Self-Scaled Barrier Functions: Decomposition and Classification

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Abstract

The theory of self-scaled conic programming provides a unified framework for the theories of linear programming, semidefinite programming and convex quadratic programming with convex quadratic constraints. Nesterov and Todd's concept of self-scaled barrier functionals allows the exploitation of algebraic and geometric properties of symmetric cones in certain variants of the barrier method applied to self-scaled conic programming problems. In a first part of this article we show that self-scaled barrier functionals can be decomposed into direct sums of self-scaled barrier functionals over the irreducible components of the underlying symmetric cone. Applying this decomposition theory in a second part, we give a complete classification of the set of self-scaled barrier functionals that are invariant under the action of the orthogonal group of their conic domain of definition (we call such functionals *isotropic*).

Key words. self-concordant barriers, symmetric cones, self-scaled cones, interior-point methods, convex programming, self-scaled programming, linear programming, semidefinite programming, second-order cone programming, quadratic programming.

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1 Introduction

Self-scaled conic programming provides a unified framework for linear- semidefinite- and second-order-cone programming. These are classes of optimization problems that play an important role in such areas as network theory, operations management, logistics and the theory of robust control, among others. The definition of the general problem of self-scaled conic programming and the first investigation of the barrier method applied to these problems is due to Nesterov and Todd. In their seminal papers [8, 9], Nesterov and Todd consider a class of self-concordant barrier functionals [7] which adequately describe the geometric and algebraic properties of the conic domain of admissibility of self-scaled conic optimization problems. Their so-called *self-scaled barrier functionals* [8] form the central object of investigation in this article. Important contributions to the theory of self-scaled barrier functionals were made by Güler [3] who showed that there exists a deep link between the theory of self-scaled conic programming and Jordan algebra theory via the notion of symmetric cones.

Let E be a n-dimensional real vector space and $E^\sharp$ its dual. Throughout this article we will reserve super- or subscript-$\sharp$ for duals, superscript-$*$ for adjoints and superscript-$\circ$ for topological interiors. Let $K \subset E$ be a regular cone (i.e. K is convex, pointed and has non-empty topological interior K°). The dual of K, defined as usual by

$$K^\sharp := \big\{ s \in E^\sharp : \langle x; s \rangle \geq 0 \quad \forall x \in K \big\},$$

is a regular cone in $E^\sharp$. $\langle \cdot ; \cdot \rangle$ denotes the canonical bilinear product defined on $E \times E^\sharp$. Let $(Y, Y^\sharp, \langle \cdot ; \cdot \rangle)$ be another bilinear system of m-dimensional real vector spaces, and let $A : E \to Y^\sharp$ be a surjective vector space homomorphism. Let $b \in Y^\sharp$ and $c \in E^\sharp$ be fixed vectors. We consider the canonical dual pair of conic optimization problems:

$$\begin{array}{llll} & \text{(P')} \quad \inf \langle x; c \rangle & \qquad & \text{(D')} \quad \sup \langle y; b \rangle \\ \text{(1.1)} & \qquad Ax = b & & \qquad A^* y + s = c \\ & \qquad x \in K & & \qquad s \in K^\sharp. \end{array}$$

Note that the general problem of linear programming is an example of a conic optimization problem, as can be seen by choosing $K = \mathbb{R}^n_+$. Conic programming is however a much more general framework. In fact, it is easy to see that *any arbitrary* convex optimization problem can be reformulated equivalently in conic form. Let us make the usual regularity assumption that there *exist* vectors $x_0 \in K^\circ$ and $(y_0, s_0) \in Y \times K^{\sharp\circ}$ such that $Ax_0 = b$ and $A^* y_0 + s_0 = c$ respectively. Let L be the linear subspace of E defined by $L := \ker(A)$ and $L^\perp := \mathrm{im}(A^*)$ its orthogonal complement with respect to the bilinear form $\langle \cdot ; \cdot \rangle$. Then it is easy to see that the pair of conic optimisation problems (P'), (D') is equivalent to the canonical primal-dual symmetric pair of conic optimization problems

$$\begin{array}{llll} \text{(1.2)} & \text{(P)} \quad \inf \langle x; s_0 \rangle & \qquad & \text{(D)} \quad \inf \langle x_0; s \rangle \\ & \qquad x \in \big(L + x_0\big) \cap K & & \qquad s \in \big(L^\perp + s_0\big) \cap K^\sharp. \end{array}$$

In this article we consider a problem pair of the form (1.2) with a choice of K that satisfies the following conditions:

1. K and $K^\sharp$ are isomorphic and

2. for all $x, z \in K^\circ$ there exists an automorphism $\theta \in \mathrm{Aut}(K)$ of K such that $\theta x = z$.

Such cones are called *symmetric* and play an important role in the theory of Jordan algebras. Three particular examples of symmetric cones are given by

1. $K = \mathbb{R}^n_+$ the positive orthant,

2. $K = \left\{ \left(\begin{smallmatrix}\tau \\ x\end{smallmatrix}\right) \in \mathbb{R}^{n+1} : \tau \geq \|x\|_2 \right\}$ the Lorentz cone and

3. $K = \mathrm{S}^{n\times n}(\mathbb{R})_+$ the cone of positive semidefinite symmetric matrices over the reals.

In the context of problem 1.2, the choice $K = \mathbb{R}^n_+$ yields the general problem of linear programming, as already mentioned above. If we choose K equal to the Lorentz cone we retrieve the general problem of second-order cone programming, and this contains the general problem of convex quadratic programming with convex quadratic constraints. Finally, under the choice $K = \mathrm{S}^{n\times n}(\mathbb{R})_+$ we obtain the general problem of semidefinite programming.

The theory of conic optimization problems over symmetric cones is called *self-scaled conic programming* and was introduced by Nesterov and Todd [8, 9]. The group

$$\mathrm{Aut}(K) := \left\{\theta \in \mathrm{Aut}(E) : \theta K = K\right\},$$

of automorphisms of a cone K defines the set of possible scale transformations of a conic programming problem in the form (P). The automorphism groups of symmetric cones are particularly rich, and the algorithms investigated by Nesterov and Todd satisfy the best possible invariance properties under such scale transformations. They achieve this by considering *self-scaled barrier functionals* over K°, which they define to be self-concordant logarithmically homogeneous barrier functionals (see [7]) that also satisfy the following two properties:

1. $F''(w)x \in K^{\sharp\circ}$ for all $x, w \in K^\circ$ and

2. $F_\sharp\big(F''(w)x\big) = F(x) - 2F(w) - \nu$ for all $x, w \in K^\circ$,

where ν is a parameter associated with F – the so-called *complexity parameter* (see Section 2.1). As usual, we endow the interior of the dual cone $K^\sharp$ with the *dual barrier functional*

$$\begin{aligned} F_\sharp : K^{\sharp\circ} &\to \mathbb{R} \\ s &\mapsto \max\{-\langle x; s\rangle - F(x) : x \in K^\circ\}. \end{aligned}$$

This is the Legendre-Fenchel transform of F with the sign of its argument inverted. If F is self-scaled then $F_\sharp$ is a self-scaled barrier functional too (see [8]). The feature that renders self-scaled barrier functionals the barriers of choice in the case where the domain of definition is the interior of a symmetric cone K is that their Hessians are closely related to the automorphism group $\mathrm{Aut}(K)$ (see Theorems 2.8 and 2.22).

Vinberg [11] gave a complete classification of symmetric cones, based on earlier work done by Cartan [1], Köcher [6] and Rothaus [10]. His classification is in terms of a unique decomposition of symmetric cones into a direct sum of irreducible parts, i.e. symmetric cones that cannot be further reduced (see Theorem 2.19 below). In Section 3 we show that any self-scaled barrier functional over a symmetric cone K decomposes into a direct sum of self-scaled barrier functionals over the irreducible parts of K, and that this decomposition is unique up to additive constants.

Self-scaled barrier functionals are characterized by a rather lengthy list of axioms. One may thus ask the question whether the set of functions that satisfy these axioms is sufficiently constrained for it to be possible to classify them by means of an explicit representation theory. In this article we consider the set of *isotropic* self-scaled barrier functionals, which we define as being self-scaled barrier functionals that are invariant under orthogonal automorphisms of their conic domain of definition. One may think of these as self-scaled barrier functionals that are rotationally invariant. This assumption is natural, since rotations change neither the geometric nor the metric aspects of an optimization problem. In our main result, Theorem 4.12, we show that the set of isotropic barrier functionals can be completely classified in terms of the decomposition theory we developed in Section 3. In fact, the irreducible parts of an isomorphic self-scaled barrier functional are isotropic themselves, and all isotropic self-scaled barrier functionals over irreducible cones can be described in terms of the characteristic function of the cone. Barrier functionals of this type were first described by Güler [3]. We will also point out how our classification might be extended to the general case (see Conjecture 4.18).

Let us finish this section with some remarks regarding notation. Often the need arises to distinguish between linear operators and their matrix representation with respect to a particular choice of coordinates. In particular, we will write $\nabla^2 F(x)$ for matrix representations of the Hessian $F''(x)$. The sets of symmetric matrices, symmetric positive semidefinite and symmetric positive definite matrices over $\mathbb{R}$ will be denoted by $\mathrm{S}^{n\times n}(\mathbb{R})$, $\mathrm{S}^{n\times n}(\mathbb{R})_+$ and $\mathrm{S}^{n\times n}(\mathbb{R})_{++}$ respectively. For the set of diagonal $n \times n$-matrices over $\mathbb{R}$ we will use the notation $\mathrm{Diag}_n(\mathbb{R})$. By $\mathrm{diag}(D)$ we denote the element of $\mathbb{R}^n$ defined by the diagonal of matrix D, and conversely we write $\mathrm{Diag}(v)$ for the diagonal $n \times n$-matrix defined by $v \in \mathbb{R}^n$. Identity maps over any space will always be denoted by the symbol $\imath$.

2 Prerequisite results, examples and additional notation

2.1 Self-concordant barrier functionals

Modern interior-point theory is based on the foundation of Nesterov & Nemirovskii's theory of self-concordant functionals. The monumental reference in this field is [7].

Definition 2.1 (Nesterov & Nemirovskii). Let $K \subset E$ be a regular cone. A *self-concordant functional* for K is a function $F \in \mathcal{C}^3(K^\circ, \mathbb{R})$ that satisfies the following properties:

$i)$ $F''(x)$ is positive definite for all $x \in K^\circ$,
$ii)$ $\{y : \langle y - x; F''(x)[y-x] \rangle < 1\} \subset K^\circ$ for all $x \in K^\circ$,
$iii)$ for all $0 \neq v \in E$ and for all $x, y \in K^\circ$ such that $\langle y - x; F''(x)[y-x] \rangle < 1$,

$$\begin{aligned} 1 - \langle y - x; F''(x)[y-x] \rangle^{1/2} \leq \\ \leq \frac{\langle v; F''(y)v \rangle^{1/2}}{\langle v; F''(x)v \rangle^{1/2}} \\ \leq \Big(1 - \langle y - x; F''(x)[y-x] \rangle^{1/2}\Big)^{-1}. \end{aligned}$$

F is called a *barrier functional* if it is a self-concordant functional and the following condition is satisfied:

$iv)$ $\nu := \sup\{\langle F''^{-1}(x)F'(x); F'(x) \rangle : x \in K^\circ\} < \infty.$

Finally, F is said to be *logarithmically homogeneous* if

$v)$ $F(\tau x) = F(x) - \nu \ln \tau$ for all $x \in K^\circ$ and $\tau > 0$.

Remark 2.2. It is straightforward to see that ν-logarithmic homogeneity alone implies that $-F'(x) = F''(x)x$ and $\langle x; -F'(x) \rangle = \nu$ for all $x \in K^\circ$. Hence, the identity

$$\nu = \langle F''^{-1}(x)F'(x); F'(x) \rangle$$

is satisfied for all $x \in K^\circ$ and condition $iv)$ in Definition 2.1 becomes obsolete, i.e. ν-logarithmically homogeneous self-concordant functionals are automatically barrier functionals.

On the interior of the dual cone $K^\sharp \subset E^\sharp$ one defines the following functional:

$$\begin{aligned} F_\sharp : K^{\sharp\circ} &\to \mathbb{R} \\ s &\mapsto \max\{-\langle x; s \rangle - F(x) : x \in K^\circ\}. \end{aligned}$$

If F is a ν-self-concordant logarithmically homogeneous barrier functional on K then $F_\sharp$ is a ν-self-concordant logarithmically homogeneous barrier functional on $K^\sharp$. We shall make frequent use of the following properties of these functionals:

Proposition 2.3 (Nesterov & Nemirovskii). *Let F be a ν-self-concordant logarithmically homogeneous barrier functional on the regular cone $K \subset E$, and let $x \in K^\circ$. Then* *i)* $-F'(x) = F''(x)x \in K^{\sharp\circ}$,

ii) $-F'_\sharp(-F'(x)) = x$,

iii) $F''_\sharp(-F'(x)) = F''(x)^{-1}$,

iv) $\langle x; -F'(x) \rangle = \nu$,

v) for all $\tau > 0 :$ $F'(\tau x) = \tau^{-1} F'(x)$,

vi) for all $\tau > 0 :$ $F''(\tau x) = \tau^{-2} F''(x)$ *and*

vii) $F_\sharp\big(-F'(x)\big) = -\nu - F(x)$. □

2.2 Self-scaled barrier functionals

The theory of self-scaled conic programming is founded on two seminal papers by Nesterov and Todd [8, 9]. The following definition and theorem are among the basic building blocks of this theory.

Definition 2.4 (Nesterov-Todd). A regular cone $K \subset E$ is *self-scaled* if it admits a ν*-self-scaled* barrier functional F, defined as follows:

i) $F \in \mathcal{C}^3(K^\circ, \mathbb{R})$ is a ν-self- concordant logarithmically homogeneous barrier functional for K,

ii) $F''(w)x \in K^{\sharp\circ}$ for all $x, w \in K^\circ$ and

iii) $F_\sharp\big(F''(w)x\big) = F(x) - 2F(w) - \nu$ for all $x, w \in K^\circ$.

(Compare with $i)$ and $vii)$ of Proposition 2.3 which show that these last two conditions hold for $w = x$ in the general case.) It can be shown that if F is self-scaled then its dual $F_\sharp$ is self-scaled as well.

Example 2.5 (Positive orthant).
We endow the spaces $E = E^\sharp = V := \mathbb{R}^n$ with their canonical basis and scalar product and choose $K = K^\sharp := \mathbb{R}^n_+$ and

$$F : x \mapsto -\sum_{i=1}^n \ln(x_i) \qquad \text{for all } x \in \mathbb{R}^n_{++}.$$

Then F is n-self-scaled and the following identities hold true:

$$\begin{aligned} &F_\sharp : s \mapsto -\sum_{i=1}^n \ln(s_i) - n \qquad \text{for all } s \in \mathbb{R}^n_{++}, \\ &F'(x) = -\big(x_1^{-1}, \dots, x_n^{-1}\big)^T \qquad \text{and} \\ &F''(x) = \mathrm{Diag}(x)^{-2}. \end{aligned}$$

Example 2.6 (Symmetric positive semidefinite cone).
We endow $E = E^\sharp = V := \mathrm{S}^{n\times n}(\mathbb{R})$ with the trace inner product

$$\begin{aligned} \langle x; s\rangle &= x \bullet s := \operatorname{tr}(xs) \qquad \text{for all } x, s \in \mathrm{S}^{n\times n}(\mathbb{R}), \\ (v_1; v_2) &= v_1 \bullet v_2 \qquad\qquad \text{for all } v_1, v_2 \in \mathrm{S}^{n\times n}(\mathbb{R}). \end{aligned}$$

On $\mathrm{S}^{n\times n}(\mathbb{R})$ let us choose the canonical basis and consider the cone $K = K^\sharp := \mathrm{S}^{n\times n}(\mathbb{R})_+ \subset E$ and the functional

$$F : x \mapsto -\ln(\det(x)) \qquad \text{for all } x \in \mathrm{S}^{n\times n}(\mathbb{R})_{++}.$$

Then F is n-self-scaled and the following identities hold true:

$$\begin{aligned} &F_\sharp : s \mapsto -\ln\big(\det(s)\big) - n \qquad \forall\, s \in \mathrm{S}^{n\times n}(\mathbb{R})_{++}, \\ &F'(x) = -x^{-1} \qquad \text{and} \\ &F''(x)h = x^{-1}hx^{-1} \qquad \forall\, h \in \mathrm{S}^{n\times n}(\mathbb{R}). \end{aligned}$$

Example 2.7 (Lorentz cone).
We endow $E = E^\sharp = V := \mathbb{R}^{n+1}$ with its canonical basis and scalar product, and define

$$\begin{aligned} &K := \{\left(\begin{smallmatrix}\tau\\ x\end{smallmatrix}\right) \in E : \tau \geq \|x\|\}, \\ &F : \left(\begin{smallmatrix}\tau\\ x\end{smallmatrix}\right) \mapsto -\ln\Big(\left(\begin{smallmatrix}\tau\\ x\end{smallmatrix}\right)^T\left(\begin{smallmatrix}\tau\\ -x\end{smallmatrix}\right)\Big) \qquad \text{for all } \left(\begin{smallmatrix}\tau\\ x\end{smallmatrix}\right) \in K^\circ. \end{aligned}$$

Then F is 2-self-scaled and the following equalities hold true:

$$\begin{aligned} &K^\sharp = \{\left(\begin{smallmatrix}\rho\\ s\end{smallmatrix}\right) \in E^\sharp : \rho \geq \|s\|\}, \\ &F_\sharp : \left(\begin{smallmatrix}\rho\\ s\end{smallmatrix}\right) \mapsto -\ln\Big(\left(\begin{smallmatrix}\rho\\ s\end{smallmatrix}\right)^T\left(\begin{smallmatrix}\rho\\ -s\end{smallmatrix}\right)\Big) - 2 + 2\ln 2 \qquad \text{for all } \left(\begin{smallmatrix}\rho\\ s\end{smallmatrix}\right) \in K^{\sharp\circ}, \\ &F'\Big(\left(\begin{smallmatrix}\tau\\ x\end{smallmatrix}\right)\Big) = -\frac{2}{\left(\begin{smallmatrix}\tau\\ x\end{smallmatrix}\right)^T\left(\begin{smallmatrix}\tau\\ -x\end{smallmatrix}\right)}\begin{pmatrix}\tau\\ -x\end{pmatrix}, \\ &F''\Big(\left(\begin{smallmatrix}\tau\\ x\end{smallmatrix}\right)\Big) = \frac{2}{\left(\begin{smallmatrix}\tau\\ x\end{smallmatrix}\right)^T\left(\begin{smallmatrix}\tau\\ -x\end{smallmatrix}\right)}\begin{pmatrix}-1 & 0\\ 0 & I_n\end{pmatrix} + \left[\frac{2}{\left(\begin{smallmatrix}\tau\\ x\end{smallmatrix}\right)^T\left(\begin{smallmatrix}\tau\\ -x\end{smallmatrix}\right)}\right]^2\begin{pmatrix}\tau\\ -x\end{pmatrix}\begin{pmatrix}\tau\\ -x\end{pmatrix}^T. \end{aligned}$$

Theorem 2.8 (Nesterov-Todd). *Let K be a self-scaled cone with ν-self-scaled barrier functional F, and suppose $x \in K^\circ$, $s \in K^{\sharp\circ}$. Then there exists a unique* scaling point *$w = w(x,s) \in K^\circ$ such that $s = F''(w)x$. Furthermore, the following properties hold:*

i) $w(-F_\sharp'(s), -F'(x)) = w(x,s)$,

ii) $F''(w)$ *is a cone isomorphism* $K \to K^\sharp$,

iii) $F''(x) = F''(w) \circ F_\sharp''(s) \circ F''(w)$, *iv) The dual scaling point $t(x,s) = -F'(w)$ satisfies $F''_\sharp(t(x,s))s = x$.* $\square$

The uniqueness part of Theorem 2.8 shows that self-scaledness is a very restrictive condition. This is further illustrated by the following corollary:

Corollary 2.9. *Let K be a self-scaled cone with ν-self-scaled barrier functional F, and suppose $x, y \in K^\circ$ are such that $F''(x)h = F''(y)h$ for some $h \in K^\circ$. Then $x = y$.* $\square$

2.3 Symmetric cones

Definition 2.10. Let K, F be self-scaled. In the interior-point literature it is customary to view E as a Euclidean space by identifying E with $E^\sharp$ via $F''(e)$ for some fixed $e \in K^\circ$, i.e. we define a scalar product $\left(\cdot;\cdot\right)_e$ by the rule

$$\left(x_1; x_2\right)_e := \left\langle x_1; F''(e)x_2\right\rangle \qquad \forall x_1, x_2 \in E.$$

We say that $\left(\cdot;\cdot\right)_e$ provides E with the *Euclidean structure defined by* e. A *frame* of K for this Euclidean structure is a subset $\left\{e_1, \dots, e_{\kappa(K)}\right\} \subset K$ of maximal cardinality whose elements e_i are mutually orthogonal. The number $\kappa(K)$ is called the *Carathéodory number* of K. If $B \supset B_K := \left\{e_1, \dots, e_{\kappa(K)}\right\}$ is an orthogonal basis of E for the Euclidean structure defined by e we say that B is a basis with frame B_K centered at e. We write $\mathcal{B}^{(e)}$ for the set of such bases, and we shall use the notation $\mathcal{B} := \bigcup_{e \in K^\circ} \mathcal{B}^{(e)}$ for the union of these sets. If E is endowed with the basis $B \in \mathcal{B}^{(e)}$ then we endow the space $E^\sharp$ with the dual basis $B^\sharp = \left\{e_1^\sharp, \dots, e_n^\sharp\right\}$, where $e_i^\sharp := F''(e)e_i$ for all $i \in \mathbb{N}_n$. Whenever we represent objects in matrix form it will notationally suffice to refer only to the basis B chosen on the primal space E. For example, the notation $\nabla_B^k F(x)$ shall refer to the matrix representation of the tensor $F^{(k)}(x) \in \otimes^k E^\sharp$ with respect to the bases B and $B^\sharp$. In particular, if $B \in \mathcal{B}^{(e)}$ then $\nabla_B^2 F(e) = I$. Once E has been endowed with a Euclidean structure, we denote the group of orthogonal transformations that map K onto itself by $\mathcal{O}(K)$ and call this the orthogonal group of K. By $\mathcal{Q}(K)$ let us denote the connected component of $\mathcal{O}(K)$ that contains the identity $\imath : E \to E$.

Let us now illustrate the notion of frame on our three primary examples:

Example 2.11 (Positive orthant). The self-scaled cone $\mathbb{R}^n_+$ has Carathéodory number equal to n because the canonical basis of $\mathbb{R}^n$ is a frame for $\mathbb{R}^n_+$.

Example 2.12 (Symmetric positive semidefinite cone). The self-scaled cone $\mathrm{S}^{n\times n}(\mathbb{R})_+$ of symmetric positive semidefinite $n\times n$-matrices over the reals has Carathéodory number equal to n. In fact, $\left\{e_1, \dots, e_n\right\}$ is a set of mutually orthonormal elements in $\mathrm{S}^{n\times n}(\mathbb{R})_+$ where e_i is the $n \times n$ matrix whose only non-zero entry is the ii component which takes the value one. All other elements of $\mathrm{S}^{n\times n}(\mathbb{R})_+$ are of the form $O\big(\sum x_i e_i\big)O^T$ for some $x \in \mathbb{R}^n_+$ and $O \in \mathcal{O}(n)$. But since $OIO^T = I$ for all $O \in \mathcal{O}(n)$ we get that

$$\begin{aligned}\big(O\big(\textstyle\sum x_i e_i\big)O^T; I\big) &= \big(\textstyle\sum x_i e_i; I\big)\\ &= \textstyle\sum x_i \geq 0,\end{aligned}$$

with equality if and only if $x = 0$. It follows that there cannot be another non-zero element of $\mathrm{S}^{n\times n}(\mathbb{R})_+$ orthogonal to all elements of $\{e_1, \ldots, e_n\}$, and hence this is a frame of the cone $\mathrm{S}^{n\times n}(\mathbb{R})_+$.

Example 2.13 (Lorentz cone). The Lorentz cone $K := \{\left(\begin{smallmatrix}\tau\\ x\end{smallmatrix}\right) \in \mathbb{R}^{n+1} : \tau \geq \|x\|\}$ has Carathéodory number equal to 2. In fact, let us choose $e = \left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right)$ and endow E with the Euclidean structure defined by e. The results of Example 2.7 imply that $\|\cdot\|_e = \sqrt{2}\,\|\cdot\|_2$ and hence the vectors

$$e_1 := \frac{1}{2}\left(\begin{smallmatrix}1\\ 1\\ 0\end{smallmatrix}\right) \qquad \text{and} \qquad e_2 := \frac{1}{2}\left(\begin{smallmatrix}1\\ -1\\ 0\end{smallmatrix}\right)$$

are orthonormal for the Euclidean structure defined by e. All other elements of K are of the form

$$\left(\begin{smallmatrix}1 & \\ & O\end{smallmatrix}\right)\big(\lambda_1 e_1 + \lambda_2 e_2\big)\left(\begin{smallmatrix}1 & \\ & O\end{smallmatrix}\right)^T$$

for some $O \in \mathcal{O}(n)$ and $\lambda_1, \lambda_2 \geq 0$, and hence there is no element of K orthogonal to $e_1 + e_2 \in \operatorname{span}\{e_1, e_2\}$, which shows that $\{e_1, e_2\}$ is a frame of K.

Although self-scaled cones were developed independently in the interior-point literature by Nesterov and Todd, it was later discovered that this concept is equivalent to the notion of *symmetric* cones (also called *self-dual homogeneous* cones) studied in Jordan algebra theory. The proof of equivalence is due to Güler and Nesterov-Todd.

Definition 2.14 (Self-Dual Homogeneous Cones). A cone $K \subset E$ is called *self-dual* if there exists a cone isomorphism $\theta : K \to K^\sharp$ between K and its dual. K is *homogeneous* if for every pair of elements $(x, s) \in K^\circ \times K^\circ$ there exists a cone automorphism $\theta_{(x,s)} : K \to K$ such that $\theta_{(x,s)} x = s$.

Theorem 2.15 (Güler). *Self-dual homogeneous cones are self-scaled.*

Proof. For regular cones K, Güler relates the universal barrier functional U of Nesterov and Nemirovskii [7] to the characteristic function φ_K of K (see Definition2.17 below). In the case where K is symmetric, certain properties of the characteristic function imply that there exists a self-scaled barrier functional of the form $c_1 U + c_0$ where $c_1 > 0$ and $c_0 \in \mathbb{R}$ are real numbers. For details see [3]. $\square$

Theorem 2.16 (Nesterov-Todd). *Self-scaled cones are self-dual homogeneous.*

Proof. Follows from Theorem 2.8. $\square$

We will henceforth use the terms "self-scaled cone", "self-dual homogeneous cone" and "symmetric cone" interchangeably. As discovered by Güler [3], self-concordant barrier functionals on symmetric cones are often most conveniently treated in terms of the characteristic function of the cone:

Definition 2.17. Let $K \subset E$ be a regular cone and $K^\sharp \subset E^\sharp$ its dual. Let $E^\sharp$ be provided with an alternating multi-linear form $\upsilon : \otimes^n E^\sharp \to \mathbb{R}$. The characteristic function $\varphi_{(K,\upsilon)}$ of K is defined as the Laplace transform of the uniform measure defined on $K^\sharp$ by υ:

$$\varphi_K : x \mapsto \int_{K^\sharp} \exp\{-\langle x; s\rangle\}\left|\upsilon(ds)\right|. \tag{2.1}$$

Remark 2.18.

i) In the case of self-scaled cones it is standard to pick an element $e \in K^\circ$ and to choose υ as the canonical alternating multi-linear form defined by the Euclidean structure defined by e: Let $B \in \mathcal{B}^{(e)}$. Then $\upsilon := e_1 \wedge \cdots \wedge e_n$.

ii) A different choice of the form υ affects $\varphi_{(K,\upsilon)}$ only via a multiplicative constant, because the space of alternating multi-linear n-forms $\mathrm{Alt}^n\big(E^\sharp\big)$ over $E^\sharp$ is one-dimensional. In particular, in the self-scaled case, let $B \in \mathcal{B}^{(e)}$. If we choose $\hat{e} \in K^\circ$ in place of e the corresponding multi-linear alternating form is given by

$$\hat{\upsilon} = \det\big(\nabla_B^2 F''(\hat{e})\big)^{-1} \cdot \upsilon.$$

iii) Güler [3] proved that the universal barrier functional U_K of a self-scaled cone K is of the form

$$U_K(x) = c_1 \ln \varphi_K + c_0$$

for some real numbers $c_1 > 0, c_2 \in \mathbb{R}$. Note that this result is independent of the choice of the alternating multi-linear form υ since only the constant c_0 is affected.

Based on earlier results by Cartan [1], Köcher [6] and Rothaus [10], Vinberg [11] gave a classification theory for self-dual homogeneous cones via an irreducible decomposition. For a systematic treatment in a single reference consult for example the book by Faraut and Korányi [2].

Theorem 2.19 (Vinberg).
Let $K \subset E$ be a self-dual homogeneous cone. Then there exists a unique set of self-dual homogeneous cones $\{K_{ij} \subset E_{ij} : (ij) \in J\}$ indexed by some index set $J = \{(ij) : i \in \mathbb{N}_l, j \in \mathbb{N}_{k(i)}\}$ such that

$$\begin{aligned} &i)\ E = \bigoplus_{i=1}^{l} \bigoplus_{j=1}^{k(i)} E_{ij} \qquad \textit{and} \\ &ii)\ K = \bigoplus_{i=1}^{l} \bigoplus_{j=1}^{k(i)} K_{ij}, \end{aligned}$$

and such that none of the cones K_{ij} can be further decomposed into a direct sum of self-dual homogeneous cones. The K_{ij} are called the irreducible parts *of K, and the direct sum ii) is called the* irreducible decomposition *of K. The first index is to distinguish sets of mutually isomorphic irreducible parts of K, i.e. $K_{i_1 j_1} \cong K_{i_2 j_2}$ if and only if $i_1 = i_2$. All irreducible parts K_{ij} are isomorphic to one of following cones for some $n \in \mathbb{N}$:*

1. $\mathbb{R}_+$ *(positive real line),*
2. $\left\{\binom{\tau}{x} \in \mathbb{R}^{n+1} : \tau \geq \|x\|_2\right\}$ *(Lorentz cone),*
3. $\mathrm{S}^{n\times n}(\mathbb{R})_+$ *(positive semidefinite symmetric matrices over the reals),*
4. $\mathrm{H}^{n\times n}(\mathbb{C})_+$ *(positive semidefinite Hermitian matrices over the complex numbers),*
5. $\mathrm{H}^{n\times n}(\mathbb{H})_+$ *(positive semidefinite Hermitian matrices over the quaternions) and*
6. $\mathrm{H}^{3\times 3}(\mathbb{O})_+$ *(positive semidefinite Hermitian 3×3-matrices over the octonions).*

□

Definition 2.20. Let $K \subset E$ be a self-scaled cone. Let $\mathrm{Aut}(E)$ be the group of automorphisms of E. The group of automorphisms of K is defined as

$$\mathrm{Aut}(K) := \{\theta \in \mathrm{Aut}(E) : \theta K = K\}.$$

$\mathrm{Aut}(K)$ is a Lie group, and Definition 2.14 requires that this group act transitively on the homogeneous manifold K°. Let the connected component of $\mathrm{Aut}(K)$ containing the identity $\imath : E \to E$ be denoted by $\mathrm{Aut}^{(\imath)}(K)$. The *stabilizers* of an element $e \in K^\circ$ in $\mathrm{Aut}(K)$ and $\mathrm{Aut}^{(\imath)}(K)$ respectively are defined as the subgroups

$$\begin{aligned} \mathrm{Aut}(K)_e &:= \{\theta \in \mathrm{Aut}(K) : \theta e = e\} \qquad \text{and} \\ \mathrm{Aut}^{(\imath)}(K)_e &:= \{\theta \in \mathrm{Aut}^{(\imath)}(K) : \theta e = e\}. \end{aligned}$$

Recall also from Definition 2.10 that we already defined the compact subgroups $\mathcal{O}(K) \subset \mathrm{Aut}\, K$ and $\mathcal{Q}(K) \subset \mathrm{Aut}^{(\imath)}(K)$.

Lemma 2.21. *Let K be an irreducible self-scaled cone with self-scaled barrier functional F. Let us fix an element $e \in K^\circ$ and endow E with the Euclidean structure defined by e. Let $B \in \mathcal{B}^{(e)}$ be a basis of E with frame B_K centered at e (see Definition 2.10). Then the orbit of the convex hull of B_K under the action of the group $\mathcal{Q}(K)$ covers all of K:*

$$K = \mathcal{Q}(K)\big[\text{cone}(B_K)\big].$$

Proof. See Faraut-Korányi [2] Corollary IV.2.7. □

Theorem 2.22 (Rothaus [10]). *Let the self-scaled barrier functional F of the self-scaled cone K be of the form $c_1 U + c_2$ where $c_1 > 0$ and $c_2 \in \mathbb{R}$ are constants and U is the universal barrier functional for K. Let E be provided with the Euclidean structure defined by a fixed vector $e \in K^\circ$. Then any cone automorphism $\theta \in \mathrm{Aut}(K)$ of K can be uniquely decomposed in the form*

$$\theta = \Omega \circ F''^{-1}(e) \circ F''(x) \tag{2.2}$$

for some $x \in K^\circ$ and $\Omega \in \mathcal{O}(K)$. The decomposition (2.2) *is called the* polar decomposition *of θ.* □

3 Decomposition of self-scaled barrier functionals

In this section we show that self-scaled barrier functionals on self-scaled cones K decompose into direct sums of self-scaled barrier functionals for the irreducible parts of K. The main result is Theorem 3.3, and it will play an important role in the next section.

Lemma 3.1. *Let $K \subset E$ be a self-scaled cone with irreducible decomposition $K = \bigoplus_J K^{(ij)}$. Consider a particular value of the index $i \in \mathbb{N}_l$. Then it is possible to choose a set of isomorphisms*

$$\big\{\imath_i^{(j_1 j_2)} \in \mathrm{Iso}\big(K^{(ij_1)}, K^{(ij_2)}\big) : (j_1 j_2) \in \mathbb{N}^2_{k(i)}\big\}$$

such that the following conditions are satisfied:

i) $\imath_i^{(j_2 j_3)} \circ \imath_i^{(j_1 j_2)} = \imath_i^{(j_1 j_3)} \quad \forall\,(j_1 j_2 j_3) \in \mathbb{N}^3_{k(i)}$ *and*

ii) $\imath_i^{(j_1 j_1)} = \imath \quad \forall\, j_1 \in \mathbb{N}_{k(i)}$.

Note that this implies in particular that $\imath_i^{(j_2 j_1)} = \big(\imath_i^{(j_1 j_2)}\big)^{-1}$ for all $(j_1 j_2) \in \mathbb{N}^2_{k(i)}$.

Proof. The proof is by recursion over the cardinality $k(i)$ of the set

$$\big\{K^{(i1)}, \ldots, K^{(ik(i))}\big\}$$

of mutually isomorphic irreducible parts of K corresponding to the index i: The case where $k(i)=1$ is trivial since we only need to choose the map $\imath_i^{(11)}=\imath$. Suppose $k(i)=k$ and that the lemma holds for $k(i)\leq k-1$. The lemma applied to the symmetric cone $\bigoplus_{j=1}^{k-1}K^{(ij)}$ says that there exists a set of isomorphisms

$$\left\{\imath_i^{(j_1j_2)}\in\operatorname{Iso}\bigl(K^{(ij_1)},K^{(ij_2)}\bigr):(j_1j_2)\in\mathbb{N}_{k-1}^2\right\}$$

that satisfies the conditions spelled out in the lemma. In order to extend this set, choose $\imath_i^{(1k)}\in\operatorname{Iso}\bigl(K^{(i1)},K^{(ik)}\bigr)$ arbitrary and set

$$\imath_i^{(jk)}:=\imath_i^{(1k)}\circ\imath_i^{(j1)}.$$

It is easily checked that the extended set of isomorphisms satisfies the conditions of the lemma. $\square$

In the sequel we will assume that such a set of isomorphisms has been fixed for each set of mutually isomorphic irreducible parts of K and we will freely use the notation $\imath_i^{(j_1j_2)}$.

Lemma 3.2. *Let $K\subset E$ be a self-scaled cone with irreducible decomposition $K=\bigoplus_J K^{(ij)}$. Then, for each cone automorphism $\theta\in\operatorname{Aut}(K)$ there exists a unique set of permutations $\bigl\{\sigma_\theta^{(i)}\in\mathcal{S}_{k(i)}:i\in\mathbb{N}_l\bigr\}$ and a unique set of cone automorphisms $\bigl\{\theta_\theta^{(ij)}\in\operatorname{Aut}\bigl(K^{(ij)}\bigr):(ij)\in J\bigr\}$ such that*

$$\theta\big|_{K^{(ij)}}=\imath_i^{j\sigma_\theta^{(i)}(j)}\circ\theta_\theta^{(ij)}$$

for all $(ij)\in J$. Conversely, all mappings of this form are automorphisms of K. Furthermore, the following maps are continuous for all $(ij)\in J$:

$$\begin{aligned}\operatorname{Aut}(K)&\to\mathcal{S}_{k(i)},\\ \theta&\mapsto\sigma_\theta^{(i)},\end{aligned}$$

and

$$\begin{aligned}\operatorname{Aut}(K)&\to\operatorname{Aut}\bigl(K^{(ij)}\bigr),\\ \theta&\mapsto\theta_\theta^{(ij)}.\end{aligned}$$

Proof. For $\theta\in\operatorname{Aut}(K)$ we have $K=\sum_J\theta\bigl(K^{(ij)}\bigr)$, and since K is full-dimensional this sum must be direct:

$$K=\bigoplus_J\theta\bigl(K^{(ij)}\bigr).\tag{3.1}$$

Thus, the identity (3.1) constitutes an irreducible decomposition of K. But the irreducible decomposition of K is unique up to indexing. Hence, for each $(ij)\in J$ there exists a unique element $\sigma_\theta^{(i)}(j)$ of $\mathbb{N}_{k(i)}$ such that

$$\theta\bigl(K^{(ij)}\bigr)=K^{(i\sigma_\theta^{(i)}(j))}.$$

Moreover, it is trivially true that

$$\theta_\theta^{(ij)} := \imath_i^{\sigma_\theta^{(i)}(j)j} \circ \theta\big|_{K^{(ij)}} \in \operatorname{Aut}\big(K^{(ij)}\big).$$

Finally, the last two statements are trivial. $\square$

Theorem 3.3 (Decomposition of Self-Scaled Barrier Functionals).
Let K be a symmetric cone provided with the ν-self-scaled barrier functional $F \in \mathcal{C}^3(K^\circ, \mathbb{R})$. Then there exist $\nu^{(ij)}$-self-scaled barrier functionals $F^{(ij)} \in \mathcal{C}^3(K^{(ij)\circ}, \mathbb{R})$ for the irreducible symmetric cones $K^{(ij)}$ $\big((ij) \in J\big)$ such that

$$F = \bigoplus_J F^{(ij)} : \bigoplus_J K^{(ij)} \to \mathbb{R},$$
$$\bigoplus_J x^{(ij)} \mapsto \sum_J F^{(ij)}\big(x^{(ij)}\big).$$

Furthermore, $\nu = \sum_J \nu^{(ij)}$ and the functionals $F^{(ij)}$ are unique up to additive constants.

Proof. Let $e \in K^\circ$ be fixed. For every $x = \bigoplus_J x^{(ij)} \in K^\circ$ we can apply Lemma 3.2 to the cone automorphism $\theta^{(x)} := F''^{-1}(e) \circ F''(x) \in \operatorname{Aut}(K)$, and we can use the shorthand notation $\sigma_x^{(i)}$ for $\sigma_{\theta^{(x)}}^{(i)}$ and $\theta_x^{(ij)}$ for $\theta_{\theta^{(x)}}^{(ij)}$ (see Lemma 3.2). In particular, $\sigma_e^{(i)} = \imath$ for all $i \in \mathbb{N}_l$ since $\theta_e = \imath$. Let x be an arbitrary element of K°, and let $\zeta \in \mathcal{C}^0\big([0,1], K^\circ\big)$ be a continuous path linking $\zeta(0) = e$ with $\zeta(1) = x$. Then, by virtue of Lemma 3.2 $t \mapsto \sigma_{\zeta(t)}^{(i)}$ is a continuous map from the unit interval $[0,1]$ to the discrete set $\mathcal{S}_{k(i)}$ for each $i \in \mathbb{N}_l$, and hence we must have $\sigma_x^{(i)} = \sigma_e^{(i)} = \imath$ for all $i \in \mathbb{N}_l$. It follows that

$$\theta^{(x)}\big|_{K^{(ij)}} = \theta_x^{(ij)} \in \operatorname{Aut}(K^{(ij)}) \tag{3.2}$$

for all $(ij) \in J$. Let $E^{(ij)\sharp} := F''(e)E^{(ij)}$ and denote the operator $F''(e) \circ \theta_x^{(ij)} \in \operatorname{Iso}\big(E^{(ij)}, E^{(ij)\sharp}\big)$ by $H^{(ij)}(x)$. Then equation (3.2) shows that

$$F''(x) = \bigoplus_J H^{(ij)}(x). \tag{3.3}$$

and $H^{(ij)} \in \operatorname{Iso}\big(K^{(ij)}, K^{(ij)\sharp}\big)$, where $K^{(ij)\sharp} := F''(e)K^{(ij)} \subset E^{(ij)\sharp}$ is the dual of $K^{(ij)}$ in the dual system $\big(E^{(ij)}, E^{(ij)\sharp}, \langle \cdot ; \cdot \rangle\big)$. Let $\pi_{E^{(ij)}} : E \to E^{(ij)}$ be the orthogonal projection of E onto $E^{(ij)}$, and $\pi_{E^{(ij)\sharp}} : E^\sharp \to E^{(ij)\sharp}$ the orthogonal projection of $E^\sharp$ onto $E^{(ij)\sharp}$. Let $v^{(ij)}(x) := \pi_{E^{(ij)\sharp}} F'(x)$ for all $x \in K^\circ$. Now, for a fixed index pair $(ij) \in J$ let $x, y \in K^\circ$ be two elements such that $\pi_{E^{(ij)}} y = \pi_{E^{(ij)}} x =: x^{(ij)}$. We contend that this implies that $v^{(ij)}(x) = v^{(ij)}(y)$. In fact, let

$$\zeta \in \mathcal{C}^1\big([0,1], K^\circ \cap \pi_{E^{(ij)}}^{-1}(\{x^{(ij)}\})\big)$$

be a differentiable path such that $\zeta(0)=x$ and $\zeta(1)=y$. Then we have

$$\begin{aligned}
v^{(ij)}(y) &= \pi_{E^{(ij)\sharp}}\Big[F'(x)+\int_0^1 F''\big(\zeta(t)\big)\dot\zeta(t)dt\Big]\\
&\overset{(3.3)}{=} v^{(ij)}(x)+\int_0^1 H^{(ij)}\big(\zeta(t)\big)\pi_{E^{(ij)}}\dot\zeta(t)dt\\
&= v^{(ij)}(x)+\int_0^1 H^{(ij)}\big(\zeta(t)\big)[0]dt = v^{(ij)}(x).
\end{aligned}$$

Hence, there are differentiable vector fields $v^{(ij)}: K^{(ij)\circ}\to E^{(ij)}$ $(ij)\in J$ such that $F'=\bigoplus_J v^{(ij)}$. Since the gradient field F' is conservative, the $v^{(ij)}$ are conservative vector fields too, i.e. they are gradient fields. Hence, there exist functions $F^{(ij)}\in\mathcal{C}^3\big(K^{(ij)\circ},\mathbb{R}\big)$ $\big((ij)\in J\big)$ and a constant $c\in\mathbb{R}$ such that

$$F=\bigoplus_J F^{(ij)}+c \qquad \text{and} \tag{3.4}$$

$$F^{(ij)\prime}(x^{(ij)}) = v^{(ij)}\big(x^{(ij)}\big) = \pi_{E^{(ij)\sharp}}F'(x). \tag{3.5}$$

The $F^{(ij)}$ are uniquely determined up to an additive constant, and we may choose these constants so that $c=0$. Moreover,

$$H^{(ij)}(x) = F^{(ij)\prime\prime}\big(x^{(ij)}\big) \tag{3.6}$$

and hence the left-hand side depends only on $x^{(ij)}$ for all $(ij)\in J$.

Using equation (3.3), it is straightforward to check that the functionals $F^{(ij)}$ are self-concordant.

Next, let us show that these functionals are logarithmically homogeneous. Since F is ν-logarithmically homogeneous, it follows from Proposition 2.3 $i)$ & $iv)$ that $\nu=\big\langle x;F''(x)x\big\rangle$ for all $x\in K^\sharp$, and hence we can write

$$\begin{aligned}
\nu &= \sup\big\{\big\langle x;F''(x)x\big\rangle : x\in K^\circ\big\}\\
&= \sum_J \sup\big\{\big\langle x^{(ij)};F^{(ij)\prime\prime}(x^{(ij)})x^{(ij)}\big\rangle : x^{(ij)}\in K^{(ij)\circ}\big\}\\
&= \sum_J \nu^{(ij)}
\end{aligned} \tag{3.7}$$

for some positive numbers $0<\nu^{(ij)}<\nu<\infty$. For all $x=\bigoplus_J x^{(ij)}\in K^\circ$ we have

$$\begin{aligned}
F^{(ij)\prime\prime}\big(x^{(ij)}\big)x^{(ij)} &= H^{(ij)}\big(x^{(ij)}\big)x^{(ij)}\\
&= \pi_{E^{(ij)\sharp}}F''(x)x\\
&= -\pi_{E^{(ij)\sharp}}F'(x)\\
&\overset{(3.5)}{=} -F^{(ij)\prime}\big(x^{(ij)}\big).
\end{aligned} \tag{3.8}$$

Therefore, the following inequality holds:

$$
\begin{aligned}
\nu &\overset{2.3iv}{=} \big\langle x; -F'(x)\big\rangle \\
&= \sum_J \big\langle x^{(ij)}; -F^{(ij)\prime}(x^{(ij)})\big\rangle \\
&\overset{(3.8),(3.7)}{\leq} \sum_J \nu^{(ij)} \\
&= \nu.
\end{aligned}
$$

But this is possible only if

$$
\big\langle x^{(ij)}; -F^{(ij)\prime}(x^{(ij)})\big\rangle = \nu^{(ij)} \tag{3.9}
$$

for all $x^{(ij)} \in K^{(ij)\circ}$ for all $(ij) \in J$. Now, Proposition 2.3 $v)$ says that for all $\tau > 0$ $F'(\tau x) = \tau^{-1}F'(x)$, and hence we get that $v^{(ij)}\big(\tau x^{(ij)}\big) = \tau^{-1}v^{(ij)}\big(x^{(ij)}\big)$. Therefore,

$$
\begin{aligned}
F^{(ij)}\big(\tau x^{(ij)}\big) &= F(x^{(ij)}) + \int_1^\tau \frac{d}{d\xi}F^{(ij)}\big(\xi x^{(ij)}\big)d\xi \\
&= F(x^{(ij)}) + \int_1^\tau \big\langle x^{(ij)}; F^{(ij)\prime}(\xi x^{(ij)})\big\rangle d\xi \\
&= F(x^{(ij)}) - \int_1^\tau \xi^{-1}\big\langle x^{(ij)}; -F^{(ij)\prime}(x^{(ij)})\big\rangle d\xi \\
&\overset{(3.9)}{=} F(x^{(ij)}) - \nu^{(ij)}\int_1^\tau \xi^{-1}d\xi \\
&= F(x^{(ij)}) - \nu^{(ij)}\ln\tau.
\end{aligned}
$$

This shows that the $F^{(ij)}$ are $\nu^{(ij)}$-logarithmically homogeneous.

Now it follows from Remark 2.2 that the $F^{(ij)}$ are in fact $\nu^{(ij)}$-self-concordant *barrier* functionals. It remains to show that the $F^{(ij)}$ are self-scaled.

Note that the functionals $F^{(ij)}$ satisfy condition $i)$ of Definition 2.4 by virtue of equation (3.6) and the fact that $H^{(ij)}(x) \in \mathrm{Iso}\big(K^{(ij)}, K^{(ij)\sharp}\big)$.

Finally, let us show that the functionals $F^{(ij)}$ satisfy condition $ii)$ of Definition 2.4: Note that for all $s = \bigoplus_J s^{(ij)}$ we have

$$
\begin{aligned}
F_\sharp(s) &= \max\Big\{-\big\langle x; s\big\rangle - F(x) : x \in K^\circ\Big\} \\
&= \max\Big\{-\sum_J\Big(\big\langle x^{(ij)}; s^{(ij)}\big\rangle + F^{(ij)}\big(x^{(ij)}\big)\Big) : x \in K^\circ\Big\} \\
&= \sum_J \max\Big\{-\big\langle x^{(ij)}; s^{(ij)}\big\rangle - F^{(ij)}\big(x^{(ij)}\big) : x^{(ij)} \in K^{(ij)\circ}\Big\} \\
&= \sum_J F_\sharp^{(ij)}\big(s^{(ij)}\big).
\end{aligned} \tag{3.10}
$$

Our fixed element e decomposes into the direct sum $e = \bigoplus_J e^{(ij)} \in K^\circ$, and the following identity holds true for all $(ij) \in J$:

$$
\begin{aligned}
& F_\sharp^{(ij)}\big(F^{(ij)\prime\prime}(e^{(ij)})e^{(ij)}\big) \overset{(3.8)}{=} \\
& \qquad = \max\{-\big\langle x^{(ij)}; -F^{(ij)\prime}(e^{(ij)})\big\rangle - F^{(ij)}(x^{(ij)}) : x^{(ij)} \in K^{(ij)\circ}\} \\
& \qquad = -\big\langle e^{(ij)}; -F^{(ij)\prime}(e^{(ij)})\big\rangle - F^{(ij)}(e^{(ij)}) \\
& \qquad \overset{(3.9)}{=} -F^{(ij)}(e^{(ij)}) - \nu^{(ij)} \\
& \qquad = F^{(ij)}(e^{(ij)}) - 2F^{(ij)}(e^{(ij)}) - \nu^{(ij)}.
\end{aligned}
\tag{3.11}
$$

Now, for a fixed index pair $(i_0 j_0) \in J$ choose arbitrary elements $w^{(i_0 j_0)}$ and $x^{(i_0 j_0)}$ in $K^{(i_0 j_0)\circ}$, and consider the vectors

$$
\begin{aligned}
w &:= \Big(\bigoplus_{J\setminus\{(i_0 j_0)\}} e^{(ij)}\Big) \oplus w^{(i_0 j_0)} \in K^\circ \qquad \text{and} \\
x &:= \Big(\bigoplus_{J\setminus\{(i_0 j_0)\}} e^{(ij)}\Big) \oplus x^{(i_0 j_0)} \in K^\circ.
\end{aligned}
$$

Then self-scaledness of F implies that

$$
\begin{aligned}
& F(x) - 2F(w) - \nu \overset{2.4ii)}{=} F_\sharp\big(F''(w)x\big) \overset{(3.10),(3.3),(3.6)}{=} \\
& \quad = \Big[\sum_{J\setminus\{(i_0 j_0)\}} F_\sharp^{(ij)}\big(F^{(ij)\prime\prime}(e^{(ij)})e^{(ij)}\big)\Big] + F_\sharp^{(i_0 j_0)}\big(F^{(i_0 j_0)\prime\prime}(w^{(i_0 j_0)})x^{(i_0 j_0)}\big) \\
& \overset{(3.11),(3.4)}{=} F(x) - 2F(w) - \nu + \\
& \qquad + F_\sharp^{(i_0 j_0)}\big(F^{(i_0 j_0)\prime\prime}(w^{(i_0 j_0)})x^{(i_0 j_0)}\big) - \Big[F^{(i_0 j_0)}\big(x^{(i_0 j_0)}\big) - 2F^{(i_0 j_0)}\big(w^{(i_0 j_0)}\big) - \nu^{(i_0 j_0)}\Big].
\end{aligned}
$$

Comparison of both sides of this equation shows that

$$
F_\sharp^{(i_0 j_0)}\big(F^{(i_0 j_0)\prime\prime}(w^{(i_0 j_0)})x^{(i_0 j_0)}\big) = F^{(i_0 j_0)}\big(x^{(i_0 j_0)}\big) - 2F^{(i_0 j_0)}\big(w^{(i_0 j_0)}\big) - \nu^{(i_0 j_0)},
$$

and hence that $F^{(i_0 j_0)}$ satisfies condition $ii)$ of Definition 2.4 indeed. □

4 Classification of isotropic self-scaled barrier functionals

We now proceed to applying the results obtained in the last section to derive a classification theory for the set of self-scaled barrier functionals which are invariant under the orthogonal group of the underlying cone. We call these barrier functionals *isotropic*. The main result is Theorem 4.12 which says that up to an additive constant each isotropic self-scaled barrier functional is a direct sum of positive multiples of the universal barrier functionals (see Nesterov-Nemirovskii [7]) of the irreducible parts of K. Isotropic barrier functionals satisfy a stronger version of self-scaledness: For all $\theta \in \mathrm{Aut}(K)$ there exists a constant $c \in \mathbb{R}$ such that $F(\theta x) \equiv F(x) + c$ (see Theorem 4.15 below). In addition, we will derive a complete classification of self-scaled barrier functionals for the positive orthant, even non-isotropic ones (see Theorem 4.4).

Definition 4.1. Let K be a self-scaled cone. A self-scaled barrier functional F for K is *isotropic* under the Euclidean structure defined by $e \in K^\circ$ if the identity

$$F \circ \theta \equiv F$$

holds true for all $\theta \in \mathcal{O}(K)$.

Example 4.2. Let K be a self-scaled cone and F a self-scaled barrier functional for K of the form $c_1 \ln \varphi_K + c_0$, where $c_1 > 0$ and c_0 are real numbers and φ_K the characteristic function of K. Then F is isotropic for all $e \in K^\circ$. In fact, let $e \in K^\circ$. Since the choice of the Euclidean structure on $E^\sharp$ in the definition of φ_K affects only the constant c_0 (see Remark 2.18 *ii*)), we may assume that the Euclidean structure used was defined by e, i.e.

$$\varphi_K : x \mapsto \int_{K^\sharp} \exp\bigl\{-\bigl(x; s\bigr)_e\bigr\} ds.$$

Let $\theta \in \mathcal{O}(K)$. Then

$$\begin{aligned}
\varphi_K(\theta x) &= \int_{K^\sharp} \exp\bigl\{-\bigl(\theta x; s\bigr)_e\bigr\} ds \\
&= \int_{K^\sharp} \exp\bigl\{-\bigl(x; \theta^* s\bigr)_e\bigr\} ds \\
&= \int_{\theta^* K^\sharp} \exp\bigl\{-\bigl(x; s\bigr)_e\bigr\} |\det \theta^*|^{-1} ds \\
&= \varphi_K(x).
\end{aligned}$$

Lemma 4.3. *All self-scaled barrier functionals for the self-scaled cone $\mathbb{R}_+$ are of the form $x \mapsto -c_1 \ln x + c_0$ for some $c_1 > 0$ and $c_0 \in \mathbb{R}$.*

Proof. Logarithmic homogeneity implies that if F is ν-self-scaled then $F(x) = -\nu \ln x + F(1)$. ☐

Theorem 4.4 (Self-scaled barriers for the positive orthant).
All self-scaled barrier functionals for the self-scaled cone $\mathbb{R}^n_+$ are of the form

$$F : \bigl(x_1, \ldots, x_n\bigr)^T \mapsto -\sum_{i=1}^{n} c_i \cdot \ln x_i + c_0$$

for some constants $c_i > 0\,(i = 1, \ldots, n)$ and $c_0 \in \mathbb{R}$, and F is isotropic for the canonical Euclidean structure defined on $\mathbb{R}^n$ if and only if $c_1 = \cdots = c_n$.

Proof. The first part follows immediately by applying Theorem 3.3 to the irreducible decomposition $\mathbb{R}^n_+ = \bigoplus_{i=1}^n \mathbb{R}_+$ while taking Lemma 4.3 into account. The second part follows from the fact that any permutation of the coordinate axes is an element of the orthogonal group of $\mathbb{R}^n_+$ under which F is invariant. $\square$

Corollary 4.5. *If the self-scaled cone $K \subset E$ is isomorphic to $\mathbb{R}^n_+$ then all isotropic self-scaled barrier functionals for K are of the form*

$$F = c_1 \ln \varphi_K + c_0$$

for some $c_1 > 0$ and $c_0 \in \mathbb{R}$, and the notion of isotropy is independent of the element $e \in K^\circ$ chosen to fix a Euclidean structure on E.

Proof. Fix $e \in K^\circ$ and choose a basis $B = \{e_1, \ldots, e_n\} \in \mathcal{B}^{(e)}$. Let $\mathcal{K} \subset \mathbb{R}^n$ be the set of column matrices that represent the elements of K with respect to the basis B. Then $\mathcal{K} = \mathbb{R}^n_+$. In fact, $\mathcal{K}$ is a polyhedral regular cone since $K \cong \mathbb{R}^n_+$, and a minimal set of generators for $\mathcal{K}$ consists of mutually orthogonal vectors in the canonical Euclidean structure of $\mathbb{R}^n$ due to self-duality of K under $(\cdot;\cdot)_e$. Without loss of generality we may assume that this Euclidean structure was also used in the definition of φ_K (see Remark 2.18). Using the fact that

$$\begin{aligned} &\mathbb{R}^n_+ \to \mathbb{R} \\ &(x_1, \ldots, x_n)^T \mapsto F\big(\sum_{i=1}^n x_i e_i\big) \end{aligned} \tag{4.1}$$

is an isomorphic self-scaled barrier functional for $\mathbb{R}^n_+$ endowed with the canonical Euclidean structure, Theorem 4.4 implies that there exist constants $c_1 > 0$ and $c_0 \in \mathbb{R}$ such that for all $x \in K^\circ$

$$\begin{aligned} F(x) &\overset{(4.1),4.4}{=} -c_1 \sum_{i=1}^n \ln\langle x; e_i^\sharp\rangle + c_0 \\ &= -c_1 \ln\Big(\prod_{i=1}^n \langle x; e_i^\sharp\rangle\Big) + c_0 \\ &= c_1 \ln\Big(\prod_{i=1}^n \int_0^\infty \exp\{-\langle x; t_i \cdot e_i^\sharp\rangle\} dt_i\Big) + c_0 \\ &= c_1 \ln\Big(\int_{K^\sharp} \exp\Big\{\Big\langle x; -\sum_{i=1}^n \langle e_i; s\rangle e_i^\sharp\Big\rangle\Big\} \cdot \big|e_1 \wedge \cdots \wedge e_n(ds)\big|\Big) + c_0 \\ &\overset{(2.1)}{=} c_1 \ln \varphi_K(x) + c_0. \end{aligned}$$

From this representation it is also clear that isotropy does not depend on the vector e chosen to endow E with a Euclidean structure. $\square$

Lemma 4.6. *Let* $K = \mathrm{S}^{n\times n}(\mathbb{R})_+ \subset E = \mathrm{S}^{n\times n}(\mathbb{R})$ *be provided with the standard barrier functional* F *from Example 2.6, and provide* E *with the standard Euclidean structure, i.e. with the Euclidean structure defined by* $e = I$. *Then the following is a group homomorphism:*

$$\mathcal{N} : \mathcal{O}(n) \to \mathcal{O}(\mathrm{S}^{n\times n}(\mathbb{R}))$$
$$\mathcal{N}(o) : h \mapsto oho^T \qquad \text{for all } h \in \mathrm{S}^{n\times n}(\mathbb{R}).$$

Moreover, $\mathcal{N}|_{\mathcal{SO}(n)} \mapsto \mathcal{Q}(\mathrm{S}^{n\times n}(\mathbb{R}))$ *is a surjective group homomorphism.*

Proof. First let us show that $\mathcal{N}$ is well defined. For all $o \in \mathcal{O}(n)$, $g, h \in \mathrm{S}^{n\times n}(\mathbb{R})$ and $x \in \mathrm{S}^{n\times n}(\mathbb{R})_+$,

$$\begin{aligned}
&\mathcal{N}(o)[g] \bullet \mathcal{N}(o)[h] = \mathrm{tr}(ogo^T oho^T) = g \bullet h,\\
&\mathcal{N}(o)[x] = oxo^T \in \mathrm{S}^{n\times n}(\mathbb{R})_+ \qquad \text{and}\\
&\mathcal{N}(o)(I) = oIo^T = I,
\end{aligned}$$

hence the image of $\mathcal{N}$ lies in the subgroup of $\mathcal{O}(K)$ that leaves I invariant. Next, we show that $\mathcal{N}$ is a group homomorphism: In fact,

$$\begin{aligned}
&\mathcal{N}(o_1 o_2) = \mathcal{N}(o_1) \circ \mathcal{N}(o_2) \qquad \forall o_1, o_2 \in \mathcal{SO}(n) \quad \text{and}\\
&\mathcal{N}(I) = \imath.
\end{aligned}$$

The image of $\mathcal{N}|_{\mathcal{SO}(n)}$ contains the identity $\imath = \mathcal{N}(I)$ and is connected by arcs, since the preimage $\mathcal{SO}(n)$ is connected by arcs (see e.g. [5]) and $\mathcal{N}$ is a continuous map. Hence it must be true that $\mathrm{im}\,\mathcal{N}|_{\mathcal{SO}(n)} \subset \mathcal{Q}(K)$. Surjectivity was shown in Faraut-Korányi [2] (see table on page 97). $\square$

Lemma 4.7. *Let* $K \subset E$ *and* $F \in \mathcal{C}^3(K^\circ, \mathbb{R})$ *be self-scaled, where* K *is irreducible and has Carathéodory number* $\kappa := \kappa(K)$. *Let us fix an element* $e \in K^\circ$ *and choose an orthogonal basis* $B \in \mathcal{B}^{(e)}$ *with frame* $B_K = \{e_1, \ldots, e_\kappa\}$ *centered at* e. *If* F *is isotropic for the Euclidean structure* $(\cdot;\cdot)_e$ *defined by* e *then its restriction* $F\big|_C$ *to the cone* $C := \mathrm{cone}(B_K)$ *generated by the frame is an isotropic self-scaled barrier functional for* $C \cong \mathbb{R}^\kappa_+$.

Proof. Let us first show that $C = K \cap \mathrm{span}(B_K)$. Clearly, $C \subset K \cap \mathrm{span}(B_K)$. On the other hand, self-duality of K under the Euclidean structure $(\cdot;\cdot)_e$ implies that

$$B_K^\sharp := \{e_1^\sharp, \ldots, e_n^\sharp\} \subset K^\sharp$$

and hence we have $\langle x; e_i^\sharp\rangle \geq 0$ for all $x \in K \cap \mathrm{span}(B_K)$ and $i \in \mathbb{N}_n$. Therefore,

$$x = \sum_{i=1}^n \langle x; e_i^\sharp\rangle \cdot e_i \in \mathrm{cone}(B_K),$$

and this shows that $K \cap \mathrm{span}(B_K) \subset C$. It follows thus that $F|_C$ is the restriction of the ν-self-concordant barrier functional F to a linear subspace, and by a well-known result of

Nesterov-Nemirovskii [7] such restrictions are ν-self-concordant barrier functionals themselves. Logarithmic homogeneity is trivially preserved.

Next, let us show that $F|_C$ satisfies conditions $i)$ and $ii)$ of Definition 2.4: Suppose x and w are two arbitrary elements of C°. Since $\operatorname{span}(B_K)$ inherits the Euclidean structure $(\cdot;\cdot)_e$ from E we can embed its dual $\operatorname{span}(B_K^\sharp)$ in $E^\sharp$:

$$\imath : \operatorname{span}(B_K^\sharp) \hookrightarrow E^\sharp.$$

Using this embedding we get

$$F''(w)|_{\operatorname{span}(B_K)} = \imath \circ F|''_C(w). \tag{4.2}$$

Combined with the fact that F is self-scaled for K this yields

$$F|''_C(w)x = F''(w)x \in \operatorname{span}(B_K^\sharp) \cap K^{\sharp\circ} = C^{\sharp\circ}, \tag{4.3}$$

where $C^{\sharp\circ}$ is the dual of C in the space $\operatorname{span}(B_K)$ endowed with the Euclidean structure $(\cdot;\cdot)_e$ it inherits from E (the last equality of equation (4.3) can be established via a repetition of the arguments that led to the equality $C = K \cap \operatorname{span}(B_K)$ above). Equation (4.3) shows that the functional $F|_C$ meets condition 2.4 $i)$. On the other hand, equation (4.2) implies that condition 2.4 $ii)$ merely describes the restriction to C° of an equation that holds for K°, by assumption of self-scaledness of F. Hence, this condition is met by $F|_C$ also.

It remains to show that if K is irreducible and F isotropic for $(\cdot;\cdot)_e$ then $F|_C$ is isotropic for the symmetric cone $C \cong \mathbb{R}^n_+$. Clearly, it suffices to show that for any permutation $\sigma \in \mathcal{S}_n$ of the basis elements in B_K there exists a transformation $\theta \in \mathcal{O}(K)$ such that $\theta|_C = \sigma : C \to C$. If the irreducible cone K is isomorphic to the ray $\mathbb{R}_+$ then nothing needs to be proven. Suppose K is isomorphic to the second-order cone $K := \{\left(\begin{smallmatrix}\tau\\ x\end{smallmatrix}\right) \in \mathbb{R}^{n+1} : \tau \geq \|x\|\}$. Without loss of generality we may assume that $B_K = \{e_1, e_2\}$, where e_1, e_2 are defined like in Example 2.11. If σ is the identity then nothing needs to be proven. If σ interchanges e_1 and e_2 then it is given as the restriction of the reflection of $\mathbb{R}^{n+1}$ in the hyperplane defined by the vectors whose second component is zero, and this reflection is clearly in $\mathcal{O}(K)$. Now assume K is isomorphic to $\mathrm{S}^{n\times n}(\mathbb{R})_+$. Without loss of generality we may assume that B_K is the frame described in Example 2.11. Then σ is simply the restriction to the set $\operatorname{Diag}(\mathbb{R}^n_{++})$ of the transformation $\mathcal{N}(\sigma) \in \mathcal{O}(K)$. The three remaining cases (see Theorem 2.19) can be treated by arguments similar to those we just gave for the positive semidefinite cone. $\square$

Theorem 4.8. *Let K be an irreducible symmetric cone and $F \in \mathcal{C}^3(K^\circ, \mathbb{R})$ a self-scaled barrier functional. Let $e \in K^\circ$ be given. Then F is isotropic for the Euclidean structure defined by e if and only if there exist constants $c_1 > 0$ and $c_0 \in \mathbb{R}$ such that*

$$F = c_1 \ln \varphi_K + c_0.$$

Moreover, the notion of isotropy is independent of the vector e chosen to endow E with a Euclidean structure.

Proof. Suppose K is irreducible and let us fix $e \in K^\circ$. Let $F \in \mathcal{C}^3(K^\circ, \mathbb{R})$ be ν-self-scaled and isotropic for $(\cdot;\cdot)_e$. Let us choose an orthogonal basis $B \in \mathcal{B}^{(e)}$ with frame B_K centered at e, and let us denote the cone generated by this frame by $C = \mathrm{cone}(B_K)$ as in Lemma 4.7. In Example 4.2 we saw that the universal barrier functional for K is the form $U = c_1 \ln \varphi_K + c_0$, and hence it is isotropic for $(\cdot;\cdot)_e$ too. It follows thus from Lemma 4.7 that the restrictions $F|_C$ and $U|_C$ are both isotropic ν-self-scaled barrier functionals for the cone C. Hence, by Theorem 4.5 both of these functionals are positive multiples of the universal barrier functional for C plus a constant, and therefore the same relationship must hold between $F|_C$ and $U|_C$ as well:

$$F|_C = \tilde{c}_1 U|_C + \tilde{c}_0$$

for some real numbers $\tilde{c}_1 > 0$ and $\tilde{c}_0$. Now let x be an arbitrary element of K°. Lemma 2.21 says that there exists $\theta \in \mathcal{O}(K)$ such that $\theta x \in C$, and hence isotropy of F and U implies that

$$\begin{aligned} F(x) = F(\theta x) &= F|_C(\theta x) \\ &= \tilde{c}_1 U|_C(\theta x) + \tilde{c}_0 \\ &= \tilde{c}_1 U(\theta x) + \tilde{c}_0 = \tilde{c}_1 U(x) + \tilde{c}_0. \end{aligned}$$

This shows that $F = \tilde{c}_1 U + \tilde{c}_0$, and since $U = c_1 \ln \varphi_K + c_0$ this proves the first claim. The rest now follows trivially from Example 4.2. $\square$

Lemma 4.9. *Let $K \subset E$ and $F \in \mathcal{C}^3(K^\circ, \mathbb{R})$ be self-scaled with irreducible decompositions $K = \bigoplus_J K^{(ij)}$ and $F = \bigoplus_J F^{(ij)}$ (see Theorems 2.19 and 3.3). Let us fix a vector $e = \bigoplus_J e^{(ij)} \in K^\circ$ and endow each irreducible part $K^{(ij)}$ of K with the Euclidean structure $(\cdot;\cdot)_{e^{(ij)}}$ defined by the Hessian of $F^{(ij)}$ at $e^{(ij)}$. Consider a particular value of the index $i \in \mathbb{N}_l$. Then it is possible to choose the set of isomorphisms $\{\imath_i^{(j_1 j_2)} : (j_1 j_2) \in \mathbb{N}^2_{k(i)}\}$ defined in Lemma 3.1 as isomorphisms of Euclidean spaces, i.e. such that*

$$\left(\imath_i^{(j_1 j_2)} z_1; \imath_i^{(j_1 j_2)} z_2\right)_{e^{(ij_2)}} = \left(z_1; z_2\right)_{e^{(ij_1)}} \qquad \forall \left(j_1 j_2\right) \in \mathbb{N}^2_{k(i)},\ z_1, z_2 \in E^{(ij_1)}.$$

Proof. The proof is by the same recursive construction as used in Lemma 3.1. The sole difference is that for each $(ij) \in \{i\} \times \mathbb{N}_{k(i)}$ we choose a basis $B^{(ij)} \in \mathcal{B}^{(e^{(ij)})}$ with frame $B_{K^{(ij)}}$ for $K^{(ij)}$ centered at $e^{(ij)}$, and in the recursion step we choose $\imath_i^{(1k)}$ such that the elements of $B^{(i1)}$ are mapped to the elements of $B^{(ik)}$ and the elements of $B_{K^{(i1)}}$ to those of $B_{K^{(ik)}}$. These maps certainly define isomorphisms between the Euclidean spaces $E^{(ij)}$ that satisfy the conditions of Lemma 3.1, and it follows from Lemma 2.21 that the $\imath_i^{(j_1 j_2)}$ so constructed are also cone isomorphisms between the irreducible parts $K^{(ij_1)}$ and $K^{(ij_2)}$ of K. $\square$

Lemma 4.10. *Let $K = \bigoplus_J K^{(ij)}$ and $F = \bigoplus_J F^{(ij)} \in \mathcal{C}^3(K^\circ, \mathbb{R})$ be self-scaled and suppose $e = \bigoplus_J e^{(ij)}$ is a fixed element in K°. Endow all the spaces $E^{(ij)}$ and E with* *the canonical Euclidean structures defined by* e*, and choose a set of isomorphisms* $\{\imath_i^{(j_1j_2)} : i \in \mathbb{N}_l, (j_1j_2) \in \mathbb{N}^2_{k(i)}\}$ *as in Lemma 4.9. Then, for each orthogonal cone automorphism* $\theta \in \mathcal{O}(K)$ *there exists a unique set of permutations* $\{\sigma_\theta^{(i)} \in \mathcal{S}_{k(i)} : i \in \mathbb{N}_l\}$ *and a unique set of orthogonal cone automorphisms* $\{\theta_\theta^{(ij)} \in \mathcal{O}(K^{(ij)}) : (ij) \in J\}$ *such that*

$$\theta\big|_{K^{(ij)}} = \imath_i^{j\sigma_\theta^{(i)}(j)} \circ \theta_\theta^{(ij)} \qquad \forall\, (ij) \in J.$$

Conversely, all mappings of this form are orthogonal automorphisms of K*.*

Proof. The proof is identical to that of Lemma 3.2, with the additional observation that the $\theta^{(ij)}$ are now orthogonal cone automorphisms, due to the special choice of the $\imath_i^{(j_1j_2)}$. $\square$

Lemma 4.11. *Let* $K = \bigoplus_J K^{(ij)}$ *and* $F = \bigoplus_J F^{(ij)} \in \mathcal{C}^3\big(K^\circ, \mathbb{R}\big)$ *be self-scaled and suppose* $e = \bigoplus_J e^{(ij)}$ *is a fixed element in* K°*. Endow all the spaces* $E^{(ij)}$ *and* E *with the canonical Euclidean structures defined by* e*. If* F *is isotropic then all the* $F^{(ij)}$ *are isotropic.*

Proof. Let $(ij) \in J$ and consider an arbitrary $\vartheta \in \mathcal{O}(K^{(ij)})$. Lemma 4.10 says that

$$\vartheta \oplus \Big(\bigoplus_{J\setminus(ij)} \imath\Big) \in \mathcal{O}(K),$$

whence F is invariant under the action of this operator. But this implies immediately that $F^{(ij)}$ is invariant under the action of ϑ. $\square$

We are finally ready to state the main theorem of this section:

Theorem 4.12 (Classification of Isotropic Self-Scaled Barriers).
Let $K = \bigoplus_J K^{(ij)}$ *and* $F = \bigoplus_J F^{(ij)} \in \mathcal{C}^3\big(K^\circ, \mathbb{R}\big)$ *be self-scaled and suppose* $e = \bigoplus_J e^{(ij)}$ *is a fixed element in* K°*. Endow all the spaces* $E^{(ij)}$ *and* E *with the canonical Euclidean structures defined by* e*. For each* $(ij) \in J$ *let* $\varphi_{K^{(ij)}}$ *be the characteristic function of* $K^{(ij)}$ *defined by the Euclidean structure we chose on* $E^{(ij)}$*. Then* F *is isotropic for the Euclidean structure defined by* e *if and only if there exist real numbers* $c_1^{(i)} > 0$*,* $c_0^{(ij)} \in \mathbb{R}$ $\big(1 \le i \le l; (ij) \in J\big)$ *such that*

$$F = \bigoplus_J \big(c_1^{(i)} \ln \varphi_{K^{(ij)}} + c_0^{(ij)}\big). \tag{4.4}$$

Moreover, the notion is isotropy is independent of the point $e \in K^\circ$ *chosen to endow* E *with a Euclidean structure.*

Proof. Let us first suppose that F is isotropic and show that the constants $c^{(i)}$ and $c^{(ij)}$ exist. By virtue of Theorems 3.3 and 4.11 there exist isotropic self-scaled barrier functionals $F^{(ij)}$ for the irreducible parts $K^{(ij)}$ of K such that F decomposes into the direct sum

$F = \bigoplus_J F^{(ij)}$. But the symmetric cones $K^{(ij)}$ are irreducible, hence Theorem 4.8 implies that there exist constants $c_1^{(ij)} > 0$ and $c_0^{(ij)} \in \mathbb{R}$ such that $F^{(ij)} = c_1^{(ij)} \ln \varphi_{K^{(ij)}} + c_0^{(ij)}$. Let us now show that $c_1^{(ij)}$ depends only on the index i. Suppose $i \in \mathbb{N}_l$ and $m, n \in \mathbb{N}_{k(i)}$. Lemma 4.10 implies that

$$\theta := \imath_i^{(mn)} \oplus \imath_i^{(nm)} \oplus \Big(\bigoplus_{J \setminus \{(im),(in)\}} \imath \Big) \in \mathcal{O}(K)$$

lies in the orthogonal group of K. Since F is isotropic this implies that for any $x = x^{(im)} \oplus x^{(in)} \oplus z \in K^\circ$ we have

$$\begin{aligned} 0 &= F\big(\theta(x)\big) - F(x) \\ &= F^{(in)}\big(\imath_i^{(mn)} x^{(im)}\big) - F^{(im)}\big(x^{(im)}\big) \\ &\quad + F^{(im)}\big(\imath_i^{(nm)} x^{(in)}\big) - F^{(in)}\big(x^{(in)}\big) \\ &= \big(F^{(in)} \circ \imath_i^{(mn)} - F^{(im)}\big)\big(x^{(im)}\big) \\ &\quad - \big(F^{(in)} \circ \imath_i^{(mn)} - F^{(im)}\big)\big(\imath_i^{(nm)} x^{(in)}\big), \end{aligned}$$

and since the vectors $x^{(im)}$ and $x^{(in)}$ were arbitrary it must therefore be true that

$$F^{(in)} \circ \imath_i^{(mn)} + cst = F^{(im)}. \tag{4.5}$$

Note also that

$$\begin{aligned} \varphi_{K^{(in)}} \circ \imath_i^{(mn)}\big(x^{(im)}\big) &= \int_{K^{(in)\sharp}} \exp\big\{-\big\langle \imath_i^{(mn)} x^{(im)}; s\big\rangle\big\} ds \\ &= \int_{K^{(in)\sharp}} \exp\big\{-\big\langle x^{(im)}; \imath_i^{(mn)*} s\big\rangle\big\} ds \\ &= \int_{K^{(im)\sharp}} \exp\big\{-\big\langle x^{(im)}; s\big\rangle\big\} \cdot \big|\det\big(\imath_i^{(mn)*}\big)^{-1}\big| ds \\ &= \varphi_{K^{(im)}}\big(x^{(im)}\big). \end{aligned} \tag{4.6}$$

Combining all the information we gathered it follows that

$$\begin{aligned} c_1^{(im)} \ln \varphi_{K^{(im)}} + c_0^{(im)} &= F^{(im)} \\ &\overset{(4.5)}{\equiv} F^{(in)} \circ \imath_i^{(mn)} + cst \\ &= c_1^{(in)} \ln\big(\varphi_{K^{(in)}} \circ \imath_i^{(mn)}\big) + cst \\ &\overset{(4.6)}{\equiv} c_1^{(in)} \ln \varphi_{K^{(im)}} + cst. \end{aligned}$$

Therefore, $c_1^{(in)} = c_1^{(im)}$ as claimed. This shows the "only if" part of the theorem.

Let us now assume that F is a self-scaled barrier functional of the described form, and let $\theta \in \mathcal{O}(K)$ be an orthogonal automorphism of K. By virtue of Lemma 4.10, θ decomposes into the direct sum

$$\theta = \bigoplus_J \imath_i^{j\sigma_\theta^{(i)}(j)} \circ \theta_\theta^{(ij)} = \Big(\bigoplus_J \imath_i^{j\sigma_\theta^{(i)}(j)}\Big) \circ \Big(\bigoplus_J \theta_\theta^{(ij)}\Big),$$

where $\theta_\theta^{(ij)} \in \mathcal{O}(K^{(ij)})$ for all $(ij) \in J$. Now, Theorem 4.8 says that the barrier functionals $c_1^{(i)} \ln \varphi_{K^{(ij)}} + c_0^{(ij)}$ are isotropic for all $(ij) \in J$, whence we have

$$F \circ \big(\bigoplus_J \theta_\theta^{(ij)}\big) = F.$$

Hence, without loss of generality we may assume that θ is simply the permutation $\theta = \bigoplus_J \imath_i^{j\sigma_\theta^{(i)}(j)}$. But then it is true that

$$\begin{aligned} F \circ \theta \quad &= \quad \bigoplus_J c_1^{(i)} \ln\big(\varphi_{K^{(i\sigma_\theta^{(i)}(j))}} \circ \imath_i^{j\sigma_\theta^{(i)}(j)}\big) + \sum_J c_0^{(i\sigma_\theta^{(i)}(j))} \\ &\overset{(4.6)}{=} \bigoplus_J c_1^{(i)} \ln \varphi_{K^{(ij)}} + \sum_J c_0^{(ij)} \\ &= F. \end{aligned}$$

This shows that F is isotropic, as claimed.

Finally, the last statement is now trivial, because changing the Euclidean structure affects only the constants $c_0^{(ij)}$. ☐

Corollary 4.13. *Let K be a symmetric cone and $F \in \mathcal{C}^3\big(K^\circ, \mathbb{R}\big)$ an isotropic self-scaled barrier functional for K. Then F is analytic.*

Proof. This follows immediately from the the fact that F can be represented in the form (4.4). ☐

Definition 4.14. A self-scaled barrier functional $F \in \mathcal{C}^3\big(K^\circ, \mathbb{R}\big)$ for a symmetric cone K is *strongly self-scaled* if for all cone automorphims $\theta \in \mathrm{Aut}(K)$ there exists a constant $c_\theta \in \mathbb{R}$ such that

$$F \circ \theta \equiv F + c_\theta.$$

Definition 4.14 captures in some sense what motivates the nomenclature "self-scaled" to begin with. This is the simplest scaling property we can assume. It turns out that strongly self-scaled barrier functionals are the same objects as isotropic self-scaled barrier functionals:

Theorem 4.15 (Strong self-scaledness).
Let $K \subset E$ and $F \in \mathcal{C}^3\big(K^\circ, \mathbb{R}\big)$ be self-scaled. Then F is strongly self-scaled if and only if it is isotropic.

Proof. Let us start with the "if" part and assume that F is isotropic. Theorem 4.12 implies that F is of the form

$$F = \bigoplus_J \big(c_1^{(i)} \ln \varphi_{K^{(ij)}} + c_0^{(ij)}\big).$$

Let $\theta \in \mathrm{Aut}(K)$ be an automorphism of K. Lemma 3.2 says that θ is of the form

$$\theta = \bigoplus_J \imath_i^{j\sigma_\theta^{(i)}(j)} \circ \theta_\theta^{(ij)} = \big(\bigoplus_J \imath_i^{j\sigma_\theta^{(i)}(j)}\big) \circ \big(\bigoplus_J \theta_\theta^{(ij)}\big).$$

Moreover, we may assume that a vector $e \in K^\circ$ has been chosen, that all the spaces $E^{(ij)}$ and E have been endowed with the canonical Euclidean structures defined by e, that these Euclidean structures have been used in the definition of the characteristic functions $\varphi_{K^{(ij)}}$ and that the set of isomorphisms $\big\{\imath_i^{(j_1 j_2)} : i \in \mathbb{N}_l, (j_1 j_2) \in \mathbb{N}^2_{k(i)}\big\}$ have been chosen the same way as in Lemma 4.9. Note that for all $x^{(ij)}$ in $K^{(ij)\circ}$,

$$\begin{aligned}
\varphi_{K^{(ij)}} \circ \theta_\theta^{(ij)}\big(x^{(ij)}\big) &= \int_{K^{(ij)\sharp}} \exp\big\{-\big\langle \theta_\theta^{(ij)} x^{(ij)}; s\big\rangle\big\} ds \\
&= \int_{K^{(ij)\sharp}} \exp\big\{-\big\langle x^{(ij)}; \theta_\theta^{(ij)*} s\big\rangle\big\} ds \\
&= \int_{K^{(ij)\sharp}} \exp\big\{-\big\langle x^{(ij)}; s\big\rangle\big\} \cdot \big|\det\big(\theta_\theta^{(ij)*}\big)^{-1}\big| ds \\
&= \Big|\det\big(\theta_\theta^{(ij)*}\big)^{-1}\Big| \cdot \varphi_{K^{(ij)}}\big(x^{(ij)}\big).
\end{aligned} \tag{4.7}$$

Therefore we have

$$\begin{aligned}
F \circ \theta &= \bigoplus_J c^{(i)} \ln\big(\varphi_{K^{(i\sigma_\theta^{(i)}(j))}} \circ \imath_i^{j\sigma_\theta^{(i)}(j)} \circ \theta_\theta^{(ij)}\big) + cst \\
&\overset{(4.6)}{=} \bigoplus_J c^{(i)} \ln\big(\varphi_{K^{(ij)}} \circ \theta_\theta^{(ij)}\big) + cst \\
&\overset{(4.7)}{=} \bigoplus_J c^{(i)} \ln \varphi_{K^{(ij)}} + cst \\
&= F + cst,
\end{aligned}$$

which shows that F is strongly self-scaled.

Conversely, suppose F is strongly self-scaled and let $\theta \in \mathcal{O}(K)$ be an orthogonal automorphism of K. We claim that $c_\theta = 0$. Suppose to the contrary that $c_\theta \neq 0$. Then we may assume without loss of generality that $c_\theta > 0$ since $c_{\theta^{-1}} = -c_\theta$. For a fixed $x \in K^\circ$, the distance to the boundary of K is constant for all the points points $\theta^n x$ on the orbit $\big\{\theta^n x : n \in \mathbb{Z}\big\}$ under the action of the group generated by θ, since all the transformations θ^n lie in the orthogonal group of K. Hence, the orbit of x is bounded away from the boundary, and yet

$$\lim_{n\to\infty} F\big(\theta^n x\big) = F(x) + \lim_{n\to\infty} n \cdot c_\theta = +\infty.$$

It is well known from the theory of self-concordant barrier functionals (see Nesterov-Nemirovskii [7]) that this is impossible. Hence $c_\theta = 0$ and F is invariant under the action of θ. $\square$

Theorem 4.16 (Polar Decomposition).
Let $K \subset E$ and $F \in \mathcal{C}^3(K^\circ, \mathbb{R})$ be self-scaled and suppose F is isotropic. Let us fix $e \in K^\circ$ and endow E with the Euclidean structure defined by this vector. Then each automorphism $\theta \in \mathrm{Aut}(K)$ of K has a unique polar decomposition

$$\theta = \Omega \circ F''^{-1}(e) \circ F''(x)$$

with $\Omega \in \mathcal{O}(K)$ and $x \in K^\circ$.

Proof. Let us adopt all the notation from Lemmas 4.9 and 4.10 and write θ in its irreducible decomposition

$$\theta = \bigoplus_J \imath_i^{j\sigma_\theta^{(i)}(j)} \circ \theta_\theta^{(ij)} = \Big(\bigoplus_J \imath_i^{j\sigma_\theta^{(i)}(j)}\Big) \circ \Big(\bigoplus_J \theta_\theta^{(ij)}\Big). \tag{4.8}$$

By virtue of Theorem 4.12, F can be decomposed into a direct sum of the form

$$F = \bigoplus_J \big(c_1^{(i)} \ln \varphi_{K^{(ij)}} + c_0^{(ij)}\big).$$

Now, Theorem 2.22 implies that there exist unique vectors $x^{(ij)} \in K^{(ij)\circ}$ and orthogonal cone automorphisms $\Omega^{(ij)} \in \mathcal{O}(K^{(ij)})$ such that

$$\theta_\theta^{(ij)} = \Omega^{(ij)} \circ F^{(ij)\prime\prime-1}\big(e^{(ij)}\big) \circ F^{(ij)\prime\prime}\big(x^{(ij)}\big) \tag{4.9}$$

for all $(ij) \in J$, and Lemma 4.10 says that the following operator is an element of the orthogonal group of K:

$$\Omega := \Big(\bigoplus_J \imath_i^{j\sigma_\theta^{(i)}(j)}\Big) \circ \Big(\bigoplus_J \Omega^{(ij)}\Big). \tag{4.10}$$

Hence we have

$$\begin{aligned}
\theta &\overset{(4.8)}{=} \Big(\bigoplus_J \imath_i^{j\sigma_\theta^{(i)}(j)}\Big) \circ \Big(\bigoplus_J \theta_\theta^{(ij)}\Big) \\
&\overset{(4.9)}{=} \Big(\bigoplus_J \imath_i^{j\sigma_\theta^{(i)}(j)}\Big) \circ \Big(\bigoplus_J \Omega^{(ij)}\Big) \circ \Big(\bigoplus_J F^{(ij)\prime\prime-1}(e^{(ij)})\Big) \circ \Big(\bigoplus_J F^{(ij)\prime\prime}(x^{(ij)})\Big) \\
&\overset{(4.10)}{=} \Omega \circ F''^{-1}(e) \circ F''\Big(\bigoplus_J x^{(ij)}\Big).
\end{aligned}$$

This settles the claim of existence of a polar decomposition of θ.

Let us now assume that $\theta = \Delta \circ F''^{-1}(e) \circ F''(y)$ is another polar decomposition of θ, where $\Delta \in \mathcal{O}(K)$ and $y \in K^\circ$ have irreducible decomposition

$$\begin{aligned}
\Delta &= \Big(\bigoplus_J \imath_i^{j\sigma_\Delta^{(i)}(j)}\Big) \circ \Big(\bigoplus_J \theta_\Delta^{(ij)}\Big) \qquad \text{(see Lemma 4.10), and} \\
y &= \bigoplus_J y^{(ij)}.
\end{aligned}$$

This implies that θ has the irreducible decomposition

$$\theta = \Big(\bigoplus_J \imath_i^{j\sigma_\Delta^{(i)}(j)}\Big) \circ \Big(\bigoplus_J \theta_\Delta^{(ij)} \circ F^{(ij)''-1}(e^{(ij)}) \circ F^{(ij)''}(y^{(ij)})\Big),$$

and it follows from the uniqueness part of Lemma 3.2 that

$$\begin{aligned}
\theta_\theta^{(ij)} &= \theta_\Delta^{(ij)} \circ F^{(ij)''-1}(e^{(ij)}) \circ F^{(ij)''}(y^{(ij)}) \qquad \text{and} \\
\imath_i^{j\sigma_\Delta^{(i)}(j)} &= \imath_i^{j\sigma_\theta^{(i)}(j)}
\end{aligned}$$

for all $(ij) \in J$. But then the uniqueness part of Theorem 2.22 implies that

$$\begin{aligned}
y^{(ij)} &= x^{(ij)} \qquad \text{and} \\
\theta_\Delta^{(ij)} &= \Omega^{(ij)}
\end{aligned}$$

for all $(ij) \in J$. It follows that the two polar decompositions of θ are identical. $\square$

Remark 4.17. Standard results on barrier functionals of the form $c_1 \ln \varphi_K + c_0$ combined with Theorem 4.12 imply that the smallest achievable value of the complexity parameter ν of a self-scaled isotropic barrier functional for a symmetric cone K equals the Carathéodory number $\kappa(K)$ of this cone. A stronger version of this result was already obtained by Güler and Tunçel [4] who used a proof technique that involves a bound on ν given by Nesterov-Nemirovskii [7] rather than a representation theory for barrier functionals.

We are still left with the problem of classifying non-isotropic self-scaled barrier functionals. To conclude this section we present a conjecture and a lemma that might be an important step toward its proof. According to the conjecture, all self-scaled barrier functionals on irreducible symmetric cones are isotropic, and hence all self-scaled barrier functionals on general symmetric cones are of the form

$$F = \bigoplus_J c_1^{(ij)} \ln \varphi_{K^{(ij)}} + c_0^{(ij)}.$$

Conjecture 4.18. *All self-scaled barrier functionals on irreducible symmetric cones are isotropic.*

Lemma 4.19. *Let $K \subset E$ and $F \in \mathcal{C}^3(K^\circ, \mathbb{R})$ be self-scaled, and let us endow E with the Euclidean structure defined by a fixed vector $e \in K^\circ$. Then the following is a mapping with image in the orthogonal group of K:*

$$
\begin{aligned}
(4.11)\qquad & K^\circ \times K^{\sharp\circ} \to \mathcal{O}(K), \\
& (x,s) \mapsto \theta_{(x,s)} := F''_\sharp\big(t(e, -F'(x))\big) \circ F''\big(w(x,s)\big) \circ F''_\sharp\big(t(e,s)\big) \circ F''(e),
\end{aligned}
$$

where w and t denote primal and dual scaling points (see Theorem 2.8). Moreover, F is invariant under the action of the images of this map.

Proof. Clearly, $\theta_{(x,s)} \in \operatorname{Aut} K$ for all x, s. Let us now show that these are orthogonal operators: First consider $\theta = F''_\sharp(v) \circ F''(u)$ for any $u \in K^\circ$, $v \in K^{\sharp\circ}$. Then for any $z \in K^\circ$ we have from Theorem 2.8 *iii)* that

$$
F''_\sharp\big(F''(u)z\big) = F''^{-1}(u) \circ F''(z) \circ F''^{-1}(u),
$$

so

$$
\begin{aligned}
F''(\theta z) &= F''^{-1}_\sharp(v) \circ F''_\sharp\big(F''(u)z\big) \circ F''^{-1}_\sharp(v) \\
&= F''^{-1}_\sharp(v) \circ F''^{-1}(u) \circ F''(z) \circ F''^{-1}(u) \circ F''^{-1}_\sharp(v) \\
&= \theta^{-*} \circ F''(z) \circ \theta^{-1}.
\end{aligned}
$$

Applying this result twice gives

$$
F''\big(\theta_{(x,s)}z\big) = \theta^{-*}_{(x,s)} \circ F''(z) \circ \theta^{-1}_{(x,s)}. \tag{4.12}
$$

Now let $z = e$, and note that $\theta_{(x,s)}e = e$ by applying Theorem 2.8 *i)* and *iii)* to (4.11). Then $F''(e) = \theta^{-*}_{(x,s)} \circ F''(e) \circ \theta^{-1}_{(x,s)}$, or

$$
\theta^*_{(x,s)} \circ F''(e) \circ \theta_{(x,s)} = F''(e),
$$

which is the same as showing that $\theta_{(x,s)}$ is orthogonal, since for any $v, w \in E$,

$$
\begin{aligned}
\big(\theta v; \theta w\big)_e &= \big\langle \theta v; F''(e) \circ \theta w \big\rangle \\
&= \big\langle v; \theta^* \circ F''(e) \circ \theta w \big\rangle \\
&= \big\langle v; F''(e) w \big\rangle \\
&= \big(v; w\big)_e.
\end{aligned}
$$

In order to show that F is invariant under the action of the $\theta_{(x,s)}$, let us note that Proposition 2.3 *i)* combined with the fact that $\theta_{(x,s)}$ is orthogonal implies

$$
\begin{aligned}
F'\big(\theta_{(x,s)}z\big) &= -F''\big(\theta_{(x,s)}z\big) \circ \theta_{(x,s)}z \\
&\overset{(4.12)}{=} \theta^{-*}_{(x,s)} F'(z) \\
&= F''(e) \circ \theta_{(x,s)} \circ F''^{-1}(e) F'(z).
\end{aligned} \tag{4.13}
$$

Now, recall that $\theta_{(x,s)}e = e$ and let z be an arbitrary point in K°. Then

$$\begin{aligned}
F\big(\theta_{(x,s)}z\big) - F\big(e\big) &= F\big(\theta_{(x,s)}z\big) - F\big(\theta_{(x,s)}e\big) \\
&= \int_0^1 \Big\langle \theta_{(x,s)}[z-e]; F'\big(\theta_{(x,s)}[\tau \cdot z + (1-\tau)\cdot e]\big)\Big\rangle d\tau \\
&\overset{(4.13)}{=} \int_0^1 \Big\langle \theta_{(x,s)}[z-e]; F''(e) \circ \theta_{(x,s)} \circ F''^{-1}(e) F'\big(\tau \cdot z + (1-\tau)\cdot e\big)\Big\rangle d\tau \\
&= \int_0^1 \Big\langle F''^{-1}(e) \circ \theta^*_{(x,s)} \circ F''(e) \circ \theta_{(x,s)}[z-e]; F'\big(\tau \cdot z + (1-\tau)\cdot e\big)\Big\rangle d\tau \\
&= \int_0^1 \Big\langle z-e; F'\big(\tau \cdot z + (1-\tau)\cdot e\big)\Big\rangle d\tau \\
&= F(z) - F(e).
\end{aligned}$$

Comparing both sides of this equation we get that F is invariant under the action of $\theta_{(x,s)}$ as claimed. □

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