Primal-Dual Symmetric Scale-Invariant Square-Root Fields for Isotropic Self-Scaled Barrier Functionals

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Abstract

Square-root fields are differentiable operator fields used in the construction of target direction fields for self-scaled conic programming, a unifying framework for primal-dual interior-point methods for linear programming, semidefinite programming and second-order cone programming. In this article we investigate square-root fields for so-called *isotropic* self-scaled barrier functionals, i.e. self-scaled barrier functionals that are invariant under rotations of their conic domain of definition. We prove a structure theorem for so-called *congruent* square-root fields for isotropic self-scaled barrier functionals in terms of the irreducible decomposition of their domain of definition. Using this structure theorem, we then investigate primal-dual symmetry and scale-invariance of such square-root fields. In our main theorem we show that these two assumptions together with one additional natural invariance property (so-called *canonical reduction*) dramatically reduce the degree of freedom in the choice of square-root fields that satisfy these properties, but that such square-root fields always exist.

Key words. symmetric cones, self-scaled cones, primal-dual symmetry, scale-invariance, interior-point methods, weighted analytic centers, target-following methods, convex programming, self-scaled programming, linear programming, semidefinite programming, quadratic programming, second-order cone programming.

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1 Introduction

Self-scaled conic programming provides a unified framework for linear- semidefinite- and second-order-cone programming and was first investigated by Nesterov-Todd [20, 21]. In [5] we began the generalization of the so-called "V-space" approach, a theory that allows the unified analysis of various important classes of primal-dual interior-point methods for linear programming (see e.g. Kojima-Mizuno-Yoshise [11], Monteiro-Adler [17], Mizuno [13], Jansen-Roos-Terlaky-Vial [8, 9], Todd [27]). Primal-dual interior point methods for convex optimization problems are designed to solve a problem and its dual jointly by making use of convex duality theory. The paradigm for such algorithms usually is to reduce the duality gap between primal and dual approximate solutions to zero. In the V-space approach, both the primal and the dual problem are mapped into some Euclidean space V lying "half-way" between the spaces E and $E^\sharp$ in which the primal and dual problems live. In [5] we introduced a class of differentiable operator fields that achieve a useful transformation into such a V-space. We called these operator fields *square-root fields.* These objects are closely related to similar transformations defined by Tunçel for self-scaled conic programming in [29] and for general convex conic programming in [30]. In [6] we used square-root fields to define *target-maps* and *target directions*, a new infinite family of Newton directions that lend themselves to the construction of so-called *target-following* algorithms and *predictor-corrector* methods. These search directions are relatives of Nesterov-Todd directions (see Nesterov-Todd [20, 21] and Todd-Toh-Tütüncü [28]).
We also showed that a certain class of target-maps generalize the notion of *weighted analytic centers* to the framework of self-scaled conic programming. Weighted centers were introduced to the linear programming literature by Kojima, Mizuno and Yoshise [11] and have been generalized to the framework of semidefinite programming by Monteiro-Pang [18], Sturm-Zhang [25] and Monteiro-Zanjacomo [16].

Let E be a n-dimensional real vector space and $E^\sharp$ its dual. Throughout this article we will reserve super- or subscript-$\sharp$ for duals, superscript-$*$ for adjoints and superscript-$\circ$ for topological interiors. Let $K \subset E$ be a regular cone in E, i.e. convex, pointed and with non-empty interior $K^\circ \neq \emptyset$. As usual, we define the dual of K as follows:

$$K^\sharp := \{ s \in E^\sharp : \langle x; s \rangle \geq 0 \quad \forall\, x \in K \}.$$

By $\langle \cdot ; \cdot \rangle$ let us denote the canonical bilinear product defined on $E \times E^\sharp$. Let $(Y, Y^\sharp, \langle \cdot ; \cdot \rangle)$ be another bilinear system of m-dimensional real vector spaces, and let $A : E \to Y^\sharp$ be a surjective vector space homomorphism. Let $b \in Y^\sharp$ and $c \in E^\sharp$ be fixed vectors. Let us make the usual regularity assumption that there *exist* vectors $x_0 \in K^\circ$ and $(y_0, s_0) \in Y \times K^{\sharp\circ}$ such that $Ax_0 = b$ and $A^* y_0 + s_0 = c$ respectively. The general primal-dual pair of self-scaled conic programming problems in primal-dual symmetric form can be defined as follows:

$$\text{(1.1)} \qquad \begin{array}{ll} \text{(P')} & \inf \langle x; c \rangle \\ & Ax = b \\ & x \in K \end{array} \qquad\qquad \begin{array}{ll} \text{(D')} & \sup \langle y; b \rangle \\ & A^* y + s = c \\ & s \in K^\sharp, \end{array}$$

where $K \subset E$ is a symmetric cone (see e.g. [10, 2, 23, 31]). Let L be the linear subspace of E defined by $L := \ker(A)$ and $L^\perp := \operatorname{im}(A^*)$ its orthogonal complement with respect to the bilinear form $\langle \cdot ; \cdot \rangle$. Then the problem pair (1.1) can be reformulated equivalently in primal-dual symmetric form as follows:

$$
\text{(1.2)} \qquad \text{(P)} \quad \begin{array}{l} \inf \langle x; s_0 \rangle \\ x \in \big(L + x_0\big) \cap K \end{array} \qquad\qquad \text{(D)} \quad \begin{array}{l} \inf \langle x_0; s \rangle \\ s \in \big(L^\perp + s_0\big) \cap K^\sharp. \end{array}
$$

Under the above-mentioned regularity assumption of primal-dual strict feasibility, self-scaled conic programming contains as special cases linear programming (by choosing $K = \mathbb{R}^n_+$), semidefinite programming (by choosing $K = \mathrm{S}^{n\times n}(\mathbb{R})_+$, the cone of positive semidefinite symmetric $n \times n$ matrices) and second-order cone programming (by choosing K equal to the $n+1$-dimensional Lorentz cone).

Let the symmetric cone K be endowed with a *self-scaled barrier functional* F (see [20]). As usual, we endow $K^\sharp$ with the *dual barrier functional*

$$
\begin{aligned}
F_\sharp : K^{\sharp\circ} &\to \mathbb{R} \\
s &\mapsto \max\{-\langle x; s \rangle - F(x) : x \in K^\circ\},
\end{aligned}
$$

which is also self-scaled (see [20]). The reader may want to think of the linear programming case, where the standard logarithmic barrier functional

$$
F(x) : x \mapsto -\sum \ln(x_i)
$$

and its dual barrier functional

$$
F_\sharp(s) : s \mapsto -\sum \ln(s_i) - n
$$

are self-scaled. Let us consider the primal-dual pair of self-scaled conic programming problems 1.2. The fact that F is self-scaled implies among other things that at any point $w \in K^\circ$ the mapping $F''(w) : E \to E^\sharp$ is an isomorphism between K and $K^\sharp$, and that for any pair of points $(x, s) \in K^\circ \times K^{\sharp\circ}$ there exists a unique *scaling point* $w(x, s) \in K^\circ$ such that $F''(w)x = s$.

In [5] we defined *square-root fields* as follows: Let V be a n--dimensional Euclidean space and let $e_V \in V$ be a fixed vector of length $\|e_V\|_2 = \nu^{1/2}$, where ν is the *complexity parameter* of F (see [19]). A square-root field for (F, e_V) is a pair (X, S) of differentiable operator fields

$$
\begin{aligned}
X :\ K^\circ \times K^{\sharp\circ} &\to \operatorname{Aut}(E^\sharp, V) \qquad \text{and} \\
S :\ K^\circ \times K^{\sharp\circ} &\to \operatorname{Aut}(E, V)
\end{aligned}
$$

such that for each pair $(x, s) \in K^\circ \times K^{\sharp\circ}$ the following diagrams become commutative:

$$
\begin{tikzcd}
E^\sharp \arrow[r, "F''_\sharp(s)"] & E \arrow[r, "F''(w(x,s))"] \arrow[d, "S(x,s)"] & E^\sharp \arrow[d, "X(x,s)"] & E \arrow[l, "F''(x)"'] \\
V \arrow[u, "S^*(x,s)"] \arrow[r, equal] & V \arrow[r, equal] & V \arrow[r, equal] & V \arrow[u, "X^*(x,s)"']
\end{tikzcd}
$$

$$
\begin{array}{ccc}
-F_\sharp'(s) & \xrightarrow{F''(w(x,s))} & -F'(x) \\
\Big\downarrow{\scriptstyle S(x,s)} & & \Big\downarrow{\scriptstyle X(x,s)} \\
e_V & = & e_V
\end{array}
$$

In [4] we defined a self-scaled barrier functional F to be *isotropic* if it is invariant under orthogonal automorphisms of its conic domain of definition. The reader may imagine these transformations as rotations that map the cone to itself. This is a natural family of barrier functionals to look at, since orthogonal transformations of the cone change neither geometric nor metric aspects of the optimization problem. In [4] we gave a complete classification and representation theory for isotropic self-scaled barrier functionals in terms of an irreducible decomposition linked to the decomposition theory for symmetric cones (see [10, 2, 23, 31]).

In the present article we will use this decomposition theory to give a structure theorem for square-root fields for isotropic self-scaled barrier functionals (see Theorem 3.4). In Section 4 we aim at localizing the square-root fields that have the nicest possible invariance properties, motivated by the paradigm of constructing algorithms that

1. do not change their behaviour if the roles of (P) and (D) are interchanged (*primal-dual symmetry*), see Figure 1,
2. do not change their behaviour if a different coordinate system is used to represent the optimization problem (*scale-invariance*), see Figure 2, and
3. whose search directions coincide with the canonical search direction defined when the algorithm is applied to a linear programming problem embedded in a self-scaled conic programming problem expressed in more complicated form (*canonical reduction*), see Figure 3.

This leads us to define corresponding invariance properties for square-root fields for isotropic self-scaled barrier functionals. Theorem 4.13, the main result of our article, shows that for any isotropic self-scaled barrier functional there always exists a primal-dual symmetric scale-invariant square-root field that reduces canonically on linear-programming sub-instances. It seems to us that the target directions corresponding to such square-root fields should be particularly promising in algorithmic applications, due to their strong geometric invariance properties.

Let us conclude this section with some remarks regarding notation. The set of *primal-dual strictly feasible solutions* and the set of *primal-dual feasible solutions* respectively are defined as follows:

$$
\begin{aligned}
\mathcal{F}^\circ(PD) &:= \Big(\big(L + x_0\big) \cap K^\circ\Big) \times \Big(\big(L^\perp + s_0\big) \cap K^{\sharp\circ}\Big) \qquad \text{and} \\
\mathcal{F}(PD) &:= \Big(\big(L + x_0\big) \cap K\Big) \times \Big(\big(L^\perp + s_0\big) \cap K^{\sharp}\Big).
\end{aligned}
$$

On the same note we define the following sets whose nomenclature is clear by analogy to what was just said:

$$\begin{aligned}
\mathcal{F}(P) &:= \big(L + x_0\big) \cap K && \textit{the primal feasible domain} \\
\mathcal{F}^\circ(P) &:= \big(L + x_0\big) \cap K^\circ && \textit{the primal strictly feasible domain} \\
\mathcal{F}(D) &:= \big(L^\perp + s_0\big) \cap K^\sharp && \textit{the dual feasible domain} \\
\mathcal{F}^\circ(D) &:= \big(L^\perp + s_0\big) \cap K^{\sharp\circ} && \textit{the dual strictly feasible domain}
\end{aligned}$$

The group of orthogonal transformations of the Euclidean space V will play an important role, and we denote it by $\mathcal{O}(V)$. The subgroup whose action leaves the element $e_V \in V$ invariant will be denoted by $\mathcal{O}(V, e_V)$. Also, we will consider a regular cone $K_V \subset V$ and denote the subgroup of $\mathcal{O}(V)$ whose elements map K_V onto itself by $\mathcal{O}(K_V)$, and the subgroup of $\mathcal{O}(K_V)$ whose action leaves e_V invariant by $\mathcal{O}(K_V, e_V)$. Often the need arises to distinguish between linear operators and their matrix representation with respect to a particular choice of coordinates. In particular, we will write $\nabla^2 F(x)$ for the matrix representation of $F''(x)$. See also Remark 2.2. The sets of symmetric matrices, symmetric positive semidefinite and symmetric positive definite matrices over $\mathbb{R}$ will be denoted by $\mathrm{S}^{n\times n}(\mathbb{R})$, $\mathrm{S}^{n\times n}(\mathbb{R})_+$ and $\mathrm{S}^{n\times n}(\mathbb{R})_{++}$ respectively. For the set of diagonal $n \times n$-matrices over $\mathbb{R}$ we will use the notation $\mathrm{Diag}_n(\mathbb{R})$. By $\mathrm{diag}(D)$ we denote the element of $\mathbb{R}^n$ defined by the diagonal of matrix D, and conversely we write $\mathrm{Diag}(v)$ for the diagonal $n \times n$-matrix defined by $v \in \mathbb{R}^n$. Identity maps will always be denoted by the symbol $\imath$. $S^{(n-1)}(H)$ denotes the unit sphere in a n-dimensional normed vector space H.

2 Prerequisite results and additional notation

The results and definitions of Hauser [4] and [5] will be heavily used. References to those papers will be indicated by the string "H1." and "H2." respectively followed by the number of the cited result, i.e. H2.3.1 refers to Lemma 3.1, [5] for example.

Modern interior-point theory is based on the foundation of Nesterov & Nemirovskii's theory of self-concordant functionals. For a definition of ν-logarithmically homogeneous self-concordant barrier functionals see [19] or [4], Definition 2.1 from which we adopt notation. A few properties of these functionals are listed in Proposition H1.2.3. The theory of self-scaled conic programming was founded in two seminal papers by Nesterov and Todd [20, 21]. In Definition H1.2.4 and Theorem H1.2.8 we reiterated the definition of self-scaled barrier functionals and properties thereof in the form we shall use in this article.

Güler [3] and Nesterov-Todd [20] showed that the concept of self-scaled cones, i.e. the class of cones that allows for a self-scaled barrier functional, is equivalent to the concept of symmetric cones studied in Jordan algebra theory (see Theorems H1.2.15 and H1.2.16). For a definition of symmetric cone see e.g. [2] or Definition H1.2.14. For three classical examples of symmetric cones and self-scaled barrier functionals (positive orthant, Lorentz cone and the cone of positive semidefinite symmetric matrices) see H1.2.5–H1.2.7.

One of the important tools of the theory of self-scaled conic programming is the notion of *scaling point* (see [20]). For a definition and examples on how to compute scaling points see e.g. H2.2.6, and H2.2.8–H2.2.10.

The automorphism groups of symmetric cones (see e.g. [2] or H1.2.20) plays an important role in their analysis. Güler [3] showed that the universal barrier functional (see [19]) of a regular cone K is of the form $c_1\varphi_K + c_0$ with $c_1 > 0$ and $c_0 \in \mathbb{R}$, and where φ_K is the characteristic function of K (see e.g. [2] or H1.2.17).

Based on earlier results by Cartan [1], Köcher [10] and Rothaus [23], Vinberg [31] gave a classification theory for self-dual homogeneous cones via an irreducible decomposition. For a systematic treatment in a single reference consult for example [2].

Theorem 2.1 (Vinberg).
Let $K \subset E$ be a self-dual homogeneous cone. Then there exists a unique set of self-dual homogeneous cones $\{K_{ij} \subset E_{ij} : (ij) \in J\}$ indexed by some index set $J = \{(ij) : i \in \mathbb{N}_l, j \in \mathbb{N}_{k(i)}\}$ such that

$$i)\ E = \bigoplus_{i=1}^{l} \bigoplus_{j=1}^{k(i)} E_{ij} \qquad \text{and}$$

$$ii)\ K = \bigoplus_{i=1}^{l} \bigoplus_{j=1}^{k(i)} K_{ij},$$

and such that none of the cones K_{ij} can be further decomposed into a direct sum of self-dual homogeneous cones. The K_{ij} are called the irreducible parts *of K, and the direct sum ii) is called the* irreducible decomposition *of K. The first index is to distinguish sets of mutually isomorphic irreducible parts of K, i.e. $K_{i_1 j_1} \cong K_{i_2 j_2}$ if and only if $i_1 = i_2$. All irreducible parts K_{ij} are isomorphic to one of following cones for some $n \in \mathbb{N}$:*

1. $\mathbb{R}_+$ *(positive real line),*
2. $\left\{ \binom{\tau}{x} \in \mathbb{R}^{n+1} : \tau \geq \|x\|_2 \right\}$ *(Lorentz cone),*
3. $\mathrm{S}^{n\times n}(\mathbb{R})_+$ *(positive semidefinite symmetric matrices over the reals),*
4. $\mathrm{H}^{n\times n}(\mathbb{C})_+$ *(positive semidefinite Hermitian matrices over the complex numbers),*
5. $\mathrm{H}^{n\times n}(\mathbb{H})_+$ *(positive semidefinite Hermitian matrices over the quaternions) and*
6. $\mathrm{H}^{3\times 3}(\mathbb{O})_+$ *(positive semidefinite Hermitian 3×3-matrices over the octonions).*

$\square$

Definition 2.2. Let K, F be self-scaled. In the interior-point literature it is customary to view E as a Euclidean space by identifying E with $E^\sharp$ via $F''(e)$ for some fixed $e \in K^\circ$, i.e. we define a scalar product $\big(\cdot;\cdot\big)_e$ by the rule

$$\big(x_1; x_2\big)_e := \big\langle x_1; F''(e)x_2\big\rangle \qquad \forall x_1, x_2 \in E.$$

We say that $\big(\cdot;\cdot\big)_e$ provides E with the *Euclidean structure defined by* e. A *frame* of K for this Euclidean structure is a subset $\big\{e_1, \dots, e_{\kappa(K)}\big\} \subset K$ of maximal cardinality whose elements e_i are mutually orthogonal. The number $\kappa(K)$ is called the *Carathéodory number* of K. If $B \supset B_K := \big\{e_1, \dots, e_{\kappa(K)}\big\}$ is an orthogonal basis of E for the Euclidean structure defined by e we say that B is a basis with frame B_K centered at e. We write $\mathcal{B}^{(e)}$ for the set of such bases, and we shall use the notation $\mathcal{B} := \bigcup_{e \in K^\circ} \mathcal{B}^{(e)}$ for the union of these sets. If E is endowed with the basis $B \in \mathcal{B}^{(e)}$ then we endow the space $E^\sharp$ with the dual basis $B^\sharp = \big\{e_1^\sharp, \dots, e_n^\sharp\big\}$, where $e_i^\sharp := F''(e)e_i$ for all $i \in \mathbb{N}_n$. Whenever we represent objects in matrix form it will notationally suffice to refer only to the basis B chosen on the primal space E. For example, the notation $\nabla_B^k F(x)$ shall refer to the matrix representation of the tensor $F^{(k)}(x) \in \otimes^k E^\sharp$ with respect to the bases B and $B^\sharp$. In particular, if $B \in \mathcal{B}^{(e)}$ then $\nabla_B^2 F(e) = I$. Once E has been endowed with a Euclidean structure, we denote the group of orthogonal transformations that map K onto itself by $\mathcal{O}(K)$ and call this the orthogonal group of K. By $\mathcal{Q}(K)$ let us denote the connected component of $\mathcal{O}(K)$ that contains the identity $\imath : E \to E$.

For examples of frames see H1.2.11–H1.2.12. The set of isomorphisms and permutations defined in H1.3.1, H1.3.2 and H1.4.10 will play an important role in our proofs. From [4] we cite the following results:

Theorem 2.3 (Decomposition of Self-Scaled Barrier Functionals).
Let K be a symmetric cone provided with the ν-self-scaled barrier functional $F \in \mathcal{C}^3(K^\circ, \mathbb{R})$. Then there exist $\nu^{(ij)}$-self-scaled barrier functionals $F^{(ij)} \in \mathcal{C}^3(K^{(ij)\circ}, \mathbb{R})$ for the irreducible symmetric cones $K^{(ij)}$ $\big((ij) \in J\big)$ such that

$$\begin{aligned} F = \bigoplus_J F^{(ij)} : \bigoplus_J K^{(ij)} &\to \mathbb{R}, \\ \bigoplus_J x^{(ij)} &\mapsto \sum_J F^{(ij)}\big(x^{(ij)}\big). \end{aligned}$$

Furthermore, $\nu = \sum_J \nu^{(ij)}$ and the functionals $F^{(ij)}$ are unique up to additive constants.

Definition 2.4. Let K be a self-scaled cone. A self-scaled barrier functional F for K is *isotropic* under the Euclidean structure defined by $e \in K^\circ$ if the identity

$$F \circ \theta \equiv F$$

holds true for all $\theta \in \mathcal{O}(K)$.

Theorem 2.5 (Classification of Isotropic Self-Scaled Barriers).
Let $K = \bigoplus_J K^{(ij)}$ and $F = \bigoplus_J F^{(ij)} \in \mathcal{C}^3\big(K^\circ, \mathbb{R}\big)$ be self-scaled and suppose $e = \bigoplus_J e^{(ij)}$ is a fixed element in K°. Endow all the spaces $E^{(ij)}$ and E with the canonical Euclidean structures defined by e. For each $(ij) \in J$ let $\varphi_{K^{(ij)}}$ be the characteristic function of $K^{(ij)}$ defined by the Euclidean structure we chose on $E^{(ij)}$. Then F is isotropic for the Euclidean structure defined by e if and only if there exist real numbers $c_1^{(i)} > 0$, $c_0^{(ij)} \in \mathbb{R}$ $\big(1 \leq i \leq l; (ij) \in J\big)$ such that

$$F = \bigoplus_J \big(c_1^{(i)} \ln \varphi_{K^{(ij)}} + c_0^{(ij)}\big). \tag{2.1}$$

Moreover, the notion is isotropy is independent of the point $e \in K^\circ$ chosen to endow E with a Euclidean structure.

Theorem 2.6 (Strong self-scaledness).
Let $K \subset E$ and $F \in \mathcal{C}^3\big(K^\circ, \mathbb{R}\big)$ be self-scaled. Then F is strongly self-scaled if and only if it is isotropic.

Theorem 2.7 (Polar Decomposition).
Let $K \subset E$ and $F \in \mathcal{C}^3\big(K^\circ, \mathbb{R}\big)$ be self-scaled and suppose F is isotropic. Let us fix $e \in K^\circ$ and endow E with the Euclidean structure defined by this vector. Then each automorphism $\theta \in \mathrm{Aut}(K)$ of K has a unique polar decomposition

$$\theta = \Omega \circ F''^{-1}(e) \circ F''(x)$$

with $\Omega \in \mathcal{O}(K)$ and $x \in K^\circ$.

From [5] we cite the following:

Remark 2.8. Some of our definitions will depend on the choice of coordinates on the spaces $E, E^\sharp$ and V. For coordinates on E and $E^\sharp$ we will stick to the conventions of Definition 2.2 by choosing a basis $B \in \mathcal{B}^{(e)}$ for E for some $e \in K^\circ$ and providing $E^\sharp$ with the dual basis $B^\sharp$. The objects defined in terms of $\big(B, B^\sharp\big)$ will usually bear a subscript B in their notation. V is always a Euclidean space of same dimension as E, and we will assume that an orthogonal basis $\big\{v_1, \ldots, v_n\big\} \subset V$ has been fixed for this space. Therefore, we will not usually refer to this basis in our notation for objects whose definitions depend on $\big\{v_1, \ldots, v_n\big\}$.

Definition 2.9. Suppose $B \in \mathcal{B}$ and $x \in K^\circ$. Let $F_B''^{1/2}(x) \in \mathrm{Lin}(E, V)$ and $F_B''^{-1/2}(x) \in \mathrm{Lin}(E^\sharp, V)$ be the unique vector space homomorphisms whose matrix representations with respect to the bases tuple $\big(B, \{v_1, \ldots, v_n\}\big)$ are $\nabla_B^2 F(x)^{1/2}$ and $\nabla_B^2 F(x)^{-1/2}$ respectively (see Definition 2.2 for notation). Let us then define the following sets of linear operators:

$$\begin{aligned} F''^{1/2}(x) &:= \{F_B''^{1/2}(x) : B \in \mathcal{B}\} \qquad \text{and} \\ F''^{-1/2}(x) &:= \{F_B''^{-1/2}(x) : B \in \mathcal{B}\}. \end{aligned}$$

Remark 2.10. Note that $\big(F_B''^{1/2}(x)\big)^{-1} : V \to E$ and $F_B''^{-1/2}(x) : E^\sharp \to V$ are two distinct operators. In fact, it is straightforward to see that these are adjoints of one another and that the following diagram is commutative:

$$\begin{tikzcd}
E \arrow[r, equal] & E \arrow[r, "F_B''(x)"] \arrow[d, "F_B''^{1/2}(x)"] & E^\sharp \arrow[r, equal] \arrow[d, "F_B''^{-1/2}(x)"] & E^\sharp \\
V \arrow[r, equal] \arrow[u, "F_B''^{-1/2}(x)^*"] & V \arrow[r, equal] & V \arrow[r, equal] & V \arrow[u, "F_B''^{1/2}(x)^*"]
\end{tikzcd}$$

Proposition 2.11. *Suppose* $B \in \mathcal{B}$ *and* $x \in K^\circ$. *Then*

$$\begin{aligned} F''^{1/2}(x) &= \mathcal{O}(V) \circ F_B''^{1/2}(x) = \{L \in \mathrm{Lin}(E, V) : L^* \circ L = F''(x)\}, \\ F''^{-1/2}(x) &= \mathcal{O}(V) \circ F_B''^{-1/2}(x) = \{L \in \mathrm{Lin}(E^\sharp, V) : L^* \circ L = F''^{-1}(x)\}. \end{aligned}$$

$\square$

Definition 2.12 (Square-root fields).
Let $K \subset E$ be a symmetric cone with ν-self-scaled barrier functional $F \in \mathcal{C}^k\big(K^\circ, \mathbb{R}\big)$, and let $e_V \in V$ be a vector of length $\|e_V\|_2 = \nu^{1/2}$. A *square-root field* for (F, e_V) consists of a pair of differentiable operator fields

$$(X, S) \in \mathcal{C}^{k-2}\big(K^\circ \times K^{\sharp\circ}, \mathrm{Lin}(E^\sharp, V)\big) \times \mathcal{C}^{k-2}\big(K^\circ \times K^{\sharp\circ}, \mathrm{Lin}(E, V)\big)$$

such that the following diagram commutes for all (x, s):

(2.2a)
$$\begin{tikzcd}
E^\sharp \arrow[r, "F_\sharp''(s)"] & E \arrow[r, "F''(w(x,s))"] \arrow[d, "S(x,s)"] & E^\sharp \arrow[d, "X(x,s)"] & E \arrow[l, "F''(x)"'] \\
V \arrow[r, equal] \arrow[u, "S^*(x,s)"] & V \arrow[r, equal] & V \arrow[r, equal] & V \arrow[u, "X^*(x,s)"']
\end{tikzcd}$$

(2.2b)
$$\begin{tikzcd}
-F_\sharp'(s) \arrow[r, "F''(w(x,s))"] \arrow[d, "S(x,s)"] & -F'(x) \arrow[d, "X(x,s)"] \\
e_V \arrow[r, equal] & e_V
\end{tikzcd}$$

Our choice of nomenclature reflects the fact that X and S are square-roots of the tensor fields $(x,s) \mapsto F''^{-1}(x)$ and $(x,s) \mapsto F''^{-1}_{\sharp}(s)$ respectively. Let $B \in \mathcal{B}$ (see Definition 2.2). Then it follows from Proposition 2.11 that there exist orthogonal fields

$$\Omega_{(x,B)}, \Omega_{(s,B)} \in \mathcal{C}^{k-2}\big(K^\circ \times K^{\sharp\circ}, \mathcal{O}(V)\big)$$

such that

$$\begin{aligned} X(x,s) &\equiv \Omega_{(x,B)}(x,s) \circ F''^{-1/2}_{B}(x), \\ S(x,s) &\equiv \Omega_{(s,B)}(x,s) \circ F''^{-1/2}_{\sharp B}(s), \\ \Omega_{(s,B)} &\equiv \Omega_{(x,B)} \circ F''^{-1/2}_{B}(x) \circ F''(w) \circ F''^{1/2}_{\sharp B}(s)^*. \end{aligned}$$

These identities combined with Corollary H3.10 show that the cones $X(x,s)K^\sharp$ are self-dual under the Euclidean structure defined on V and mutually congruent for all values of (x,s). If all of these cones coincide in a fixed cone $K_V \subset V$ then we say that (X,S) is a *congruent* square-root field for $\big(F, e_V, K_V\big)$. Finally, we shall refer to a square-root field (X,S) as *homogeneous* if $X(\tau x, \eta s) = \tau X(x,s)$ for all $x \in K^\circ, s \in K^{\sharp\circ}$ and $\tau, \eta > 0$ (compare with Proposition H3.13*iii*)). Since $w(\tau x, \eta s) = \tau^{1/2}\eta^{-1/2}w(x,s)$ it follows that $S(\tau x, \eta s) = \eta S(x,s)$ in this case. By abuse of language we shall often simply speak of a (congruent) square-root field for F, and it is then tacitly understood that a vector $e_V \in V$ (and a cone $K_V \subset V$) has been fixed.

Theorem 2.13 (Existence of square-root fields).
Let $K \subset E$ and $F \in \mathcal{C}^k\big(K^\circ, \mathbb{R}\big)$ be self-scaled. Then the following applies:

i) The set of congruent square root fields for F is nonempty.

ii) If (X,S) is a square-root field for $\big(F, e_V\big)$ then any pair (Y,T) of operator fields is a square-root field for (F, e_V) if and only if there exists a mapping $\Omega \in \mathcal{C}^{k-2}\big(K^\circ \times K^{\sharp\circ}, \mathcal{O}(V, e_V)\big)$ such that $Y \equiv \Omega \circ X$ and $T \equiv \Omega \circ S$.

iii) If (X,S) is a congruent square-root field for $\big(F, e_V, K_V\big)$ then any pair (Y,T) of operator fields is a congruent square-root field for (F, e_V, K_V) if and only if there exists a mapping $\Omega \in \mathcal{C}^{k-2}\big(K^\circ \times K^{\sharp\circ}, \mathcal{O}(K_V, e_V)\big)$ such that $Y \equiv \Omega \circ X$ and $T \equiv \Omega \circ S$.

□

Square-root fields for direct sums of symmetric cones can be constructed as direct sums of square-root fields for their constituent cones (see Theorem H2.3.17). The converse of this result does not hold true in general. For a partial inverse see Theorem 4.11 below. For examples of square-root fields see H2.3.19 and H2.3.21. Let us cite the following two examples explicitly:

Example 2.14 (Linear Programming). Let us construct a square-root field for the self-scaled cone and barrier functional from Example H1.2.5: We may choose $e = e_V := (1, \ldots, 1)^T$ and $R := I \in F''^{-1/2}(e)$. Consider the square-root field (X, S) that was constructed in the proof of Theorem 2.13:

$$X : (x, s) \mapsto R \circ F''^{-1}\big(w(e, -F'(x))\big) = F''^{-1}(\sqrt{x}) = \operatorname{Diag}(x),$$
$$S = X^\sharp : (x, s) \mapsto X \circ F''(w(x, s)) = \operatorname{Diag}(x) \operatorname{Diag}(x)^{-1} \operatorname{Diag}(s) = \operatorname{Diag}(s).$$

Note that this is the canonical square-root field for linear programming that was already mentioned in the introduction. Since this is the example that motivated the definition of square-root fields at the outset we will henceforth refer to it as the *primordial square-root field*. From Theorem 2.13 and the fact that $\mathcal{O}(V, \mathbb{R}^n_+)$ is the group of coordinate permutations it follows that for a fixed K_V there are only $n!$ distinct congruent square-root fields for (F, e_V). The "theory" of square-root fields is trivial in this case due to the polyhedral nature of the cone. $\square$

Example 2.15 (Semidefinite Programming II). This example was found by Mike Todd. We will make use of Theorem 2.13 and Lemma H1.4.6 to compose Example H2.3.19 with an orthogonal field

$$\Omega \in \mathcal{C}^{k-2}\big(K^\circ \times K^{\sharp\circ}, \mathcal{O}(V, I, \mathrm{S}^{n\times n}(\mathbb{R})_{++})\big)$$

so as to retrieve a primal-dual symmetric square-root field for F as a result. For $x, s \in \mathrm{S}^{n\times n}(\mathbb{R})_{++}$ let us define $v(x,s) := w(x,s)^{-1/2} x w(x,s)^{-1/2}$. (This concept is originally due to Sturm-Zhang [25], see also Todd [26]. For a generalization to general instances of self-scaled conic programming, see Equation 4.6 below.) The map $v : \mathrm{S}^{n\times n}(\mathbb{R})^2_{++} \to V$ is primal-dual symmetric. In fact, since the dual scaling point satisfies the equation $t(x,s) = w(x,s)^{-1}$ we have

$$v(x,s) = w^{-1/2} x w^{-1/2} = t^{-1/2} w^{-1} x w^{-1} t^{-1/2} = t^{-1/2} s t^{-1/2} = v(s,x).$$

Let $n(x,s) := v^{-1/2} w^{-1/2} x^{1/2}$. Then $n(x,s) n(x,s)^T = v^{-1/2} w^{-1/2} x w^{-1/2} v^{-1/2} = I$, i.e. $n(x,s) \in \mathcal{O}(n)$. Let (X, S) denote the square-root field found in Example H2.3.19, define

$$\Omega := \mathcal{N} \circ n : \mathrm{S}^{n\times n}(\mathbb{R})^2_{++} \to \mathcal{O}(V, I, \mathrm{S}^{n\times n}(\mathbb{R})_+)$$

and consider the the tuple $(\Omega \circ X, \Omega \circ S)$ which according to Theorem 2.13 is a congruent square-root field for (F, I). We contend that it is primal-dual symmetric. In fact, for all $h \in \mathrm{S}^{n\times n}(\mathbb{R})$,

$$\begin{aligned}
&\big(\Omega \circ X\big)(x,s)h = v^{-1/2} w^{-1/2} x h x w^{-1/2} v^{-1/2},\\
&\big(\Omega \circ S\big)(x,s)h = \\
&\qquad = v^{-1/2} w^{-1/2} x^{1/2} \big(x^{1/2} s x^{1/2}\big)^{1/2} x^{-1/2} h x^{-1/2} \big(x^{1/2} s x^{1/2}\big)^{1/2} x^{1/2} w^{-1/2} v^{-1/2}\\
&\qquad = v^{-1/2} w^{-1/2} x w^{-1} h w^{-1} x w^{-1/2} v^{-1/2}\\
&\qquad = v^{-1/2} t^{-1/2} w^{-1} x w^{-1} h w^{-1} x w^{-1} t^{-1/2} v^{-1/2}\\
&\qquad = v^{-1/2} t^{-1/2} s h s t^{-1/2} v^{-1/2}.
\end{aligned}$$

$\square$

3 Congruent square-root fields for isotropic barriers

The purpose of this section is to give a structure theorem for congruent square-root fields for symmetric cones endowed with isotropic self-scaled barrier functionals. We shall see in particular that the vector e_V is entirely determined by the cone K_V and the two groups $\mathcal{O}(K_V, e_V)$ and $\mathcal{O}(K_V)$ coincide.

Lemma 3.1. *Let K be an irreducible self-scaled cone with self-scaled isotropic barrier functional F. Let us fix a vector $e \in K^\circ$ and endow E with the Euclidean structure defined by e. Let $B \in \mathcal{B}^{(e)}$ be a basis with frame $B_K = \{e_1, \ldots . e_q\}$ centered at e (see Definition 2.2). Then*

i) e *lies in the unique direction invariant under* $\mathcal{Q}(K)$*,*

ii) $e \sim \sum_{i=1}^{q} e_i$*,*

iii) $\mathrm{Aut}^{(\imath)}(K)_e = \mathcal{Q}(K)$ *and*

iv) $\mathrm{Aut}(K)_e = \mathcal{O}(K)$*.*

Proof. Recall from Theorem H1.4.8 that F is of the form $F = c_1 \ln \varphi_K + c_0$. Everything now follows from the results of Chapters I, III and IV of Farraut-Korányi [2], Propositions I.1.9, I.4.3, Theorem III.1.2 and Corollary IV.2.7. $\square$

Lemma 3.2. *Let K be an irreducible symmetric cone in a Euclidean space E. Then a frame $B_K = \{e_1, \ldots, e_\kappa\}$ of K is enough to determine K uniquely.*

Proof. The vector $e := \sum_{i=1}^{\kappa} e_i$ uniquely determines the group $\mathcal{Q}(K)$ by means of its stabilizer in $\mathrm{Aut}(K)$, i.e. the subgroup $\mathrm{Aut}(K)_e$ of $\mathrm{Aut}(K)$ that leaves e invariant. In fact, Lemma 3.1 *iii)* implies that the connected component of $\mathrm{Aut}(K)_e$ that contains the identity map is equal to $\mathcal{Q}(K)$:

$$\mathrm{Aut}^{(\imath)}(K)_v = \mathcal{Q}(K).$$

Finally, Lemma H1.2.21 shows that $K = \mathcal{Q}\big[\mathrm{cone}\{e_1, \ldots, e_\kappa\}\big]$ is entirely determined by B_K. $\square$

Recall our usage of the notation $F''^{1/2}_B(x)$ from Definition 2.9. In the following results we need to refer explicitly to the orthonormal basis $D = \{v_1, \ldots, v_n\}$ chosen in V, and we will thus revert to the more cumbersome notation $F''^{1/2}_{(B,D)}(x)$.

Lemma 3.3. *Let $K \subset E$ be a symmetric cone, and let $F \in \mathcal{C}^k\big(K^\circ, \mathbb{R}\big)$ be a self-scaled isotropic barrier functional for K with $k \geq 3$. Consider a vector $e \in K^\circ$ and a basis $B \in \mathcal{B}^{(e)}$ with frame B_K centered at e. As always, let us assume that an orthonormal basis D has been fixed in V, and Let K_V be the cone in V defined by $K_V := F''^{1/2}_{(B,D)}(e)K = F''^{-1/2}_{(B,D)}(e)K^\sharp$. Then $F''^{1/2}_{(B,D)}(x) \in \mathrm{Iso}\big(K, K_V\big)$ for all $x \in K^\circ$.*

Proof. Let us consider the linear operator

$$\theta := F''^{-1/2}_{(B,D)}(e) \circ F''\big(w(x,-F'(e))\big)$$

Then the following identity holds true:

$$\begin{aligned}\theta^* \circ \theta &\overset{2.10}{=} F''\big(w(x,-F'(e))\big) \circ F''^{-1}(e) \circ F''\big(w(x,-F'(e))\big)\\ &\overset{H1.2.3iii)}{=} F''\big(w(x,-F'(e))\big) \circ F''_\sharp\big(-F'(e)\big) \circ F''\big(w(x,-F'(e))\big)\\ &\overset{H1.2.8iii)}{=} F''(x).\end{aligned}$$

This shows that $\theta \in F''^{1/2}(x)$. But the matrix representation of θ is symmetric, since it is given by

$$I \cdot \nabla^2_B F\big(w(x,-F'(e))\big).$$

Hence, in fact $F''^{1/2}_{(B,D)}(x) = \theta$ and $F''^{1/2}_{(B,D)}(x)K = F''^{-1/2}_{(B,D)}(e)K^\sharp = K_V$. □

Note that Lemma 3.2 shows that the cone K_V from Lemma 3.3 is almost entirely determined by the choice of basis in V. In fact, once such a basis D has been fixed, the operator $F''^{1/2}_{(B,D)}(e)$ maps the frame B_K onto a subset $D_K \subset D$ which has the same cardinality as B_K and is a frame of K_V. Moreover, there is a partitioning of D_K into subsets $D_K^{(i)} \subset D_K$ that constitute frames for the irreducible parts $K_V^{(i)}$ of K_V. By Lemma 3.2 the $K_V^{(i)}$ are entirely determined by $D_K^{(i)}$ and the Euclidean structure defined by D. But there are only finitely many subsets $D_K \subset D$ and partitionings thereof that exhibit the correct cardinalities. Hence, once D is fixed, there are only finitely many choices left for the cone K_V.

Theorem 3.4. *Let $K = \bigoplus_J K^{(ij)} \subset \bigoplus_J E^{(ij)} = E$ be a symmetric cone with ν-self-scaled isotropic barrier functional $F = \bigoplus_J F^{(ij)}$, and let $e = \bigoplus_J e^{(ij)} \in K^\circ$ be fixed. Let us endow each space $E^{(ij)}$ with the Euclidean structure defined by $(F^{(ij)}, e^{(ij)})$, and let $B^{(ij)} \in \mathcal{B}(e^{(ij)})$ be an orthonormal basis with frame $B_K^{(ij)}$ centered at $e^{(ij)}$. Finally, let (X,S) be a congruent square-root field for $\big(F, e_V, K_V\big)$ for some $e_V \in K_V \subset V$. Then the following hold:*

i) The vector $e_V = \bigoplus_J e_V^{(ij)} \in K_V = \bigoplus_J K_V^{(ij)}$ is completely determined by K_V. In fact, $e_V^{(ij)} \in K^{(ij)\circ}$, lies in the unique direction under the action of the group $\mathcal{O}(K^{(ij)})$ and is of length $\|e_V^{(ij)}\|_2 = \big(\nu^{(ij)}\big)^{1/2}$ for all $(ij) \in J$.

ii) There exists an orthonormal basis $D \in V$ such that $S\big(e,-F'(e)\big) = F''^{1/2}_{(B,D)}(e)$. Consider the partition $D = \bigcup_J D^{(ij)}$ of D defined by $D^{(ij)} = F''^{1/2}_{(B,D)}(e)B^{(ij)}$ and let the indexing of the irreducible decompositions of K and K_V be consistent, i.e.

$K_V^{(ij)}$ is generated by the frame $D_K^{(ij)} := B_K^{(ij)}$ (see Lemma 3.2). Then $(X,S) = \bigoplus_J \big(X^{(ij)}, S^{(ij)}\big)$ where

$$\begin{aligned}
&X^{(ij)}(x,s) = \Omega_x^{(ij)} \circ F_{(B,D)}^{(ij)\prime\prime-1/2}\big(x^{(ij)}\big),\\
&S^{(ij)}(x,s) = \Omega_s^{(ij)} \circ F_{\sharp(B,D)}^{(ij)\prime\prime-1/2}\big(s^{(ij)}\big) = X^{(ij)}(x,s) \circ F^{(ij)\prime\prime}\big(w^{(ij)}(x,s)\big),\\
&\Omega_x^{(ij)}, \Omega_s^{(ij)} \in \mathcal{C}^{k-2}\big(K^\circ \times K^{\sharp\circ}, \mathcal{Q}(K_V^{(ij)})\big) \qquad \textit{and}\\
&X^{(ij)}(x,s)\big(-F^{(ij)\prime}(x^{(ij)})\big) \equiv e_V^{(ij)}.
\end{aligned}$$

iii) All other tuples (Y,T) are congruent square-root fields for $\big(F, e_V, K_V\big)$ if and only if there exists an orthogonal field $\Omega \in \mathcal{C}^{k-2}\big(K^\circ \times K^{\sharp\circ}, \mathcal{O}(K)\big)$ such that $Y \equiv \Omega \circ X$ and $T \equiv \Omega \circ S$.

Proof. $i)$ It follows from Lemma H2.3.7 that $D := S\big(e, -F'(e)\big)B$ is an orthonormal basis of V, and clearly we then have $S\big(e, -F'(e)\big) = F_{(B,D)}^{\prime\prime 1/2}(e)$. In particular, $D_K^{(ij)} := S\big(e, -F'(e)\big)B_K^{(ij)}$ is a frame for $K_V^{(ij)} := S\big(e, -F'(e)\big)K^{(ij)}$ and

$$\begin{aligned}
e_V &= S\big(e, -F'(e)\big)\left[-F'_\sharp\big(-F'(e)\big)\right] \overset{H1.2.3ii)}{=} S\big(e, -F'(e)\big)e\\
&= S\big(e, -F'(e)\big)\Big[\bigoplus_J \sum_{z \in B_K^{(ij)}} z\Big] = \bigoplus_J \sum_{v \in D_K^{(ij)}} v.
\end{aligned}$$

By Lemma 3.1 this means that $e_V^{(ij)}$ lies in the invariant direction under $\mathcal{O}(K_V^{(ij)})$ for all $(ij) \in J$. Moreover, again by Lemma H2.3.7 we have

$$\begin{aligned}
\|e_V^{(ij)}\|_2 &= \|e^{(ij)}\|_e = \|e^{(ij)}\|_{e^{(ij)}}\\
&\overset{H1.2.3i),iv)}{=} \big(\nu^{(ij)}\big)^{1/2}.
\end{aligned}$$

This settles the first claim.

$ii)$ Again we use the basis D defined in the first part. Lemma 3.3 then says that for all (x,s) both $X(x,s)$ and $F_{(B,D)}^{\prime\prime-1/2}(x)$ lie in $F^{\prime\prime-1/2}(x) \cap \mathrm{Iso}\big(K^\sharp, K_V\big)$, and hence by Lemma 2.11 there exists an operator field $\Omega_x \in \mathcal{C}^{k-2}\big(K^\circ \times K^{\sharp\circ}, \mathcal{O}(K_V)\big)$ such that $X(x,s) \equiv \Omega_x(x,s) \circ F_{(B,D)}^{\prime\prime-1/2}(x)$. Let us choose a set of isomorphisms $\big\{\imath_i^{(j_1 j_2)} : i \in \mathbb{N}_l, (j_1 j_2) \in \mathbb{N}_{k(i)}^2\big\}$ between the mutually isomorphic irreducible parts of K_V as in the proof of Lemma H1.4.9. Lemma H1.4.10 implies that Ω_x decomposes into a direct sum

$$\Omega_x = \Big(\bigoplus_J \imath_i^{(j\sigma_{\Omega_x}^{(i)}(j))}\Big) \circ \Big(\bigoplus_J \theta_{\Omega_x}^{(ij)}\Big).$$

By continuity of Ω_x, the fact that $\sigma_{\Omega_x(e,-F'(e))}^{(i)} = \imath$, and the discreteness of the group $\mathcal{S}_{k(i)}$ we get $\sigma_{\Omega_x}^{(i)} \equiv \imath$ for all $i \in \mathbb{N}_l$ and all (x,s). Therefore, $\Omega_x = \bigoplus_J \theta_{\Omega_x}^{(ij)}$. Everything else is trivial if we take into account Theorem 2.3 by which we have $F''(w) = \bigoplus_J F^{(ij)\prime\prime}\big(w^{(ij)}\big)$, and Lemma 3.1.

iii) The last part follows trivially from the first part, Lemma 3.1 and Theorem 2.13. ☐

The reader should be careful to note that Theorem 3.4 *does not* imply that (X, S) decomposes into a direct sum of square-root fields over the irreducible parts of K, since it may not be true that the orthogonal fields $\Omega_x^{(ij)}$ and $\Omega_s^{(ij)}$ depend only on the projection $\pi^{(ij)}(x,s) := \big(x^{(ij)}, s^{(ij)}\big)$ of the argument into the subspace $E^{(ij)} \times E^{\sharp(ij)}$ for all $(ij) \in J$.

4 Scale-invariance and primal-dual symmetry

Note that we formulated the optimization problems (P), (D) (1.2) purely in terms of geometric objects, without any reference to a particular coordinate system. For the construction of algorithms to solve these problems we would like to think of a computer that can perform operations directly in the underlying vector spaces. The sequence of intermediate results computed by such an algorithm should thus in principle be independent of coordinate systems. There is a passive interpretation of an automorphism of the cone K as a changes of coordinates. One may thus wish that if we run our algorithm on transformed data the intermediate results produced should simply transform accordingly. This idea is captured in the concepts of *scale-invariance* and *primal-dual symmetry.* If the construction of such an algorithm is to succeed on the basis of target maps then the underlying square-root fields should satisfy appropriate scaling properties also. These square-root fields are the generalization of Example 2.15 to arbitrary self-scaled cones and isotropic self-scaled barrier functionals. In this section we show that such square-root fields exist.

Consider the problem pair (1.1) and suppose that we have fixed a particular coordinate system in order to represent (P) and (D) in matrix form:

$$
\begin{array}{llcll}
\text{(P)} & \min c^T x & \qquad\qquad & \text{(D)} & \max b^T y \\
& Ax = b & & & A^T y + s = c \\
& x \in \mathcal{K} & & & s \in \mathcal{K}^\sharp.
\end{array}
$$

Here $\mathcal{K}$ and $\mathcal{K}^\sharp$ denote the sets of column vectors that represent the elements of K and $K^\sharp$ respectively. Let $T \in Gl_n(\mathbb{R})$ be a matrix transformation that leaves the set $\mathcal{K}$ invariant, i.e. $\{Tx : x \in \mathcal{K}\} = \mathcal{K}$, and let $Q \in Gl_m(\mathbb{R})$ be an arbitrary invertible matrix. Suppose we transform the problem data with T and Q:

$$
\begin{array}{llcll}
\text{(P')} & \min (T^{-T}c)^T x & \qquad\qquad & \text{(D')} & \min (Qb)^T y \\
& (QAT^{-1})x = (Qb) & & & (QAT^{-1})^T y + s = T^{-T}c \\
& x \in \mathcal{K} & & & s \in \mathcal{K}^\sharp.
\end{array}
$$

We may interpret this new problem pair in two distinct ways: T and Q can be seen as the matrices that describe a change of coordinates. A particular choice of coordinates tells us how to measure distances and angles, i.e. this is merely a matter of convention. One may thus wish to construct algorithms that evolve along the same intermediate points irrespectively of the chosen coordinate system. Second, we may interpret (P') and (D') as a new pair of optimization problems, represented in our old coordinate system, i.e. we have made a different choice for the homomorphism $A : E \to Y^\sharp$ and for the vectors $b \in Y^\sharp$ and $c \in E^\sharp$. The problem tuples (P, D) and (P', D') are relatives of each other because all their geometric properties are equivalent. In fact, all objects in the primal space E are transformed with some cone automorphism $\theta \in \mathrm{Aut}(K)$, and all objects in the dual space $E^\sharp$ are transformed with $\theta^{-*} \in \mathrm{Aut}\big(K^\sharp\big)$ (see Figure 1).

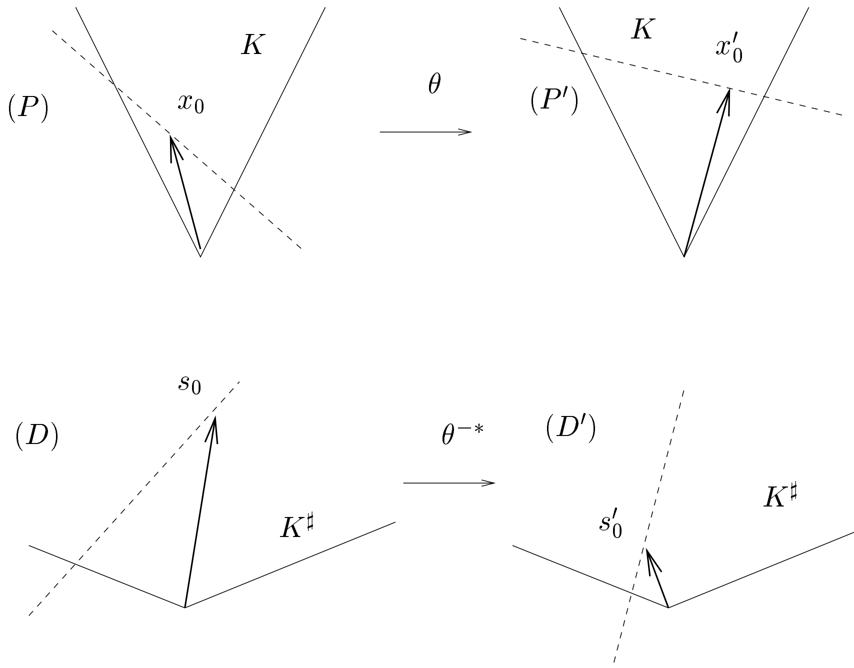

Figure 1: Rescaling of a primal-dual SSCP problem pair via a cone automorphism $\theta \in \mathrm{Aut}(K)$.

For consistency, if we want our algorithm to be independent of the chosen coordinate system, we also need to require that it behave equivalently on the problem pairs (P, D) and (P', D'), i.e. that the intermediate results computed by the algorithm when running on both problem pairs relate to each other via the same transformations as the input data. It is important to note that while algorithms defined in coordinate-free fashion automatically satisfy the first condition, the second condition - *scale-invariance* - is violated in general.

Note also that our primal and dual problems are of the same format, and hence we could interchange their roles and run our algorithm on the new problem pair. One may wish that the new sequence of intermediate solutions produced on the new problem pair be related to the sequence produced by the old problem pair simply by an interchange of primal and dual coordinates. Algorithms that satisfy this property are called primal-dual symmetric (see Figure 2).

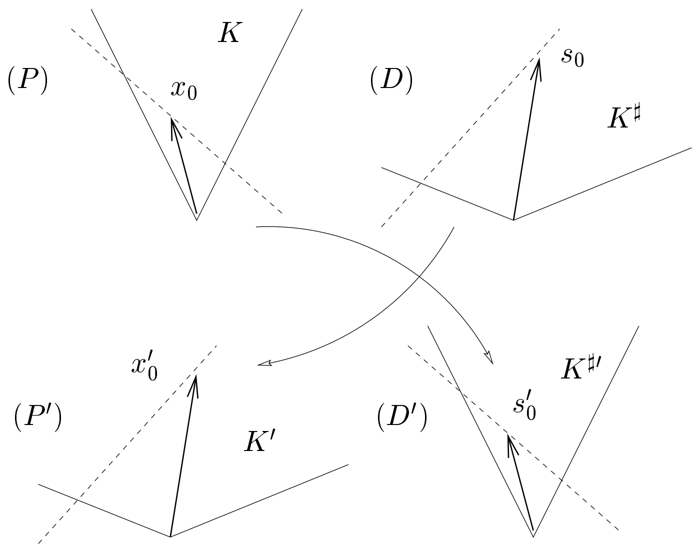

Figure 2: Exchanging the roles of the primal and dual problems.

Let us now formulate the above remarks in more precise mathematical terms. Consider the set of all dual pairs of optimization problems of the form (1.2), i.e. for all possible choices of the input data (E, K, L, x_0, s_0). Under the assumption of the real-number model, suppose $\texttt{Algo}$ is an algorithm which takes the input (E, K, L, x_0, s_0) and computes a finite or infinite sequence of feasible points

$$\big(\texttt{Algo}(E, K, L, x_0, s_0)_k\big)_{\mathbb{N}} = \big((x_k, s_k)\big)_{\mathbb{N}} \subset \mathcal{F}(P) \times \mathcal{F}(D)$$

which converges to the optimal solution (x_*, s_*) of the problem pair (P), (D) corresponding to the input. We say that $\texttt{Algo}$ is *scale-invariant* if for all choices of E and K in the input and for all cone automorphisms $\theta \in \mathrm{Aut}(K)$ the following identity holds for all $k \in \mathbb{N}$:

$$\texttt{Algo}(E, K, \theta L, \theta x_0, \theta^{-*} s_0)_k = (\theta x_k, \theta^{-*} s_k),$$

where $(x_k, s_k) = \texttt{Algo}(E, K, L, x_0, s_0)_k$.

Note also that since $K^\sharp = K$ we may interchange the roles of (P) and (D) and apply the same algorithm $\texttt{Algo}$ to the new problem pair. $\texttt{Algo}$ is called *primal-dual symmetric* if the following identity applies for all $k \in \mathbb{N}$:

$$\texttt{Algo}(E^\sharp, K^\sharp, L^\perp, s_0, x_0)_k = (s_k, x_k),$$

where $(x_k, s_k) = \texttt{Algo}(E, K, L, x_0, s_0)_k$.

For the purposes of this section we shall only consider *isotropic* self-scaled barrier functionals. Recall that by virtue of Theorem 2.6 this is equivalent to the assumption that F be *strongly* self-scaled, i.e. that the identity

$$F\big(\theta x\big) \equiv F(x) + cst \tag{4.1}$$

applies for all cone automorphisms $\theta \in \mathrm{Aut}(K)$. The elements of $\mathrm{Aut}(K)$ therefore act like a change of coordinates on $F^{(k)}$, $k \geq 1$, and this implies in particular that

$$w\big(\theta x, \theta^{-*} s\big) = \theta w(x,s) \quad \text{and} \tag{4.2}$$

$$F''\big(\theta^{-1} w(x,s)\big) = \theta^* \circ F''\big(w(x,s)\big) \circ \theta. \tag{4.3}$$

for all $(x,s) \in K^\circ \times K^{\sharp\circ}$. Barrier methods generally exploit the tensor fields F, F' and F''. Isotropic self-scaled barrier functionals therefore lend themselves to the construction of scale-invariant algorithms.

We shall now proceed to develop a notion of scale-invariance and primal-dual symmetry for square-root fields. Let (X,S) be a square-root field for the isotropic barrier functional $F \in \mathcal{C}^3(K^\circ, \mathbb{R})$ on K. Since objects in E transform with θ and objects in $E^\sharp$ with θ^{-*}, a straightforward approach would be to require that

$$X(\theta x, \theta^{-*} s) \equiv X(x,s) \circ \theta^* \tag{4.4}$$

for all $(x,s) \in K^\circ \times K^{\sharp\circ}$. This would of course imply that

$$S(\theta x, \theta^{-*} s) = X(\theta x, \theta^{-*} s) \circ F''(\theta w(x,s)) \equiv S(x,s) \circ \theta^{-1},$$

and hence the definition would be primal-dual symmetric. Note however that in the case of linear programming and $\theta \in \mathcal{O}(\mathbb{R}^n_+)$ chosen as the orthogonal automorphism defined by a permutation $\sigma \in \mathcal{S}_n$ the notion of scale-invariance as defined by equation (4.4) fails, since we have

$$X(\theta x) = \mathrm{Diag}\big(x_{\sigma(1)} \dots x_{\sigma(n)}\big) = \theta \circ X(x) \circ \theta^*.$$

In order to arrive at a notion of scale-invariance that extends the LP case we must therefore involve an orthogonal transformation of V-space.

For the purposes of our first lemma it is necessary to explicitly refer to the basis chosen in V, and hence we shall use the slightly cumbersome notation $F''_{(B,D)}$ instead of F''_B (see Remark 2.8).

Lemma 4.1. *Let $K \subset E$ be a symmetric cone endowed with a self-scaled isotropic barrier functional $F \in \mathcal{C}^k\big(K^\circ, \mathbb{R}\big)$, where $k \geq 3$. For a fixed vector $e \in K^\circ$ let $B \in \mathcal{B}^{(e)}$ be an orthogonal basis for E with frame B_K centered at e. Let V be endowed with an orthogonal basis $D := \big\{v_1, \dots, v_n\big\}$ and consider the cone $K_V := F''^{1/2}_{(B,D)}(e)K$. Then there exists a mapping*

$$\Omega^{(B,D)} \in \mathcal{C}^{k-2}\big(\mathrm{Aut}(K) \times K^\circ, \mathcal{O}(K_V)\big),$$
$$\Omega^{(B,D)} : (\theta, x) \mapsto \Omega^{(B,D)}_{(\theta,x)}$$

such that for all $\theta \in \mathrm{Aut}(K)$ *and* $x \in K^\circ$ *the following diagram commutes:*

$$\begin{array}{ccccccc} K & \xleftarrow{\theta^{-1}} & K & \xrightarrow{F''(\theta x)} & K^\sharp & \xleftarrow{\theta^{-*}} & K \\ \Big\downarrow{\scriptstyle F_B''^{1/2}(x)} & & \Big\downarrow{\scriptstyle F_B''^{1/2}(\theta x)} & & \Big\uparrow{\scriptstyle F_B''^{1/2}(\theta x)^*} & & \Big\uparrow{\scriptstyle F_B''^{1/2}(x)^*} \\ K_V & \xrightarrow{\Omega^{(B,D)}_{(\theta,x)}} & K_V & = & K_V & \xrightarrow{\Omega^{(B,D)*}_{(\theta,x)}} & K_V \end{array} \tag{4.5}$$

This mapping has the following properties:

i) $\Omega^{(B,D)}_{(\theta,\cdot)} \equiv F_B''^{1/2}(e) \circ \theta \circ F_B''^{-1/2}(e)^*$ *for* $\theta \in \mathcal{O}(K)$.

ii) $\Omega^{(B,D)}_{(\theta,x)} = \imath$ *for* $\theta = F''^{-1}(e) \circ F''(x)$.

iii) $\Omega^{(B,D)}_{(\theta,e)} = F_B''^{1/2}(e) \circ \vartheta \circ F_B''^{-1/2}(e)^*$ *for* $\theta = \vartheta \circ F''^{-1}(e) \circ F''(z)$, *where* $\vartheta \in \mathcal{O}(K)$.

iv) $\Omega_{(\theta_2 \circ \theta_1, x)} = \Omega_{(\theta_2, \theta_1 x)} \circ \Omega_{(\theta_1, x)}$ *for all* $x \in K^\circ, \theta_1, \theta_2 \in \mathrm{Aut}(K)$.

v) $\Omega_{(\tau\theta,x)} = \Omega_{(\theta,x)} = \Omega_{(\theta,\tau x)}$ *for all* $\tau > 0$.

Proof. Let $\theta \in \mathrm{Aut}(K)$. Since F is isotropic it follows from equation (4.1) that $F''(\theta x) = \theta^{-*} \circ F''(x) \circ \theta^{-1}$. Proposition 2.11 thus implies that $F_B''^{1/2}(x) \circ \theta^{-1} \in F''^{1/2}(\theta x)$ and that there exists an orthogonal transformation $\Omega^{(B,D)}_{(\theta,x)} \in \mathcal{O}(V)$ such that $\Omega^{(B,D)}_{(\theta,x)} \circ F_B''^{1/2}(x) \circ \theta^{-1} = F_B''^{1/2}(\theta x)$. It follows from Lemmas H2.3.1 and H2.3.2 that $\Omega^{(B,D)}$ is of class $\in \mathcal{C}^{k-2}$, and Lemma 3.3 says that

$$\Omega^{(B,D)}_{(\theta,x)} = F_B''^{1/2}(\theta x) \circ \theta \circ \big(F_B''^{1/2}(x)\big)^{-1} \in \mathrm{Aut}\big(K_V\big).$$

Therefore, $\Omega^{(B,D)}_{(\theta,x)} \in \mathcal{O}(K_V)$ as claimed. Let $\theta \in \mathcal{O}(K)$ and let us denote its matrix representation with respect to B by Θ. Then the fact that $\Theta \nabla_B^2 F(x)^{1/2} \Theta^T$ is symmetric implies that $\Omega^{(B,D)}_{(\theta,x)}$ has matrix representation Θ, and this gives property $i)$. Now suppose $\theta = F''^{-1}(e) \circ F''(x)$. Then

$$F_B''^{1/2}(x) \circ \theta^{-1} = F_B''^{-1/2}(x) \circ F''(e)$$

has the symmetric matrix representation $\big(\nabla_B^2 F(x)\big)^{1/2}$, and this implies that $\Omega^{(B,D)}_{(\theta,x)} = \imath$, which establishes property $ii)$. The composition rule $iv)$ is trivial, and combined with part $i)$ it shows that we only need to treat the special case $\vartheta = \imath$ for property $iii)$.
But this case follows readily from the fact that the following matrix is symmetric:

$$\nabla_B^2 F(e)^{1/2} \cdot \big[\nabla_B^2 F(e)^{-1} \cdot \nabla_B^2 F(z)\big]^{-1} = \big[\nabla_B^2 F(z)\big]^{-1}.$$

Finally, part $v)$ is trivial to prove. $\square$

Note that the mapping $F''(e)$ not only depends on the choice of the bases B and D but also on the indexing of their elements. The next lemma is going to tell us how the definition of $\Omega_{(\cdot,\cdot)}$ is affected under the operation of taking direct sums.

Lemma 4.2. *Let us make all the assumptions of Lemma 4.1, and let the setup be as in Theorem 3.4. Moreover, let the orthogonal basis $D \subset V$ and the set of norm-preserving isomorphisms $\{\imath_i^{(j_1 j_2)} : i \in \mathbb{N}_l, (j_1 j_2) \in \mathbb{N}^2_{k(i)}\}$ be chosen as in the proof of Theorem 3.4. Then, we have*

$$\Omega^{(B,D)}_{(\theta,x)} = F''^{1/2}_{(B,D)}(e) \circ \Big(\bigoplus_J \imath_i^{j\sigma_\theta^{(i)}(j)}\Big) \circ F''^{1/2}_{(B,D)}(e)^* \circ \Big(\bigoplus_J \Omega^{(B^{(ij)},D^{(ij)})}_{(\theta_\theta^{(ij)},x^{(ij)})}\Big)$$

Proof. This follows from Lemma 4.1 $i)$ & $iv)$. ∎

Next, we are getting ready to define scale-invariance and primal-dual symmetry for square-root fields. To make the notation simpler in the sequel we shall often tacitly assume that two bases $B \subset E$ and $D \subset V$ and an indexing of their elements have been fixed, and we then refer to the mapping $\Omega^{(B,D)}$ defined in Lemma 4.1 simply as Ω.

In analogy to semidefinite programming we can define a V-valued mapping

$$\begin{aligned} &v : K^\circ \times K^{\sharp\circ} \to V \\ &\qquad (x,s) \mapsto v(x,s) := F''^{1/2}_B(w(x,s))x = F''^{1/2}_{\sharp B}(t(x,s))s. \end{aligned} \tag{4.6}$$

Let θ be an automorphism of K. Then it follows from Diagram (4.5) that

$$v(\theta x, \theta^{-*}s) = \Omega_{(\theta,w(x,s))}v(x,s). \tag{4.7}$$

Thus, when objects α and β living in E and $E^\sharp$ respectively are related to one another via $F''(w)$ and transform with θ and θ^{-*} respectively, then the object in V-space defined by $F''^{1/2}_B(w)\alpha = F''^{-1/2}_B(w)\beta$ transforms with $\Omega_{(\theta,w)}$. Square-root fields are scale-invariant if they comply with this rule:

Definition 4.3. Let us choose K, F, e and B as in Lemma 4.1, and let (X,S) be a congruent square-root field for (F, e_V, K_V), for some choice of $e_V \in K_V \subset V$. (X,S) is *scale-invariant* for (e,B) if there exists a orthonormal basis D for V with frame for K_V such that for every cone automorphism $\theta \in \mathrm{Aut}(K)$ the following diagram commutes:

$$\begin{array}{ccccccc} E & \xleftarrow{\theta^{-1}} & E & \xrightarrow{F''(\theta w)} & E^\sharp & \xrightarrow{\theta^*} & E^\sharp \\ \downarrow{\scriptstyle S(x,s)} & & \downarrow{\scriptstyle S(\theta x,\theta^{-*}s)} & & \downarrow{\scriptstyle X(\theta x,\theta^{-*}s)} & & \downarrow{\scriptstyle X(x,s)} \\ V & \xrightarrow{\Omega_{(\theta,w)}} & V & = & V & \xleftarrow{\Omega_{(\theta,w)}} & V \end{array} \tag{4.8}$$

(X, S) is *primal-dual symmetric* in the Euclidean structure defined by e if for all $(x, s) \in K^\circ \times K^{\sharp\circ}$ it is true that

$$X\big(F''^{-1}(e)s, F''(e)x\big) = S(x, s) \circ F''^{-1}(e).$$

In the sequel we may at times explicitly mention the basis D whose existence is required in Definition 4.3 and say that (X, S) is scale-invariant for (e, B, D). Note that scale-invariance is compatible with congruence, since $\mathrm{Aut}(K)$ acts transitively on K°, $\theta^* \in \mathrm{Aut}(K^\sharp)$ and $\Omega_{(\theta,x)} \in \mathcal{O}(K_V)$ for all $\theta \in \mathrm{Aut}(K)$ and $x \in K^\circ$. Next, we are going to show that primal-dual symmetry comes for free in scale-invariant square-root fields.

Theorem 4.4. *If the square-root field (X, S) for F is scale-invariant for (e, B, D) then it is also primal-dual symmetric.*

Proof. Let $(x, s) \in K^\circ \times K^{\sharp\circ}$ and consider $w := w(x, s)$, $t := t(x, s)$ and $\theta := F''^{-1}(e) \circ F''(w)$. Then $\theta \in \mathrm{Aut}(K)$ and $\theta w = F''^{-1}(e)t$. The matrix representation of $F_B''^{1/2}(w) \circ \theta^{-1}$ is given by the symmetric matrix

$$\nabla_B^2 F(w)^{1/2} \nabla_B^2 F(w)^{-1} I = \nabla_B^2 F(w)^{-1/2}.$$

Moreover, we have

$$F''(\theta w) = F''\big(F''^{-1}(e)t\big) \overset{H1.2.8iii)}{=} F''(e) \circ F_\sharp''(t) \circ F''(e) = F''(e) \circ F''^{-1}(w) \circ F''(e),$$

and the matrix representation of this operator is given by $\nabla_B^2 F(w)^{-1}$. Hence,

$$F_B''^{1/2}(w) \circ \theta^{-1} = F_B''^{1/2}(\theta w)$$

and $\Omega_{(\theta,w)} = \imath$. Since (X, S), is scale-invariant it follows that

$$\begin{aligned} X\big(F''^{-1}(e)s, F''(e)x\big) &= X\big(\theta x, \theta^{-*}s\big) = \imath \circ X(x, s) \circ \theta^* \\ &= X(x, s) \circ F''(w) \circ F''^{-1}(e) = S(x, s) \circ F''^{-1}(e). \end{aligned}$$

□

The next result is going to show how scale-invariant square-root fields can be recovered when rescaling is applied to the underlying barrier functional.

Lemma 4.5. *Let $K \subset E$ be a symmetric cone with ν-self-scaled barrier functional $F \in \mathcal{C}^3(K^\circ, \mathbb{R})$. Let us choose (e, B, D) as in Lemma 4.1, and suppose $H = c_1 F + c_2$ is another self-scaled barrier functional for K, where $c_1 > 0$ and $c_2 \in \mathbb{R}$. If (X, S) is a square-root field for F then $\big(c_1^{-1/2} X, c_1^{-1/2} S\big)$ is a square-root field for H. Moreover, if F is isotropic and if (X, S) is scale-invariant for (e, B, D) then $(c_1^{-1/2} X, c_1^{-1/2} S)$ is scale-invariant for $(e, c_1^{-1/2} B, D)$.*

Proof. By virtue of Proposition H1.2.3 *iv*), choosing an arbitrary $x \in K^\circ$ we get that the complexity value of H must be equal to $\langle x; -H'(x) \rangle = c_1 \nu$. Clearly,

$$c_1^{-1/2} X(x,s) \in H''^{-1/2}(x) = c_1^{-1/2} F''^{-1/2}(x)$$

for all $(x,s) \in K^\circ \times K^{\sharp\circ}$, and

$$c_1^{-1/2} X(x,s)[-H'(x)] = c_1^{1/2} e_V.$$

Moreover, $c_1^{1/2} e_V$ complies with the condition that it be of norm $\|c_1^{1/2} e_V\| = c_1^{1/2} \nu^{1/2}$, i.e. the square-root of the complexity value. Finally, let $u(x,s)$ be the (primal) scaling point of x and s for H. Since

$$\{F''(z) : z \in K^\circ\} = \{H''(z) : z \in K^\circ\} \subset \mathrm{Iso}(E, E^\sharp),$$

and because of Corollary H1.2.9, we must have that $H''\big(u(x,s)\big) = F''\big(w(x,s)\big)$, i.e. $u = c_1^{1/2} w$ and $c_1^{-1/2} S \equiv c_1^{-1/2} X \circ H''\big(u(x,s)\big)$. For the last statement, observe that $\nabla_B^2 F(x) = \nabla_{c_1^{-1/2} B} H(x)$ for all $x \in K^\circ$, and use the polar decomposition introduced in Theorem H1.2.22 and 2.7. □

We are now ready to introduce our last invariance property for square-root fields: Canonical reduction. This notion is motivated by the requirement that when a linear programming instance is formulated as a more general self-scaled program, then an ideal target-following algorithm applied to this optimization problem should reduce to a target-following algorithm as defined classically for LP.

Definition 4.6. Let K be a symmetric cone with ν-self-scaled isotropic barrier functional F, and let $B \in \mathcal{B}^{(e)}$ be a basis with frame $B_K = \{e_1, \ldots, e_q\}$ centered at some vector $e \in K^\circ$. Note that the conic hull $C := \mathrm{cone}\big(B_K\big) \cong \mathbb{R}_+^n$ of B_K allows us to embed q-dimensional linear programming instances in the framework (P) over K. Let $G := \mathrm{span}\big(B_K\big)$ and $G^\sharp := \mathrm{span}\big(B_K^\sharp\big)$. A square-root field (X,S) for F *reduces canonically* on LP sub-instances if there exists a q-dimensional linear subspace $W \subset V$ such that $\mathrm{im}\big(X(x,s)|_{G^\sharp}\big) \subset W$ for all $(x,s) \in C^\circ \times C^{\sharp\circ}$ (see Figure 3).

The next two propositions are to shed some light on our notion of canonical reduction.

Proposition 4.7. *Under the assumptions of Definition 4.6, let (X,S) be a square-root field for F that reduces canonically on LP sub-instances. Then*

i) $\mathrm{im}\big(S(x,s)|_G\big) = \mathrm{im}\big(X(x,s)|_{G^\sharp}\big) \subset W$ *for all* $(x,s) \in C^\circ \times C^{\sharp\circ}$,

ii) $(X|_{G^\sharp}, S|_G)$ *is a square-root field for the self-scaled barrier functional* $F|_{C^\circ}$.

iii) If (X,S) *is congruent then* $(X|_{G^\sharp}, S|_G)$ *is congruent too.*

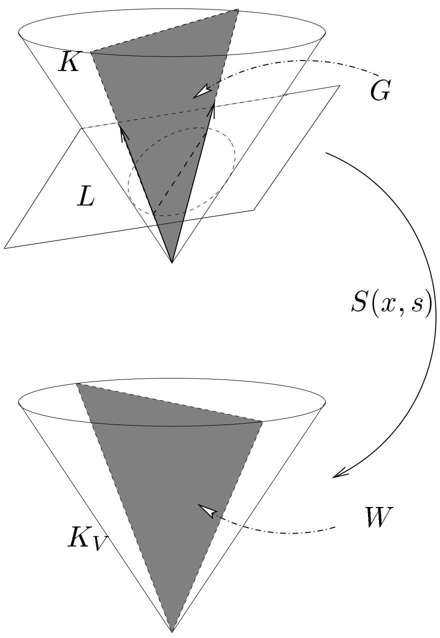

Figure 3: Linear programming sub-instances live in G.

Proof. $i)$ Consider a point $(x, s) \in C^\circ \times C^{\sharp\circ}$, and denote the scaling point of x and s for the self-scaled barrier functional $F|_{C^\circ}$ by $\bar{w}(x, s) \in C^\circ$. Since $F''(\bar{w})|_G \equiv F|''_{C^\circ}(\bar{w})$ holds true we must have $F''(\bar{w})x = s$, and by the uniqueness of scaling points for F we get $w(x, s) = \bar{w}(x, s)$. Therefore,

$$\begin{aligned}
\mathrm{im}\big(S(x,s)|_G\big) &= X(x,s) \circ F''\big(w(x,s)\big)[G] \\
&= X(x,s) \circ F|''_{C^\circ}\big(\bar{w}(x,s)\big)[G] = X(x,s)\big[G^\sharp\big] \\
&= \mathrm{im}\big(X(x,s)|_{G^\sharp}\big) \subset W.
\end{aligned}$$

$ii)$ Moreover, since F is isotropic we have $F'(x) \in G^\sharp$, and since $-F'(x) \in K^{\sharp\circ}$ this implies that $-F'(x) = -F|'_{C^\circ}(x) \in C^{\sharp\circ}$. Therefore,

$$e_V = X(x,s)\big[-F'(x)\big] = X(x,s)|_{G^\sharp}\big[-F|'_{C^\circ}(x)\big] \in W.$$

In Lemma H1.4.7 we saw that $F|_{C^\circ}$ has the same complexity value ν as F, and hence $\|e_V\|$ has length equal to the square-root of the complexity value of $F|_{C^\circ}$. Furthermore,

$$\begin{aligned}
\big(X(x,s)|_{G^\sharp}\big)^* \circ X(x,s)|_{G^\sharp} &\equiv \pi_G \circ X(x,s)^*|_W \circ \pi_W \circ X(x,s)|_{G^\sharp} \\
&\equiv \pi_G \circ X(x,s)^* \circ X(x,s)|_{G^\sharp} \\
&= \pi_G F''^{-1}(x)|_{G^\sharp} = F''^{-1}_{C^\circ}(x)
\end{aligned}$$

for any projections π_G and π_W into G and W respectively. Finally, $S(x,s)|_G = X(x,s)|_{G^\sharp} \circ F|''_{C^\circ}\big(\bar{w}(x,s)\big)$ as we have seen in part $i)$ already. Hence, this proves that $\big(X|_{G^\sharp}, S|_G\big)$ is a square-root field for $F|_{C^\circ}$ as claimed.

iii) For all $(x,s) \in C^\circ \times C^{\sharp\circ}$ the operator $S(x,s)$ is non-singular and $S|_G(x,s) \in \mathrm{Iso}(G,W)$. Together with the condition $S(x,s)K \equiv K_V$ this implies that $S|_G(x,s)C \equiv K_V \cap W$. $\square$

Proposition 4.8. *Under the assumptions of Definition 4.6 suppose K is irreducible and (X,S) is a congruent square-root field for F that reduces canonically on LP sub-instances. Let C and $W \subset V$ be as in Definition 4.6. Then there exists a constant $c_1 > 0$ and an orthogonal basis $D_K \subset W$ such that*

$$\left(c_1^{-1/2} X|_{G^\sharp}, c_1^{-1/2} S|_G\right)$$

expressed in coordinates with respect to B_K and D_K is the primordial square-root field $\left(\hat{X}, \hat{S}\right)$ from Example 2.14 for the standard log-barrier functional on C.

Proof. We express all the objects in G in coordinates with respect to B and identify C with $\mathbb{R}^q_+$. Corollary H1.4.5 and Lemma H1.4.7 imply that there exist constants $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that $U = c_1 F|_{C^\circ} + c_2$ is the standard log-barrier functional on $C \cong \mathbb{R}^q_+$. By virtue of Lemmas 4.5 and 4.7 $\left(c_1^{-1/2} X|_{G^\sharp}, c_1^{-1/2} S|_G\right)$ is a congruent square-root field for U. But Theorem 2.13 says that all congruent square-root fields for U must be of the form $\left(\Omega \circ \hat{X}, \Omega \circ \hat{S}\right)$ for some $\mathcal{C}^1$-operator field $\Omega : C^\circ \times C^{\sharp\circ} \to \mathcal{O}(C_W)$. But there are only $q!$ elements in $\mathcal{O}(C_W)$, and hence we must have $\Omega \equiv cst$. A relabelling of the basis elements chosen for W achieves thus that $\Omega \equiv \imath$ and this proves the claim. $\square$

The next result shows that the class of scale-invariant square-root fields that reduce canonically is closed under the operation of taking direct sums.

Theorem 4.9. *Under the setup of Theorem H2.3.17, if the $\left(X^{(ij)}, S^{(ij)}\right)$ are scale-invariant for $\left(e^{(ij)}, B^{(ij)}\right)$ and reduce canonically on LP sub-instances then (X,S) is scale-invariant for (e,B) and reduces canonically on LP sub-instances.*

Proof. Consider $W := \bigoplus_J W^{(ij)}$ and take Lemma 4.2 into account. The proof then becomes trivial. $\square$

The next lemma is going to be a useful tool in the proofs of the subsequent theorems.

Lemma 4.10. *Let K be irreducible and not isomorphic to a Lorentz cone and consider a basis B with frame B_K for (K,E). Let us define $G := \mathrm{span}\left(B_K\right)$ and let $\theta \in \mathcal{Q}(K)$ be such that $\theta\big|_G = \imath$. Then $\theta\left(e_i\right) = \pm e_i$ for all $e_i \in B$.*

Proof. We will show this only for the cone of positive semidefinite matrices. The case $K = \mathbb{R}_+$ is trivial, and all other cases can be proven exactly the same way as the positive semidefinite case, with orthogonal matrices being replaced by unitary matrices. After introducing coordinates we may assume without loss of generality that B is the canonical basis for $\mathrm{S}^{n\times n}(\mathbb{R})_+$ and B_K is the frame from Example H1.2.12. Let $\theta \in \mathcal{Q}(K)$ be such that $\theta\big|_G = \imath$. By virtue of Lemma H1.4.6 there exists a matrix $O \in \mathcal{O}(n)$ such that $\theta = \mathcal{N}(O)$. The condition $\theta\big|_G = \imath$ then translates into

$$O^T E_{ii} O = E_{ii} \qquad \text{for all } i \in \mathbb{N}_n, \tag{4.9}$$

where E_{ii} is the $n \times n$-matrix whose ii-entry equals 1 and all other entries are zero. Let us denote the i^{th} row of the matrix O by O_i. Then condition (4.9) can be restated in terms of the exterior products of these rows as follows:

$$O_i^T \cdot O_i = E_{ii} \qquad \text{for all } i \in \mathbb{N}_n,$$

But this implies that $O_i = \pm E_i$ for all i, where E_i is the row vector whose i^{th} entry equals 1 and all other entries are zero. The claim now follows immediately. □

The next theorem is a partial converse of Theorems 4.9 and H2.3.17.

Theorem 4.11. *Under the setup of Theorem 3.4, suppose none of the irreducible components $K^{(ij)}$ of K is isomorphic to a Lorentz cone. If (X,S) is scale-invariant for (e,B) and reduces canonically on LP sub-instances then it decomposes into a direct sum*

$$(X,S) = \Big(\bigoplus_J X^{(ij)}, \bigoplus_J S^{(ij)}\Big) \tag{4.10}$$

of square-root fields $\big(X^{(ij)}, S^{(ij)}\big)$ for $F^{(ij)}$ which are scale-invariant for the structures $\big(e^{(ij)}, B^{(ij)}\big)$ and reduce canonically on LP sub-instances.

Proof. Theorem 3.4 says that we can write (X,S) as a direct sum (4.10) of operator fields $\big(X^{(ij)}, S^{(ij)}\big)$ which possess all the required properties for congruent square-root fields for the tuples $\big(F^{(ij)}, e_V^{(ij)}, K_V^{(ij)}\big)$ with the sole exception that it may not necessarily be the case that the $X^{(ij)}(x,s)$ depend only on the projection $\big(x^{(ij)}, s^{(ij)}\big)$ of (x,s) into the subspace $E^{(ij)} \times E^{\sharp(ij)}$. Let us now show that this property holds however true in the presence of scale-invariance and canonical reduction. Let us use the notation $C^{(ij)} := \mathrm{cone}\big(B_K^{(ij)}\big)$, $G^{(ij)} := \mathrm{span}\big(B_K^{(ij)}\big)$, $C := \bigoplus_J C^{(ij)} = \mathrm{cone}\big(B_K\big)$ and $G := \bigoplus_J G^{(ij)} = \mathrm{span}\big(B_K\big)$ as in Definition 4.6. Likewise, we shall use the notation $C^{\sharp(ij)}, G^{\sharp(ij)}, C^\sharp$ and $G^\sharp$. Since (X,S) reduces canonically on LP sub-instances, there exists a $\kappa(K)$-dimensional linear subspace $W \subset V$ such that $X\big[G^\sharp\big] \equiv W \equiv S[G]$. Hence, for all $(x,s) \in C^\circ \times C^{\sharp\circ}$ we have

$$\begin{aligned} X^{(ij)}|_{G^{\sharp(ij)}}(x,s)\big[C^{\sharp(ij)}\big] &= K_V^{(ij)} \cap W \\ &=: C_V^{(ij)}. \end{aligned} \tag{4.11}$$

Let us now fix a vector $x^{(ij)} \in C^{(ij)\circ}$ and consider the set $P := \big(C^\circ \cap \pi_{E^{(ij)}}^{-1}\big(x^{(ij)}\big)\big) \times C^{\sharp\circ}$ of vectors in $C^\circ \times C^{\sharp\circ}$ whose $E^{(ij)}$-component is $x^{(ij)}$. Let

$$\begin{aligned} \hat{x} &:= \Big(\bigoplus_{J\setminus\{(ij)\}} e^{(mn)}\Big) \oplus x^{(ij)} \qquad \text{and} \\ \hat{s} &:= -F'(e). \end{aligned}$$

Then clearly $(\hat{x},\hat{s}) \in P$. By virtue of Theorem 3.4 $X^{(ij)}(x,s)$ is a square-root of $F^{(ij)\prime\prime-1}\big(x^{(ij)}\big)$ for all $(x,s) \in P$, and hence there exists a continuous orthogonal field $\Omega : P \to \mathcal{O}(K_V^{(ij)})$ such that

$$X^{(ij)}(x,s) = \Omega(x,s) \circ X^{(ij)}\big(\hat{x},\hat{s}\big) \tag{4.12}$$

for all $(x,s) \in P$. Now, canonical reducibility implies that

$$\begin{aligned} C_V^{(ij)} &\equiv X^{(ij)}(x,s)\big[C^{\sharp(ij)}\big] \\ &\overset{(4.12)}{=} \Omega(x,s) \circ X^{(ij)}\big(\hat{x},\hat{s}\big)\big[C^{\sharp(ij)}\big] \\ &= \Omega(x,s)\big[C_V^{(ij)}\big], \end{aligned}$$

and hence the following is a well defined continuous map:

$$\begin{aligned} P &\to \mathcal{O}(C_V^{(ij)}), \\ (x,s) &\mapsto \Omega(x,s)\big|_{C_V^{(ij)}}. \end{aligned}$$

However, $\mathcal{O}(C_V^{(ij)})$ is a finite group, since $C_V^{(ij)} \cong C^{(ij)}$ is polyhedral, and hence $\Omega|_{C_V^{(ij)}}$ is constant and equal to $\Omega\big(\hat{x},\hat{s}\big)|_{C_V^{(ij)}} = \imath$. Lemma 4.10 combined with the facts that Ω is continuous and $\Omega\big(\hat{x},\hat{s}\big) = \imath$ then imply that $\Omega \equiv \imath$ too. Substituting this equality in equation (4.12) we get

$$X^{(ij)}(x,s) \equiv X^{(ij)}\big(\hat{x},\hat{s}\big) \qquad \forall\ (x,s) \in P. \tag{4.13}$$

Thus, so far we have proven that on the set $C^\circ \times C^{\sharp\circ}$ the operator field $X^{(ij)}(x,s)$ depends only on the projection $x^{(ij)}$ of x into the subspace $E^{(ij)}$.

Now, for points $(x,s) \in K^\circ \times K^{\sharp\circ}$ in general position consider the map

$$\theta^{-*} := F''(e) \circ F''_\sharp\big(t(e,s)\big) \in \operatorname{Aut}\big(K^\sharp\big).$$

Then it follows from equation (H1.3.2) that

$$\begin{aligned} \theta &= \bigoplus_J F^{(mn)\prime\prime-1}\big(e^{(mn)}\big) \circ F_\sharp^{(mn)\prime\prime-1}\big(t^{(mn)}(e^{(mn)}, s^{(mn)})\big) \\ &=: \bigoplus_J \theta^{(mn)}, \end{aligned}$$

and Lemma 4.2 implies that

$$X^{(ij)}(x,s) = \Omega^{(ij)-1}_{\left(\theta^{(ij)},w^{(ij)}(x^{(ij)},s^{(ij)})\right)} \circ X^{(ij)}\big(\theta x, -F'(e)\big) \circ \theta^{(ij)-*}. \tag{4.14}$$

But by Lemma H1.2.21 there exists an orthogonal operator

$$O = \bigoplus_J O^{(mn)} \in \bigoplus_J \mathcal{Q}\big(K_V^{(ij)}\big) = \mathcal{Q}(K)$$

such that $O[\theta x] \in C^\circ$. Lemma 3.1 implies that $-F'(e)$ is invariant under the action of the group $\mathcal{O}(K^\sharp)$. Combining this fact with Lemma 4.2 we get

$$\begin{aligned} X^{(ij)}\big(\theta x, -F'(e)\big) &= \\ &= \Omega^{(ij)-1}_{\left(O^{(ij)}, w^{(ij)}\left(\theta^{(ij)}x^{(ij)}, -F^{(ij)\prime}(e^{(ij)})\right)\right)} \circ X^{(ij)}\big(O[\theta x], -F'(e)\big) \circ O^{(ij)-*}. \end{aligned} \tag{4.15}$$

Finally, note that equation (4.13) implies that $X^{(ij)}\big(O[\theta x], e\big)$ depends only on

$$\pi_{E^{(ij)}} O[\theta x] = O^{(ij)}\big[\theta^{(ij)}x^{(ij)}\big],$$

and $O^{(ij)}$ depends only on $x^{(ij)}$ and $\theta^{(ij)}$, which in turn depends only on $s^{(ij)}$. Therefore, equations (4.14) and (4.15) imply that $X^{(ij)}$ depends only on $\big(x^{(ij)}, s^{(ij)}\big) \in K^{(ij)\circ} \times K^{(ij)\sharp\circ}$. Hence, we have shown that the decomposition $(X, S) = \bigoplus_J \big(X^{(ij)}, S^{(ij)}\big)$ exists. All other claims follow easily. $\square$

Next, we are going to present a proposition that yields a characterization of canonical reduction in a very useful algebraic form: In part $i)$ we will see that a sufficient condition for the canonical reducibility of scale-invariant square-root fields is that $X(e, s)$ be independent of s (with e fixed!). Moreover, in part $ii)$ we will see that this condition is also necessary in the cases where the symmetric cone does not contain any irreducible component isomorphic to a Lorentz cone.

Proposition 4.12. *Let K, F be self-scaled and assume F is isotropic. Let us choose a fixed vector $e \in K^\circ$ and a basis $B \in \mathcal{B}^{(e)}$ with frame centered at e. Let (X, S) be a square-root field for F which is scale-invariant for (e, B, D) where D is some orthonormal basis of V.*

- *i) If $X(e, s) \equiv \Omega \circ F_B''^{-1/2}(e)$ for all $s \in K^{\sharp\circ}$ and for some fixed operator $\Omega \in \mathcal{O}(K_V)$, then (X, S) reduces canonically on LP sub-instances.*
- *ii) If none of the irreducible parts $K^{(ij)}$ of K are isomorphic to a Lorentz cone and if (X, S) reduces canonically on LP sub-instances, then there exists an orthogonal operator $\Omega \in \mathcal{O}(K_V)$ such that $X(e, s) \equiv \Omega \circ F_B''^{-1/2}(e)$ for all $s \in K^{\sharp\circ}$. Moreover, (X, S) is homogeneous (see Definition 2.12).*
- *iii) The property that (X, S) reduces canonically on LP sub-instances is independent of the choice of $B \in \mathcal{B}^{(e)}$.*

Proof. $i)$ Because of Theorem 4.9 it is sufficient to prove this claim for K irreducible. Without loss of generality we may assume that $\Omega = \imath$. Let $C \subset G \subset E$, D_K and $W := \operatorname{span}\big(D_K\big)$ be defined as usual. In the derivations below, keep in mind that the following sets of matrices are all equal:

$$\begin{aligned} \{\nabla_B^2 F(x) : x \in K^\circ\} &= \{\nabla_B^2 F(x)^{1/2} : x \in K^\circ\} \\ &= \{\nabla_B^2 F_\sharp(s)^{1/2} : s \in K^{\sharp\circ}\} \\ &= \{\nabla_B^2 F_\sharp(s) : s \in K^{\sharp\circ}\}. \end{aligned} \tag{4.16}$$

In fact, in the proof of Lemma 3.3 we showed that $\nabla_B^2 F(x)^{1/2} = \nabla_B^2 F\big(w(x,-F'(e))\big)$ for all $x \in K^\circ$, and it follows from H2.2.6 *iii)* that $F''(x) = F''(e) \circ F''_\sharp\big(F''(e)x\big) \circ F''(e)$ and hence,

$$\nabla_B^2 F(x) = I \cdot \nabla_B^2 F_\sharp\big(F''(e)x\big) \cdot I.$$

Recall that in equation (4.6) we defined the map $(x,s) \mapsto v(x,s)$. Let $(x,s) \in C^\circ \times C^{\sharp\circ}$ and $z := F''^{-1/2}_{(B,D)}(e)^* v(x,s)$. Since $w(x,s) \in C^\circ$ and $F''(w(x,s))|_G \in \mathrm{Iso}(G, G^\sharp)$ we have that

$$v(x,s) = F''^{1/2}_{(B,D)}(w(x,s))x \in W := F''^{1/2}_{(B,D)}(e)G,$$

and hence $z \in C^\circ$. Consider the following two automorphisms of K:

$$\begin{aligned} \theta_1 &:= F''^{-1}(e) \circ F''\big(w(z, F''(e)x)\big) \quad \text{and} \\ \theta_2 &:= F_B''^{-1/2}(w(x,s))^* \circ F_B''^{1/2}(e). \end{aligned}$$

Then $\theta_1 z = \theta_2 z = x$. If $O \circ F''^{-1}(e) \circ F''(y)$ is the polar decomposition of θ_2, then it follows from equation (4.16) that we must have $O = \imath$. Together with the fact that scaling points are unique this implies that $y = w(z, F''(e)x)$ and $\theta_1 = \theta_2$. In particular, we get that

$$\theta_1^* s = F_B''^{1/2}(e)^* \circ F_B''^{-1/2}(w(x,s))s = F''(e)z.$$

Therefore,

$$X(x,s) = X\big(\theta_1 z, \theta_1^{-*}\theta_1^* s\big) = \Omega_{(\theta_1, w(z, F''(e)z))} \circ X(z, F''(e)z) \circ \theta_1^*.$$

But $w(z, F''(e)z) = e$ and Lemma 4.1 *iii)* says that $\Omega_{(\theta_1, e)} = \imath$. Hence, we get

$$X(x,s) = X(z, F''(e)z) \circ \theta_1^*. \tag{4.17}$$

Next, consider

$$\theta_3 := F''^{-1}(e) \circ F''\big(w\big(e, F''(e)z\big)\big)$$

and note that $\theta_3 e = z$. Let $b := \theta_3^* \circ F''(e)z$. Then

1. $b \in C^{\sharp\circ}$,
2. $\theta_3 w(e,b) = e$,
3. $\theta_3^2 = F''^{-1}(e) \circ F''\big(w(e,b)\big)$.

In fact, Property 1 holds true since $z \in C^\circ \subset G$, $F''(e)|_G \in \mathrm{Iso}(G, G^\sharp)$ and $\theta_3|_G \in \mathrm{Aut}(G)$. Properties 2 and 3 can be checked in coordinates with respect to $B_K = \{e_1, \ldots, e_\kappa\}$, and since $e, w(e,b) \in C^\circ$ and $b \in C^{\sharp\circ}$ we can work with $F|_{C^\circ}$: Since K was assumed irreducible, Lemma H1.4.7 applies, and it follows that there exist constants $c_1 > 0$ and

$c_0 \in \mathbb{R}$ such that $F|_{C^\circ} = c_1 U + c_0$, where U is the standard logarithmic barrier functional defined by the basis $\{e_1, \ldots, e_\kappa\}$ of G:

$$U : \sum_{i=1}^{\kappa} x_i e_i \mapsto -\sum_{i=1}^{\kappa} \ln x_i.$$

Consider the vector $\hat{e} = \sum_{i=1}^{\kappa} e_i$. Then $\nabla_B^2 U(\hat{e}) = I_\kappa$. Moreover, we know from Lemma 3.1 *ii)* that there exist a $\lambda > 0$ such that $e = \lambda \cdot \hat{e}$. Hence,

$$I_q = \nabla_B^2 U(\hat{e}) \overset{H1.2.3vi)}{=} c_1^{-1} \lambda^2 \nabla_B^2 F(e) = c_1^{-1} \lambda^2 \cdot I_q$$

and this shows that $c_1 = \lambda^2$. Let us denote scaling points with respect to U by $\hat{w}$. Then $w(x,s) = \lambda \hat{w}(x,s)$ for all $(x,s) \in C^\circ \times C^{\sharp\circ}$. It is now straightforward to verify the following equations, where we denote the correspondence between tensors and their matrix representation with respect to B_K by $\simeq$ (the reader may compute all these matrices using the formulas provided in Example H2.2.3, and by making use of the scaling defined by $F = \lambda^2 U + c_0$):

$$\begin{aligned}
(\theta_3)\big|_{C^\circ} &\simeq \lambda^{-1} \operatorname{Diag}(z_1, \ldots, z_\kappa), \\
b &\simeq \lambda^{-1} \operatorname{Diag}(z_1, \ldots, z_\kappa) \cdot (z_1, \ldots, z_\kappa)^T = \lambda^{-1}(z_1^2, \ldots, z_\kappa^2)^T, \\
w(e,b) &= \lambda \hat{w}(\lambda \hat{e}, b) \simeq \lambda^2 (z_1^{-1}, \ldots, z_\kappa^{-1})^T, \\
\theta_3 w(e,b) &\simeq \lambda (1, \ldots, 1)^T \simeq e, \\
(\theta_3^2)\big|_{C^\circ} &\simeq \lambda^{-2} \operatorname{Diag}(z_1^2, \ldots, z_\kappa^2) \\
&\simeq F''^{-1}(e) \circ \lambda^2 U''\big(w(e,b)\big) = F''^{-1}(e) \circ F|''_{C^\circ}\big(w(e,b)\big).
\end{aligned}$$

The third correspondence proves that property 2 holds, and the last one tells us that property 3 holds on the subspace G. Therefore, the polar decomposition of θ_3^2 must be of the form

$$\theta_3^2 = \Omega \circ F''^{-1}(e) \circ F''\big(w(e,b)\big),$$

with $\Omega \in \mathcal{O}(K)$ such that $\Omega|_G = \imath$. But since Theorem H1.2.8 *iii)* implies that

$$\begin{aligned}
\theta_3^2 &= F''^{-1}(e) \circ F''\big(w(e, F''(e)z)\big) \circ F''_\sharp\big(-F'(e)\big) \circ F''\big(w(e, F''(e)z)\big) \\
&= F''^{-1}(e) \circ F''\big(F''^{-1}(w(e, F''(e)z))[-F'(e)]\big),
\end{aligned}$$

we know in fact that $\Omega = \imath$ by uniqueness of the polar decomposition. Hence, property 3 holds as claimed. Combined with Lemma 4.1, properties 1, 2, and 3 imply that

$$\begin{aligned}
\imath &\overset{4.1ii)}{=} \Omega_{\left(\theta_3^2, w(e,b)\right)} \\
&\overset{4.1iv)}{=} \Omega_{\left(\theta_3, \theta_3 w(e,b)\right)} \circ \Omega_{\left(\theta_3, w(e,b)\right)} \\
&\overset{2}{=} \Omega_{\left(\theta_3, e\right)} \circ \Omega_{\left(\theta_3, w(e,b)\right)} \\
&\overset{4.1iii)}{=} \Omega_{\left(\theta_3, w(e,b)\right)}.
\end{aligned}$$

Therefore we have

$$\begin{aligned} X\big(z, F''(e)z\big) &= X\big(\theta_3 e, \theta_3^{-*} b\big) \\ &= \Omega_{\big(\theta_3, w(e,b)\big)} \circ X(e,b) \circ \theta_3^* \\ &= F_B''^{-1/2}(e) \circ \theta_3^*. \end{aligned}$$

Together with equation (4.17) this yields

$$X(x,s) = F_B''^{-1/2}(e) \circ \theta_3^* \circ \theta_1^*.$$

But $\theta_1|_G, \theta_3|_G \in \mathrm{Aut}(G)$, since the vectors e, $w(z, F''(e)x)$ and $w\big(e, F''(e)z\big)$ all lie in C°. Hence, $\mathrm{im}\big(X(x,s)|_{C^{\sharp\circ}}\big) \subset \mathrm{im}\big(F_B''^{-1/2}(e)|_{C^{\sharp\circ}}\big)$ for all $(x,s) \in C^\circ \times C^{\sharp\circ}$, and this proves the claim.

ii) Because of Theorem 4.11 it suffices to prove this claim for irreducible K. Let $\kappa := \kappa(K)$ be the Carathéodory number of K, and consider the vector $\hat{e} = \sum_{i=1}^{\kappa} e_i$. Let U be the standard logarithmic barrier functional defined by the basis $\{e_1, \ldots, e_\kappa\}$ of $\mathrm{span}(e_1, \ldots, e_\kappa)$ on $C^\circ = \mathrm{cone}\big(e_1, \ldots, e_\kappa\big)^\circ$:

$$U : \sum_{i=1}^{\kappa} x_i e_i \mapsto -\sum_{i=1}^{\kappa} \ln x_i.$$

Lemma 3.1 implies that there exists a constant $\lambda > 0$ such that $e = \lambda\hat{e}$. As in the proof of part *i)* one can show that together with the fact that F is isotropic this implies that there exists a constant $c_0 \in \mathbb{R}$ such that $F|_{C^\circ} = \lambda^2 U + c_0$. Proposition 4.8 *iii)* implies that there exists an orthogonal transformation $O \in \mathcal{O}(V)$ such that for all $x = \sum_{i=1}^{\kappa} \eta_i e_i \in C^\circ$ and $s \in C^{\sharp\circ}$ we have

$$\lambda O \circ X(x,s)|_{C^\circ} : \sum_{i=1}^{\kappa} \mu_i e_i \mapsto \sum_{i=1}^{\kappa} (\eta_i \mu_i) e_i.$$

In particular, for $x = e = \lambda\hat{e}$ and $s \in C^{\sharp\circ}$ we get

$$O \circ X(e,s)|_{C^\circ} : \sum_{i=1}^{\kappa} \mu_i e_i \mapsto \sum_{i=1}^{\kappa} \mu_i e_i, \tag{4.18}$$

which is the same as to say that

$$X(e,s)|_{C^{\sharp\circ}} \equiv O \circ F_B''^{-1/2}(e)|_{C^{\sharp\circ}} \qquad \forall\, s \in C^{\sharp\circ}.$$

Without loss of generality we may assume that the orthogonal basis D for V was chosen such that $O = \imath$. Now, for $s \in K^{\sharp\circ}$ in general position, Lemma H1.2.21 says that there exists a $\theta \in \mathcal{O}(K)$ such that $\theta^* s \in C^{\sharp\circ}$, and Lemma 4.1 implies that

$$\begin{aligned} X(e,s) &= X\big(\theta\theta^{-1} e, \theta^{-*}\theta^* s\big) \\ &= \Omega_{(\theta, w(e, \theta^* s))} \circ X\big(e, \theta^* s\big) \circ \theta^* \\ &\overset{4.1\,i)}{=} F_B''^{1/2}(e) \circ \theta \circ F_B''^{-1/2}(e)^* \circ X\big(e, \theta^* s\big) \circ \theta^*. \end{aligned}$$

Hence, the restriction of this operator to $\theta^{-*}C^{\sharp\circ}$ is given by

$$
\begin{aligned}
X(e,s)|_{\theta^{-*}C^{\sharp\circ}} &= F_B''^{1/2}(e)\circ\theta\circ F_B''^{-1/2}(e)^*\circ X\big(e,\theta^*s\big)|_{C^{\sharp\circ}}\circ\theta^*|_{\theta^{-*}C^{\sharp\circ}}\\
&= F_B''^{1/2}(e)\circ\theta\circ F_B''^{-1/2}(e)^*\circ F_B''^{-1/2}(e)\circ\theta^*|_{\theta^{-*}C^{\sharp\circ}}\\
&= F_B''^{1/2}(e)\circ\theta\circ F_B''^{-1}(e)\circ\theta^*|_{\theta^{-*}C^{\sharp\circ}}\\
&= F_B''^{-1/2}(e)|_{\theta^{-*}C^{\circ}}. \qquad (4.19)
\end{aligned}
$$

We know from Definition H2.3.15 that there exists a continuous operator field

$$
\begin{aligned}
\Omega : K^{\sharp\circ} &\to \mathcal{O}(K_V),\\
s &\mapsto \Omega(s)
\end{aligned}
$$

such that $X(e,s)\equiv\Omega(s)\circ F_B''^{-1/2}(e)$. For all $s\in K^{\sharp\circ}$ let $W(s):=F_B''^{-1/2}(e)\circ\theta^{-*}C^{\circ}$ where $\theta\in\mathcal{O}(K)$ is defined as above. Equation (4.19) then implies that $\Omega(s)|_{W(s)}=\imath$ for all s. Finally, continuity of Ω and Lemma 4.10 imply that $\Omega\equiv\Omega\big(-F'(e)\big)$ is constant, and this proves the first claim of part *ii*).

Let us now show homogeneity. In Definition 2.12 we already remarked that

$$
w(\tau x,\eta s)=\tau^{1/2}\eta^{-1/2}w(x,s)\qquad\text{for all }\tau,\eta>0. \qquad (4.20)
$$

For all $(x,s)\in K^{\circ}\times K^{\sharp\circ}$ and $\tau,\eta>0$ consider the transformations

$$
\begin{aligned}
&\psi^{-1}:=F''^{-1}(e)\circ F''\Big(w\big(\tau x,-F'(e)\big)\Big)\in\operatorname{Aut}(K)\qquad\text{and}\\
&\phi^{-1}=\tau^{-1}\psi\in\operatorname{Aut}(K).
\end{aligned}
$$

Using scale-invariance of (X,S) we get

$$
\begin{aligned}
X\big(\tau x,\eta s\big) &= X\big(\psi e,\eta\,\psi^{-*}\circ\psi^*s\big)\\
&\overset{(4.8)}{=} \Omega_{(\psi,w(e,\eta\,\psi^*s))}\circ X\big(e,\eta\,\psi^*s\big)\circ\psi^*,\\
&= \Omega_{(\psi,w(e,\eta\,\psi^*s))}\circ X\big(e,\tau^{-1}\psi^*s\big)\circ\psi^*\\
&= \Omega_{(\psi,w(e,\eta\,\psi^*s))}\circ X\big(\phi x,\phi^{-*}s\big)\circ\psi^* \overset{(4.20),4.1\,v),(4.8)}{=}\\
&= \Omega_{(\psi,w(e,\psi^*s))}\circ\Omega_{(\phi,w(x,s))}\circ X(x,s)\circ\phi^*\circ\psi^*. \qquad (4.21)
\end{aligned}
$$

But now we can substitute the identity $\phi^*\circ\psi^*=\tau\cdot\imath$ in equation (4.21). Note also that equation (4.3) implies that $F''\big(\psi^{-1}w(x,s)\big)e=\tau\psi^*s$, which shows that

$$
\psi^{-1}w(x,s)=w(e,\tau\psi^*s)\overset{(4.20)}{=}\tau^{-1/2}w(e,\psi^*s).
$$

But then we get

$$
\begin{aligned}
\imath &= \Omega_{\left(\imath, w(x,s)\right)} \\
&= \Omega_{\left(\tau^{-1}\cdot\psi\circ\phi, w(x,s)\right)} \\
&\overset{4.1\,v)}{=} \Omega_{\left(\psi\circ\phi, w(x,s)\right)} \\
&\overset{4.1\,iv)}{=} \Omega_{\left(\psi, \phi\, w(x,s)\right)} \circ \Omega_{\left(\phi, w(x,s)\right)} \\
&= \Omega_{\left(\psi, \tau\cdot\psi^{-1} w(x,s)\right)} \circ \Omega_{\left(\phi, w(x,s)\right)} \\
&\overset{4.1\,v)}{=} \Omega_{\left(\psi, \psi^{-1} w(x,s)\right)} \circ \Omega_{\left(\phi, w(x,s)\right)} \overset{(4.20)}{=} \\
&= \Omega_{\left(\psi, \tau^{1/2} w(e,\psi^* s)\right)} \circ \Omega_{\left(\phi, w(x,s)\right)} \overset{(4.20),4.1\,v)}{=} \\
&= \Omega_{\left(\psi, w(e,\psi^* s)\right)} \circ \Omega_{\left(\phi, w(x,s)\right)}.
\end{aligned} \tag{4.22}
$$

Substituting (4.22) in (4.21) we get $X\left(\tau x, \eta s\right) = \tau X(x,s)$, i.e. the square-root field (X,S) is homogeneous as claimed.

iii) Let $B, B' \in \mathcal{B}^{(e)}$. Then clearly, $\theta := F_{B'}^{\prime\prime -1/2}(e)^* \circ F_B^{\prime\prime 1/2}(e) \in \mathcal{O}(K)$ and $\varpi := F_B^{\prime\prime 1/2}(e) \circ F_{B'}^{\prime\prime -1/2}(e)^* \in \mathcal{O}(K_V)$. Suppose (X,S) is scale-invariant and reduces canonically with respect to the basis $B \in \mathcal{B}^{(e)}$. Note that the definition of the operators $F_B^{\prime\prime 1/2}(e)$ and $F_{B'}^{\prime\prime 1/2}(e)$ (that depends on the basis D chosen in V) implies the existence of an indexation of $B = \left\{e_1, \ldots, e_n\right\}$ and $B' = \left\{e_1', \ldots, e_n'\right\}$ such that $\theta : e_i \mapsto e_i'$ for all i, and such that $B_K = \left\{e_1, \ldots, e_\kappa\right\}$ and $B_K' = \left\{e_1', \ldots, e_\kappa'\right\}$ are the unique frames of K defined by B and B' respectively. Let us use the notation $G := \operatorname{span}\left(B_K\right)$, $G^\sharp := \operatorname{span}\left(B_K^\sharp\right)$, $G' := \operatorname{span}\left(B_K'\right)$, and $G'^\sharp := \operatorname{span}\left(B_K'^\sharp\right)$. Likewise, we use $C, C', C^\sharp$ and $C'^\sharp$ for the conic hulls of the two frames and their duals. For all $(x', s') \in C'^\circ \times C'^{\sharp\circ}$ there exists a pair $(x,s) \in C^\circ \times C^{\sharp\circ}$ such that $x' = \theta x$ and $s' = \theta^{-*} s$. Therefore,

$$
\begin{aligned}
X\left(x', s'\right) &= X\left(\theta x, \theta^{-*} s\right) \\
&= \Omega_{\left(\theta, w(x,s)\right)} \circ X(x,s) \circ \theta^* \\
&\overset{4.1i)}{=} F_B^{\prime\prime 1/2}(e) \circ \theta \circ F_B^{\prime\prime -1/2}(e)^* \circ X(x,s) \circ \theta^* \\
&= \varpi \circ X(x,s) \circ \theta^*.
\end{aligned}
$$

But since $\theta^* : G'^\sharp \to G^\sharp$ we get that $\operatorname{im}\left(X(x', s'\right) \equiv W' := \varpi W$ for all $(x', s') \in C'^\circ \times C'^{\sharp\circ}$. Hence, the square-root field (X,S) reduces canonically with respect to B' too. □

We are finally ready to state our main theorem, which guarantees that scale-invariant primal-dual symmetric square-root fields that reduce canonically exist for every symmetric cone provided with an isotropic self-scaled barrier functional. The construction is based on the algebraic characterization presented in Proposition 4.12. It thus follows that in the cases where the underlying symmetric cone does not contain any irreducible component that is isomorphic to a Lorentz cone, scale-invariant square-root field that reduce canonically on LP subinstances are essentially unique.

Theorem 4.13. *Let $K \subset E$ be a symmetric cone with self-scaled isotropic barrier functional F. Let us fix a vector $e \in K^\circ$ and a basis $B \in \mathcal{B}^{(e)}$ with frame B_K centered at e. Let D be an orthogonal basis for V. Then there exists a square-root field (X, S) for F that is primal-dual symmetric, scale-invariant for (e, B, D) and reduces canonically on LP sub-instances.*

Proof. Note that by virtue of Theorem 4.4 primal-dual symmetry is implied automatically if the other properties can be established. For all $(x, s) \in K^\circ \times K^{\sharp\circ}$ consider $\theta := F''^{-1}\big(w(x, -F'(e)\big) \circ F''(e) \in \operatorname{Aut} K$ so that $x = \theta e$, and let us define

$$X(x, s) := \Omega_{(\theta,\theta^{-1}w(x,s))} \circ F''^{-1/2}_{(B,D)}(e) \circ \theta^* \tag{4.23}$$

and $S(x, s) := X^\sharp(x, s)$. First let us show that (X, S) is a square-root field for F.

$$\begin{aligned}
X^*(x,s)\circ X(x,s) &= \\
&= \theta \circ F''^{-1/2}_{(B,D)}(e)^* \circ \Omega^*_{\left(\theta,\theta^{-1}w(x,s)\right)} \circ \Omega_{\left(\theta,\theta^{-1}w(x,s)\right)} \circ F''^{-1/2}_{(B,D)}(e) \circ \theta^* \\
&= \theta \circ F''^{-1}(e) \circ \theta^* \\
&= F''_\sharp\big(t(x, -F'(e))\big) \circ F''(e) \circ F''^{-1}(e) \circ F''(e) \circ F''_\sharp\big(t(x, -F'(e))\big) \\
&\overset{H1.2.8i)}{=} F''_\sharp\big(t(e, -F'(x))\big) \circ F''(e) \circ F''_\sharp\big(t(e, -F'(x))\big) \\
&\overset{H1.2.8iii)}{=} F''^{-1}(x).
\end{aligned}$$

Hence, $X(x, s) \in F''^{-1/2}(x)$. Moreover,

$$\begin{aligned}
X(x,s)[-F'(x)] &= \\
&= \Omega_{(\theta,\theta^{-1}w(x,s))} \circ F''^{-1/2}_{(B,D)}(e) \circ F''(e) \circ F''_\sharp\big(t(e, -F'(x))\big)[-F'(x)] \\
&= \Omega_{(\theta,\theta^{-1}w(x,s))} \circ F''^{-1/2}_{(B,D)}(e)[-F'(e)] \\
&= e_V.
\end{aligned}$$

It is also clearly true that $(X, S) \in \mathcal{C}^{k-2}$.

Next, let us show that (X, S) is scale-invariant for (e, B, D). Let us therefore compute $X(\psi x, \psi^{-*}s)$ where $\psi \in \operatorname{Aut}(K)$. Let θ be as above and define $\tilde{\theta} := \psi \circ \theta$. Then, since $\tilde{\theta}e = \psi x$, uniqueness of the polar decomposition of $\tilde{\theta}^{-1}$ (see Theorem H1.2.22) implies that there exists an operator $\Omega \in \mathcal{O}(K)$ such that

$$\tilde{\theta} = F''^{-1}\big(w(\psi x, -F'(e)\big) \circ F''(e) \circ \Omega. \tag{4.24}$$

Therefore,

$$
\begin{aligned}
X(\psi x,\psi^{-*}s) &\overset{(4.23),(4.24)}{=} \\
&= \Omega_{\left(\psi\theta\Omega^{-1},\Omega\theta^{-1}\psi^{-1}w(\psi x,\psi^{-*}s)\right)} \circ F''^{-1/2}_{(B,D)}(e) \circ \Omega^{-*} \circ \theta^* \circ \psi^* \\
&\overset{(4.2)}{=} \Omega_{\left(\psi\theta\Omega^{-1},\Omega\theta^{-1}w(x,s)\right)} \circ F''^{-1/2}_{(B,D)}(e) \circ \Omega^{-*} \circ \theta^* \circ \psi^* \\
&\overset{4.1\, iv)}{=} \Omega_{\left(\psi,w(x,s)\right)} \circ \Omega_{\left(\theta,\theta^{-1}w(x,s)\right)} \circ \Omega_{\left(\Omega^{-1},\Omega\theta^{-1}w(x,s)\right)} \circ F''^{-1/2}_{(B,D)}(e) \circ \Omega^{-*} \circ \theta^* \circ \psi^* \\
&\overset{4.1\, i)}{=} \Omega_{\left(\psi,w(x,s)\right)} \circ \Omega_{\left(\theta,\theta^{-1}w(x,s)\right)} \circ F''^{-1/2}_{(B,D)}(e) \circ \theta^* \circ \psi^* \\
&\overset{(4.23)}{=} \Omega_{\left(\psi,w(x,s)\right)} \circ X(x,s) \circ \psi^*.
\end{aligned}
$$

This shows that the scaling properties of X are in accordance with the rules of Definition 4.3, and then S satisfies them automatically also.

Finally, since equation (4.23) implies that $X(e,s) = F''^{-1/2}_{(B,D)}(e)$, it follows from Proposition 4.12 $i)$ that (X,S) reduces canonically on LP sub-instances. $\square$

Example 4.14 (Linear Programming). Let the positive orthant $\mathbb{R}^n_+$ be provided with the standard logarithmic barrier functional F and the canonical Euclidean structure defined by $e = (1\dots 1)^T$. Then the canonical square-root field given in Example 2.14 is scale-invariant and reduces canonically. In fact, the polar decomposition $\theta = \Omega \circ F''(y)$ of $\theta \in \mathrm{Aut}(\mathbb{R}^n_+)$ is of the form $\Omega = \sigma \in \mathfrak{S}_n$, $F''(y) = \mathrm{Diag}(z)$ for some $z \in \mathbb{R}^n_{++}$. But $\Omega_{(\sigma,w)} \equiv \sigma$ for all $w \in \mathbb{R}^n_{++}$ (see Lemma 4.1 $i)$), and $F''(\mathrm{Diag}(z)w) = \mathrm{Diag}\big((z_1w_1\dots z_nw_n)^T\big)^{-2}$, i.e.

$$
F''^{1/2}(\mathrm{Diag}(z)w) = \mathrm{Diag}\big((z_1w_1\dots z_nw_n)^T\big)^{-1} = F''^{1/2}(w) \circ \mathrm{Diag}(z)^{-1},
$$

which implies that $\Omega_{(\mathrm{Diag}(z),w)} = \imath$. It follows by Lemma 4.1 $iv)$ that $\Omega_{(\theta,w)} = \sigma$. Hence,

$$
\begin{aligned}
X\big(\theta x,\theta^{-*}s\big) = X(\theta x) &= \mathrm{Diag}\Big(\big(z_{\sigma(1)}x_{\sigma(1)}\dots z_{\sigma(n)}x_{\sigma(n)}\big)^T\Big) \\
&= \sigma \circ X(x) \circ \mathrm{Diag}(z) \circ \sigma^{-1} = \Omega_{(\theta,w(x,s))} \circ X(x,s) \circ \theta^*.
\end{aligned}
$$

Trivially, this square-root field reduces canonically. $\square$

Example 4.15 (Semidefinite Programming). Consider the semidefinite cone $\mathrm{S}^{n\times n}(\mathbb{R})_+$ and provide it with the standard logarithmic barrier functional F. Let us choose the Euclidean structure defined by $e = I$ on $\mathrm{S}^{n\times n}(\mathbb{R})$, i.e. the standard scalar product that corresponds to the Frobenius norm (see Example H1.2.12). Then the square-root field given in Example 2.15 is scale-invariant and reduces canonically on LP sub-instances. In fact, let $\theta \in \mathrm{Aut}(\mathrm{S}^{n\times n}(\mathbb{R})_+)$, $x, s \in \mathrm{S}^{n\times n}(\mathbb{R})_{++}$, $w = w(x,s)$ and $v = v(x,s)$ (see Example 2.15). It follows from Lemma H1.4.6 that $\Omega_{(\theta,w)} = \mathcal{N}(o)$ for some $o \in \mathcal{O}(K)$. Using equations (4.5) and (4.7) we get

$$\begin{aligned}
X(\theta x, \theta^{-*}s) &= (o^T v o)^{-1/2} o^T \Big(w^{-1/2} \big(\theta^{-1} F''^{-1}(\theta x)[\cdot] \big) w^{-1/2} \Big) o (o^T v o)^{-1/2} \\
&= o^T v^{-1/2} w^{-1/2} \big(\theta^{-1} \theta F''^{-1}(x) \theta^* [\cdot] \big) w^{-1/2} v^{-1/2} o \\
&= o^T v^{-1/2} w^{-1/2} x \big(\theta^* [\cdot] \big) x w^{-1/2} v^{-1/2} o \\
&= \Omega_{(\theta,w)} \circ X(x,s) \circ \theta^*.
\end{aligned}$$

This combined with the fact that $S(x,s) = X(x,s) \circ F''(w)$ implies that the square-root field (X, S) renders Diagram (4.8) commutative. All that is left to show is that (X, S) reduces canonically. But for all $s \in \mathrm{S}^{n\times n}(\mathbb{R})_{++}$ it is true that

$$\begin{aligned}
w(e,s) &= e^{1/2} \big(e^{1/2} s e^{1/2} \big)^{-1/2} e^{1/2} = s^{-1/2}, \\
v(e,s) &= w(e,s)^{-1/2} e w(e,s)^{-1/2} = s^{1/2} = w(x,s)^{-1}, \\
X(e,s) &= v(e,s)^{-1/2} w(x,s)^{-1/2} e[\cdot] e w(x,s)^{-1/2} v(e,s)^{-1/2} = e[\cdot]e = F''^{-1/2}(e).
\end{aligned}$$

Hence, by Proposition 4.12 i), we are done. $\square$

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