **Expanding Tweak**.

A one-sided event will have one of these situations, and need not be considered in detail separately.

The combination of a small tweak with a final collapsing tweak or of an expanding tweak with a subsequent small tweak is termed a **Large Tweak**.

To achieve a desired effect may require a sequence of large tweaks.

Because the set of cells involved changes during each large tweak, it may be necessary for the sequence to be controlled one large tweak at a time.

A third desirable symmetry now becomes evident.

3) The inverse of an expanding tweak should be achievable by a collapsing tweak.

[6] If 0 and 1 are events, this description is sufficient. If, however, 0, for example, is not an event, the first event being at $t_0$, the interval $[0 \ldots t_0)$ is regarded as being open. If there are no events, then the entire interval $[0 \ldots 1]$ is regarded as being open. In fact these are indeed open relative to $[0 \ldots 1]$, though not relative to the entire real line.

Clearly the transformation will be the inverse, but there are also relationships between the various sets used to specify the tweak. These relationships are not trivial, because the cells before and after are not in 1:1 correspondence, but identifying them is an important step in understanding.

## 5.3 Determination of the Border and Static sets

The notions of Move, Border and Static sets do not depend on the type, and so at first we ignore issues arising from collapse and expansion.

We wish to specify a tweak by identifying

– a transform

– the move set, of cells whose carriers are to be transformed as prescribed.

and we hope that the border and static sets can be deduced from this information.

There are two distinct cases. In one case the move set is a cell complex, in which case the static set is also, and we have all the symmetries we desire. In the other case the move set is not a cell complex and we are not able to specify the tweak only by the move set.

### 5.3.1 Move set is a subcomplex of $C$

This means that boundary of every moved cell is also in $M$. We find that the minimal border set is exactly $\mathbf{Star}(M) - M$, and that $S$ is also a subcomplex of $C$. The border set is also $\mathbf{Star}(S)$ and so our required symmetry is achieved.

Furthermore, the tweak requires only the ability to modify the carriers of the border cells to interpolate the moved and static cells.

- Theorem: If Move is a complex, then Border = Star(Move)-Move

Let $m$ be a cell in $M$, and let $b$ be a cell in $\mathbf{Star}(m) - M$. The dimensionality of $b$ must be higher than that of $m$. Thus its incidence relationship can be achieved by interpolation. It can also achieve its incidence relationships with the other cells in its frontier by interpolation, and so there is no requirement for the change of $M$ to propagate further.

- End of Theorem

**Notes:**

- We assume that interpolation of a set of cells forming a frontier is always possible, provided that those cells themselves satisfy their mutual incidence relationships. The soap bubble is our proof of existence. [7]

However, the interpolation we rely on here is slightly more subtle. We expect an interpolated cell to be a continuous deformation of its initial ($t = 0$) geometry.

- The above implies that we may compute the interpolating cells in ascending sequence of dimension.

### 5.3.2 Move set does not form a subcomplex of $C$

There is a boundary cell $X$ of a cell $M_i$ in the move set which is NOT also in the move set.

$X$ is clearly in the border;

The carrier of this cell will have to be recomputed using intersection of the moved cell and carriers in its star. In this case, the Border is a proper subset of Star(Closure(Move))-Move

- Theorem: Border $\subset$ Star(Closure(Move))-Move

1) Suppose that an additional cell is added to the move set. This cannot cause the union of border and move to shrink. Adding all frontier cells reduces to case (a), and so the border of the original Move cannot contain cells not in the Star of the closed Move.

2) The carrier of $X$ is defined by intersection of $M_i$ with some cell or cells of $S$. Therefore some cell or cells of $S$ must be within the star of $X$. Thus these cells (within Star(Closure(Move))) must be in $S$ and therefore not in $B$. Thus $B$ is a proper subset.

- end of theorem.

[7] This assumption may not hold if carriers are restricted to specific types or orders of smoothness. For example a polyhedral modeller would need to permit multi-facet carriers to support it.

Determining the carriers of the border cells requires recomputing intersection, which causes at least two difficulties:

– multiple roots and

– possible indeterminacy of intersection.

These appear to be the most significant limitations of small tweaks.

**Notes:**

- one approach to multiple roots is to appeal to continuity, though this is not necessarily straightforward to implement.

- there is a break in the duality symmetry, in the sense that a face can interpolate any number of boundaries, which need only to meet in their common vertices, whereas faces which intersect in a single vertex are strongly constrained.

- indeterminacy of intersection demands that further information be added to the specification. Sharp identification of at least part of the static set appears to be required.

- there are situations where there happens to be no indeterminacy (e.g. trivalent vertex cases)(general position), and this may be used to avoid additional specification on a heuristic basis.

- Another possible way of avoiding specification of $S$ as well as $M$ is to use some kind of least perturbation principle, so that instead of any one cell being identified as being in the static set, all candidates are treated as border, and the intersection is with some average.

## 5.4 Small Tweaks

### 5.4.1 specification

Subcomplex Case: the tweak is specified by a transform and a subcomplex $M$ of $C$

General Case: the tweak is specified by a transform, a subset $M$ of $C$ and a subset of $S$ containing enough cells to resolve exactly how each cell in $\mathbf{Clo}(M)$ should be intersected.

### 5.4.2 semantics

The carriers of the cells in $M$ as specified are transformed by $T$ as specified.

The carriers of any cells in $B \cap \mathbf{Clo}(M)$ are derived by intersection of the cells with which they have to maintain incidence.

The carriers of all other cells in $B$ are derived by interpolation of their frontier cells to maintain incidence.

### 5.4.3 validity conditions

The necessary conditions for small tweaks are in the Raghothama+Shapiro paper, but the interpretation is slightly different. The necessary conditions should apply not just to the cells, but to the specific carriers of these cells.Thus, these conditions can be reinterpreted in terms of intersection and/or interpolation requirements stated above.

### 5.4.4 limitations

## 5.5 Collapsing Tweaks

Collapsing tweaks are defined as what happens at the limit of small, and are signalled by detecting collisions between cells under the transformation.

Thus, specification is identical to the case of small tweak; the necessary conditions are also well understood. Clearly, all limitations of small tweaks apply.

The collision might take the form of an intersection of cells in any of the pairs:

(Move, Border), (Move, Static), (Border, Static), (Border, Border)

When a collision is detected, between, say, two cells, $A$ and $B$, we need to prepare for the collapse by setting up a 1:1 correspondence between cells that are to combine. We do this

(1) by replacing the cells forming the closure of $A$ by the cells of $\mathbf{Clo}(A)|\mathbf{Clo}(B)$ and

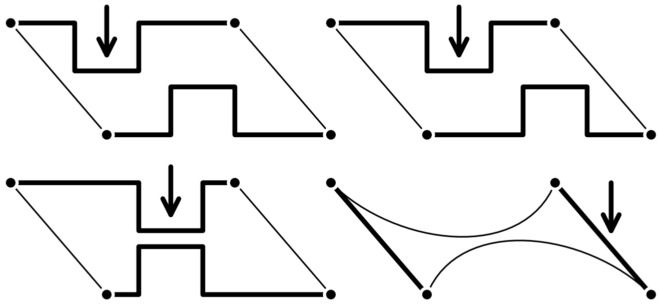

Figure 4: Various collision possibilities.

(2) by replacing the cells forming the closure of $B$ by the cells of $\mathbf{Clo}(B)|\mathbf{Clo}(A)$.

The collapse is then captured as a mapping between the cells resulting from (1) and those resulting from (2).

- splitting and stratification are separate issues. Splitting has much in common with the partitioning of two complexes by each other.

We observed that collapses are instantaneous many to one events, and proper record of collapses must precisely specify the mappings between oriented cells before and after the collapse.

- the information returned by a collapse is presumably whatever is required to determine its inverse expansion which is not present in the final state of the complex.

### 5.5.1 Detection of Collapse

Detection of collision might be achieved by constructing the continued union of the tweak (leaving behind all the places it has been), and testing for overlaps. (This is conservative: the projection of a set will have overlaps if the set has overlaps, and may have overlaps even if the set does not. However, it may also give some evidence as to the range of $t$ where a collision might occur, allowing a finer construction to home in on the region of interest.) This is a standard collision detection technique.

### 5.5.2 specification

As for small tweaks.

### 5.5.3 semantics

A collapsing tweak has two logical stages.

In the first stage the moving part and the static part imprint upon each other their cellular structures. This causes some previous cells to be split.

In the second part the cells which are now to coincide are collapsed together, together with any cells which have shrunk to contain no points as a result of the movement. This causes a merging of cells.

The splitting and the merging can both be reported back in terms of a relation, keyed on Old Cell and New Cell, linking Old Cell, New Cell and their relative Orientation. Because two cells resulting from splitting the same cell will not be merged together, no information is lost by returning a single relation[8] instead of separate relations for the splitting and the merging.

### 5.5.4 validity conditions

### 5.5.5 limitations

## 5.6 Expanding Tweaks

We argue that expanding tweaks are needed to provide the inverse of collapse, and they appear to be useful in feature creations and parametric editing (more on this below). Expansions appear to have a lot in common in sweeps, but they are not the same because they may follow existing carriers.

During the merge part of a collapsing tweak, information about cellularity is lost. Different configurations can collapse to the same complex. An expanding tweak therefore needs additional information to distinguish which of many possible outcomes is required.

By definition, expansions are instantaneous one-to-many mappings (followed by a small tweak) that have a unique inverse (collapse).

It became clear that there are many different ways a complex may be expanded, and specifying precisely which expansion is to take place is a complex task that requires additional non-trivial information. The issues are addressed one at a time

[8] The join over the intermediate cells in $A|B$ and $B|A$

in the following subsections.

The identifying information needed for expanding also tells us what information should be returned to the application from a collapsing tweak.

### 5.6.1 The Expand Set, $E$

In order to distinguish a required expanding tweak from a small tweak with the same transformation and move set, we need to identify which cells are to be expanded. These are the cells which, in the inverse collapse, resulted from the merging of cells. They are specified by the provision of an **expand set** to the operation.

The expand set must be a valid, at most $(\bar{d}-1)$-dimensional, cell complex. At most two-dimensional because the carrier of a $\bar{d}$ dimensional cell is $E_{\bar{d}}$, which is incident on everything, and so there is no need for expansion: a cell complex, because if a cell is expanded so must its frontier be.

There are other constraints on $E$ in relation to $M$. For example, $E$ cannot form just part of the subcomplex which separates $M$ from the remainder of $C$. The case where the move set consists of part of the boundary of a solid, or part of a surface, and where the expand set is the frontier of the move set, is well-behaved. In this case the expand set is the frontier between the move set and the static set, both of which contain cells in the image of $E$ after the expansion.

### 5.6.2 Move/Border/Static Assignment

In general, each cell in the expand set becomes three[9] cells in the configuration immediately after expansion. We need to specify which set each of the three is to be assigned to. This is not trivial because collisions can occur between a pair of cells in the

(Move, Border), (Move, Static), (Border, Static), (Border, Border)

sets.

Thus each of the parts of an expanding cell needs to be allocated to one of the

[9] The normal case has two 'ends', joined by a cell of higher dimension, but the inverse collapse might not have had the joining cell. Clearly the two-cell situation is simpler than the three-cell one. We consider this aspect under Retention.

three sets.

The simplest situation is when each cell in the expand set results in three pieces, one in each of the three sets. In this case, one 'end' lies in the move set, the other in the static set, and the join in the border. Further, the static set then contains the static ends of the expand cells and all cells of the original complex not in $M$.

### 5.6.3 Retention

Where a cell forming the collapse was created by the imprinting process just before the collapse, the inverse process should either delete the cell resulting from the expansion, or merge it with its neighbours.

However, it is always possible for this deletion/merging to be carried out subsequent to the completion of the large tweak.

### 5.6.4 Carrier Assignment

An expanded cell has three carriers. In the simple (Move/Border/Static) case the end carriers are the carrier of the original cell and its image under the transformation, and the carrier of the join is the carrier generated by a sweep using the same transformation.

In other situations, we need to identify the carrier of each component. In addition to the first three possibilities we have the cases:-

- carrier defined by intersection (when an end is in the border set)
- carrier defined to be the same as some other cell
- carrier explicitly defined.

There is a further complication in the fact that in cases where, for example, a carrier line initially tangent to a circle moves to become a chord, we also need to specify which root is to be tracked. Homotopy is not a solution here, because both branches start at the same initial point.

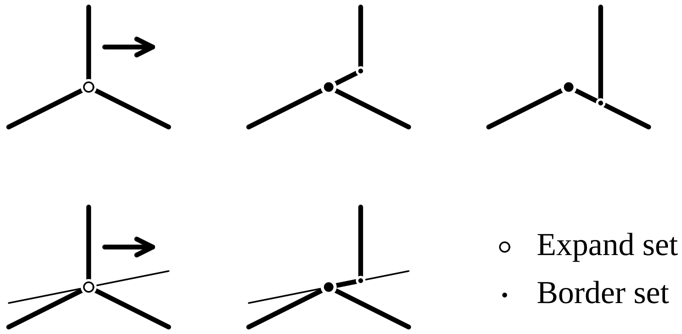


Figure 5: Various possible static carriers.

### 5.6.5 Incidence Sharing

– Incidence sharing addresses how the initially incident star-cells are distributed between the two (or more) 'ends' of the expanding item for modification during the subsequent small tweaking. This is an issue when those star cells are part of the border during that subsequent processing.

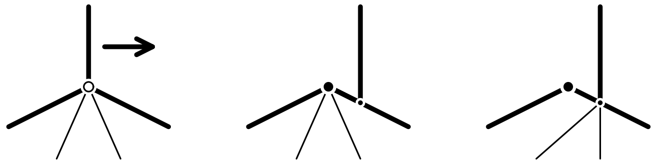

Figure 6: Different incidence allocations.

Non-interference can be used as a principle that the distribution of star-cells to the two ends of an expanding cell probably has $O(n)$ possibilities, rather then $O(2^n)$. (Counter-examples welcome)

This is all very ugly indeed, especially as in many situations much of the possible information is unnecessary.

### 5.6.6 specification

Here we need to specify not only the basic Move set, but also

- the Expand set

- for each cell in the expand set which set each of its three image components should lie in, and whether it should be retained,

- for each cell in the expand set in the move or static sets, a carrier in which it is to lie,

- for each cell initially incident on a cell in the expand set, which of the three image components it should remain incident to,

at very minimum. There may be more information needed in some cases which we have not yet identified.

### 5.6.7 semantics

The cells in the expand set are split into three image components, and incidence relationships and carriers associated with the three. It is then possible for a subsequent small tweak to continue.

### 5.6.8 validity conditions

### 5.6.9 limitations

# 6 Speculations

## 6.1 Parametric updates

This may be the most interesting part, because the parametrics was the primary motivation for our effort, but did not seem to always separate parametric issues from the general issues. A parametric edit specifies the parameter range [a,b], and the first thing we do is partition the parameter range into a sequence of small-collapsing-expanding tweaks ... subject to all the difficulties mentioned above.

Note that a parametric change should NOT create any new information (such as new carriers); rather it may lead to repartitioning of the existing carriers. This means that a new cellular structure may indeed appear during the update, but no new carriers should be created.

It does appear logical that each parametric edit can be expressed in terms of a MOVE / BORDER / STATIC sets of cells. But the cells should be allowed to be "macro" (i.e. they may have an internal cellular structure). We then consider small (and possibly ! collapsing) tweaks for these macro-cellular structures, but stratifying the result to the finest existing level. This will allow using all the developed machinery of small tweaks for more general edits, while producing the effect of collapses and expansions on the lowest cellular level.

Expanding tweaks may be useful somewhere else, but for parametric edits they do not seem to be required.

- If we pose that a 'large' parametric edit is expressed in this way, we have to face the fact that the complex on which the edit is operating changes its structure at each event. How then can the separate tweaks which form it be regarded as the same edit ? The macro-cellular structure is an approach to this.

- The information required to specify an expansion tells us what information a collapsing tweak must return to the application. The incidence structure before the collapse and the carriers of any cells which have disappeared during the collapse are the obvious examples.

## 6.2 Macro-cellular structures

One way of avoiding some complexity in specification of an expanding tweak is to reduce the number of independent cells. This might be done by combining a collection of cells whose interior structure we wish, temporarily, to ignore, into a macro-cell.

Given a cell complex $K$, a macro-cell $\bar{C}$ is a union of cells in $C$ such that

– $\bar{C}$ is homogeneous in dimension

– $\bar{C}$ is relatively open

– $\bar{C}$ is connected (but not necessarily simply connected)

This implies that macro-cells are manifold, which immediately translates into a combinatorial requirement that every cell of dimension $\bar{d} - 1$ is incident to no more than 2 $\bar{d}$-cells.

A macro-complex $\tilde{C}$ is a collection of macro-cells[10] satisfying the usual requirements: macro-cells must be disjoint and satisfy the frontier condition.

We observe that a macro-cell is not a subcomplex of $K$. This is to be expected if we insist that a (closed) complex is made up from open cells. [11]

This leaves the open question of how to reimpose a structure ignored during a tweak by using macro-cells?

It is also possible for the macrocell to be used only to define a carrier, shared by all its cells of full dimension. This would seem to avoid a lot of the reimposition problem. It would also mean that we could define the macro-carrier as a manifold subcomplex, since carriers need to be closed.

## 6.3 Tweaks by replacement of carrier rather than by transform.

The transform is only actually applied to the (carrier of the) move set, and so replacing that carrier by an alternative is OK if we can identify a continuous transformation of just that point set which takes the original carrier into the new one. We don't actually have to apply the transform to the whole of space,

[10] A single cell is also an example of a macro-cell, being a collection with just one member.

[11] An equivalent (but perhaps more symmetric) definition would define $C$ to be an assembly of closed cells such that intersection of any two cells is another (lower-dimensional) cell. Thus, a macro-cell could be defined as sub-complex of $C$, while a macro-complex would require that intersection of any two macro-cells is another lower-dimensional macro-cell. However, this approach would lose the valuable property that every point lies in at most one cell.

though the definition of a tweak in terms of point to point transforms does have the advantage that the geometric frontier condition is automatically maintained within the move set.

If the move set carrier is a parametric entity, and we can map it into a new parametric entity on the same domain, then linear interpolation between initial and final carriers will suffice

$$\begin{aligned} P(u,t) &= (1-t)P(u,0) + tP(u,1) \\ &= (1-t)P_0(u) + tP_1(u) \end{aligned}$$

provided it never causes any local collapsing within the move set. This is not totally painless, since the same map has to apply to all elements of the move set. To move a trimmed nurbs in this way, as a complex, demands that the Pcurves before and after be the same.

If the carrier of a cell in the move set is defined by a potential function before and after, then apparently this can again be linearly interpolated

$$\begin{aligned} \phi(P,t) &= (1-t)\phi(P,0) + t\phi(P,1) \\ &= (1-t)\phi_0(P) + t\phi_1(P) \end{aligned}$$

but this does not give a point to point mapping, and so it is harder to identify exactly what is happening to the frontier of a surface. It would be unusual for a linear interpolation of a parametric boundary to be compatible with a linear interpolation of this kind between two functions.

Examples such as the mapping from a plane face to a face with a handle suggest that we need to be very careful indeed as to how this is defined.

## 6.4 Inference and deduction

**inference** can happen when a list of candidates for some role has only one member (like the three-valent vertex examples)

**heuristics** can be used to reduce the amount specified by the application to the kernel. Examples of heuristics are:-

– carriers whose cells are frontiers of OUT are more likely to be selected than purely internal ones

– use the sense of the direction of expansion. truncatees have priority over extendees.

# 7 Conclusions

We have explored the issues in implementing small tweaks in a totally reliable, incremental way and shown that this is feasible. Collapsing tweaks are also relatively straightforward, although there is a lot of detail which might need to be reported back to the application. Expanding tweaks look really messy, and it would seem wise to try to find ways of avoiding them, possibly by using the concept of macro-carriers.