Nesterov-Todd Directions are Newton Directions

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December 18, 1999

Abstract

The theory of self-scaled conic programming provides a unified framework for the theories of linear programming, semidefinite programming and convex quadratic programming with convex quadratic constraints. The standard search directions for interior-point methods applied to self-scaled conic programming problems are the so-called Nesterov-Todd directions. In this article we show that these direction fields are special cases of so-called *target directions*, a unifying concept for primal-dual interior point methods for self-scaled conic programming. In particular, this implies that Nesterov-Todd directions derive from a Newton system.

Key words. symmetric cones, self-scaled cones, Newton method, search directions, Nesterov-Todd direction, interior-point methods, self-scaled programming, linear programming, semidefinite programming, quadratic programming, second-order cone programming, Magnus expansion, orthogonal flow, Lie-group methods, asymptotic analysis.

AMS 1991 subject classification. Primary: 90C25, 34E10, 34B15. Secondary: 90C60, 90C05, 90C20.

*Research supported by the Swiss National Science Foundation through project "Newton Directions for Interior-Point Methods in Self-Scaled Convex Programming", by the National Science Foundation through grant DMS-9505155 and by the Office of Naval Research through grant N00014-96-1-0050.

1 Introduction

Self-scaled conic programming provides a unified framework for linear- semidefinite- and second-order-cone programming and was first investigated by Nesterov-Todd [28, 29]. The standard search direction fields used in so-called primal-dual interior-point methods for self-scaled conic programming are the so-called Nesterov-Todd (or N-T) directions (see Nesterov-Todd [28, 29] and equation (1.6) below).

In [10] we began the generalization of the so-called "V-space" approach for primal-dual interior-point methods from linear programming to self-scaled conic programming. Primal-dual interior point methods for convex optimization problems are designed to solve a problem and its dual jointly by making use of convex duality theory. The paradigm for such algorithms usually is to reduce the duality gap between primal and dual approximate solutions to zero. In the V-space approach, both the primal and the dual problem are mapped into some Euclidean space V lying "half-way" between the spaces E and $E^\sharp$ in which the primal and dual problems live. In [10] we introduced a class of differentiable operator fields that achieve a useful transformation into such a V-space. In [11] we used these objects which we call *square root fields* to define *target-maps* and *target directions*, a new infinite family of Newton directions that lend themselves to the construction of so-called *target-following* algorithms and *predictor-corrector* methods. We also showed that a certain class of target-maps generalize the notion of *weighted analytic centers* to the framework of self-scaled conic programming.

In the linear programming literature target directions and weighted analytic centers have led to a unified theory for the analysis of various important classes of primal-dual interior-point methods. (see e.g. Kojima-Mizuno-Yoshise [18], Monteiro-Adler [25], Mizuno [21], Jansen-Roos-Terlaky-Vial [15, 16]). For various generalizations of some aspects of this theory and for other related material see also Monteiro-Pang [26], Sturm-Zhang [33], Monteiro-Zanjacomo [24], Tunçel [36, 37], Todd [34] and Burer-Monteiro [1].

target directions are thus a promising family of search directions for use in a unifying theory of primal-dual interior-point methods for a large class of convex optimization problems including linear, semidefinite and second-order cone programming. Target directions are a rather flexible tool and can be designed to satisfy certain geometric properties. In [12] for instance we showed that for a special class of self-scaled barrier functionals (so-called *isotropic* self-scaled barrier functionals) there always exists a square-root field that leads to a primal-dual symmetric scale-invariant target direction. The purpose of the present article is to show that the family of target direction fields contains as special cases the standard Nesterov-Todd search-direction fields used in primal-dual interior-point methods for self-scaled conic programming. This further illustrates the promise of target directions as a unifying concept. In [10], Theorem 3.16 we showed that square-root fields can be manipulated via differentiable fields of orthogonal operators acting on V. Using this principle and starting from a given square-root field, we construct an orthogonal field such that the resulting target direction field coincides with the Nesterov-Todd field for a fixed parameter value μ that determines a point on the central path (see below). Since target directions are

Newton directions this implies that N-T directions are genuine Newton directions. This is a direct extension of the corresponding result for the special case of linear programming, developed by Kojima-Mizuno-Yoshise [18]. However, since practically all of the difficulties that come up in the theory of square-root fields collapse in the linear programming case (see [10], Section 5), a completely different set of analytic tools had to be developed in this paper to tackle with the general case. For the special case of semidefinite programming, Todd-Toh-Tütüncü [35] showed that the Nesterov-Todd direction can be interpreted as a em local Newton direction, i.e. for every primal-dual feasible point (x,s) they consider a different Newton system. The approach of Todd-Toh-Tütüncü to showing this result is entirely different from ours, and our result is an extension of theirs in that we show that Nesterov-Todd directions are *global* Newton directions. Let us also mention that it was already known that the the Nesterov-Todd process converges at a quadratic rate asymptotically (see Nesterov-Todd [29] and Hauser [11], Theorem 4.1 for two entirely different proofs). The results from this article combined with the standard results from the theory of the Newton process (see e.g. [4]) constitute a third independent approach to proving quadratic convergence rates.

Let E be a n-dimensional real vector space and $E^\sharp$ its dual. Throughout this article we will reserve super- or subscript-$\sharp$ for duals, superscript-$*$ for adjoints and superscript-$\circ$ for topological interiors. Let $K \subset E$ be a symmetric cone in E (see e.g. [17, 5, 31, 38]), and let the dual of K be defined as usual:

$$K^\sharp := \{ s \in E^\sharp : \langle x; s \rangle \geq 0 \quad \forall x \in K \}.$$

By $\langle \cdot ; \cdot \rangle$ let us denote the canonical bilinear product defined on $E \times E^\sharp$. Let L be a linear subspace of E and $L^\perp := \mathrm{im}(A^*)$ its orthogonal complement with respect to the bilinear form $\langle \cdot ; \cdot \rangle$. Let us make the regularity assumption that there exist vectors $x_0 \in K^\circ \cap L \neq \emptyset$ and $s_0 \in K^{\sharp\circ} \cap L^\perp \neq \emptyset$. Then we consider the following primal-dual pair of convex optimization problems:

$$\text{(1.1)} \qquad \text{(P)} \quad \begin{array}{l} \inf \langle x; s_0 \rangle \\ x \in (L + x_0) \cap K \end{array} \qquad\qquad \text{(D)} \quad \begin{array}{l} \inf \langle x_0; s \rangle \\ s \in (L^\perp + s_0) \cap K^\sharp. \end{array}$$

Problems of this form are called *self-scaled conic programming* problems. Under the abovementioned regularity assumption of primal-dual strict feasibility, self-scaled conic programming contains as special cases linear programming (by choosing $K = \mathbb{R}^n_+$), semidefinite programming (by choosing $K = \mathrm{S}^{n\times n}(\mathbb{R})_+$, the cone of positive semidefinite symmetric $n \times n$ matrices) and second-order cone programming (by choosing K equal to the $n+1$-dimensional Lorentz cone). Second-order cone programming is a convenient formulation of convex quadratic programming with convex quadratic constraints. The full extent of problems covered by the framework (1.1) with symmetric K is known, since Vinberg [38] gave a complete classification of symmetric cones in terms of an decomposition theory. His work was based on earlier results by Cartan [3], Köcher [17] and Rothaus [31]. See also [5].

We endow the interior of the cone K with a *self-scaled* barrier functional (see [28]), a special class of self-concordant barrier functionals (see [27]) that reflects the group action of the automorphism group $\mathrm{Aut}(K)$ of K (see e.g. [5]) through the Hessians. In fact, if a barrier functional F is self-scaled then this implies among other things that at any point $w \in K^\circ$ the mapping $F''(w) : E \to E^\sharp$ is an isomorphism between K and $K^\sharp$, and that for any pair of points $(x,s) \in K^\circ \times K^{\sharp\circ}$ there exists a unique *scaling point* $w(x,s) \in K^\circ$ such that $F''(w)x = s$. Self-scaled barrier functionals that are invariant under the orthogonal group of their conic domain of definition (think of rotational invariance!) have been completely classified by Hauser [9] in terms of their characteristic function (see e.g. [5] and [7]). In particular, these results imply that modulo an additive constant all such barrier functionals for the positive orthant $\mathbb{R}^n_+$ are positive multiples of the standard logarithmic barrier functional $F(x) : x \mapsto -\sum \ln(x_i)$. Likewise, modulo an additive constant all such barrier functionals for the positive semidefinite symmetric cone $\mathrm{S}^{n\times n}(\mathbb{R})_+$ are positive multiples of the standard logarithmic barrier functional $F : x \mapsto -\ln(\det(x))$. Finally, modulo an additive constant all such barrier functionals for the Lorentz cone $K = \{\left(\begin{smallmatrix}\tau\\ x\end{smallmatrix}\right) \in \mathbb{R}^{n+1} : \tau \geq \|x\|\}$ are positive multiples of the standard logarithmic barrier functional $F : \left(\begin{smallmatrix}\tau\\ x\end{smallmatrix}\right) \mapsto -\ln\left(\left(\begin{smallmatrix}\tau\\ x\end{smallmatrix}\right)^T\left(\begin{smallmatrix}\tau\\ -x\end{smallmatrix}\right)\right)$. As usual, we endow $K^\sharp$ with the *dual barrier functional*

$$\begin{aligned} F_\sharp &: K^{\sharp\circ} \to \mathbb{R}\\ &s \mapsto \max\{-\langle x; s\rangle - F(x) : x \in K^\circ\}, \end{aligned}$$

which is also self-scaled (see [28]).

We call $Z := L \oplus L^\perp$ the *primal-dual domain* and consider the two projections $\pi_L : Z \to L$ and $\pi_{L^\perp} = \imath - \pi_L$ of Z onto L and $L^\perp$ along $L^\perp$ and L respectively. For each $e \in K^\circ$ let us define the following scalar product on Z

$$\begin{aligned} (\cdot;\cdot)_e &: Z \times Z \to \mathbb{R}\\ &(z_1, z_2) \mapsto \langle \pi_L z_2; F''(e)\pi_L z_1\rangle + \langle F''^{-1}(e)\pi_{L^\perp} z_2; \pi_{L^\perp} z_1\rangle, \end{aligned}$$

and let us call this the *Euclidean structure on* Z *defined by* e. Note that in particular L and $L^\perp$ are mutually orthogonal in the Euclidean structure defined by e on E. We call

$$\begin{aligned} x &: Z \to E & \text{and} \qquad & s : Z \to E^\sharp\\ &z \mapsto x_0 + \pi_L z & & z \mapsto s_0 + \pi_{L^\perp} z \end{aligned}$$

the *primal* and *dual* parts of z respectively. The following linear functional on Z is called the *duality gap* of $z \in Z$:

$$\begin{aligned} \delta &: Z \to \mathbb{R}_+\\ &z \mapsto \langle x(z); s(z)\rangle. \end{aligned}$$

We call the sets

$$\begin{aligned} \mathcal{F}(PD) &:= \{z \in Z : x(z) \in K \cap L,\, s(z) \in L^\perp \cap K^\sharp\} \qquad \text{and}\\ \mathcal{F}^\circ(PD) &:= \{z \in Z : x(z) \in K \cap L,\, s(z) \in L^\perp \cap K^\sharp\} \end{aligned}$$

the primal-dual *feasible domain* and the primal-dual *strictly feasible domain* respectively. The mapping

$$\begin{aligned}\mathfrak{F} : \mathcal{F}^\circ(PD) &\to \mathbb{R}\\ z &\mapsto F\big(x(z)\big) + F_\sharp\big(s(z)\big)\end{aligned}$$

is a ν-self-concordant barrier functional for the convex set $\mathcal{F}^\circ(PD)$. Finally, the following primal-dual optimization problem is equivalent to the problems *(P)* and *(D)* from (1.1):

$$\text{(PD)} \qquad \inf\{\delta(z) : z \in \mathcal{F}(PD)\}.$$

Moreover, if η is a positive real number, then $\mathcal{F}(PD) \cap \delta^{-1}\big([0,\eta]\big)$ and $\mathcal{F}(PD) \cap \delta^{-1}\big(\{\eta\}\big)$ are compact subsets of Z. The guiding principle of the barrier method is to replace the constrained problem (PD) by the directed set of perturbed unconstrained problems $\{(\text{PD})_\mu : \mu > 0\}$ where

$$(\text{PD})_\mu \qquad \min\{\delta(z) + \mu\mathfrak{F}(z) : z \in Z\}.$$

As usual, the directed set of optimal solutions of $\{(\text{PD})_\mu : \mu > 0\}$ parameterized by μ, $\mu \mapsto z_\mu$ is called the *central path* for the tuple $\big((\text{PD}), \mathfrak{F}\big)$.

In [10] we defined *square-root fields* as follows: Let V be a n--dimensional Euclidean space and let $e_V \in V$ be a fixed vector of length $\|e_V\|_2 = \nu^{1/2}$, where ν is the *complexity parameter* of F (see [27]). Suppose $F \in \mathcal{C}^k\big(K^\circ, \mathbb{R}\big)$. A square-root field for (F, e_V) is a pair (X, S) of $\mathcal{C}^{k-2}$-operator fields

$$\begin{aligned} X :\ K^\circ \times K^{\sharp\circ} &\to \operatorname{Aut}(E^\sharp, V) \qquad \text{and}\\ S :\ K^\circ \times K^{\sharp\circ} &\to \operatorname{Aut}(E, V)\end{aligned}$$

such that for each pair $(x, s) \in K^\circ \times K^{\sharp\circ}$ the following diagrams become commutative:

$$\begin{tikzcd} E^\sharp \arrow[r, "F_\sharp''(s)"] & E \arrow[r, "F''(w(x,s))"] \arrow[d, "S(x,s)"] & E^\sharp \arrow[d, "X(x,s)"] & E \arrow[l, "F''(x)"'] \\ V \arrow[u, "S^*(x,s)"] \arrow[r, equal] & V \arrow[r, equal] & V \arrow[r, equal] & V \arrow[u, "X^*(x,s)"'] \end{tikzcd}$$

$$\begin{tikzcd} -F_\sharp'(s) \arrow[r, "F''(w(x,s))"] \arrow[d, "S(x,s)"] & -F'(x) \arrow[d, "X(x,s)"] \\ e_V \arrow[r, equal] & e_V \end{tikzcd}$$

Suppose (X, S) a square root field for the self-scaled barrier functional $F \in \mathcal{C}^k\big(K^\circ, \mathbb{R}\big)$. We define

$$\begin{aligned} X : \mathcal{F}^\circ(PD) &\to \operatorname{Lin}\big(E^\sharp, V\big) \\ z &\mapsto X\big(x(z), s(z)\big)\end{aligned} \qquad \text{and} \qquad \begin{aligned} S : \mathcal{F}^\circ(PD) &\to \operatorname{Lin}\big(E, V\big) \\ z &\mapsto S\big(x(z), s(z)\big).\end{aligned}$$

In [11] we defined the *target map* associated with (X,S) as the function

$$\begin{aligned}\gamma : \mathcal{F}^\circ(PD) &\to V\\ z &\mapsto X(z)s(z) = S(z)x(z).\end{aligned}$$

Then we showed that solving the perturbed problem $\{(\mathrm{PD})_\mu : \mu > 0\}$ is equivalent to solving the so-called *central path equation*

$$\gamma(z) = \mu e_V,$$

a non-linear equation which has a unique solution for each fixed $\mu > 0$. The corresponding Newton equation

$$\gamma'(z)d_z^{(\mu)} = \mu e_V - \gamma(z), \qquad (d_z^{(\mu)} \in Z) \tag{1.2}$$

determines the *target direction* field $d^{(\mu)}$ defined by (X,S) and μ. We will hold (X,S) and $\mu > 0$ fixed and hence we will usually drop reference to these objects in our notation. In [11], Definition 3.8 we defined operator fields $M, R \in \mathcal{C}^{k-2}\big(\mathcal{F}^\circ(PD), \mathrm{Lin}(Z,V)\big)$ as by

$$\begin{aligned}M(z) : d_z &\mapsto S(z)\circ\pi_L[d_z] + X(z)\circ\pi_{L^\perp}[d_z] \qquad \forall d_z \in Z,\\ R(z) : d_z &\mapsto \big(\Omega'_{(s,B)}(z)[\pi_L d_z]\circ\Omega^*_{(s,B)}(z) + \Omega'_{(x,B)}(z)[\pi_{L^\perp} d_z]\circ\Omega^*_{(x,B)}(z)\big)\gamma(z),\end{aligned}$$

so that $\gamma' \equiv M + R$. We call M and R the main and residual part of the Jacobian γ' respectively. Hence, equation (1.2) can be rewritten as

$$\big(M(z) + R(z)\big)d_z^{(\mu)} = \mu e_V - \gamma(z), \quad (d_z^{(\mu)} \in Z). \tag{1.3}$$

For $z \in \mathcal{F}^\circ(PD)$ let us define

$$\mu(z) := \nu^{-1}\delta(z) \tag{1.4}$$

Then $\mu(z)$ is the unique value of the parameter $\mu > 0$ such that the corresponding point on the central path satisfies $\delta(z_\mu) = \delta(z)$.

In order to solve the perturbed problem $\{(\mathrm{PD})_\mu : \mu > 0\}$ Nesterov-Todd [28] propose to use the search-direction that results as the solution to the system of so-called *NT equations*:

$$F''\big(w(x,s)\big)d_x + d_s = -\mu F'(x) - s, \tag{1.5a}$$

$$d_x \in L, \tag{1.5b}$$

$$d_s \in L^\perp. \tag{1.5c}$$

This linear system is non-singular everywhere (see [28] and [11] Proposition 3.9). Formulated in the language we introduced above, the N-T system is equivalent to the following equation:

$$M(z)d_z^{(\mu)} = \mu e_V - \gamma(z), \quad (d_z^{(\mu)} \in Z). \tag{1.6}$$

Hence, the Nesterov-Todd system differs from the Newton equation (1.3) only in the residual term $R(z)d_z^{(\mu)}$. In [10], Section 5 we showed that for general self-scaled barrier functionals there cannot exist a square-root field such that the corresponding operator field R is identically equal to the zero-operator. For example, the standard log-barriers for the positive semidefinite symmetric cone and for the Lorentz cone do not allow such square-root fields (see [10], Theorems 4.3 and 4.4). However, in [11] we show the following:

Proposition 1.1. *For all $z \in \mathcal{F}^{\circ}(PD)$ it is true that*

i) $M(z)$ *is a non-singular operator,*

ii) $\mathrm{im}\big(R(z)\big) \subseteq \mathrm{span}\{e_V, \gamma(z)\}^{\perp}$,

iii) if z lies on the central path, then $R(z) = 0$,

iv) $\dim\big(\ker R(z)\big) \geq 2$.

□

In Section 3 we will show that, given any self-scaled barrier functional over a symmetric cone, there always exists a square-root field such that the N-T direction resulting from system (1.6) lies in in the kernel of $R(z)$ at every point $z \in \mathcal{F}^{\circ}(PD)$, and hence it also solves system (1.2) and is a genuine target direction. In Theorem 3.8 we reduce this problem to showing that a certain boundary value problem with an isolated boundary point allows for a solution whose continuous extension to this boundary point is twice continuously differentiable. Our proof uses techniques from the theory of dynamical systems that evolve in Lie groups.

Let us conclude this section with some remarks regarding notation. The group of orthogonal transformations of the Euclidean space V will play an important role, and we denote it by $\mathcal{O}(V)$. The subgroup whose action leaves the element $e_V \in V$ invariant will be denoted by $\mathcal{O}(V, e_V)$. Also, we will consider a regular cone $K_V \subset V$ and denote the subgroup of $\mathcal{O}(V)$ whose elements map K_V onto itself by $\mathcal{O}(K_V)$, and the subgroup of $\mathcal{O}(K_V)$ whose action leaves e_V invariant by $\mathcal{O}(K_V, e_V)$. Often the need arises to distinguish between linear operators and their matrix representation with respect to a particular choice of coordinates. In particular, we will write $\nabla^2 F(x)$ for the matrix representation of $F''(x)$. See also Remark 2.3. Identity maps will always be denoted by the symbol $\imath$, irrespective of the set over which they are defined. $S^{(n-1)}(H)$ denotes the unit sphere in a n-dimensional normed vector space H.

2 Prerequisite results and additional notation

The results and definitions of Hauser [10] and [11] will be heavily used. References to those papers will be indicated by the string "H2." and"H3." respectively followed by the number of the cited result, i.e. H2.3.1 refers to Lemma 3.1, [10] for example.

Modern interior-point theory is based on the foundation of Nesterov & Nemirovskii's theory of self-concordant functionals. For a definition of ν-logarithmically homogeneous self-concordant barrier functionals see [27] or [9], Definition 2.1 from which we adopt notation. We will frequently use the following properties:

Proposition 2.1 (Nesterov & Nemirovskii). *Let F be a ν-self-concordant logarithmically homogeneous barrier functional on the regular cone $K \subset E$, and let $x \in K^\circ$. Then*

i) $-F'(x) = F''(x)x \in K^{\sharp\circ}$,

ii) $-F'_\sharp(-F'(x)) = x$,

iii) $F''_\sharp(-F'(x)) = F''(x)^{-1}$,

iv) $\langle x; -F'(x)\rangle = \nu$,

v) for all $\tau > 0 : F'(\tau x) = \tau^{-1}F'(x)$,

vi) for all $\tau > 0 : F''(\tau x) = \tau^{-2}F''(x)$ *and*

vii) $F_\sharp\big(-F'(x)\big) = -\nu - F(x)$. □

The foundations of the theory of self-scaled conic programming were laid in two seminal papers by Nesterov and Todd [28, 29]. In Definition H1.2.4 and Theorem H1.2.8 we reiterated the definition of self-scaled barrier functionals and properties thereof in the form we shall use in this article.

Güler [7] and Nesterov-Todd [28] showed that the concept of self-scaled cones, i.e. the class of cones that allows for a self-scaled barrier functional, is equivalent to the concept of symmetric cones studied in Jordan algebra theory (see Theorems H1.2.15 and H1.2.16). Symmetric cones are defined as being self-dual and homogeneous, i.e. their automorphism group

$$\mathrm{Aut}(K) := \big\{\theta \in \mathrm{Aut}(E) : \theta K = K\big\}$$

acts transitively on K (see e.g. [5]). For three classical examples of symmetric cones and self-scaled barrier functionals (positive orthant, Lorentz cone and the cone of positive semidefinite symmetric matrices) see H1.2.5–H1.2.7.

One of the important tools of the theory of self-scaled conic programming is the notion of *scaling point* introduced in Nesterov-Todd [28]. Self-scaled barrier functionals are ideal for use in the construction of interior-point methods for self-scaled conic programming, because scaling points tie such barrier functionals F to the algebraic and analytic structures of the automorphism group $\mathrm{Aut}(K)$ of the symmetric cones K that constitute the closure of their domains of definition:

Theorem 2.2 (Nesterov-Todd). *Let K be a symmetric cone with ν-self-scaled barrier functional F, and suppose $x \in K^\circ$, $s \in K^{\sharp\circ}$. Then there exists a unique* scaling point *$w = w(x,s) \in K^\circ$ such that $s = F''(w)x$. Furthermore, the following properties hold:*

i) $w(-F'_\sharp(s), -F'(x)) = w(x,s)$,

ii) $F''(w)$ *is a cone isomorphism* $K \to K^\sharp$,

iii) $F''(x) = F''(w) \circ F''_\sharp(s) \circ F''(w)$,

iv) The dual scaling point $t(x,s) = -F'(w)$ *satisfies* $F''_\sharp(t(x,s))s = x$. □

See H2.2.8–H2.2.10 for explicit formulas of the scaling point for the three classical examples H1.2.5–H1.2.7.

Remark 2.3. Wherever it useful to introduce coordinates we assume that a dual pair of bases $B = \{e_1, \dots, e_n\} \subset E$ and $B^\sharp = \{e^\sharp_1, \dots, e^\sharp_n\} \subset E^\sharp$ was introduced, i.e. $\langle e_i; e^\sharp_j \rangle = \delta^j_i$ where δ is the Kronecker symbol. We also assume that an orthogonal basis $\{v_1, \dots, v_n\} \subset V$ has been fixed for the Euclidean space V. We define $F''^{1/2}_B(x) \in \mathrm{Lin}(E, V)$ and $F''^{-1/2}_B(x) \in \mathrm{Lin}(E^\sharp, V)$ be the unique vector space homomorphisms whose matrix representations with respect to the bases tuple $\big(B, B^\sharp, \{v_1, \dots, v_n\}\big)$ are $\nabla^2_B F(x)^{1/2}$ and $\nabla^2_B F(x)^{-1/2}$, where $\nabla^2_B F(x)$ is the matrix representation of $F''(x) \in \mathrm{Lin}\big(E, E^\sharp\big)$ with respect to the bases B and $B^\sharp$.

Remark 2.4. Note that $\big(F''^{1/2}_B(x)\big)^{-1} : V \to E$ and $F''^{-1/2}_B(x) : E^\sharp \to V$ are two distinct operators. In fact, it is straightforward to see that these are adjoints of one another and that the following diagram is commutative:

$$\begin{tikzcd}
E \arrow[r, equal] & E \arrow[r, "F''_B(x)"] \arrow[d, "F''^{1/2}_B(x)"] & E^\sharp \arrow[r, equal] \arrow[d, "F''^{-1/2}_B(x)"] & E^\sharp \\
V \arrow[r, equal] \arrow[u, "F''^{-1/2}_B(x)^*"] & V \arrow[r, equal] & V \arrow[r, equal] & V \arrow[u, "F''^{1/2}_B(x)^*"]
\end{tikzcd}$$

Definition 2.5.
Let $K \subset E$ be a symmetric cone with ν-self-scaled barrier functional $F \in \mathcal{C}^k\big(K^\circ, \mathbb{R}\big)$, and let $e_V \in V$ be a vector of length $\|e_V\|_2 = \nu^{1/2}$. Let (X, S) be a square-root field for (F, e_V) (see introduction and H2.3.15). In H2.3.6 we showed that there exist orthogonal fields

$$\Omega_{(x,B)}, \Omega_{(s,B)} \in \mathcal{C}^{k-2}\big(K^\circ \times K^{\sharp\circ}, \mathcal{O}(V)\big)$$

such that

$$\begin{aligned}
X(x,s) &\equiv \Omega_{(x,B)}(x,s) \circ F''^{-1/2}_B(x), \\
S(x,s) &\equiv \Omega_{(s,B)}(x,s) \circ F''^{-1/2}_{\sharp B}(s), \\
\Omega_{(s,B)} &\equiv \Omega_{(x,B)} \circ F''^{-1/2}_B(x) \circ F''(w) \circ F''^{1/2}_{\sharp B}(s)^*.
\end{aligned}$$

In [10] we showed:

Theorem 2.6 (Existence of square-root fields).
Let $K \subset E$ *and* $F \in \mathcal{C}^k\big(K^\circ, \mathbb{R}\big)$ *be self-scaled. Then the following applies:*

i) The set of square-root fields for F *is nonempty.*

ii) If (X, S) *is a square-root field for* $\big(F, e_V\big)$ *then any pair* (Y, T) *of operator fields is a square-root field for* (F, e_V) *if and only if there exists a mapping* $\Omega \in \mathcal{C}^{k-2}\big(K^\circ \times K^{\sharp\circ}, \mathcal{O}(V, e_V)\big)$ *such that* $Y \equiv \Omega \circ X$ *and* $T \equiv \Omega \circ S$.

□

The proof of Theorem 2.6 is constructive. For specific examples of square-root fields see [10].

Next, let us cite the theorems of existence and uniqueness of solutions for dynamical systems evolving on differentiable manifolds in the form we will use them in Section 3:

Theorem 2.7 (Local integration of vector fields). *Let* $k \in \mathbb{N}$ *and suppose* $\mathcal{M}$ *is a finite-dimensional* $\mathcal{C}^k$*-manifold. Let* f *be a* $\mathcal{C}^k$*-vector field on* $\mathcal{M}$*, i.e. a mapping* $f \in \mathcal{C}^k\big(\mathcal{M}, T\mathcal{M}\big)$ *from* $\mathcal{M}$ *to the tangent bundle* $T\mathcal{M}$ *such that* $f(x) \in T_x\mathcal{M}$ *for all* $x \in \mathcal{M}$*. Then the following holds true:*

i) For all $\tilde{x} \in \mathcal{M}$ *there exists an open neighborhood* $N_{\tilde{x}} \subseteq \mathcal{M}$ *of* $\tilde{x}$*, a positive real number* $\epsilon_{\tilde{x}} > 0$ *and a* $\mathcal{C}^{k+1}$*-mapping*

$$\Phi_{\tilde{x}} : \big(-\epsilon_{\tilde{x}}, +\epsilon_{\tilde{x}}\big) \times N_{\tilde{x}} \to \mathcal{M}$$

such that for all $x \in N_{\tilde{x}}$ *and for all* $-\epsilon_{\tilde{x}} < t < +\epsilon_{\tilde{x}}$ *it is true that*

$$\Phi_{\tilde{x}}(0, x) = x \qquad \textit{and} \tag{2.1}$$

$$\frac{\partial}{\partial t}\Phi_{\tilde{x}}(t, x) = f\big(\Phi_{\tilde{x}}(t, x)\big). \tag{2.2}$$

ii) Let $\big(a^{(i)}, b^{(i)}\big) \subseteq \mathbb{R}$ *for* $(i = 1, 2)$*, and suppose* $\alpha^{(i)} \in \mathcal{C}^1\big((a^{(i)}, b^{(i)}), \mathcal{M}\big)$ *are differentiable curves such that*

$$\frac{d}{dt}\alpha^{(i)}(t) = f\big(\alpha^{(i)}(t)\big)$$

for all $a^{(i)} < t < b^{(i)}$*. Suppose that there exists* $t_0 \in (a^{(1)}, b^{(1)}) \cap (a^{(2)}, b^{(2)})$ *such that* $\alpha^{(1)}(t_0) = \alpha^{(2)}(t_0)$*. Then,* $\alpha^{(1)}(t) \equiv \alpha^{(2)}(t)$ *for all* $t \in (a^{(1)}, b^{(1)}) \cap (a^{(2)}, b^{(2)})$*.*

Proof. See the literature, e.g. [2], Satz 8.10. □

It follows from part *ii)* of Theorem 2.7 that

$$\Phi_x\big(t_0 + t, x\big) = \Phi_x\big(t, \Phi_x(t_0, x)\big) \tag{2.3}$$

for all t_0, t for which both sides are defined. This is called the *conservation of flow* property.

Definition 2.8 (Local flows). A *local flow* on $\mathcal{M}$ with phase velocity field f is a mapping $\Phi : W \to \mathcal{M}$ such that

1. $W \subset \mathbb{R} \times \mathcal{M}$ is an open set containing $\{0\} \times \mathcal{M}$ and $I_x := \big\{t : (t,x) \in W\big\}$ is an open interval for all $x \in \mathcal{M}$,
2. Φ satisfies the conditions

$$\Phi(0,x) = x \qquad \text{for all } x \in \mathcal{M}, \tag{2.4}$$

$$\frac{\partial}{\partial t}\Phi(t,x) = f\big(\Phi(t,x)\big) \quad \text{for all } (t,x) \in W, \quad \text{and} \tag{2.5}$$

$$\Phi\big(t_0 + t, x\big) = \Phi\big(t, \Phi(t_0,x)\big) \tag{2.6}$$

wherever both sides of the last equation are well defined.

Φ is called a *maximal local flow* if there is no local flow on $\mathcal{M}$ whose phase velocity field is f and whose domain of definition is strictly larger than W.

Theorem 2.9 (Theorem of the integration of vector fields). *Let $k \in \mathbb{N}$, suppose $\mathcal{M}$ is a finite-dimensional $\mathcal{C}^k$-manifold and let f be a $\mathcal{C}^k$-vector field on $\mathcal{M}$. Then there exists a unique maximal local flow $\Phi \in \mathcal{C}^{k+1}(W,\mathcal{M})$ on $\mathcal{M}$ with phase velocity field f.*

Proof. See the literature, e.g. [2], Satz 8.10. □

Next we will cite a few results from the theory of dynamical systems evolving on Lie groups. We adhere to the notation introduced in Iserles-Nørsett [14].

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Let $a : \mathbb{R}^+ \to \mathfrak{g}$ be Lipschitz continuous at every point $t \geq 0$ and let $y_0 \in G$. Consider the initial value problem

$$\begin{aligned} y : \mathbb{R}^+ &\to G \qquad \text{s.t.} \\ y(0) &= y_0 \qquad \text{and} \\ \dot{y}(t) &= a(t)y(t) \qquad \text{for all } t \geq 0. \end{aligned} \tag{2.7}$$

Since $a(t) \in \mathfrak{g} = T_0 G$ we have $\dot{y}(t) \in T_{y(t)}G$ for all $t \geq 0$, as should be expected. Let $\mathrm{ad}^k(v,u) := (-\mathrm{ad}\, u)^k[v]$, i.e.

$$\mathrm{ad}^k(v,u) := \begin{cases} v & \text{if } k = 0, \\ \big[\mathrm{ad}^{k-1}(v,u), u\big] & \text{for } k \in \mathbb{N}. \end{cases}$$

In order to solve (2.7) Hausdorff [8] makes the following ansatz:

$$y(t) = \exp\big(\sigma(t)\big) \cdot y_0 \qquad \text{for all } 0 \leq t \leq t^* \tag{2.8}$$

for some real number $t^* > 0$, where

$$\sigma \in \mathcal{C}^1\big([0,t^*],\mathfrak{g}\big), \qquad \sigma(0) = 0.$$

Theorem 2.10 (Hausdorff). *For t^* small enough the unique solution to* (2.7) *on the interval* $[0,t^*]$ *is given by* (2.8) *when* σ *is the solution to the the so called dexpinv-equation*

$$\dot{\sigma} = \sum_{m=0}^{\infty} h_m \mathrm{ad}^m(a,\sigma), \qquad 0 \le t \le t^*, \qquad \sigma(0) = 0. \tag{2.9}$$

Here $\big(h_m\big)_{\mathbb{N}_0}$ *denotes the sequence of coefficients in the Taylor series expansion of the function*

$$\begin{aligned} h &: \mathbb{C} \to \mathbb{C}, \\ z &\mapsto \frac{z}{e^z - 1} + z. \end{aligned} \tag{2.10}$$

Proof. See e.g. [14]. □

Using Picard-Lindelöf iteration it is possible to explicitly determine more and more terms of a series development of the solution to the dexpinv equation (2.9). Magnus [20] derived the first four terms of this series:

$$\begin{aligned} \sigma(t) = \int_0^t a(\kappa)d\kappa &+ \frac{1}{2}\int_0^t \big[a(\kappa), \int_0^\kappa a(\xi)d\xi\big]\,d\kappa \\ &+ \frac{1}{4}\int_0^t \Big[a(\kappa), \int_0^\kappa \big[a(\xi), \int_0^\xi a(\eta)d\eta\big]\,d\xi\Big]\,d\kappa \\ &+ \frac{1}{12}\int_0^t \Big[\big[a(\kappa), \int_0^\kappa a(\xi)d\xi\big], \int_0^\kappa a(\eta)d\eta\Big]\,d\kappa + \dots . \end{aligned} \tag{2.11}$$

The general term of the this series was characterized by Iserles-Nørsett [14] as follows:

Consider the set of functions $\mathcal{E} \subset \mathcal{F}\big([0,t^*],\mathfrak{g}\big)$ for which membership is defined by recursive application of the following rules:

1. $a \in \mathcal{E}$,
2. if $p,\ q \in \mathcal{E}$ then $t \mapsto \big[p, \int_0^t q(\kappa)d\kappa\big] \in \mathcal{E}$.

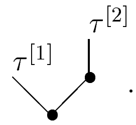
$\mathcal{E}$ can be indexed by a set of oriented rooted trees by recursive application of the following rules:

1. $t \mapsto a(t)$ is associated with the tree consisting of a single node.

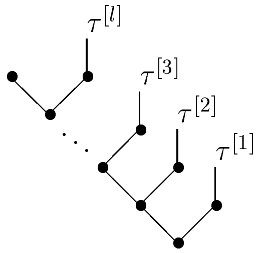
$$a \sim \tau_0 := \bullet$$

2. If $p, q \in \mathcal{E}$ are associated with the trees $\tau^{[1]}$ and $\tau^{[2]}$ respectively, then the function $t \mapsto \big[p, \int_0^t q(\kappa)d\kappa\big] \in \mathcal{E}$ is associated with the tree obtained by appending a new root to $\tau^{[2]}$ and joining the resulting tree with $\tau^{[1]}$ via a new root on the left:

$$p \sim \tau^{[1]}, q \sim \tau^{[2]} \Rightarrow \Big[p(\cdot), \int_0^{\cdot} q(\kappa)d\kappa\Big] \quad \sim$$



Let $\mathcal{T}$ be the set of all trees obtained in this fashion, and for every $k \in \mathbb{N}$ let $\mathcal{T}_k \subset \mathcal{T}$ be the subset consisting of the elements of $\mathcal{T}$ of order k, i.e. containing k nodes. For $\tau \in \mathcal{T}$ let H_τ denote the unique element of $\mathcal{E}$ such that $H_\tau \sim \tau$. Each $\tau \in \mathcal{T}$ can be written in a unique fashion in the form



for some $\tau^{[1]}, \ldots, \tau^{[l]} \in \mathcal{T}$ of order strictly less than k and where $l \in \mathbb{N}_0$. Let this relationship between τ and $\tau^{[1]}, \ldots, \tau^{[l]} \in \mathcal{T}$ be captured notationally by writing $\tau = \mathcal{R}(\tau^{[1]}, \ldots, \tau^{[l]})$. Recall that we reserved the notation τ_0 for the tree $\bullet$. For later convenience, let us reserve the notation τ_1 for the tree $\mathcal{R}(\tau_0)$.

Consider the set of real numbers indexed by $\mathcal{T}$ and defined recursively as follows:

$$\begin{aligned} \alpha_{\tau_0} &= h_0, \\ \alpha_{\mathcal{R}(\tau^{[1]}, \ldots, \tau^{[l]})} &= h_l \prod_{i=1}^{l} \alpha_{\tau^{[i]}}. \end{aligned} \tag{2.12}$$

Thus, the coefficients α_τ depend only on the coefficients h_m of the Taylor expansion (2.10) of h, and since the series (2.10) has a radius of convergence strictly greater than 1 it follows that

$$|\alpha_\tau| \leq 1 \tag{2.13}$$

for all τ of sufficiently high order (actually for all $\tau \in \mathcal{T}$).

Since G was supposed to be a matrix Lie group we may endow the matrix Lie algebra $\mathfrak{g}$ with a matrix norm $\|\cdot\|$.

Theorem 2.11 (Iserles-Nørsett).
Suppose $\|a(t)\| \leq \rho t^\lambda$ for all $0 \leq t \leq t^*$ where $\lambda \in \mathbb{N} \cup \{0\}$ and $\rho \in \mathbb{R}_{++}$. For each $H \in \mathcal{E}$ let $\alpha_H := \alpha_\tau$, where $\tau \in \mathcal{T}$ is the oriented rooted tree corresponding to H. Then both

$$\sigma(t) = \sum_{H \in \mathcal{E}} \alpha_H \int_0^t H(\kappa) d\kappa. \tag{2.14}$$

and the term-wise derivative of this series with respect to t converge absolutely for

$$0 \leq t \leq \min\Big\{t^*, \Big(\frac{\lambda+1}{8\rho}\Big)^{1/(\lambda+1)}\Big\}, \tag{2.15}$$

and (2.14) *constitutes the unique solution of the dexpinv-equation* (2.9) *on this interval.*

Proof. See [14]. $\square$

(2.14) is called the *Magnus series* of problem (2.7). We will also need the following result of Iserles-Nørsett:

Lemma 2.12 (Iserles-Nørsett).

$$\#\mathcal{T}_{3k+1} = \frac{(2k)!}{k!(k+1)!} \qquad \forall\, k \in \mathbb{N} \cup \{0\}.$$

Proof. See [14]. $\square$

3 Nesterov-Todd directions in the target framework

Let $E, E^\sharp, V$ and K be as in the introduction. Let $F \in \mathcal{C}^k\big(K^\circ, \mathbb{R}\big)$ be a ν--self-scaled barrier functional with $k \geq 3$, and let $\mu > 0$ be fixed. Let us denote the Nesterov-Todd field $d^{(\mu)}$ corresponding to (F, μ) simply by $d : \mathcal{F}^\circ(PD) \to Z$. Let $e_V \in V$ be a fixed vector of length $\|e_V\|_2 = \nu^{1/2}$. The subject of this section is the following conjecture:

Conjecture 3.1. *If $F^{(k)}$ is Lipschitz continuous everywhere then there exists a square-root field $(X, S) \in \mathcal{C}^{k-2}$ for (F, e_V) and correspondingly a target map $\gamma \in \mathcal{C}^{k-2}\big(\mathcal{F}^\circ(PD), V\big)$ such that the the Nesterov-Todd field for F satisfies the target equation*

$$\gamma'(z)d(z) = \mu e_V - \gamma(z) \tag{3.1}$$

at every point $z \in \mathcal{F}^\circ(PD)$.

Instead of proving this conjecture in the general case we consider only the case $k = 4$ and replace the Lipschitz assumption by $F \in \mathcal{C}^5$. Hence, we will prove:

Theorem 3.2. *If $F \in \mathcal{C}^5$ then there exists a square-root field $(X, S) \in \mathcal{C}^2$ for (F, e_V) and correspondingly a target map $\gamma \in \mathcal{C}^2\big(\mathcal{F}^\circ(PD), V\big)$ such that the the Nesterov-Todd field for F satisfies the target equation* (3.1) *at every primal-dual strictly feasible point $z \in \mathcal{F}^\circ(PD)$.*

Our proof of Theorem 3.2 will make clear how Conjecture 3.1 could be proven in the general case at least in principle. The case where $\gamma \in \mathcal{C}^2$ is of particular interest to us, because this shows that Nesterov-Todd directions derive from the Newton system (3.1) which has the non-singular Jacobian $\gamma'(z_\mu) = M(z_\mu)$ at the unique root z_μ of the equation

$$\gamma(z) = \mu e_V$$

(see [11] or Section 1 for all of these statements), and the Jacobian γ' is Lipschitz-continuous in a neighborhood of z_μ since it is continuously differentiable. It follows from the standard results on local convergence of the Newton process (see e.g. [4], Theorem 5.2.1) that the Nesterov-Todd process converges at a quadratic rate, and that upper bounds on the coefficients of convergence can be obtained from the first two derivatives of the target map γ in a neighborhood of z_μ. Such bounds can easily be obtained from our results. Hence, we get the following

Corollary 3.3. *If $F \in \mathcal{C}^5$ then the Nesterov-Todd directions defined by F and $\mu > 0$ are Newton directions, and the corresponding Newton process converges at a quadratic rate.* □

We are now going to show how the statement of Theorem 3.2 can be reduced to a boundary value problem:

Lemma 3.4. *Let $\mu > 0$ be fixed as usual. If $F \in \mathcal{C}^4\big(K^\circ, RN\big)$ then the Nesterov-Todd vector field $d \in \mathcal{C}^2\big(\mathcal{F}^\circ(PD), E\big)$ is twice continuously differentiable.*

Proof. Let $(X, S) \in \mathcal{C}^2$ be an arbitrary square-root field for (F, e_V) and let $M \in \mathcal{C}^2$ denote the main part (see Section 1) of the Jacobian γ' of the target map $\gamma \in \mathcal{C}^2$ corresponding to (X, S). From (1.6) we know that

$$d : z \mapsto M(z)^{-1}\big[\mu e_V - \gamma(z)\big]. \tag{3.2}$$

Since M is nonsingular everywhere we have $M^{-1} \in \mathcal{C}^2$, and hence $d \in \mathcal{C}^2$. □

In order to prove Theorem 3.2 we have to prove that the following problem has a solution:

Problem 3.5. Let $F \in \mathcal{C}^5\big(K^\circ, \mathbb{R}\big)$ be a ν-self-scaled barrier functional for the symmetric cone K, and let $\mu > 0$ and $e_V \in V$ be fixed with $\|e_V\|_2 = \nu^{1/2}$. Find a square-root field $(X, S) \in \mathcal{C}^2$ for (F, e_V) such that

$$R(z)[d(z)] \equiv 0 \qquad \forall\, z \in \mathcal{F}^\circ(PD), \tag{3.3}$$

where $d(z)$ is the Nesterov-Todd vector field defined by F and the fixed parameter value $\mu > 0$, and where R denotes the residual part (see Section 1) of the Jacobian of the target map $\gamma \in \mathcal{C}^2\big(\mathcal{F}^\circ(PD), V\big)$ corresponding to (X, S).

We propose to solve Problem 3.5 via the following line of attack: We choose an arbitrary square-root field $\big(\tilde{X}, \tilde{S}\big) \in \mathcal{C}^2$ and denote the associated target map by $\tilde{\gamma}$. We fix a dual basis pair on the primal and dual spaces E and $E^\sharp$. Since this basis remains fixed throughout the chapter we will drop all references to B in expressions such as $F_B''^{1/2}$, $\Omega_{(x,B)}$ (see Definition 2.5) etc.. In particular, we are going to denote the orthogonal fields associated with (X,S) by $\tilde{\Omega}_x$ and $\tilde{\Omega}_s$ respectively. By virtue of Theorem 2.6 we may express the desired square-root field (X,S) in the form

$$
\begin{aligned}
X(x,s) &\equiv \big(\Omega(x,s)\big)^* \circ \tilde{X}(x,s), \\
S(x,s) &\equiv \big(\Omega(x,s)\big)^* \circ \tilde{S}(x,s)
\end{aligned}
\tag{3.4}
$$

for some $\mathcal{C}^2$-orthogonal field $\Omega : \mathcal{F}^\circ(PD) \to \mathcal{O}(V, e_V)$. The adjoint notation in equation (3.4) is introduced for later convenience. Let Ω_x and Ω_s be the orthogonal fields associated with X and S respectively. Then it is simple to check that

$$
\begin{aligned}
\Omega_x &\equiv \Omega^* \circ \tilde{\Omega}_x \qquad \text{and} \\
\Omega_s &\equiv \Omega^* \circ \tilde{\Omega}_s.
\end{aligned}
$$

Using this notation we may reformulate Problem 3.5 as follows:

Problem 3.6. Under the assumptions of Problem 3.5, find $\Omega \in \mathcal{C}^2\big(\mathcal{F}^\circ(PD), \mathcal{O}(V, e_V)\big)$ such that

$$
\big(\Omega^{*\prime}[d]\big)[\tilde{\gamma}] \equiv -\Omega^* \circ \big(\tilde{\Omega}_s'[\pi_L d] \circ \tilde{\Omega}_s^* + \tilde{\Omega}_x'[\pi_{L^\perp} d] \circ \tilde{\Omega}_x^*\big)[\tilde{\gamma}]
\tag{3.5}
$$

holds true on $\mathcal{F}^\circ(PD) \setminus \{z_\mu\}$.

Let us now replace equation (3.5) with the stronger condition

$$
\Omega^{*\prime}[d] \equiv -\Omega^* \circ \big(\tilde{\Omega}_s'[\pi_L d] \circ \tilde{\Omega}_s^* + \tilde{\Omega}_x'[\pi_{L^\perp} d] \circ \tilde{\Omega}_x^*\big)
$$

and take the transpose of this equation. Note that this implies a change of sign because $\big(\tilde{\Omega}_s'[\pi_L d] \circ \tilde{\Omega}_s^* + \tilde{\Omega}_x'[\pi_{L^\perp} d] \circ \tilde{\Omega}_x^*\big)$ is skew-symmetric. To fix ideas let us require the arbitrary boundary condition $\Omega\big(z_\mu\big) = \imath$. Our transformed problem is thus the following:

Problem 3.7. Under the assumptions of Problem 3.5, find $\Omega \in \mathcal{C}^2\big(\mathcal{F}^\circ(PD), \mathcal{O}(V, e_V)\big)$ such that

$$
\begin{aligned}
&\Omega\big(z_\mu\big) = \imath \qquad \text{and} \\
&\Omega'[d] \equiv \big(\tilde{\Omega}_s'[\pi_L d] \circ \tilde{\Omega}_s^* + \tilde{\Omega}_x'[\pi_{L^\perp} d] \circ \tilde{\Omega}_x^*\big) \circ \Omega
\end{aligned}
\tag{3.6}
$$

holds true on $\mathcal{F}^\circ(PD) \setminus \{z_\mu\}$.

Note that in particular we require twofold continuous differentiability of Ω at the point $z = z_\mu$, and hence solving Problem 3.7 is stronger than showing that the boundary value problem (3.6) has a solution. Any solution to Problem 3.7 provides a solution to Problem 3.6 and hence to Problem 3.5.

Theorem 3.8. *There exists a unique orthogonal field Ω that solves Problem 3.7.*

The remainder of this and the next section consists of the proof of Theorem 3.8. After a rectification of the Nesterov-Todd field, existence and uniqueness of a solution to the boundary value problem (3.6) reduce to standard results for ordinary differential equations. However, the true bottleneck of the proof consists of showing twofold continuous differentiability of this solution at the boundary point z_μ. The strategy of proof is as follows: We introduce curvilinear coordinates defined in terms of the flux-lines of the maximal local flow ϕ for the Nesterov-Todd field d corresponding to a fixed value of the parameter $\mu > 0$. Away from the boundary point z_μ this allows us to rewrite Problem 3.7 as a linear system of ODE's and to apply the standard results that guarantee existence, uniqueness and differentiability of local solutions to such problems. It then remains to show that Problem 3.7 has a solution in a neighborhood of z_μ. Let $z \neq z_\mu$ be a fixed point in a sufficiently small neighborhood of z_μ and consider the restriction of the boundary value problem (3.6) to the flux-line of ϕ passing through z. This yields a linear differential equation on the Lie group $\mathcal{O}(V, e_V)$. The solution to this ODE can be represented "explicitly" in terms of a *Magnus series* development (see Iserles-Nørsett [14]). We can then take the first two derivatives of this series and compute their limits for $z \to z_\mu$. Having said all of this, let us proceed with the details of the proof.

Let us first apply Theorems 2.7 and 2.9 in the context of Problem 3.7: Let $D := \mathcal{F}^\circ(PD) \setminus \{z_\mu\}$ and let this set be endowed with the $\mathcal{C}^\infty$-structure inherited from the n-dimensional normed vector space Z. In particular there is an atlas $\mathcal{A}_1$ consisting of a single chart given by the embedding of D as an open subset of Z. Let the Lie group $\mathcal{O}(V, e_V)$ be endowed with its canonical differentiable structure, i.e. the structure $\mathcal{A}_2$ it inherits as a closed subgroup of $\mathcal{O}(V)$. Clearly, $\mathcal{O}(V, e_V)$ is isomorphic to $\mathcal{O}(\langle e_V\rangle^\perp)$ and hence its Lie algebra

$$L\big(\mathcal{O}(V, e_V)\big) = \big\{g \in \operatorname{Aut} V : g^* = -g, g[e_V] = 0\big\} \tag{3.7}$$

is isomorphic to the set of skew-symmetric $(n-1) \times (n-1)$-matrices, the Lie algebra of $\mathcal{SO}_{n-1}$. It follows that $\mathcal{O}(V, e_V)$ is a compact $\mathcal{C}^\infty$-manifold of dimension $(n-2)(n-1)/2$ (see e.g. [13] Lemma I.1.17, Satz I.3.3 and Beispiel I.3.16 for proofs of these remarks). Now, let $\mathcal{M}$ be the differentiable product of the $\mathcal{C}^\infty$-manifolds D and $\mathcal{O}(V, e_V)$, i.e. this is the Cartesian product $D \times \mathcal{O}(V, e_V)$ endowed with the differentiable structure $\mathcal{A} := \mathcal{A}_1 \otimes \mathcal{A}_2$ whose elements are the tensor products of elements of $\mathcal{A}_1$ and $\mathcal{A}_2$. The tangent bundle of $\mathcal{M}$ is the direct sum $T\mathcal{M} = TD \oplus T\mathcal{O}(V, e_V)$, i.e. the fibers of this bundle are defined as the direct sum of the corresponding fibers of TD and $T\mathcal{O}(V, e_V)$:

$$T_{(z,\Omega)}\mathcal{M} = T_z D \oplus T_\Omega \mathcal{O}(V, e_V) \qquad \text{for all } (z, \Omega) \in \mathcal{M}.$$

By π_1 and π_2 we shall denote the canonical projections of $\mathcal{M}$ onto D and $\mathcal{O}(V, e_V)$ respectively. The following is a $\mathcal{C}^{k-2}$-vector-field on $\mathcal{M}$:

$$\begin{aligned} &f : \mathcal{M} \to T\mathcal{M}, \\ &(z, \Omega) \mapsto d(z) \oplus \big(\tilde{\Omega}_s'\big[\pi_L d(z)\big] \circ \tilde{\Omega}_s^* + \tilde{\Omega}_x'\big[\pi_{L^\perp} d(z)\big] \circ \tilde{\Omega}_x^*\big) \circ \Omega \in T_{(z,\Omega)}\mathcal{M}. \end{aligned} \tag{3.8}$$

In fact, in order to show that the right hand side lies indeed in $T_{(z,\Omega)}\mathcal{M}$ it suffices to prove that

$$\big(\tilde{\Omega}_s'[\pi_L d] \circ \tilde{\Omega}_s^* + \tilde{\Omega}_x'[\pi_{L^\perp} d] \circ \tilde{\Omega}_x^*\big) \in L(\mathcal{O}(V, e_V)).$$

But the left-hand side is skew-symmetric, since $\tilde{\Omega}_x$ and $\tilde{\Omega}_s$ lie in $\mathcal{O}(V)$. Moreover, it is true that

$$\begin{aligned}
&\big(\tilde{\Omega}_s'[\pi_L d] \circ \tilde{\Omega}_s^* + \tilde{\Omega}_x'[\pi_{L^\perp} d] \circ \tilde{\Omega}_x^*\big)[e_V] = \\
&\quad = \tilde{\Omega}_s'[\pi_L d] \circ F_\sharp''^{-1/2}\big(s(z)\big)\big[-F_\sharp'\big(s(z)\big)\big] + \tilde{\Omega}_x'[\pi_{L^\perp} d] \circ F''^{-1/2}\big(x(z)\big)\big[-F'(x(z))\big] \\
&\quad = \big(\tilde{S}\big[-F_\sharp'(s(z))\big]\big)'[\pi_L d] + \big(\tilde{X}\big[-F'(x(z))\big]\big)'[\pi_{L^\perp} d] = \frac{d}{dz} e_V[d] = 0.
\end{aligned}$$

This shows that the vector-field f is well-defined. Differentiability is immediate. Now, Theorem 2.9 says that there exists a maximal local flow $\Phi : W \to \mathcal{M}$ on $\mathcal{M}$ with phase velocity field f.

Let $\varphi\big(t,(z,\Omega)\big) := \pi_1 \circ \Phi\big(t,(z,\Omega)\big)$ be the projection of Φ into D, and let

$$W_1 := \big\{\big(t, \pi_1(z,\Omega)\big) : \big(t,(z,\Omega)\big) \in W\big\} \subseteq \mathbb{R} \times D$$

be the projection of W into $\mathbb{R} \times D$. We are now going to show that $\varphi\big(t,(z,\Omega)\big)$ does not depend on Ω. Let the mapping $\mathcal{O}(V,e_V) \to \mathcal{O}(V,e_V)$ given by right multiplication with a fixed element $\Omega \in \mathcal{O}(V,e_V)$ be denoted by

$$\begin{aligned}
\varrho(\Omega) : \mathcal{O}(V,e_V) &\to \mathcal{O}(V,e_V), \\
\breve{\Omega} &\mapsto \breve{\Omega} \circ \Omega.
\end{aligned}$$

Then, for all $z \in D$ and $\Omega_1, \Omega_2 \in \mathcal{O}(V,e_V)$ we have

$$\begin{aligned}
\big(\imath \otimes \varrho(\Omega_2)\big) \circ \big(\imath \otimes \varrho(\Omega_1^*)\big) \circ \Phi\big(0,(z,\Omega_1)\big) &= \big(\imath \circ \imath(z), \Omega_1 \Omega_1^* \Omega_2\big) \\
&= \big(z, \Omega_2\big) = \Phi\big(0,(z,\Omega_2)\big) \qquad \text{and} \\
\frac{\partial}{\partial t}\Big[\big(\imath \otimes \varrho(\Omega_2)\big) \circ \big(\imath \otimes \varrho(\Omega_1^*)\big) \circ \Phi\big(t,(z,\Omega_1)\big)\Big] &= \\
&= d(z) \oplus \big(\tilde{\Omega}_s'[\pi_L d] \circ \tilde{\Omega}_s^* + \tilde{\Omega}_x'[\pi_{L^\perp} d] \circ \tilde{\Omega}_x^*\big) \circ \Omega_1 \circ \Omega_1^* \circ \Omega_2 \\
&= d(z) \oplus \big(\tilde{\Omega}_s'[\pi_L d] \circ \tilde{\Omega}_s^* + \tilde{\Omega}_x'[\pi_{L^\perp} d] \circ \tilde{\Omega}_x^*\big) \circ \Omega_2 \\
&= \frac{\partial}{\partial t} \Phi\big(t,(z,\Omega_2)\big).
\end{aligned}$$

It follows from part *ii)* of Theorem 2.7 and from maximality of Φ that

$$\begin{aligned}
\big(\imath \otimes \varrho(\Omega_2)\big) \circ \big(\imath \otimes \varrho(\Omega_1^*)\big) \circ \Phi\big(t,(z,\Omega_1)\big) &\equiv \Phi\big(t,(z,\Omega_2)\big) \qquad \text{and} \\
\big\{t \in \mathbb{R} : \big(t,(z,\Omega_1)\big) \in W\big\} &= \big\{t \in \mathbb{R} : \big(t,(z,\Omega_2)\big) \in W\big\}.
\end{aligned} \tag{3.9}$$

Therefore, $\varphi\big(t,(z,\Omega)\big)$ is indeed independent of Ω and it makes sense to use the notation

$$\begin{aligned}
\varphi : W_1 &\to D \\
(t,z) &\mapsto \varphi(t,z) := \varphi\big(t, \pi_1^{-1}(z)\big).
\end{aligned}$$

Moreover, it follows that $I_{(z,\Omega_1)} = I_{(z,\Omega_2)}$ and it makes sense to refer to this interval as I_z instead.

Lemma 3.9. *φ is the maximal local flow on D corresponding to the phase velocity field d.*

Proof. Clear from the remarks made above. □

Remark 3.10. In the sequel we shall frequently use the results and equations developed in Section 4, and the reader may at this point want to familiarize herself with the kind of questions answered there.

In order to simplify notation we will henceforth write $\|\cdot\|$ for any occurring norm, it being understood that the norm we are referring to is clear from the context. The same rule applies to scalar products which we shall invariably denote by $(\cdot;\cdot)$ without reference to the space on which they are defined.

Lemma 3.11. *Problem 3.7 has a solution if and only if it has a solution in a neighborhood of z_μ.*

Proof. The only if part is of course trivially true. Let us therefore assume that there exists an open neighborhood $\mathcal{N} \subset \mathcal{F}^\circ(PD)$ of z_μ and a mapping $\breve{\Omega} \in \mathcal{C}^{k-2}\big(\mathcal{N}, \mathcal{O}(V, e_V)\big)$ such that

$$\breve{\Omega}'[d] \equiv \big(\tilde{\Omega}'_s\big[\pi_L d\big] \circ \tilde{\Omega}^*_s + \tilde{\Omega}'_x\big[\pi_{L^\perp} d\big] \circ \tilde{\Omega}^*_x\big) \circ \breve{\Omega} \qquad \text{on } \mathcal{N} \setminus \{z_\mu\}.$$

We will now extend this solution outside of the neighborhood $\mathcal{N}$ by integrating the vector field f in reverse direction starting from the initial points $\big(z, \breve{\Omega}(z)\big)$ with $z \in \mathcal{N}$ (see Figure 1). Because of Proposition 1.1 and the Inverse Function Theorem there exist open neighborhoods $\mathcal{N}_Z \subseteq \mathcal{N}$ and $\mathcal{N}_V \subset V$ of z_μ and μe_V respectively such that $\gamma|_{\mathcal{N}_Z} : \mathcal{N}_Z \leftrightarrow \mathcal{N}_V$ is $\mathcal{C}^2$-diffeomorphism. Let $\delta > 0$ be a real number such that $B_\delta(\mu e_V) \subset \mathcal{N}_V$. For $z \in D$ let

$$t_\delta(z) := \ln \|\mu e_V - \gamma(z)\| - \ln \delta.$$

If $\gamma(z) \notin B_\delta(\mu e_V)$ then $t_\delta(z) > 0$ and it follows from Lemma 4.2 that $t_\delta(z) \in I_z$. On the other hand, if $\gamma(z) \in B_\delta(\mu e_V)$ then it follows from Lemma 4.1 that $\varphi(t, z) \in B_\delta(\mu e_V)$ for all $t \in I_z \cap [t_\delta(z), 0]$, and hence we must have $t_\delta(z) \in I_z$ by maximality of φ. Therefore,

$$\|\mu e_V - \gamma \circ \varphi\big(t_\delta(z), z\big)\| \overset{(4.1)}{=} \|\mu e_V - \gamma(z)\| e^{-t_\delta(z)} = \delta,$$

and the following mapping $\Omega : \mathcal{F}^\circ(PD) \to \mathcal{O}(V, e_V)$ is well-defined:

$$\Omega : z \mapsto \begin{cases} \imath & \text{if } z = z_\mu, \\ \pi_2\big[\Phi\big(t_\delta(z), (z, \imath)\big)\big]^* \circ \breve{\Omega}\big(\varphi(t_\delta(z), z)\big) & \text{if } z \in D. \end{cases}$$

We claim that this is a solution to Problem 3.7. For all $z \in D$ we have

$$t_\delta\big(\varphi(t, z)\big) = t_\delta(z) - t, \tag{3.10}$$

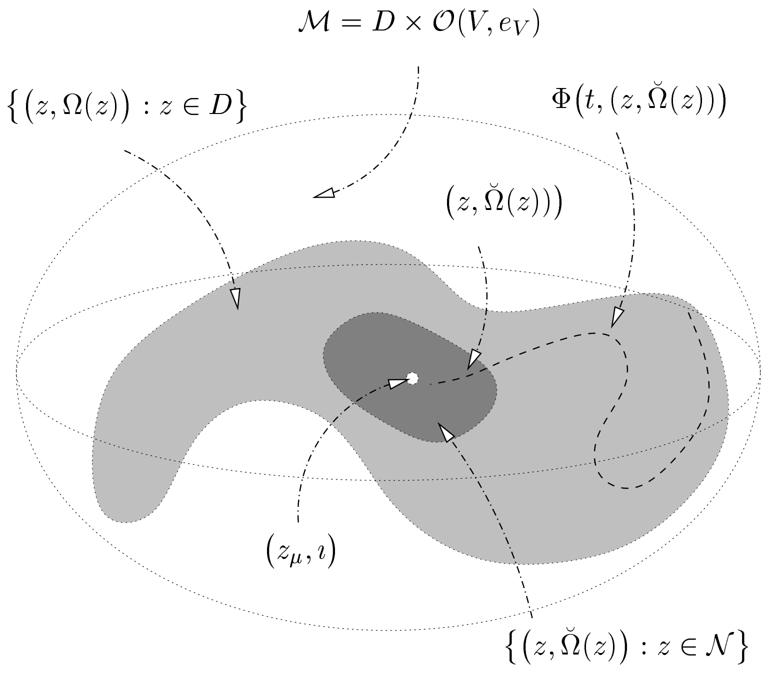


Figure 1: Extending a local solution.

and for $t \in I_z$ and $\hat{\Omega} \in \mathcal{O}(V, e_V)$ we have from (3.9) that

$$\pi_2\big[\Phi\big(t, (z, \hat{\Omega})\big)\big] = \pi_2\big[\Phi\big(t, (z, \imath)\big)\big] \circ \hat{\Omega}. \tag{3.11}$$

Therefore we get

$$\begin{aligned}
\Omega\big(\varphi(t,z)\big) &= \pi_2\big[\Phi\big(t_\delta(\varphi(t,z)), (\varphi(t,z), \imath)\big)\big]^* \circ \breve{\Omega}\big(\varphi\big(t_\delta(\varphi(t,z)), \varphi(t,z)\big)\big) \\
&\overset{(3.10),(3.11)}{=} \Big(\pi_2\big[\Phi\big(t_\delta(z) - t, \big(\varphi(t,z), \pi_2[\Phi(t,(z,\imath))]\big)\big)\big] \circ \\
&\qquad \circ \pi_2\big[\Phi\big(t, (z, \imath)\big)\big]^*\Big)^* \circ \breve{\Omega}\big(\varphi\big(t_\delta(z) - t, \varphi(t,z)\big)\big) \\
&\overset{(2.6)}{=} \Big(\pi_2\big[\Phi\big(t_\delta(z) - t + t, (z, \imath)\big)\big] \circ \pi_2\big[\Phi\big(t, (z,\imath)\big)\big]^*\Big)^* \circ \\
&\qquad \circ \breve{\Omega}\big(\varphi\big(t_\delta(z) - t + t, z\big)\big) \\
&= \pi_2\big[\Phi\big(t, (z,\imath)\big)\big] \circ \pi_2\big[\Phi\big(t_\delta(z), (z,\imath)\big)\big]^* \circ \breve{\Omega}\big(\varphi\big(t_\delta(z), z\big)\big) \\
&= \pi_2\big[\Phi\big(t, (z,\imath)\big)\big] \circ \Omega(z). \qquad (3.12)
\end{aligned}$$

Now, since $\frac{\partial}{\partial t}|_{t=0}\varphi(t,z) = d(z)$ for all $z \in D$, equation (3.12) implies that

$$\begin{aligned}
\Omega'(z)\big[d(z)\big] &= \frac{\partial}{\partial t}\Big|_{t=0} \Omega\big(\varphi(t,z)\big) \\
&= \frac{\partial}{\partial t}\Big|_{t=0} \Big(\pi_2\big[\Phi\big(t,(z,\imath)\big)\big] \circ \Omega(z)\Big) \\
&= \Big(\tilde{\Omega}'_s(z)\big[\pi_L d\big] \circ \tilde{\Omega}^*_s(z) + \tilde{\Omega}'_x(z)\big[\pi_{L^\perp} d\big] \circ \tilde{\Omega}^*_x(z)\Big) \circ \Omega(z). \qquad (3.13)
\end{aligned}$$

The orthogonal fields Ω and $\breve{\Omega}$ agree on $B_\delta(\mu e_V)$. In fact, they clearly agree at the point z_μ since $\breve{\Omega}(z_\mu) = \imath = \Omega(z_\mu)$. On the other hand, for $z \in B_\delta(\mu e_V) \setminus \{z_\mu\}$ equation (3.13) shows that both $y_1 : t \mapsto \breve{\Omega}\big(\varphi(t,z)\big)$ and $y_2 : t \mapsto \Omega\big(\varphi(t,z)\big)$ are solutions to the initial value problem

$$\begin{aligned} y : I_z \to \mathcal{O}(V, e_V) \qquad & \text{s.t.} \\ y\big(t_\delta(z)\big) = \breve{\Omega}\big(t_\delta(z), z\big), \qquad & \text{and} \\ \dot{y}(t) = \Big(\tilde{\Omega}_s'\big(\varphi(t,z)\big)\big[\pi_L d\big] \circ \tilde{\Omega}_s^*\big(\varphi(t,z)\big) & \\ + \tilde{\Omega}_x'\big(\varphi(t,z)\big)\big[\pi_{L^\perp} d\big] \circ \tilde{\Omega}_x^*\big(\varphi(t,z)\big)\Big) \circ y(t). & \end{aligned}$$

Hence, $y_1 \equiv y_2$ on I_z, and in particular

$$\breve{\Omega}(z) = y_1(0) = y_2(0) = \Omega(z).$$

This shows that $\Omega|_{B_\delta(\mu e_V)} \equiv \breve{\Omega}|_{B_\delta(\mu e_V)}$ as claimed, and it follows in particular that $\Omega|_{B_\delta(\mu e_V)}$ is in $\mathcal{C}^2$. Clearly, from the definition of Ω it follows that $\Omega|_D$ is also in $\mathcal{C}^2$, and hence Ω is twice continuously differentiable everywhere. Finally, it follows from equation (3.13) that $\Omega|_D$ solves the boundary value problem (3.6). $\square$

The rest of this section will be devoted to showing that Problem 3.7 admits a solution in a neighborhood of the boundary point z_μ.

For the time being, let us regard $z \in D$ as a fixed parameter and consider the following coordinate change in the time domain:

$$\begin{aligned} \psi : I_z \cup \{+\infty\} \to \big[0, e^{-a_z}\big), & \\ t \mapsto \psi(t) := \begin{cases} e^{-t} & \text{if } t \in (a_z, +\infty), \\ 0 & \text{for } t = +\infty, \end{cases} \end{aligned}$$

where a_z is defined as in Lemma 4.2. Note that since $a_z < 0$ we have $[0,1] \subset \mathrm{im}(\psi)$. We will denote the inverse of the mapping ψ by t, i.e. for corresponding values of $t \in I_z \cup \{+\infty\}$ and $\psi \in [0, e^{-a_z})$ we write $\psi = \psi(t)$ and $t = t(\psi)$. Let us also define the following function:

$$\begin{aligned} \beta : \big[0, e^{-a_z}\big) \to \mathcal{F}^\circ(PD), & \\ \psi \mapsto \begin{cases} \varphi\big(t(\psi), z\big) & \text{if } \psi > 0, \\ z_\mu & \text{for } \psi = 0. \end{cases} \end{aligned}$$

We contend that $\beta \in \mathcal{C}^1$. In fact, β is continuous everywhere,

$$\begin{aligned} \frac{d}{d\psi}\beta(\psi) = \frac{\partial}{\partial t}\varphi(t,z) \cdot \frac{dt}{d\psi} = d(\varphi(t,z)) \cdot \Big(-\frac{1}{\psi}\Big) = -e^t d\big(\varphi(t,z)\big), \\ \text{and} \qquad \lim_{\psi \to 0} \frac{d}{d\psi}\beta(\psi) = -\lim_{t \to +\infty} e^t d\big(\varphi(t,z)\big) \end{aligned}$$

exists by Lemma 4.10. Let $G := \mathcal{O}(V, e_V)$ and let $\mathfrak{g} := L\big(\mathcal{O}(V, e_V)\big)$ be its associated Lie algebra.

Remark 3.12 (Notation).
For ease of notation in the rest of this chapter all expressions involving $\tilde{\Omega}$ and π should be read as the sum of the corresponding expressions obtained from replacing $\tilde{\Omega}$ by $\tilde{\Omega}_s$ and π by π_L for the first summand, and $\tilde{\Omega}$ by $\tilde{\Omega}_x$ and π by $\pi_{L^\perp}$ for the second summand. For instance, we shall write $\tilde{\Omega}(z)[\pi d(z)] \circ \tilde{\Omega}^*(z)$ in lieu of

$$\tilde{\Omega}_s(z)[\pi_L d(z)] \circ \tilde{\Omega}_s^*(z) + \tilde{\Omega}_x(z)[\pi_{L^\perp} d(z)] \circ \tilde{\Omega}_x^*(z),$$

and $\tilde{\Omega}''(z)[\pi w_1, w_2] \circ \tilde{\Omega}^{*\prime}(z)[w_3]$ in place of

$$\tilde{\Omega}_s''(z)[\pi_L w_1, w_2] \circ \tilde{\Omega}_s^{*\prime}(z)[w_3] + \tilde{\Omega}_x''(z)[\pi_{L^\perp} w_1, w_2] \circ \tilde{\Omega}_x^{*\prime}(z)[w_3].$$

Let us now consider the mapping

$$\begin{aligned} y : \big[0, e^{-a_z}\big) &\to \mathcal{O}(V, e_V), \\ \psi &\mapsto \Omega\big(\beta(\psi)\big), \end{aligned}$$

where Ω is a hypothetical solution to Problem 3.7. If such an Ω exists, then for all $\psi \neq 0$ we must have

$$\begin{aligned} y'(\psi) &= \Omega'\big(\beta(\psi)\big)\left[\beta'(\psi)\right] \\ &= \Omega'\big(\beta(\psi)\big)\Big[\frac{\partial}{\partial t}\varphi(t(\psi), z) \cdot \frac{d}{d\psi} t(\psi)\Big] \\ &= -\Omega'\big(\beta(\psi)\big)\left[\psi^{-1} d\big(\beta(\psi)\big)\right] \\ &\overset{(3.6)}{=} -\tilde{\Omega}'\big(\beta(\psi)\big)\left[\pi\psi^{-1} d\big(\beta(\psi)\big)\right] \circ \tilde{\Omega}^*\big(\beta(\psi)\big) \circ \Omega\big(\beta(\psi)\big) \\ &= a(\psi) y(\psi), \end{aligned}$$

where the mapping a is defined as follows:

$$\begin{aligned} a : \big(0, e^{-a_z}\big) &\to \mathfrak{g}, \\ \psi &\mapsto -\tilde{\Omega}'\big(\beta(\psi)\big)\left[\pi\psi^{-1} d\big(\beta(\psi)\big)\right] \circ \tilde{\Omega}^*\big(\beta(\psi)\big). \end{aligned}$$

Clearly, if our barrier functional is of class $\mathcal{C}^4$ then $a \in \mathcal{C}^1\big((0, e^{-a_z}), \mathfrak{g}\big)$. Moreover, $\lim_{\psi \to 0} a(\psi)$ exists because

$$\lim_{\psi \to 0} \psi^{-1} d\big(\beta(\psi)\big) = \lim_{t \to +\infty} e^t d\big(\varphi(t, z)\big),$$

and the expression on the right-hand side exists by virtue of Lemma 4.10. Finally, Lemma 4.13 says that the following limit exists:

$$\begin{aligned} \lim_{\psi \to 0} \frac{d}{d\psi}\Big(\psi^{-1} d\big(\beta(\psi)\big)\Big) &= \lim_{t \to +\infty} \frac{\partial}{\partial t}\Big(e^t d\big(\varphi(t, z)\big)\Big) \cdot \frac{dt}{d\psi} \\ &= -\lim_{t \to +\infty} e^t \frac{\partial}{\partial t}\Big(e^t d\big(\varphi(t, z)\big)\Big). \end{aligned}$$

Hence, the domain of definition of a can be extended to include the value $\psi = 0$, and if we denote the resulting map by a again then it is true that $a \in \mathcal{C}^1\big([0, e^{-a_z}), \mathfrak{g}\big)$. In particular, a is Lipschitz continuous on any compact subset of $[0, e^{-a_z})$ and it follows from Theorem 2.11 that the initial value problem

$$\begin{aligned} y : \big[0, e^{-a_z}\big) &\to \mathcal{O}(V, e_V), \\ y'(\psi) &= a(\psi) y(\psi), \qquad \forall t \in \big[0, e^{-a_z}\big), \\ y(0) &= \imath \end{aligned} \tag{3.14}$$

has a *unique* solution at least in the interval where its Magnus series and its term-wise derivative converges absolutely. Let us denote this interval by J_z. Then for all $\psi \in J_z$ we have

$$\begin{aligned} y(\psi) &= \exp\big\{\sigma(\psi)\big\}, \qquad \text{where} \\ \sigma(\psi) &= \sum_{H \in \mathcal{E}} \alpha_H \int_0^{\psi} H(\kappa) d\kappa. \end{aligned}$$

Note however that $\Omega \circ \beta$ is a solution to problem (3.14), and by the uniqueness part of Theorem 2.11 we must have $\Omega \circ \beta \equiv y$ on J_z. It follows from Lemmas 4.3 and 4.11 and from continuity of the operator $\tilde{\Omega}'(z) \circ \tilde{\Omega}^*(z)$ at z_μ that

$$\|a(\psi)\| = O\big(\|z - z_\mu\|\big) \cdot \psi^0 \qquad \forall\, 0 \le \psi \le 1,$$

and it follows from Theorem 2.11 that for z in a neighborhood of z_μ we have $[0,1] \subset J_z$. For such z we therefore have

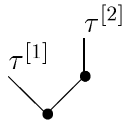
$$\begin{aligned} \Omega(z) &= \Omega\big(\varphi(0, z)\big) = \Omega\big(\beta(1)\big) = y(1) \\ &= \exp\Big\{\sum_{H \in \mathcal{E}} \alpha_H \int_0^1 H(\kappa) d\kappa\Big\}. \end{aligned} \tag{3.15}$$

We only referred to Theorem 2.11 to assure the reader that it is sensible to make the ansatz (3.15). In fact, for the purposes of this section we need a stronger convergence result which we shall derive without the use of Theorem 2.11 (see Lemma 3.14 below).

At this point let us revert to our old parameterization of the time domain. Because of the identity

$$\int_0^{\psi(T)} H(\psi) d\psi = \int_{+\infty}^{T} \frac{d\psi(t)}{dt} H\big(\psi(t)\big) dt = \int_T^{+\infty} e^{-t} H\big(e^{-t}\big) dt$$

it would be useful to have a recursive definition of the expressions $e^{-t}H\big(e^{-t}\big)$ for all $H \in \mathcal{E}$. Let $\tau, \tau^{[1]}, \tau^{[2]} \in \mathcal{T}$ be such that

$\tau =$ 

Then we have

$$
\begin{aligned}
e^{-t} H_\tau\big(e^{-t}\big) &= e^{-t}\Big[H_{\tau^{[1]}}\big(e^{-t}\big), \int_0^{e^{-t}} H_{\tau^{[2]}}(\psi)d\psi\Big] \\
&= \Big[e^{-t} H_{\tau^{[1]}}\big(e^{-t}\big), \int_{+\infty}^{t} \frac{d\psi(\xi)}{d\xi} H_{\tau^{[2]}}\big(\psi(\xi)\big)d\xi\Big] \\
&= \Big[e^{-t} H_{\tau^{[1]}}\big(e^{-t}\big), \int_t^{+\infty} e^{-\xi} H_{\tau^{[2]}}\big(e^{-\xi}\big)d\xi\Big].
\end{aligned}
$$

Moreover, for $\tau = \tau_0$ we have

$$
\begin{aligned}
e^{-t} H_{\tau_0}\big(e^{-t}\big) &= e^{-t} a\big(e^{-t}\big) \\
&= -\tilde{\Omega}'\big(\varphi(t,z)\big)\big[\pi d\big(\varphi(t,z)\big)\big] \circ \tilde{\Omega}^*\big(\varphi(t,z)\big).
\end{aligned}
$$

It follows that the set of functions

$$
\tilde{\mathcal{E}} := \big\{t \mapsto L_\tau(t) := e^{-t} H_\tau\big(e^{-t}\big) : \tau \in \mathcal{T}\big\}
$$

can be defined by recursive application of the following rules:

1. $L_{\tau_0}(t) := -\tilde{\Omega}'\big(\varphi(t,z)\big)\big[\pi d\big(\varphi(t,z)\big)\big] \circ \tilde{\Omega}^*\big(\varphi(t,z)\big)$ for all $t \in I_z \cup \{+\infty\}$.
2. For $\tau, \tau^{[1]}, \tau^{[2]} \in \mathcal{T}$ as above and $t \in I_z \cup \{+\infty\}$,

$$
L_\tau(t) := \Big[L_{\tau^{[1]}}(t), \int_t^{+\infty} L_{\tau^{[2]}}(\xi)d\xi\Big].
$$

Therefore, $\Omega(z)$ can be represented in the form $\Omega(z) = \exp\big\{\varsigma(z)\big\}$, where

$$
\begin{aligned}
\varsigma(z) &:= \sigma(1) \\
&= \sum_{\tau \in \mathcal{T}} \alpha_\tau \int_0^1 H_\tau(\kappa)d\kappa \\
&= \sum_{\tau \in \mathcal{T}} \alpha_\tau \int_0^\infty L_\tau\big(\varphi(t,z)\big)dt. \qquad (3.16)
\end{aligned}
$$

Note that the integrals appearing in this Magnus series are taken over a fixed time horizon. On the other hand, the definition of the functions L_τ depends on the "fixed" parameter z by means of the path $\varphi(\cdot, z)$. Since we would like to analyze the dependence of Ω on the spatial variable z in a neighborhood of z_μ it is henceforth more convenient to write as above $L_\tau\big(\varphi(t,z)\big)$ instead of $L_\tau(t)$.

For $k \in \mathbb{N} \cup \{0\}$ let us denote the subset of $\mathcal{T}$ consisting of those members that are of order $3k+1$ by $\mathcal{T}_{3k+1}$. It is easy to show recursively that all the elements of $\mathcal{T}$ are of order $\equiv 1$ mod 3, and hence $\mathcal{T} = \bigcup_{\mathbb{N}_0} \mathcal{T}_{3k+1}$.

At this point the reader is advised to consult the cornucopia of lemmas from Section 4. Note that these results combined with the fact that $\tilde{\Omega}$ is threefold continuously differentiable imply that it is possible to choose real numbers $Q > 0$ and $\varrho > 0$ such that for all $z \in B_\varrho(z_\mu) \setminus \{z_\mu\}$ and $t \geq 0$ the following inequalities are satisfied:

1. $\big\|\tilde{\Omega}'(z)[\pi\cdot]\tilde{\Omega}^*(z)\big\| \le Q$,
2. $\big\|\tilde{\Omega}''(z)[\pi\cdot,\cdot]\tilde{\Omega}^*(z)\big\| \le Q$,
3. $\big\|\tilde{\Omega}'(z)[\pi\cdot]\tilde{\Omega}^{*\prime}(z)[\cdot]\big\| \le Q$,
4. $\big\|\tilde{\Omega}'''(z)[\pi\cdot,\cdot,\cdot]\tilde{\Omega}^*(z)\big\| \le Q$,
5. $\big\|\tilde{\Omega}''(z)[\pi\cdot,\cdot]\tilde{\Omega}^{*\prime}(z)[\cdot]\big\| \le Q$
6. $\big\|\tilde{\Omega}'(z)[\pi\cdot]\tilde{\Omega}^{*\prime\prime}(z)[\cdot,\cdot]\big\| \le Q$,
7. $\big\|d\big(\varphi(t,z)\big)\big\| \le Q\|z_\mu - z\| \cdot e^{-t}$ (see Lemmas 4.3 and 4.11),
8. $\big\|z_\mu - \varphi(t,z)\big\| \le \|z_\mu - z\|$ (see Lemma 4.12),
9. $\big\|\frac{\partial}{\partial z}\varphi(t,z)\big\| \le 1$ (see equation (4.21) in Lemma 4.7),
10. $\big\|\frac{\partial^2}{\partial z^2}\varphi(t,z)\big\| \le Q$ (see equation (4.29)),
11. $\int_0^\infty \big\|\frac{\partial}{\partial z}d\big(\varphi(t,z)\big)\big\|dt \le 1 + Q\|z_\mu - z\|$ (see Lemma 4.7),
12. $\int_0^\infty \big\|\frac{\partial^2}{\partial z^2}d\big(\varphi(t,z)\big)\big\|dt \le Q\big(1 + Q\|z_\mu - z\|\big)$ (see Lemma 4.8).

Lemma 3.13. *Let $Q, \varrho > 0$ be real numbers that satisfy conditions $1-12$ above, and let $k \in \mathbb{N}\cup\{0\}$. Then, for $z \in B_\varrho(z_\mu)\setminus\{z_\mu\}$ and for $\tau \in \mathcal{T}_{3k+1}$ it is true that*

$$
\begin{aligned}
&i) && \int_0^\infty \big\|L_\tau\big(\varphi(t,z)\big)\big\|dt &&\le \|z_\mu - z\|^{k+1}Q^{2k+2}2^k,\\
&ii) && \int_0^\infty \big\|\frac{\partial}{\partial z}L_\tau\big(\varphi(t,z)\big)\big\|dt &&\le \|z_\mu - z\|^{k}Q^{2k+1}2^{2k}\\
&&&&& \quad + \|z_\mu - z\|^{k+1}Q^{2k+2}3^{2k+1},\\
&iii) && \int_0^\infty \big\|\frac{\partial^2}{\partial z^2}L_\tau\big(\varphi(t,z)\big)\big\|dt &&\le \|z_\mu - z\|^{k-1}Q^{2k}\chi(k)\\
&&&&& \quad + \|z_\mu - z\|^{k}Q^{2k+1}\big(3^{2k+2} + Q3^{2k}\big)\\
&&&&& \quad + \|z_\mu - z\|^{k+1}Q^{2k+2}\big(3^{2k+3} + Q3^{2k+1}\big),
\end{aligned}
$$

where $$\chi(k) := \begin{cases} 0 & \text{if } k = 0,\\ 2^{2(k+1)} & \text{if } k \ge 1.\end{cases}$$

A proof of this lemma is given in the appendix.

Lemma 3.14. *There exists a radius $\rho > 0$ such that on $z \in B_\rho(z_\mu)\setminus\{z_\mu\}$ the functions $\varsigma(z)$ (see (3.16)), $\frac{\partial}{\partial z}\varsigma(z)$ and $\frac{\partial^2}{\partial z^2}\varsigma(z)$ exist and are continuous. Moreover,* *i)* $\varsigma(z) = O\big(\|z_\mu - z\|\big)$,

ii) $\frac{\partial}{\partial z}\varsigma(z) = \int_0^\infty \frac{\partial}{\partial z} L_{\tau_0}\big(\varphi(t,z)\big)\,dt + O\big(\|z_\mu - z\|\big)$,

iii) $\frac{\partial^2}{\partial z^2}\varsigma(z) = \int_0^\infty \frac{\partial^2}{\partial z^2} L_{\tau_0}\big(\varphi(t,z)\big)\,dt + \frac{1}{2}\int_0^\infty \frac{\partial^2}{\partial z^2} L_{\tau_1}\big(\varphi(t,z)\big)\,dt + O\big(\|z_\mu - z\|\big)$.

Proof. Let $Q, \varrho > 0$ be real numbers that satisfy conditions $1-12$ above. Note that in condition 7 we made the tacit assumption that $Q \geq \sigma$ and $\varrho < \delta$, where σ and δ are chosen as in Lemma 4.3. Let $\tau \in \mathcal{T}_{3k+1}$. It follows from Lemma 3.13 that all of the integrals $\int_0^\infty L_\tau\big(\varphi(t,z)\big)\,dt$, $\int_0^\infty \frac{\partial}{\partial z} L_\tau\big(\varphi(t,z)\big)\,dt$ and $\int_0^\infty \frac{\partial^2}{\partial z^2} L_\tau\big(\varphi(t,z)\big)\,dt$ exist. Moreover, for all $z \in B_\varrho(z_\mu)$ we have

$$
\begin{aligned}
&\Big\|\int_0^\infty \frac{\partial}{\partial z} L_\tau\big(\varphi(t,z)\big)\,dt - \frac{\partial}{\partial z}\int_0^\xi L_\tau\big(\varphi(t,z)\big)\,dt\Big\| \\
&\quad= \Big\|\int_0^\infty \frac{\partial}{\partial z} L_\tau\big(\varphi(t,z)\big)\,dt - \int_0^\xi \frac{\partial}{\partial z} L_\tau\big(\varphi(t,z)\big)\,dt\Big\| \\
&\quad\overset{(2.3)}{=} \Big\|\int_0^\infty \frac{\partial}{\partial z} L_\tau\big(\varphi\big(t,\varphi(\xi,z)\big)\big)\,dt\Big\| \\
&\quad\leq \int_0^\infty \Big\|\frac{\partial}{\partial \tilde z} L_\tau\big(\varphi(t,\tilde z)\big)\Big|_{\tilde z=\varphi(\xi,z)}\Big\| \cdot \Big\|\frac{\partial}{\partial z}\varphi(\xi,z)\Big\|\,dt \\
&\quad\overset{(4.21)}{\leq} \Big(\int_0^\infty \Big\|\frac{\partial}{\partial \tilde z} L_\tau\big(\varphi(t,\tilde z)\big)\Big|_{\tilde z=\varphi(\xi,z)}\Big\|\,dt\Big) \cdot \exp\big\{-\xi\big(1 - \big|O(\|z_\mu - z\|)\big|\big)\big\} \\
&\quad\overset{3.13\,ii)}{\leq} \Big(2^{2k}Q^{2k+1}\big\|z_\mu - \varphi(\xi,z)\big\|^k + 3^{2k+1}Q^{2k+2}\big\|z_\mu - \varphi(\xi,z)\big\|^{k+1}\Big) \cdot \\
&\qquad\qquad \cdot \exp\big\{-\xi\big(1 - \big|O(\|z_\mu - z\|)\big|\big)\big\} \\
&\quad\overset{4.3}{\leq} \Big(2^{2k}Q^{3k+1}\|z_\mu - z\|^k e^{-k\xi} + 3^{2k+1}Q^{3k+3}\|z_\mu - z\|^{k+1} e^{-(k+1)\xi}\Big) \cdot \\
&\qquad\qquad \cdot \exp\big\{-\xi\big(1 - \big|O(\|z_\mu - z\|)\big|\big)\big\}.
\end{aligned}
\tag{3.17}
$$

For the function $O(\|z_\mu - z\|)$ that appears in (3.17) let us choose $0 < \rho_1 \leq \varrho$ such that $\big|O(\|z_\mu - z\|)\big| < 1$ for all $z \in B_{\rho_1}(z_\mu)$. Then inequality (3.17) shows that we have

$$
\begin{aligned}
\frac{\partial}{\partial z}\int_0^\infty L_\tau\big(\varphi(t,z)\big)\,dt &= \frac{\partial}{\partial z}\lim_{\xi\to\infty}\int_0^\xi L_\tau\big(\varphi(t,z)\big)\,dt \\
&= \lim_{\xi\to\infty}\frac{\partial}{\partial z}\int_0^\xi L_\tau\big(\varphi(t,z)\big)\,dt \\
&= \int_0^\infty \frac{\partial}{\partial z} L_\tau\big(\varphi(t,z)\big)\,dt
\end{aligned}
$$

for all $z \in B_{\rho_1}(z_\mu) \setminus \{z_\mu\}$ and for all $\tau \in \mathcal{T}$.

Similarly, for $z \in B_\varrho(z_\mu) \setminus \{z_\mu\}$ we have

$$\begin{aligned}
&\Big\| \int_0^\infty \frac{\partial^2}{\partial z^2} L_\tau\big(\varphi(t,z)\big)\,dt - \frac{\partial^2}{\partial z^2} \int_0^\xi L_\tau\big(\varphi(t,z)\big)\,dt \Big\| \\
&\quad = \quad \Big\| \int_0^\infty \frac{\partial^2}{\partial z^2} L_\tau\big(\varphi(t,z)\big)\,dt - \int_0^\xi \frac{\partial^2}{\partial z^2} L_\tau\big(\varphi(t,z)\big)\,dt \Big\| \\
&\quad \overset{(2.3)}{=} \quad \Big\| \int_0^\infty \frac{\partial^2}{\partial z^2} L_\tau\big(\varphi\big(t,\varphi(\xi,z)\big)\big)\,dt \Big\| \\
&\quad \leq \quad \int_0^\infty \Big\| \frac{\partial^2}{\partial \tilde z^2} L_\tau\big(\varphi(t,\tilde z)\big)\Big|_{\tilde z=\varphi(\xi,z)} \Big\| \cdot \Big\| \frac{\partial}{\partial z}\varphi(\xi,z) \Big\|^2 dt \\
&\qquad\qquad + \int_0^\infty \Big\| \frac{\partial}{\partial \tilde z} L_\tau\big(\varphi(t,\tilde z)\big)\Big|_{\tilde z=\varphi(\xi,z)} \Big\| \cdot \Big\| \frac{\partial^2}{\partial z^2}\varphi(\xi,z) \Big\| dt \\
&\quad \overset{(4.21),(4.29)}{\leq} \quad \Big(\int_0^\infty \Big\| \frac{\partial^2}{\partial \tilde z^2} L_\tau\big(\varphi(t,\tilde z)\big)\Big|_{\tilde z=\varphi(\xi,z)} \Big\| dt + \\
&\qquad\qquad\qquad + \int_0^\infty \Big\| \frac{\partial}{\partial \tilde z} L_\tau\big(\varphi(t,\tilde z)\big)\Big|_{\tilde z=\varphi(\xi,z)} \Big\| dt \Big) \cdot \\
&\qquad\qquad\qquad\qquad \cdot \exp\big\{ -\xi\big(1 - \big|O(\|z_\mu - z\|)\big|\big)\big\} \cdot O(1) \\
&\quad \overset{3.13ii),iii)}{\leq} \quad \Big(\chi(k) Q^{2k} \big\| z_\mu - \varphi(\xi,z) \big\|^{k-1} + \\
&\qquad\qquad + \big(3^{2k+2} + Q 3^{2k}\big) Q^{2k+1} \big\| z_\mu - \varphi(\xi,z) \big\|^{k} \\
&\qquad\qquad + \big(3^{2k+3} + Q 3^{2k+1}\big) Q^{2k+2} \big\| z_\mu - \varphi(\xi,z) \big\|^{k+1} \\
&\qquad\qquad + 2^{2k} Q^{2k+2} \big\| z_\mu - \varphi(\xi,z) \big\|^{k} \\
&\qquad\qquad + 3^{2k+1} Q^{2k+3} \big\| z_\mu - \varphi(\xi,z) \big\|^{k+1} \Big) \cdot \\
&\qquad\qquad\qquad\qquad \cdot \exp\big\{ -\xi\big(1 - \big|O(\|z_\mu - z\|)\big|\big)\big\} \cdot O(1) \\
&\quad \overset{4.3}{\leq} \quad \Big(\chi(k) Q^{3k-1} \|z_\mu - z\|^{k-1} e^{-(k-1)\xi} \\
&\qquad\qquad + \big(3^{2k+2} + Q \cdot (3^{2k} + 2^{2k})\big) \cdot Q^{3k+1} \|z_\mu - z\|^{k} e^{-k\xi} \\
&\qquad\qquad + \big(3^{2k+3} + Q \cdot 2 \cdot 3^{2k+1}\big) \cdot Q^{3k+3} \|z_\mu - z\|^{k+1} e^{-(k+1)\xi} \Big) \cdot \\
&\qquad\qquad\qquad\qquad \cdot \exp\big\{ -\xi\big(1 - \big|O(\|z_\mu - z\|)\big|\big)\big\} \cdot O(1).
\end{aligned} \tag{3.18}$$

For the function $O(\|z_\mu - z\|)$ that appears in (3.18) let us choose $0 < \rho_2 \leq \rho_1$ such that $\big|O(\|z_\mu - z\|)\big| < 1$ for all $z \in B_{\rho_2}(z_\mu)$. Then inequality (3.18) combined with the fact that

$\chi(0) = 0$ implies that

$$\begin{aligned}\frac{\partial^2}{\partial z^2}\int_0^\infty L_\tau\big(\varphi(t,z)\big)\,dt &= \frac{\partial^2}{\partial z^2}\lim_{\xi\to\infty}\int_0^\xi L_\tau\big(\varphi(t,z)\big)\,dt\\ &= \lim_{\xi\to\infty}\frac{\partial^2}{\partial z^2}\int_0^\xi L_\tau\big(\varphi(t,z)\big)\,dt\\ &= \int_0^\infty \frac{\partial^2}{\partial z^2}L_\tau\big(\varphi(t,z)\big)\,dt\end{aligned}$$

for all $z \in B_{\rho_2}(z_\mu)\setminus\{z_\mu\}$ and for all $\tau \in \mathcal{T}$.

Note that Lemma 3.13 implies that there exists a radius $0 < \rho < \rho_2$ such that for $\|z_\mu - z\| < \rho$ the following inequalities hold true for all $k \in \mathbb{N}$:

$$\begin{aligned}&\int_0^\infty \big\|L_\tau\big(\varphi(t,z)\big)\big\|\,dt < \big(1/4\big)^{k+1},\\ &\int_0^\infty \Big\|\frac{\partial}{\partial z}L_\tau\big(\varphi(t,z)\big)\Big\|\,dt < \big(1/4\big)^{k}, \qquad \text{and}\\ &\int_0^\infty \Big\|\frac{\partial^2}{\partial z^2}L_\tau\big(\varphi(t,z)\big)\Big\|\,dt < \big(1/4\big)^{k-1}.\end{aligned} \tag{3.19}$$

Let $\Theta(z) := \sum_{k=0}^\infty \big(\#\mathcal{T}_{3k+1}\big)z^k$ be the generating function of the sequence $\big(\#\mathcal{T}_{3k+1}\big)_{k\in\mathbb{N}\cup\{0\}}$. It follows from Lemma 2.12 that

$$\Theta(z) = \frac{1-\sqrt{1-4z}}{2z} = \sum_{k=0}^\infty \frac{(2k)!}{k!(k+1)!}z^k.$$

Since this function is analytic in $B_{1/4}(0) \subset \mathbb{C}$, equations (2.13) and (3.19) imply that the series

$$\begin{aligned}&\sum_{\tau\in\mathcal{T}}\alpha_\tau\int_0^\infty L_\tau\big(\varphi(t,z)\big)\,dt,\\ &\sum_{\tau\in\mathcal{T}}\alpha_\tau\int_0^\infty \frac{\partial}{\partial z}L_\tau\big(\varphi(t,z)\big)\,dt \qquad \text{and}\\ &\sum_{\tau\in\mathcal{T}}\alpha_\tau\int_0^\infty \frac{\partial^2}{\partial z^2}L_\tau\big(\varphi(t,z)\big)\,dt\end{aligned}$$

converge absolutely in $B_\rho(z_\mu)\setminus\{z_\mu\}$. Hence, all of the following series converge and satisfy the following identities in $B_\rho(z_\mu)\setminus\{z_\mu\}$:

i) $$\varsigma(z) = \sum_{\tau\in\mathcal{T}}\alpha_\tau\int_0^\infty L_\tau\big(\varphi(t,z)\big)\,dt,$$

ii)
$$\begin{aligned}\frac{\partial}{\partial z}\varsigma(z) &= \frac{\partial}{\partial z}\sum_{\tau\in\mathcal{T}}\alpha_\tau\int_0^\infty L_\tau\big(\varphi(t,z)\big)\,dt\\ &= \sum_{\tau\in\mathcal{T}}\alpha_\tau\frac{\partial}{\partial z}\int_0^\infty L_\tau\big(\varphi(t,z)\big)\,dt\\ &= \sum_{\tau\in\mathcal{T}}\alpha_\tau\int_0^\infty \frac{\partial}{\partial z}L_\tau\big(\varphi(t,z)\big)\,dt, \qquad \text{and}\end{aligned}$$

iii)
$$\begin{aligned}\frac{\partial^2}{\partial z^2}\varsigma(z) &= \frac{\partial^2}{\partial z^2}\sum_{\tau\in\mathcal{T}}\alpha_\tau\int_0^\infty L_\tau\big(\varphi(t,z)\big)\,dt\\ &= \sum_{\tau\in\mathcal{T}}\alpha_\tau\frac{\partial^2}{\partial z^2}\int_0^\infty L_\tau\big(\varphi(t,z)\big)\,dt\\ &= \sum_{\tau\in\mathcal{T}}\alpha_\tau\int_0^\infty \frac{\partial^2}{\partial z^2}L_\tau\big(\varphi(t,z)\big)\,dt.\end{aligned}$$

The other statements now follow from Lemma 3.13 combined with the facts that $\mathcal{T}_1 = \{\tau_0\}$ and $\mathcal{T}_3 = \{\tau_1\}$ are singletons and that $\alpha_{\tau_0} = 1$ and $\alpha_{\tau_1} = 1/2$ (see equation (2.11)). □

We are finally ready to attack the final piece in our puzzle:

Theorem 3.15 (Local solution to Problem 3.7).
Let us choose $\rho > 0$ as in Lemma 3.14, so that the functions ς, ς' and ς'' are well defined on $B_\rho(z_\mu)\setminus\{z_\mu\}$. Let us extend ς to $B_\rho(z_\mu)$ by setting $\varsigma(z_\mu) := 0$. Then the following applies:

i) The extended function ς is twice continuously differentiable on $B_\rho(z_\mu)$, and at the point z_μ its derivatives are given by

$$\varsigma'(z_\mu)[w] = \tilde{\Omega}'(z_\mu)[\pi w]\circ\tilde{\Omega}^*(z_\mu) \quad \text{for } w\in Z. \tag{3.20}$$

$$\begin{aligned}\varsigma''(z_\mu)[w,v] = \frac{1}{2}\Big(&\tilde{\Omega}''(z_\mu)[\pi v,w]\circ\tilde{\Omega}^*(z_\mu) +\\ &+\tilde{\Omega}''(z_\mu)[\pi w,v]\circ\tilde{\Omega}^*(z_\mu)\\ &+\tilde{\Omega}'(z_\mu)[\pi w]\circ\tilde{\Omega}^{*\prime}(z_\mu)[v]\\ &+\tilde{\Omega}'(z_\mu)[\pi v]\circ\tilde{\Omega}^{*\prime}(z_\mu)[w]\Big)\\ &\text{for all } v,w\in Z.\end{aligned} \tag{3.21}$$

ii) The function $\Omega := \exp\circ\varsigma$ is a local solution to Problem 3.7 in the neighborhood $B_\rho(z_\mu)$ of z_μ, and at the point z_μ its derivatives are given by

$$\Omega'(z_\mu)[w] = \varsigma'(z_\mu)[w] \quad \text{for } w\in Z, \tag{3.22}$$

$$\Omega''(z_\mu)[w,v] = \varsigma''(z_\mu)[w,v] + \frac{1}{2}\Big(\varsigma'(z_\mu)[w]\varsigma'(z_\mu)[v] + \varsigma'(z_\mu)[v]\varsigma'(z_\mu)[w]\Big) \quad \text{for all } v,w \in Z. \tag{3.23}$$

Proof. i) We already know from Lemma 3.14 that ς is twice continuously differentiable on $B_\rho(z_\mu)\setminus\{z_\mu\}$. In order to prove twofold continuous differentiability of ς at the point z_μ it suffices to show that $\lim_{z\to z_\mu}\varsigma(z)$, $\lim_{z\to z_\mu}\varsigma'(z)$ and $\lim_{z\to z_\mu}\varsigma''(z)$ exist. Lemma 3.14 shows that

$$\lim_{z\to z_\mu}\varsigma(z) = 0$$

$$\lim_{z\to z_\mu}\varsigma'(z) = \lim_{z\to z_\mu}\int_0^\infty \frac{\partial}{\partial z}L_{\tau_0}\big(\varphi(t,z)\big)\,dt \tag{3.24}$$

$$\lim_{z\to z_\mu}\varsigma''(z) = \lim_{z\to z_\mu}\Big(\int_0^\infty \frac{\partial^2}{\partial z^2}L_{\tau_0}\big(\varphi(t,z)\big)\,dt + \frac{1}{2}\int_0^\infty \frac{\partial^2}{\partial z^2}L_{\tau_1}\big(\varphi(t,z)\big)\,dt\Big); \tag{3.25}$$

hence all we need to show is that the expressions on the right hand side of equations (3.24) and (3.25) exist and that we retrieve the expressions given in (3.20) and (3.21). Recall that $\alpha_0 = 1$ and $\alpha_1 = 1/2$. For $w \in Z$ we have

$$\begin{aligned}
\lim_{z\to z_\mu}\varsigma'(z)[w] &= -\lim_{z\to z_\mu}\int_0^\infty \frac{\partial}{\partial z}\Big(\tilde\Omega'\big(\varphi(t,z)\big)\big[\pi d\big(\varphi(t,z)\big)\big]\circ\tilde\Omega^*\big(\varphi(t,z)\big)\Big)[w]dt \\
&= -\lim_{z\to z_\mu}\int_0^\infty \tilde\Omega''(\varphi)\big[\pi d, \frac{\partial}{\partial z}\varphi[w]\big]\circ\tilde\Omega^*(\varphi)dt && \text{(3.26a)}\\
&\quad -\lim_{z\to z_\mu}\int_0^\infty \tilde\Omega'(\varphi)\big[\pi\frac{\partial}{\partial z}d(\varphi)[w]\big]\circ\tilde\Omega^*(\varphi)dt && \text{(3.26b)}\\
&\quad -\lim_{z\to z_\mu}\int_0^\infty \tilde\Omega'(\varphi)[\pi d]\circ\tilde\Omega^{*\prime}(\varphi)\big[\frac{\partial}{\partial z}\varphi[w]\big]\,dt, && \text{(3.26c)}
\end{aligned}$$

as long as all the limits in (3.26a)–(3.26c) exist. But using twofold continuous differentiability of $\tilde\Omega$ as well as Lemmas 4.2, 4.5, 4.7 and 4.9 we can compute these limits as follows:

$$\begin{aligned}
&\lim_{z\to z_\mu}\Big\|\int_0^\infty \tilde\Omega''(\varphi)\big[\pi d, \frac{\partial}{\partial z}\varphi[w]\big]\circ\tilde\Omega^*(\varphi)dt\Big\| \\
&\quad\leq \lim_{z\to z_\mu}\int_0^\infty \big\|\tilde\Omega''(\varphi)\big[\pi\cdot, \frac{\partial}{\partial z}\varphi[w]\big]\circ\tilde\Omega^*(\varphi)\big\|\cdot\big\|d(\varphi)\big\|dt \\
&\quad\overset{(4.2),(4.21)}{\leq} \big\|\tilde\Omega''\big(z_\mu\big)\big[\pi\cdot,\cdot\big]\circ\tilde\Omega^*\big(z_\mu\big)\big\|\cdot\|w\|\lim_{z\to z_\mu}\int_0^\infty\big\|d\big(\varphi(t,z)\big)\big\|dt \\
&\quad\overset{4.5}{=} 0, && \text{(3.27)}
\end{aligned}$$

$$
\begin{aligned}
&\lim_{z\to z_\mu}\int_0^\infty \tilde\Omega'(\varphi)\big[\pi\frac{\partial}{\partial z}d(\varphi)[w]\big]\circ\tilde\Omega^*(\varphi)\,dt\\
&\qquad\overset{(4.2)}{=}\tilde\Omega'\big(z_\mu\big)\Big[\pi\lim_{z\to z_\mu}\int_0^\infty\frac{\partial}{\partial z}d(\varphi)[w]dt\Big]\circ\tilde\Omega^*\big(z_\mu\big)\\
&\qquad\overset{4.9}{=}-\tilde\Omega'\big(z_\mu\big)[\pi w]\circ\tilde\Omega^*\big(z_\mu\big),\qquad\text{and}
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{z\to z_\mu}\Big\|\int_0^\infty\tilde\Omega'(\varphi)[\pi d]\circ\tilde\Omega^{*\prime}(\varphi)\big[\frac{\partial}{\partial z}\varphi[w]\big]\,dt\Big\|\\
&\qquad\leq\quad\lim_{z\to z_\mu}\int_0^\infty\big\|\tilde\Omega'(\varphi)[\pi\cdot]\circ\tilde\Omega^{*\prime}(\varphi)\big[\frac{\partial}{\partial z}\varphi[w]\big]\big\|\cdot\big\|d(\varphi)\big\|dt\\
&\qquad\overset{(4.2),(4.21)}{\leq}\big\|\tilde\Omega'\big(z_\mu\big)[\pi\cdot]\circ\tilde\Omega^{*\prime}\big(z_\mu\big)\big\|\cdot\|w\|\lim_{z\to z_\mu}\int_0^\infty\big\|d\big(\varphi(t,z)\big)\big\|dt\\
&\qquad\overset{4.5}{=}\quad 0.
\end{aligned}
$$

Therefore, the limit $\lim_{z\to z_\mu}\varsigma'(z)[w]=\tilde\Omega'\big(z_\mu\big)[\pi w]\circ\tilde\Omega^*\big(z_\mu\big)$ exists and equation (3.20) holds true.

Now, for $v, w \in Z$ we have

$$\lim_{z\to z_\mu}\int_0^\infty \frac{\partial^2}{\partial z^2} L_{\tau_0}\big(\varphi(t,z)\big)[v,w]dt$$

$$= -\lim_{z\to z_\mu}\int_0^\infty \frac{\partial^2}{\partial z^2}\Big(\tilde\Omega'\big(\varphi(t,z)\big)\big[\pi d\big(\varphi(t,z)\big)\big]\circ\tilde\Omega^*\big(\varphi(t,z)\big)\Big)[v,w]dt$$

$$= -\lim_{z\to z_\mu}\int_0^\infty \tilde\Omega'''(\varphi)\Big[\pi d(\varphi), \frac{\partial}{\partial z}\varphi[w], \frac{\partial}{\partial z}\varphi[v]\Big]\circ\tilde\Omega^*(\varphi)dt \tag{3.28}$$

$$-\lim_{z\to z_\mu}\int_0^\infty \tilde\Omega''(\varphi)\Big[\pi\frac{\partial}{\partial z}d(\varphi)[v], \frac{\partial}{\partial z}\varphi[w]\Big]\circ\tilde\Omega^*(\varphi)dt \tag{3.29}$$

$$-\lim_{z\to z_\mu}\int_0^\infty \tilde\Omega''(\varphi)\Big[\pi d(\varphi), \frac{\partial^2}{\partial z^2}\varphi[w,v]\Big]\circ\tilde\Omega^*(\varphi)dt \tag{3.30}$$

$$-\lim_{z\to z_\mu}\int_0^\infty \tilde\Omega''(\varphi)\Big[\pi d(\varphi), \frac{\partial}{\partial z}\varphi[w]\Big]\circ\tilde\Omega^{*\prime}(\varphi)\Big[\frac{\partial}{\partial z}\varphi[v]\Big]dt \tag{3.31}$$

$$-\lim_{z\to z_\mu}\int_0^\infty \tilde\Omega''(\varphi)\Big[\pi\frac{\partial}{\partial z}d(\varphi)[w], \frac{\partial}{\partial z}\varphi[v]\Big]\circ\tilde\Omega^*(\varphi)dt \tag{3.32}$$

$$-\lim_{z\to z_\mu}\int_0^\infty \tilde\Omega'(\varphi)\Big[\pi\frac{\partial^2}{\partial z^2}d(\varphi)[w,v]\Big]\circ\tilde\Omega^*(\varphi)dt \tag{3.33}$$

$$-\lim_{z\to z_\mu}\int_0^\infty \tilde\Omega'(\varphi)\Big[\pi\frac{\partial}{\partial z}d(\varphi)[w]\Big]\circ\tilde\Omega^{*\prime}(\varphi)\Big[\frac{\partial}{\partial z}\varphi[v]\Big]dt \tag{3.34}$$

$$-\lim_{z\to z_\mu}\int_0^\infty \tilde\Omega''(\varphi)\Big[\pi d(\varphi), \frac{\partial}{\partial z}\varphi[v]\Big]\circ\tilde\Omega^{*\prime}(\varphi)\Big[\frac{\partial}{\partial z}\varphi[w]\Big]dt \tag{3.35}$$

$$-\lim_{z\to z_\mu}\int_0^\infty \tilde\Omega'(\varphi)\Big[\pi\frac{\partial}{\partial z}d(\varphi)[v]\Big]\circ\tilde\Omega^{*\prime}(\varphi)\Big[\frac{\partial}{\partial z}\varphi[w]\Big]dt \tag{3.36}$$

$$-\lim_{z\to z_\mu}\int_0^\infty \tilde\Omega'(\varphi)\big[\pi d(\varphi)\big]\circ\tilde\Omega^{*\prime\prime}(\varphi)\Big[\frac{\partial}{\partial z}\varphi[w], \frac{\partial}{\partial z}\varphi[v]\Big]dt \tag{3.37}$$

$$-\lim_{z\to z_\mu}\int_0^\infty \tilde\Omega'(\varphi)\big[\pi d(\varphi)\big]\circ\tilde\Omega^{*\prime}(\varphi)\Big[\frac{\partial^2}{\partial z^2}\varphi[w,v]\Big]dt, \tag{3.38}$$

as long as the limits of the expressions (3.28)–(3.38) exist. Expressions (3.28), (3.31), (3.35) and (3.37) can be shown to be equal to zero in much the same way as in the arguments that led to equation (3.27), that is by relying on threefold continuous differentiability of $\tilde\Omega$ as well as on Lemma 4.2 and equation (4.21) in Lemma 4.7 from Section 4. Likewise, (3.30) and (3.38) are equal to zero. The argument is almost identical for both expressions, and we will show it for (3.30) only.

$$\lim_{z\to z_\mu}\Big\|\int_0^\infty \tilde\Omega''(\varphi)\Big[\pi d(\varphi), \frac{\partial^2}{\partial z^2}\varphi[w,v]\Big]\circ\tilde\Omega^*(\varphi)dt\Big\|$$

$$\le \lim_{z\to z_\mu}\int_0^\infty \Big\|\tilde\Omega''(\varphi)\Big[\pi\cdot, \frac{\partial^2}{\partial z^2}\varphi[w,v]\Big]\circ\tilde\Omega^*(\varphi)\Big\|\cdot\big\|d(\varphi)\big\|dt$$

$$\overset{(4.2),10}{\le} \big\|\tilde\Omega''\big(z_\mu\big)[\pi\cdot,\cdot]\circ\tilde\Omega^*\big(z_\mu\big)\big\|\cdot Q\cdot\|w\|\cdot\|v\|\cdot\lim_{z\to z_\mu}\int_0^\infty\big\|d\big(\varphi(t,z)\big)\big\|dt$$

$$\overset{4.5}{=} 0.$$

Expression (3.33) is also equal to zero, as can be seen from the following argument based on Lemmas 4.8 and 4.9:

$$\begin{aligned}
&\lim_{z\to z_\mu}\Big\|\int_0^\infty \tilde{\Omega}'(\varphi)\Big[\pi\frac{\partial^2}{\partial z^2}d(\varphi)[w,v]\Big]\circ\tilde{\Omega}^*(\varphi)dt\Big\| \\
&\quad\le \lim_{z\to z_\mu}\int_0^\infty \big\|\tilde{\Omega}'(\varphi)[\pi\cdot]\circ\tilde{\Omega}^*(\varphi)-\tilde{\Omega}'\big(z_\mu\big)[\pi\cdot]\circ\tilde{\Omega}^*\big(z_\mu\big)\big\|\cdot\Big\|\frac{\partial^2}{\partial z^2}d(\varphi)[w,v]\Big\|dt \\
&\qquad + \lim_{z\to z_\mu}\Big\|\tilde{\Omega}'\big(z_\mu\big)\Big[\pi\Big(\int_0^\infty\frac{\partial^2}{\partial z^2}d(\varphi)[w,v]dt\Big)\Big]\circ\tilde{\Omega}^*\big(z_\mu\big)\Big\| \\
&\quad\overset{4.8,(4.30),4.9}{=} 0.
\end{aligned}$$

Next, let us take the limit in expression (3.29) by involving Lemma 4.14 from Section 4:

$$\begin{aligned}
&\lim_{z\to z_\mu}\int_0^\infty \tilde{\Omega}''(\varphi)\Big[\pi\frac{\partial}{\partial z}d(\varphi)[v],\frac{\partial}{\partial z}\varphi[w]\Big]\circ\tilde{\Omega}^*(\varphi)dt \\
&\qquad=\lim_{z\to z_\mu}\int_0^\infty -e^{-2t}\tilde{\Omega}''(\varphi)\Big[-\pi e^t\frac{\partial}{\partial z}d(\varphi)[v],e^t\frac{\partial}{\partial z}\varphi[w]\Big]\circ\tilde{\Omega}^*(\varphi)dt \\
&\qquad\overset{4.14}{=}\tilde{\Omega}''\big(z_\mu\big)[\pi v,w]\circ\tilde{\Omega}^*\big(z_\mu\big)\int_0^\infty -e^{-2t}dt \\
&\qquad=-\frac{1}{2}\tilde{\Omega}''\big(z_\mu\big)[\pi v,w]\circ\tilde{\Omega}^*\big(z_\mu\big).
\end{aligned}$$

Similar arguments can be applied to expressions (3.32), (3.34) and (3.36), and this yields

$$\lim_{z\to z_\mu}\int_0^\infty \tilde{\Omega}''(\varphi)\Big[\pi\frac{\partial}{\partial z}d(\varphi)[w],\frac{\partial}{\partial z}\varphi[v]\Big]\circ\tilde{\Omega}^*(\varphi)dt=-\frac{1}{2}\tilde{\Omega}''\big(z_\mu\big)[\pi w,v]\circ\tilde{\Omega}^*\big(z_\mu\big),$$

$$\lim_{z\to z_\mu}\int_0^\infty \tilde{\Omega}'(\varphi)\Big[\pi\frac{\partial}{\partial z}d(\varphi)[w]\Big]\circ\tilde{\Omega}^{*\prime}(\varphi)\Big[\frac{\partial}{\partial z}\varphi[v]\Big]dt=-\frac{1}{2}\tilde{\Omega}'\big(z_\mu\big)[\pi w]\circ\tilde{\Omega}^{*\prime}\big(z_\mu\big)[v]$$

and

$$\lim_{z\to z_\mu}\int_0^\infty \tilde{\Omega}'(\varphi)\Big[\pi\frac{\partial}{\partial z}d(\varphi)[v]\Big]\circ\tilde{\Omega}^{*\prime}(\varphi)\Big[\frac{\partial}{\partial z}\varphi[w]\Big]dt=-\frac{1}{2}\tilde{\Omega}'\big(z_\mu\big)[\pi v]\circ\tilde{\Omega}^{*\prime}\big(z_\mu\big)[w].$$

In summary we get that for all $v,w\in Z$,

$$\begin{aligned}
\lim_{z\to z_\mu}\int_0^\infty\frac{\partial^2}{\partial z^2}L_{\tau_0}(\varphi)[w,v]dt=\frac{1}{2}\Big(&\tilde{\Omega}''(z_\mu)[\pi v,w]\circ\tilde{\Omega}^*(z_\mu)+ \\
&+\tilde{\Omega}''(z_\mu)[\pi w,v]\circ\tilde{\Omega}^*(z_\mu) \\
&+\tilde{\Omega}'(z_\mu)[\pi w]\circ\tilde{\Omega}^{*\prime}(z_\mu)[v] \\
&+\tilde{\Omega}'(z_\mu)[\pi v]\circ\tilde{\Omega}^{*\prime}(z_\mu)[w]\Big).
\end{aligned}\tag{3.39}$$

This is the first term of the expression on the right-hand side of equation (3.25), and we have shown it equal to the right-hand side of equation (3.21). Let us now show that the second term on the right-hand side of (3.25) is equal to zero.

For all $v, w \in Z$

$$\lim_{z\to z_\mu}\frac{1}{2}\int_0^\infty \frac{\partial^2}{\partial z^2}L_{\tau_1}(\varphi)[w,v]dt = \frac{1}{2}\lim_{z\to z_\mu}\int_0^\infty \frac{\partial^2}{\partial z^2}\Big[\tilde\Omega'\big(\varphi(t,z)\big)[\pi d]\circ\tilde\Omega^*\big(\varphi(t,z)\big), \int_t^\infty \tilde\Omega'\big(\varphi(\theta,z)\big)\big[\pi d\big(\varphi(\theta,z)\big)\big]\circ\tilde\Omega^*\big(\varphi(\theta,z)\big)d\theta\Big][w,v]dt$$

$$= \frac{1}{2}\lim_{z\to z_\mu}\int_0^\infty\Big[\frac{\partial^2}{\partial z^2}\Big(\tilde\Omega'[\pi d]\circ\tilde\Omega^*\Big)[w,v], \int_t^\infty \tilde\Omega'[\pi d]\circ\tilde\Omega^* d\theta\Big]dt \tag{3.40}$$

$$+\frac{1}{2}\lim_{z\to z_\mu}\int_0^\infty\Big[\tilde\Omega'[\pi d]\circ\tilde\Omega^*, \int_t^\infty \frac{\partial^2}{\partial z^2}\Big(\tilde\Omega'[\pi d]\circ\tilde\Omega^*\Big)[w,v]d\theta\Big]dt \tag{3.41}$$

$$+\frac{1}{2}\lim_{z\to z_\mu}\int_0^\infty\Big[\frac{\partial}{\partial z}\Big(\tilde\Omega'[\pi d]\circ\tilde\Omega^*\Big)[w], \int_t^\infty \frac{\partial}{\partial z}\Big(\tilde\Omega'[\pi d]\circ\tilde\Omega^*\Big)[v]d\theta\Big]dt \tag{3.42}$$

$$+\frac{1}{2}\lim_{z\to z_\mu}\int_0^\infty\Big[\frac{\partial}{\partial z}\Big(\tilde\Omega'[\pi d]\circ\tilde\Omega^*\Big)[v], \int_t^\infty \frac{\partial}{\partial z}\Big(\tilde\Omega'[\pi d]\circ\tilde\Omega^*\Big)[w]d\theta\Big]dt \tag{3.43}$$

if all these limits exist. Note that for all z sufficiently close to z_μ

$$\Big\|\int_t^\infty \tilde\Omega'\big(\varphi(\theta,z)\big)\big[\pi d\big(\varphi(\theta,z)\big)\big]\circ\tilde\Omega^*\big(\varphi(\theta,z)\big)d\theta\Big\|$$
$$\le \int_t^\infty \big\|\tilde\Omega'\big(\varphi(\theta,z)\big)[\pi\cdot]\circ\tilde\Omega^*\big(\varphi(\theta,z)\big)\big\|\cdot\big\|d\big(\varphi(\theta,z)\big)\big\|d\theta$$
$$\overset{(4.18)}{=} O(1)\cdot\big\|\tilde\Omega'\big(z_\mu\big)[\pi\cdot]\circ\tilde\Omega^*\big(z_\mu\big)\big\|\cdot\int_t^\infty \|z_\mu - z\|\cdot e^{-\theta}d\theta$$
$$= O(1)\cdot\|z_\mu - z\|e^{-t}. \tag{3.44}$$

Moreover, for z close enough to z_μ,

$$\Big\|\frac{\partial^2}{\partial z^2}\Big(\tilde\Omega'\big(\varphi(t,z)\big)\big[\pi d\big(\varphi(t,z)\big)\big]\circ\tilde\Omega^*\big(\varphi(t,z)\big)\Big)\Big\| = O(1)\cdot e^{-\frac{t}{2}}. \tag{3.45}$$

In fact, this follows from our analysis of expressions (3.28)–(3.32) and (3.34)–(3.38), using equations (4.18) and (4.24), and from the following revised analysis of expression (3.33) by taking into account inequality (4.39) from Section 4:

$$\Big\|\tilde\Omega'(\varphi)\Big[\pi\frac{\partial^2}{\partial z^2}d(\varphi)[w,v]\Big]\circ\tilde\Omega^*(\varphi)\Big\|$$
$$= O(1)\cdot\big\|\tilde\Omega'\big(z_\mu\big)[\pi\cdot]\circ\tilde\Omega^*\big(z_\mu\big)\big\|\cdot\Big\|\frac{\partial^2}{\partial z^2}d(\varphi)[w,v]\Big\|$$
$$\overset{(4.39)}{=} O(1)\cdot\big\|\tilde\Omega'\big(z_\mu\big)[\pi\cdot]\circ\tilde\Omega^*\big(z_\mu\big)\big\|\cdot\|w\|\cdot\|v\|e^{-t(1-\frac{1}{2})}.$$

Therefore, expressions (3.40) and (3.41) equal zero. In fact,

$$\begin{aligned}
&\lim_{z\to z_\mu}\Big\|\int_0^\infty \Big[\frac{\partial^2}{\partial z^2}\big(\tilde\Omega'[\pi d]\circ\tilde\Omega^*\big)[w,v],\int_t^\infty \tilde\Omega'[\pi d]\circ\tilde\Omega^* d\theta\Big]dt\Big\|\\
&\qquad\le \lim_{z\to z_\mu}\int_0^\infty 2\Big\|\frac{\partial^2}{\partial z^2}\big(\tilde\Omega'[\pi d]\circ\tilde\Omega^*\big)[w,v]\Big\|\cdot\Big\|\int_t^\infty \tilde\Omega'[\pi d]\circ\tilde\Omega^* d\theta\Big\|dt\\
&\qquad\overset{(3.45),(3.44)}{\le} \lim_{z\to z_\mu} O(1)\cdot\|z-z_\mu\|\int_0^\infty e^{-\frac{t}{2}}\cdot e^{-t}dt=0,
\end{aligned}$$

$$\begin{aligned}
&\lim_{z\to z_\mu}\Big\|\int_0^\infty \Big[\tilde\Omega'[\pi d]\circ\tilde\Omega^*,\int_t^\infty \frac{\partial^2}{\partial z^2}\tilde\Omega'[\pi d]\circ\tilde\Omega^*[w,v]d\theta\Big]dt\Big\|\\
&\qquad\le \lim_{z\to z_\mu}\int_0^\infty 2\big\|\tilde\Omega'[\pi d]\circ\tilde\Omega^*\big\|\cdot\\
&\qquad\qquad\cdot\int_t^\infty \Big\|\frac{\partial^2}{\partial z^2}\Big(\tilde\Omega'\big(\varphi(\theta,z)\big)\big[\pi d\big(\varphi(\theta,z)\big)\big]\circ\tilde\Omega^*\big(\varphi(\theta,z)\big)\Big)[w,v]\Big\|d\theta dt\\
&\qquad\overset{(3.45)}{\le} \lim_{z\to z_\mu}\int_0^\infty O(1)\cdot\big\|\tilde\Omega'\big(z_\mu\big)[\pi\cdot]\circ\tilde\Omega^*\big(z_\mu\big)\big\|\cdot\big\|d\big(\varphi(t,z)\big)\big\|\cdot\int_t^\infty \cdot e^{-\frac{\theta}{2}}d\theta dt\\
&\qquad\overset{(4.16)}{\le} \lim_{z\to z_\mu} O(1)\cdot\big\|\tilde\Omega'\big(z_\mu\big)[\pi\cdot]\circ\tilde\Omega^*\big(z_\mu\big)\big\|\cdot\|z-z_\mu\|\cdot\int_0^\infty e^{-\frac{t}{2}}dt\\
&\qquad= \quad 0.
\end{aligned}$$

Let us now revisit our analysis of expressions (3.26a)–(3.26c) and take into account equations (4.18) and (4.21) in Lemma 4.7 as well as Lemma 4.14. We get

$$\begin{aligned}
&\Big\|e^t\cdot\frac{\partial}{\partial z}\Big(\tilde\Omega'(\varphi)\big[\pi d(\varphi)\big]\circ\tilde\Omega^*(\varphi)\Big)[w]+\tilde\Omega'\big(z_\mu\big)[\pi w]\circ\tilde\Omega^*\big(z_\mu\big)\Big\|\\
&\quad\le\quad e^t\cdot\Big\|\tilde\Omega''(\varphi)\Big[\pi d(\varphi),\frac{\partial}{\partial z}\varphi[w]\Big]\circ\tilde\Omega^*(\varphi)\Big\|\\
&\qquad+\Big\|\tilde\Omega'(\varphi)\Big[\pi e^t\cdot\frac{\partial}{\partial z}d(\varphi)[w]\Big]\circ\tilde\Omega^*(\varphi)-\tilde\Omega'\big(z_\mu\big)\Big[\pi e^t\cdot\frac{\partial}{\partial z}d(\varphi)[w]\Big]\circ\tilde\Omega^*\big(z_\mu\big)\Big\|\\
&\qquad+\big\|\tilde\Omega'\big(z_\mu\big)[\pi\cdot]\circ\tilde\Omega^*\big(z_\mu\big)\big\|\cdot\Big\|e^t\cdot\frac{\partial}{\partial z}d(\varphi)[w]+w\Big\|\\
&\qquad+e^t\cdot\Big\|\tilde\Omega'(\varphi)\big[\pi d(\varphi)\big]\circ\tilde\Omega^{*\prime}(\varphi)\big[\frac{\partial}{\partial z}\varphi[w]\big]\Big\|\overset{(4.18),(4.21),4.14}{=}\\
&\quad=\quad O\big(\|z-z_\mu\|\big)\cdot e^{-\frac{t}{2}}\cdot\|w\|\\
&\qquad+\big\|\tilde\Omega'(\varphi)[\pi\cdot]\circ\tilde\Omega^*(\varphi)-\tilde\Omega'\big(z_\mu\big)[\pi\cdot]\circ\tilde\Omega^*\big(z_\mu\big)\big\|\cdot\Big\|e^t\cdot\frac{\partial}{\partial z}d(\varphi)[w]\Big\|\\
&\qquad+\big\|\tilde\Omega'\big(z_\mu\big)[\pi\cdot]\circ\tilde\Omega^*\big(z_\mu\big)\big\|O\big(\|z-z_\mu\|\big)\\
&\qquad+O\big(\|z-z_\mu\|\big)\cdot e^{-\frac{t}{2}}\cdot\|w\|\\
&\quad=\quad O\big(\|z-z_\mu\|\big).
\end{aligned}$$

(3.46)

The last equality holds because Lemmas 4.12 and 4.14 combined with the fact that $\tilde{\Omega}'(z)[\pi\cdot]\circ\tilde{\Omega}^*(z)$ is continuously differentiable on $\mathcal{F}^\circ(PD)$ imply that the second term is also of order $O\big(\|z-z_\mu\|\big)$. Now, equation (3.46) implies that

$$\begin{aligned}(3.47)\qquad &\frac{\partial}{\partial z}\Big(\tilde{\Omega}'\big(\varphi(t,z)\big)\big[\pi d\big(\varphi(t,z)\big)\big]\circ\tilde{\Omega}^*\big(\varphi(t,z)\big)\Big)[w]\\ &\qquad = -e^{-t}\cdot\tilde{\Omega}'\big(z_\mu\big)[\pi w]\circ\tilde{\Omega}^*\big(z_\mu\big)+e^{-t}\cdot O\big(\|z-z_\mu\|\big).\end{aligned}$$

This finally allows us to compute expressions (3.42) and (3.43):

$$\begin{aligned}&\lim_{z\to z_\mu}\int_0^\infty\Big[\frac{\partial}{\partial z}\Big(\tilde{\Omega}'\big(\varphi(t,z)\big)[\pi d]\circ\tilde{\Omega}^*\big(\varphi(t,z)\big)\Big)[w],\\ &\qquad\qquad\int_t^\infty\frac{\partial}{\partial z}\Big(\tilde{\Omega}'\big(\varphi(\theta,z)\big)[\pi d]\circ\tilde{\Omega}^*\big(\varphi(\theta,z)\big)\Big)[v]d\theta\Big]dt\\ &\overset{(3.47)}{=}\lim_{z\to z_\mu}\int_0^\infty\Big[-e^{-t}\tilde{\Omega}'\big(z_\mu\big)[\pi w]\circ\tilde{\Omega}^*\big(z_\mu\big)+e^{-t}O\big(\|z-z_\mu\|\big),\\ &\qquad\qquad\int_t^\infty -e^{-\theta}\cdot\tilde{\Omega}'\big(z_\mu\big)[\pi v]\circ\tilde{\Omega}^*\big(z_\mu\big)+e^{-\theta}O\big(\|z-z_\mu\|\big)d\theta\Big]dt\\ &= \lim_{z\to z_\mu}\Big[\tilde{\Omega}'\big(z_\mu\big)[\pi w]\circ\tilde{\Omega}^*\big(z_\mu\big),\tilde{\Omega}'\big(z_\mu\big)[\pi v]\circ\tilde{\Omega}^*\big(z_\mu\big)\Big]\cdot\\ &\qquad\qquad\cdot\int_0^\infty e^{-t}\int_t^\infty e^{-\theta}d\theta dt+\lim_{z\to z_\mu}O\big(\|z-z_\mu\|\big)\\ &= \frac{1}{2}\Big[\tilde{\Omega}'\big(z_\mu\big)[\pi w]\circ\tilde{\Omega}^*\big(z_\mu\big),\tilde{\Omega}'\big(z_\mu\big)[\pi v]\circ\tilde{\Omega}^*\big(z_\mu\big)\Big],\end{aligned}$$

and likewise one computes

$$\begin{aligned}&\lim_{z\to z_\mu}\int_0^\infty\Big[\frac{\partial}{\partial z}\Big(\tilde{\Omega}'\big(\varphi(t,z)\big)[\pi d]\circ\tilde{\Omega}^*\big(\varphi(t,z)\big)\Big)[v],\\ &\qquad\qquad\int_t^\infty\frac{\partial}{\partial z}\Big(\tilde{\Omega}'\big(\varphi(\theta,z)\big)[\pi d]\circ\tilde{\Omega}^*\big(\varphi(\theta,z)\big)\Big)[w]d\theta\Big]dt\\ &= \frac{1}{2}\Big[\tilde{\Omega}'\big(z_\mu\big)[\pi v]\circ\tilde{\Omega}^*\big(z_\mu\big),\tilde{\Omega}'\big(z_\mu\big)[\pi w]\circ\tilde{\Omega}^*\big(z_\mu\big)\Big]\\ &= -\frac{1}{2}\Big[\tilde{\Omega}'\big(z_\mu\big)[\pi w]\circ\tilde{\Omega}^*\big(z_\mu\big),\tilde{\Omega}'\big(z_\mu\big)[\pi v]\circ\tilde{\Omega}^*\big(z_\mu\big)\Big].\end{aligned}$$

Hence, expressions (3.42) and (3.43) cancel each other out, and we have

$$(3.48)\qquad \lim_{z\to z_\mu}\frac{1}{2}\int_0^\infty\frac{\partial^2}{\partial z^2}L_{\tau_1}\big(\varphi(t,z)\big)[w,v]dt=0.$$

Using equations (3.39) and (3.48) in (3.25) we finally get that $\lim_{z\to z_\mu}\varsigma'(z)$ exists and that equation (3.21) holds. This concludes the proof of part i).

ii) The first statement of the second part of the theorem is now clear from part i), the fact that the exponential mapping is analytic and the developments that led to equation (3.16).

In order to prove the second statement, note that at the origin the first and second derivatives of the exponential mapping

$$\exp : M_{n\times n}(\mathbb{R}) \to Gl_n(\mathbb{R})$$

are given by the following formulas: For all $W, V \in M_{n\times n}(\mathbb{R})$,

$$\exp'(0)[W] = W, \tag{3.49}$$

$$\exp''(0)\big[(W,V)\big] = \frac{1}{2}(WV + VW), \tag{3.50}$$

where we have denoted the second derivative of exp in the directions W and V by $\exp''(0)\big[(W,V)\big]$ in order to avoid any confusion with the commutator $[W,V]$ of W and V. Formulas (3.49) and (3.50) can be checked very easily, since the exponential mapping defined on matrix Lie algebras is simply given by its Taylor expansion about the origin, i.e. by the absolutely convergent series

$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$

Therefore, for all $v, w \in Z$ we have

$$\begin{aligned}
\Omega'(z_\mu)[w] &= \big(\exp\circ\varsigma\big)'(z_\mu)[w] \\
&= \exp'\big(\varsigma(z_\mu)\big)\big[\varsigma'(z_\mu)[w]\big] \\
&= \exp'(0)\big[\varsigma'(z_\mu)[w]\big] \\
&\overset{(3.49)}{=} \varsigma'(z_\mu)[w],
\end{aligned}$$

and

$$\begin{aligned}
\Omega''\big(z_\mu\big)[w,v] &= \big(\exp\circ\varsigma\big)''\big(z_\mu\big)[w,v] \\
&= \exp'\big(\varsigma(z_\mu)\big)\big[\varsigma''(z_\mu)[w,v]\big] + \exp''\big(\varsigma(z_\mu)\big)\big[\big(\varsigma'(z_\mu)[w], \varsigma'(z_\mu)[v]\big)\big] \\
&= \exp'(0)\big[\varsigma''(z_\mu)[w,v]\big] + \exp''(0)\big[\big(\varsigma'(z_\mu)[w], \varsigma'(z_\mu)[v]\big)\big] \\
&\overset{(3.50)}{=} \varsigma''\big(z_\mu\big)[w,v] + \frac{1}{2}\big(\varsigma'(z_\mu)[w]\varsigma'(z_\mu)[v] + \varsigma'(z_\mu)[v]\varsigma'(z_\mu)[w]\big).
\end{aligned}$$

This proves all the claims of the theorem. □

The preceding theorem shows that Problem 3.7 admits a local solution in a neighborhood of z_μ. Together with Lemma 3.11 this completes the proof Theorem 3.8. Hence, this also concludes the proofs of Theorem 3.2 and Corollary 3.3.

4 Sensitivity analysis of the Nesterov-Todd flow

Short-step methods are close relatives of numerical algorithms for the integration of vector fields, and elements of the analysis of the latter can be useful in the analysis of the first.

For example, consider a short-step algorithm passing at the intermediate solution z during a given phase, and consider the maximal local flow φ defined by the search-direction vector field. Then the inverse of the curvature of the fluxline $\varphi(z,\cdot)$ at z is a measure that quantifies what should be a "good" stepsize for our algorithm. It is therefore interesting to investigate the geometric and variational properties of the dynamical systems defined by the search-direction vector fields used in such algorithms, and since we are interested in curvature, the analysis should be of second order. In this section we propose to undertake such an investigation in the case of the Nesterov-Todd flow in a neighborhood of the central path. The results we develop here are especially designed for use in the proofs of the previous section of this thesis. In a nutshell, one might summarize our results by saying that near z_μ the renormalized Nesterov-Todd field is up to second order terms equal to the normalized radial vector field $\|z_\mu - z\|^{-1}\cdot\big(z_\mu - z\big)$. This implies that under a magnifying glass the flux-lines of the N-T field would look like straight rays in a neighborhood of z_μ.

None of the results derived below depend on any of the material of Section 3 past Remark 3.10 where we started to rely on the present section. However, we shall use the notation introduced prior to Remark 3.10, in particular with regards to the maximal local flow defined by the Nesterov-Todd vector field defined by the fixed parameter value $\mu > 0$. Various results we present here will be strengthened as we move along, earlier versions being exploited in the proofs of the later ones. Let us start by describing the interval I_z equal to the the domain of definition of the flux-line for d that flows through $z \in D$ (see the paragraph prior to Remark 3.10).

Lemma 4.1. *For all $z \in D$ and for all $t \in I_z$ it is true that*

$$\big\|\mu e_V - \gamma\big(\varphi(t,z)\big)\big\| = \big\|\mu e_V - \gamma(z)\big\|\cdot e^{-t}. \tag{4.1}$$

Proof. Let $t \in I_z$ and suppose $t > 0$. Then $\varphi(\xi,z)$ is well defined and lies in D for all $\xi \in [0,t]$, and we have

$$\begin{aligned}
&\frac{\partial}{\partial\xi}\big\|\mu e_V - \gamma\big(\varphi(\xi,z)\big)\big\| \\
&\quad= \quad \big\|\mu e_V - \gamma\big(\varphi(\xi,z)\big)\big\|^{-1}\cdot\Big(\mu e_V - \gamma\big(\varphi(\xi,z)\big); -\gamma'\big(\varphi(\xi,z)\big)\Big[\frac{\partial}{\partial\xi}\varphi(\xi,z)\Big]\Big) \\
&\quad\overset{(2.5),3.9}{=} \big\|\mu e_V - \gamma\big(\varphi(\xi,z)\big)\big\|^{-1}\cdot\Big(\mu e_V - \gamma\big(\varphi(\xi,z)\big); -\gamma'\big(\varphi(\xi,z)\big)\big[d\big(\varphi(\xi,z)\big)\big]\Big) \\
&\quad= \quad \big\|\mu e_V - \gamma\big(\varphi(\xi,z)\big)\big\|^{-1}\cdot \\
&\qquad\quad\cdot\Big(\mu e_V - \gamma\big(\varphi(\xi,z)\big); -M\big(\varphi(\xi,z)\big)\big[d\big(\varphi(\xi,z)\big)\big] - R\big(\varphi(\xi,z)\big)\big[d\big(\varphi(\xi,z)\big)\big]\Big) \\
&\quad\overset{1.1\,ii),(1.6)}{=} \big\|\mu e_V - \gamma\big(\varphi(\xi,z)\big)\big\|^{-1}\cdot\Big(\mu e_V - \gamma\big(\varphi(\xi,z)\big); -\mu e_V + \gamma\big(\varphi(\xi,z)\big)\Big) \\
&\quad= \quad -\big\|\mu e_V - \gamma\big(\varphi(\xi,z)\big)\big\|.
\end{aligned}$$

Therefore, the function $\xi \mapsto \big\|\mu e_V - \gamma\big(\varphi(\xi,z)\big)\big\|$ is given by the solution to the initial value problem

$$\begin{aligned}
\dot{y}(\xi) &= -y(\xi), \qquad \forall \xi \in [0,t], \\
y(0) &= \big\|\mu e_V - \gamma(z)\big\|.
\end{aligned}$$

Hence, $\big\|\mu e_V - \gamma\big(\varphi(\xi,z)\big)\big\| = \big\|\mu e_V - \gamma(z)\big\| \cdot e^{-\xi}$ for all $\xi \in [0,t]$. The proof for $0 > t \in I_z$ is analogous. $\square$

Lemma 4.2. *For all $z \in D$ there exists a strictly negative real number $-\infty \leq a_z < 0$ such that $I_z = \big(a_z, +\infty\big)$, and we have*

$$\lim_{t\to+\infty} \varphi(t,z) = z_\mu, \tag{4.2}$$

$$\lim_{t\to a_z} \Big(\delta\big(\varphi(t,z)\big) + \mu/2 \cdot \mathfrak{F}\big(\varphi(t,z)\big)\Big) = +\infty, \tag{4.3}$$

where $\mathfrak{F} : z \mapsto F\big(x(z)\big) + F_\sharp\big(s(z)\big)$ is as defined in Section 1.

Proof. Consider the function

$$\begin{aligned}
g : \mathcal{F}^\circ(PD) &\to \mathbb{R} \\
z &\mapsto \delta(z) + \mu \cdot \mathfrak{F}(z).
\end{aligned}$$

Let $(z,\Omega) \in \mathcal{M} := D \times \mathcal{O}(V, e_V)$ as defined in Section 3. For $t \in I_z$ we have

$$\begin{aligned}
\frac{\partial}{\partial t} g\big(\varphi(t,z)\big) &= g'\big(\varphi(t,z)\big)\Big[\frac{\partial}{\partial t}\varphi(t,z)\Big] \\
&\overset{3.9}{=} g'\big(\varphi(t,z)\big)\big[d\big(\varphi(t,z)\big)\big] \\
&= \frac{\partial}{\partial x} g\Big(x\big(\varphi(t,z)\big), s\big(\varphi(t,z)\big)\Big)\Big[\pi_L\big[d\big(\varphi(t,z)\big)\big]\Big] \\
&\qquad + \frac{\partial}{\partial s} g\Big(x\big(\varphi(t,z)\big), s\big(\varphi(t,z)\big)\Big)\Big[\pi_{L^\perp}\big[d\big(\varphi(t,z)\big)\big]\Big] \\
&= \Big\langle \pi_L\big[d\big(\varphi(t,z)\big)\big] ; s\big(\varphi(t,z)\big) + \mu F'\big(x\big(\varphi(t,z)\big)\big)\Big\rangle \\
&\qquad + \Big\langle x\big(\varphi(t,z)\big) + \mu F_\sharp'\big(s\big(\varphi(t,z)\big)\big) ; \pi_{L^\perp}\big[d\big(\varphi(t,z)\big)\big]\Big\rangle \\
&\overset{(1.5a)}{=} -\Big\langle \pi_L\big[d\big(\varphi(t,z)\big)\big] ; \Big(F''\big(w\big(\varphi(t,z)\big)\big)\pi_L + \pi_{L^\perp}\Big)\big[d\big(\varphi(t,z)\big)\big]\Big\rangle \\
&\qquad - \Big\langle \Big(\pi_L + F''^{-1}\big(w\big(\varphi(t,z)\big)\big)\pi_{L^\perp}\Big)\big[d\big(\varphi(t,z)\big)\big] ; \pi_{L^\perp}\big[d\big(\varphi(t,z)\big)\big]\Big\rangle \\
&= -\Big\langle \pi_L\big[d\big(\varphi(t,z)\big)\big] ; F''\big(w\big(\varphi(t,z)\big)\big)\pi_L\big[d\big(\varphi(t,z)\big)\big]\Big\rangle \\
&\qquad - \Big\langle F''^{-1}\big(w\big(\varphi(t,z)\big)\big)\pi_{L^\perp}\big[d\big(\varphi(t,z)\big)\big] ; \pi_{L^\perp}\big[d\big(\varphi(t,z)\big)\big]\Big\rangle \\
&\leq 0.
\end{aligned} \tag{4.4}$$

Furthermore, recalling Definition (1.4) we find that for $t \in I_z$,

$$\begin{aligned}
\frac{\partial}{\partial t}\mu\big(\varphi(t,z)\big) &= \nu^{-1}\Big(\big\langle \pi_L\big[d\big(\varphi(t,z)\big)\big]; s\big(\varphi(t,z)\big)\big\rangle + \big\langle x\big(\varphi(t,z)\big); \pi_{L^\perp}\big[d\big(\varphi(t,z)\big)\big]\big\rangle\Big) \\
&= \nu^{-1}\Big(\Big\langle \pi_L\big[d\big(\varphi(t,z)\big)\big]; F''\big(w\big(\varphi(t,z)\big)\big)x\big(\varphi(t,z)\big)\Big\rangle + \\
&\qquad\qquad + \big\langle x\big(\varphi(t,z)\big); \pi_{L^\perp}\big[d\big(\varphi(t,z)\big)\big]\big\rangle\Big) \\
&= \nu^{-1}\cdot\Big\langle x\big(\varphi(t,z)\big); F''\big(w\big(\varphi(t,z)\big)\big)\pi_L\big[d\big(\varphi(t,z)\big)\big] + \pi_{L^\perp}\big[d\big(\varphi(t,z)\big)\big]\Big\rangle \\
&\overset{(1.5a)}{=} -\nu^{-1}\cdot\Big\langle x\big(\varphi(t,z)\big); s\big(\varphi(t,z)\big) + \mu F'\big(x\big(\varphi(t,z)\big)\big)\Big\rangle \\
&\overset{2.1\,iv)}{=} \mu - \mu\big(\varphi(t,z)\big).
\end{aligned}$$

This shows that the function $t \mapsto \mu\big(\varphi(t,z)\big)$ is the unique solution to the initial value problem

$$\begin{aligned}
\dot{y}(t) &= \mu - y(t), \qquad \forall t \in I_z, \\
y(0) &= \mu(z),
\end{aligned}$$

and hence we have

$$\mu\big(\varphi(t,z)\big) = \mu + \big(\mu(z) - \mu\big)\cdot e^{-t} \qquad \text{for all } t \in I_z. \tag{4.5}$$

We will now proceed to show that $[0,+\infty) \subset I_z$. Let

$$t := \sup\big\{\xi > 0 : \xi \in I_z\big\}.$$

The claim is that $t = +\infty$. Suppose to the contrary that this is not the case. Since g is self-concordant on $\mathcal{F}^\circ(PD)$ we must have that

$$\lim_{\breve{z}\to\partial\mathcal{F}^\circ(PD)} g\big(\breve{z}\big) = +\infty.$$

Therefore it is true that

$$Q_1 := \big\{\breve{z} \in \mathcal{F}^\circ(PD) : g\big(\breve{z}\big) \le g(z)\big\} \subset \mathcal{F}^\circ(PD)$$

and Q_1 is closed as a subset of Z. It follows from Equation (4.4) that for all $0 \le \xi < t$ we have

$$g\big(\varphi(\xi,z)\big) = g(z) + \int_0^\xi \frac{\partial}{\partial\tau} g\big(\varphi(\tau,z)\big)\,d\tau \le g(z),$$

and hence $\varphi(\xi,z) \in Q_1$. Moreover, equation (4.5) shows that for all $0 \le \xi < t$ the following holds true:

$$\varphi(\xi,z) \in Q_2 := \begin{cases} \mu^{-1}\big([\mu,\mu(z)]\big) & \text{if } \mu(z) \ge \mu, \\ \mu^{-1}\big([\mu(z),\mu]\big) & \text{if } \mu(z) < \mu. \end{cases}$$

But by virtue of what was said in Section 1, Q_2 is a compact subset of Z, and hence $Q_1 \cap Q_2$ is a compact subset of $\mathcal{F}^\circ(PD)$. Since $\mathcal{O}(V, e_V)$ is a compact manifold we get that $Q_3 := Q_1 \cap Q_2 \times \mathcal{O}(V, e_V)$ is a compact subset of $\mathcal{F}^\circ(PD) \times \mathcal{O}(V, e_V)$, and by construction of Q_1 and Q_2 it must be true that

$$\Phi\big(\xi, (z, \Omega)\big) \in Q_3 \qquad \text{for all } 0 \leq \xi < t.$$

Now, for each $k \in \mathbb{N}$ let us set $\xi_k := t - 1/k$. Since Q_3 is compact, the sequence

$$\Big(\Phi\big(\xi_k, (z, \Omega)\big)\Big)_{\mathbb{N}} \subset Q_3$$

has an accumulation point, say at $\big(\tilde{z}, \tilde{\Omega}\big) \in Q_3$. Let us replace $\big(\xi_k\big)_{\mathbb{N}}$ by a subsequence such that

$$\lim_{k \to \infty} \Phi\big(\xi_k, (z, \Omega)\big) = \big(\tilde{z}, \tilde{\Omega}\big),$$

and by abuse of language let us denote this subsequence by $\big(\xi_k\big)_{\mathbb{N}}$ again. Then it follows from continuity of γ and $\|\cdot\|$ that

$$\begin{aligned} \big\|\mu e_V - \gamma\big(\tilde{z}\big)\big\| &= \lim_{k \to \infty} \big\|\mu e_V - \gamma\big(\varphi(\xi_k, z)\big)\big\| \\ &\overset{(4.1)}{=} \big\|\mu e_V - \gamma(z)\big\| \cdot \lim_{k \to \infty} e^{-\xi_k} \\ &= \big\|\mu e_V - \gamma(z)\big\| \cdot e^{-t}. \end{aligned}$$

Now, since $t < +\infty$, Lemma H3.3.5 (alternatively see remark made in Section 1 about the uniqueness of the solution to the central path equation) implies that $\tilde{z} \neq z_\mu$, and hence $\big(\tilde{z}, \tilde{\Omega}\big) \in \mathcal{M}$. By Theorem 2.7 $i)$ there exists a real number $\tilde{\epsilon}$, a neighborhood $\tilde{N} \subset M$ of $\big(\tilde{z}, \tilde{\Omega}\big)$ and a mapping

$$\tilde{\Phi} : \big(-\tilde{\epsilon}, +\tilde{\epsilon}\big) \times \tilde{N} \to M$$

such that for all $(t, x) \in \big(-\tilde{\epsilon}, +\tilde{\epsilon}\big) \times \tilde{N}$ we have

$$\begin{aligned} \tilde{\Phi}(0, x) &= x, \\ \frac{\partial}{\partial t}\tilde{\Phi}(t, x) &= f\big(\tilde{\Phi}(t, x)\big). \end{aligned}$$

Let k_0 be large enough so that $\Phi\big(\xi_{k_0}, (z, \Omega)\big) \in \tilde{N}$ and $t - \xi_{k_0} < \tilde{\epsilon}/2$. It follows from part $ii)$ of Theorem 2.7 that $\tilde{\Phi}$ and Φ coincide on $\Big(\big(-\tilde{\epsilon}, +\tilde{\epsilon}\big) \times \tilde{N}\Big) \cap W$, and hence we get that the following is a local flow on $\mathcal{M}$ with phase velocity field f and domain of definition $W \cup \Big(\big(-\tilde{\epsilon}, +\tilde{\epsilon}\big) \times \tilde{N}\Big)$:

$$\breve{\Phi} : \big(\xi, (\breve{z}, \breve{\Omega})\big) \mapsto \begin{cases} \Phi\big(\xi, (\breve{z}, \breve{\Omega})\big) & \text{if } \big(\xi, (\breve{z}, \breve{\Omega})\big) \in W, \\ \tilde{\Phi}\big(\xi, (\breve{z}, \breve{\Omega})\big) & \text{if } \big(\xi, (\breve{z}, \breve{\Omega})\big) \in \big(-\tilde{\epsilon}, +\tilde{\epsilon}\big) \times \tilde{N}. \end{cases}$$

But maximality of Φ implies that $\left(-\tilde{\epsilon},+\tilde{\epsilon}\right)\times\tilde{N}\subset W$, and in particular that

$$\left[0,\tilde{\epsilon}\right)\times\left\{\Phi\left(\xi_{n_0},(z,\Omega)\right)\right\}\subset W.$$

Combined with equation (2.6) this implies that

$$\left[0,\xi_{n_0}+\tilde{\epsilon}\right)\times\left\{(z,\Omega)\right\}\subset W.$$

However, $\xi_{n_0}+\tilde{\epsilon}>t+\tilde{\epsilon}/2$, and this contradicts the choice of t. Therefore the case $t<+\infty$ is impossible. Since I_z is an open interval containing 0 the first statement thus follows.

Equation (4.2) follows immediately from equation (4.1) and Proposition H3.3.5.

Finally, let us show that equation (4.3) holds true. Suppose $\mu(z)>\mu$ and $a_z=-\infty$. Then for all $t<0$ we have

$$\begin{aligned} g\left(\varphi(t,z)\right) &= g(z)+\int_0^t\frac{\partial}{\partial\xi}g\left(\varphi(\xi,z)\right)d\xi \\ &\overset{(4.4)}{\geq} g(z), \end{aligned}$$

and

$$\begin{aligned} \lim_{t\to-\infty}\delta\left(\varphi(t,z)\right) &= \nu^{-1}\cdot\lim_{t\to-\infty}\mu\left(\varphi(t,z)\right) \\ &\overset{(4.5)}{=} \nu^{-1}\cdot\left(\mu+\left(\mu(z)-\mu\right)\cdot\lim_{t\to-\infty}e^{-t}\right) \\ &= +\infty. \end{aligned}$$

It follows that

$$\begin{aligned} &\lim_{t\to-\infty}\left(\delta\left(\varphi(t,z)\right)+\frac{\mu}{2}\cdot\mathfrak{F}\left(\varphi(t,z)\right)\right) \\ &\qquad=\frac{1}{2}\cdot\lim_{t\to-\infty}\delta\left(\varphi(t,z)\right)+\frac{1}{2}\cdot\lim_{t\to-\infty}\left(\delta\left(\varphi(t,z)\right)+\mu\cdot\mathfrak{F}\left(\varphi(t,z)\right)\right) \\ &\qquad=\frac{1}{2}\cdot\lim_{t\to-\infty}\delta\left(\varphi(t,z)\right)+\frac{1}{2}\cdot\lim_{t\to-\infty}\left(g\left(\varphi(t,z)\right)\right) \\ &\qquad=+\infty. \end{aligned}$$

This proves the claim in the case where $\mu(z)>\mu$ and $a_z=-\infty$.

Now suppose $-\infty<a_z$ or $a_z=-\infty$ and $\mu(z)\leq\mu$. Then equation (4.5) implies that $\mu\left(\varphi(t,z)\right)$ and $\delta\left(\varphi(t,z)\right)=\nu^{-1}\mu\left(\varphi(t,z)\right)$ remain bounded for all $t\in I_z$. We claim that in this case

$$\lim_{t\to-\infty}\mathfrak{F}\left(\varphi(t,z)\right)=+\infty.$$

In fact, suppose to the contrary that there exists a real number $B>0$ such that

$$\mathfrak{F}\left(\varphi(t,z)\right)\leq B\qquad\text{for all } t\in I_z. \tag{4.6}$$

Then $\varphi(t,z)$ lies in the compact set

$$Q_4 := \{z \in \mathcal{F}^\circ(PD) : \mathfrak{F}(z) \leq B\} \cap \\ \cap\, \delta^{-1}\Big(\big[0, \max\big(\sup\{\nu^{-1}\mu\big(\varphi(t,z)\big) : t \in I_z\}, \nu^{-1}\mu\big)\big]\Big) \subset D$$

for all $t \in I_z$, and likewise $\Phi\big(t,(z,\Omega)\big)$ lies in the compact set $Q_4 \times \mathcal{O}(V,e_V) \subset \mathcal{M}$. Reasoning along the same lines as above one can argue that Φ cannot be the maximal local flow with phase velocity field f, thus showing that a constant $B > 0$ such that (4.6) holds cannot exist. $\square$

Lemma 4.3. *There exist real numbers $\delta > 0$ and $\sigma > 1$ such that the following inequalities hold true for all $z \in D \cap B_\delta(z_\mu)$ and $t \in [0,+\infty)$:*

$$\sigma^{-1}\|z_\mu - z\| \cdot e^{-t} \leq \|z_\mu - \varphi(t,z)\| \leq \sigma \cdot \|z_\mu - z\| \cdot e^{-t}.$$

Proof. Proposition 1.1 combined with the Inverse Function Theorem and the fact that $\gamma \in \mathcal{C}^1$ imply that there exists an open convex neighborhood $Q_Z \subset D$ of z_μ and an open neighborhood $Q_V \subset V$ of μe_V such that the following are true:

1. $\gamma\big|_{Q_Z} : Q_Z \to Q_V$ is a $\mathcal{C}^1$-diffeomorphism,
2. $\|\gamma'(z)\| \leq 2 \cdot \|\gamma'(z_\mu)\|$ for all $z \in Q_Z$,
3. $\|(\gamma^{-1})'(v)\| \leq 2 \cdot \|(\gamma^{-1})'(\mu e_V)\|$ for all $v \in Q_V$.

In the rest of this proof γ refers to $\gamma\big|_{Q_Z}$. Let $\rho > 0$ be a real number such that

$$\hat{Q}_V := \{v \in V : \|\mu e_V - v\| < \rho\} \subseteq Q_V,$$

and let $\hat{Q}_Z := \gamma^{-1}(\hat{Q}_V)$. Note that

$$\operatorname{conv}(\hat{Q}_Z) \subseteq Q_Z, \qquad \text{and} \tag{4.7}$$

$$\operatorname{conv}(\hat{Q}_V) = \hat{Q}_V \subseteq Q_V. \tag{4.8}$$

Since $\gamma^{-1} \in \mathcal{C}^1$ we have

$$\begin{aligned} z_\mu - z &= \gamma^{-1}(\mu e_V) - \gamma^{-1}\big(\gamma(z)\big) \\ &= \int_0^1 (\gamma^{-1})'\big(\lambda\mu e_V + (1-\lambda)\gamma(z)\big)\,[\mu e_V - \gamma(z)]\,d\lambda \end{aligned} \tag{4.9}$$

for all $z \in \hat{Q}_Z$. But $\lambda\mu e_V + (1-\lambda)\cdot\gamma(z) \in Q_V$ because of (4.8). Taking norms on both sides of (4.9) and using property 3 from above, we get

$$\|z_\mu - z\| \leq 2 \cdot \|(\gamma^{-1})'(\mu e_V)\| \cdot \|\mu e_V - \gamma(z)\| \qquad \forall z \in \hat{Q}_Z. \tag{4.10}$$

Proceeding similarly with γ instead of γ^{-1}, we obtain

$$\|\mu e_V - v\| \le 2 \cdot \|\gamma'(z_\mu)\| \cdot \|z_\mu - \gamma^{-1}(v)\| \qquad \forall v \in \hat{Q}_V. \tag{4.11}$$

Now, equation (4.1) implies that

$$\varphi(t,z) \in \hat{Q}_Z \quad \forall t \ge 0, z \in \hat{Q}_Z. \tag{4.12}$$

Therefore, for $z \in \hat{Q}_Z$ and $t \ge 0$ we obtain

$$\begin{aligned} \|z_\mu - \varphi(t,z)\| &\overset{(4.12),(4.10)}{\le} 2 \cdot \|(\gamma^{-1})'(\mu e_V)\| \cdot \|\mu e_V - \gamma(\varphi(t,z))\| \\ &\overset{(4.1)}{=} 2 \cdot \|(\gamma^{-1})'(\mu e_V)\| \cdot \|\mu e_V - \gamma(z)\| \cdot e^{-t} \\ &\overset{(4.11)}{\le} 4 \cdot \|(\gamma^{-1})'(\mu e_V)\| \cdot \|\gamma'(z_\mu)\| \cdot \|z_\mu - z\| \cdot e^{-t}. \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} \|z_\mu - \varphi(t,z)\| &\overset{(4.12),(4.11)}{\ge} \Big(2 \cdot \|\gamma'(z_\mu)\|\Big)^{-1} \|\mu e_V - \gamma(\varphi(t,z))\| \\ &\overset{(4.1)}{=} \Big(2 \cdot \|\gamma'(z_\mu)\|\Big)^{-1} \|\mu e_V - \gamma(z)\| \cdot e^{-t} \\ &\overset{(4.10)}{\ge} \Big(4 \cdot \|(\gamma^{-1})'(\mu e_V)\| \cdot \|\gamma'(z_\mu)\|\Big)^{-1} \|z_\mu - z\| \cdot e^{-t}. \end{aligned} \tag{4.14}$$

It suffices thus to choose $\delta > 0$ such that

$$\{z \in D : \|z_\mu - z\| < \delta\} \subseteq \hat{Q}_Z$$

and to set $\sigma := 4 \cdot \|(\gamma^{-1})'(\mu e_V)\| \cdot \|\gamma'(z_\mu)\|$ for the result to follow from inequalities (4.13) and (4.14). □

Remark 4.4. As in the previous lemma, we will often restrict γ to the neighborhood Q_Z of z_μ and refer to this restriction as γ by abuse of language. This is to ensure that γ is single valued in the domain we consider.

Lemma 4.5. *For all $z \in D$ the integral $\int_0^{+\infty} d(\varphi(t,z))dt$ is absolutely convergent and we have*

$$\int_0^{+\infty} \|d(\varphi(t,z))\| dt = O(\|z_\mu - z\|).$$

Proof. Let δ and σ be as in Lemma 4.3. Note that since $\gamma \in \mathcal{C}^1$, the following holds:

$$\begin{aligned} d(z) &= M^{-1}(z)\big[\gamma(z_\mu) - \gamma(z)\big] \\ &= M^{-1}(z) \circ \int_0^1 \gamma'\big(\lambda z_\mu + (1-\lambda)\cdot z\big)\big[z_\mu - z\big] d\lambda. \end{aligned} \tag{4.15}$$

The facts that $\gamma' \in \mathcal{C}^0$, $M^{-1} \in \mathcal{C}^0$ and $\gamma'(z_\mu) = M(z_\mu)$ imply that there exists some real number $0 < \tilde{\delta} \le \delta$ such that $B_{\tilde{\delta}}(z_\mu) \subset D$ and such that for all $z \in B_{\tilde{\delta}}(z_\mu)$ we have $\|M^{-1}(z)\| \le 2 \cdot \|M^{-1}(z_\mu)\|$. Combined with property 2 from Lemma (4.3), equation (4.15) implies

$$\begin{aligned}
\|d(z)\| &\le \|M^{-1}(z)\| \cdot \int_0^1 \|\gamma'(\lambda z_\mu + (1-\lambda)z)\| \cdot \|z_\mu - z\| d\lambda \\
&\le 4 \cdot \|M^{-1}(z_\mu)\| \cdot \|M(z_\mu)\| \cdot \|z_\mu - z\| \\
&= \hat{\sigma} \cdot \|z_\mu - z\| \qquad (4.16)
\end{aligned}$$

for all $z \in B_{\tilde{\delta}}(z_\mu)$, where $\hat{\sigma} := 4 \cdot \|M^{-1}(z_\mu)\| \cdot \|M(z_\mu)\|$. Now, for any $z \in B_{\tilde{\delta}/\sigma}(z_\mu)$ and for all $t \ge 0$ we have

$$\|z_\mu - \varphi(t,z)\| \overset{4.3}{\le} \sigma \cdot \|z_\mu - z\| \cdot e^{-t} \le \tilde{\delta},$$

and hence

$$\varphi(t,z) \in B_{\tilde{\delta}}(z_\mu). \qquad (4.17)$$

Therefore, for all $z \in B_{\tilde{\delta}/\sigma}(z_\mu)$ we have

$$\begin{aligned}
\|d(\varphi(t,z))\| &\overset{(4.17),(4.16)}{\le} \hat{\sigma} \cdot \|z_\mu - \varphi(t,z)\| \\
&\overset{(4.17),4.3}{\le} \hat{\sigma} \cdot \sigma \cdot \|z_\mu - z\| \cdot e^{-t}, \qquad \text{and} \qquad (4.18)
\end{aligned}$$

$$\int_0^{+\infty} \|d(\varphi(t,z))\| dt \quad \le \quad \hat{\sigma} \cdot \sigma \cdot \|z_\mu - z\|. \qquad (4.19)$$

This proves the second statement of the lemma.

The first statement now follows easily. In fact, by virtue of equation (4.2) we have $\lim_{t \to +\infty} \varphi(t,z) = z_\mu$ for all $z \in D$, and hence there exists a $t_0 > 0$ such that $\|z_\mu - \varphi(t_0, z)\| < \tilde{\delta}/\sigma$. But this implies that

$$\begin{aligned}
\int_0^{+\infty} \|d(\varphi(t,z))\| dt &= \int_0^{t_0} \|d(\varphi(t,z))\| dt + \int_{t_0}^{+\infty} \|d(\varphi(t,z))\| dt \\
&\overset{(2.6)}{=} \int_0^{t_0} \|d(\varphi(t,z))\| dt + \int_0^{+\infty} \|d(\varphi(t, \varphi(t_0,z)))\| dt \\
&\overset{(4.19)}{\le} \int_0^{t_0} \|d(\varphi(t,z))\| dt + \hat{\sigma} \cdot \sigma \cdot \|z_\mu - \varphi(t_0,z)\| \\
&< +\infty.
\end{aligned}$$

□

Lemma 4.6. *The following holds true:*

$$d'(z) = -\imath + O(\|z_\mu - z\|).$$

Proof. By definition of d we have

$$d'(z)[\cdot] = \Big(\big(M^{-1}\big)'(z)[\cdot]\Big)\,[\mu e_V - \gamma(z)] - M^{-1}(z) \circ \gamma'(z)[\cdot],$$

and it follows from Proposition 1.1 *iii)* and from the fact that $M^{-1} \circ \gamma' \in \mathcal{C}^1$ that we have

$$\begin{aligned} M^{-1}(z) \circ \gamma'(z) &= M^{-1}\big(z_\mu\big) \circ \gamma'\big(z_\mu\big) + O\big(\|z_\mu - z\|\big) \\ &= \imath + O\big(\|z_\mu - z\|\big). \end{aligned}$$

Moreover,

$$\mu e_V - \gamma(z) = \gamma\big(z_\mu\big) - \gamma(z) \overset{\gamma \in \mathcal{C}^1}{=} O\big(\|z_\mu - z\|\big), \tag{4.20}$$

and since $\big(M^{-1}\big)'(z) \in \mathcal{C}^0$ it follows that

$$\Big(\big(M^{-1}\big)'(z)[\cdot]\Big)\,[\mu e_V - \gamma(z)] = O\big(\|z_\mu - z\|\big).$$

□

Lemma 4.7. *For $t \geq 0$ we have*

$$\begin{aligned} &\|v\| \cdot \exp\big\{-t\big(1 + \big|O(\|z_\mu - z\|)\big|\big)\big\} \\ &\qquad \leq \Big\|\frac{\partial}{\partial z}\varphi(t,z)[v]\Big\| \\ &\qquad\qquad \leq \|v\| \cdot \exp\big\{-t\big(1 - \big|O(\|z_\mu - z\|)\big|\big)\big\}. \end{aligned} \tag{4.21}$$

Moreover, for all $z \in D$ the integral $\int_0^{+\infty} \frac{\partial}{\partial z} d\big(\varphi(t,z)\big)[\cdot]dt$ converges absolutely and we have

$$\int_0^{+\infty} \Big\|\frac{\partial}{\partial z} d\big(\varphi(t,z)\big)[\cdot]\Big\| dt = 1 + O\big(\|z_\mu - z\|\big). \tag{4.22}$$

Proof. For all $v \in Z$ we have

$$\begin{aligned} \frac{\partial}{\partial z} d\big(\varphi(t,z)\big)[v] &= d'\big(\varphi(t,z)\big)\Big[\frac{\partial}{\partial z}\varphi(t,z)[v]\Big] \\ &\overset{4.6}{=} \Big(-\imath + O\big(\|z_\mu - \varphi(t,z)\|\big)\Big) \circ \frac{\partial}{\partial z}\varphi(t,z)[v]. \end{aligned} \tag{4.23}$$

Using Lemma 4.3 and taking norms on both sides of equation (4.23) we get

$$\Big\|\frac{\partial}{\partial z} d\big(\varphi(t,z)\big)[v]\Big\| = \Big(1 + O\big(\|z_\mu - z\|\big)\Big) \cdot \Big\|\frac{\partial}{\partial z}\varphi(t,z)[v]\Big\|. \tag{4.24}$$

Note that

$$
\begin{aligned}
&\frac{\partial}{\partial t}\Big\|\frac{\partial}{\partial z}\varphi(t,z)[v]\Big\| \\
&\quad = \quad \Big\|\frac{\partial}{\partial z}\varphi(t,z)[v]\Big\|^{-1}\cdot\Big(\frac{\partial}{\partial z}\varphi(t,z)[v];\frac{\partial^2}{\partial t\partial z}\varphi(t,z)[v]\Big) \\
&\quad \overset{\varphi\in\mathcal{C}^2}{=} \quad \Big\|\frac{\partial}{\partial z}\varphi(t,z)[v]\Big\|^{-1}\cdot\Big(\frac{\partial}{\partial z}\varphi(t,z)[v];\frac{\partial}{\partial z}\big(\frac{\partial}{\partial t}\varphi(t,z)[v]\big)\Big) \\
&\quad \overset{3.9,(2.5)}{=} \Big\|\frac{\partial}{\partial z}\varphi(t,z)[v]\Big\|^{-1}\cdot\Big(\frac{\partial}{\partial z}\varphi(t,z)[v];\frac{\partial}{\partial z}d\big(\varphi(t,z)[v]\big)\Big) \\
&\quad \overset{(4.23)}{=} \Big\|\frac{\partial}{\partial z}\varphi(t,z)[v]\Big\|^{-1}\cdot\Big(\frac{\partial}{\partial z}\varphi(t,z)[v];\big(-\imath+O(\|z_\mu-\varphi(t,z)\|)\big)\cdot\frac{\partial}{\partial z}\varphi(t,z)[v]\Big),
\end{aligned}
$$

and applying the Cauchy-Schwarz inequality to the last expression we get

$$
\begin{aligned}
&\frac{\partial}{\partial t}\Big\|\frac{\partial}{\partial z}\varphi(t,z)[v]\Big\| \\
&\quad = \quad -\Big\|\frac{\partial}{\partial z}\varphi(t,z)[v]\Big\|\cdot\Big(1+O\big(\|z_\mu-\varphi(t,z)\|\big)\Big) \qquad (4.25)\\
&\quad \overset{4.3}{=} \quad -\Big\|\frac{\partial}{\partial z}\varphi(t,z)[v]\Big\|\cdot\Big(1+O_2\big(\|z_\mu-z\|\big)\Big). \qquad (4.26)
\end{aligned}
$$

Now let $0<\epsilon<1$ be a real number and suppose z lies close enough to z_μ such that $\big|O_2(\|z_\mu-z\|)\big|<\epsilon$. Then it follows from equation (4.26) that for all $t\geq 0$ the following inequality holds true:

$$
-\Big\|\frac{\partial}{\partial z}\varphi(t,z)[v]\Big\|\cdot(1+\epsilon)\leq\frac{\partial}{\partial t}\Big\|\frac{\partial}{\partial z}\varphi(t,z)[v]\Big\|\leq-\Big\|\frac{\partial}{\partial z}\varphi(t,z)[v]\Big\|(1-\epsilon). \qquad (4.27)
$$

But note that

$$
\Big\|\frac{\partial}{\partial z}\varphi(0,z)[v]\Big\|=\Big\|\frac{\partial}{\partial z}z[v]\Big\|=\|v\|.
$$

Hence, upper and lower bounds on $\big\|\frac{\partial}{\partial z}\varphi(t,z)[v]\big\|$ can be obtained from integrating the initial value problems

$$
\begin{aligned}
&\dot y(t)=-y(t)\cdot(1-\epsilon), \qquad \forall t\geq 0,\\
&y(0)=\|v\|,
\end{aligned}
$$

and

$$
\begin{aligned}
&\dot y(t)=-y(t)\cdot(1+\epsilon), \qquad \forall t\geq 0,\\
&y(0)=\|v\|.
\end{aligned}
$$

Solving these equations yields the following inequality:

$$
\|v\|\cdot\exp\big\{-t(1+\epsilon)\big\}\leq\Big\|\frac{\partial}{\partial z}\varphi(t,z)[v]\Big\|\leq\|v\|\cdot\exp\big\{-t(1-\epsilon)\big\}. \qquad (4.28)
$$

This shows that equation (4.21) holds true. Substituting this result in equation (4.24) we now get

$$\begin{aligned}\big(1-\big|O\big(\|z_\mu-z\|\big)\big|\big)\cdot\|v\|\cdot\exp\big\{-t(1+\epsilon)\big\}\\ \leq\Big\|\frac{\partial}{\partial z}d\big(\varphi(t,z)\big)[v]\Big\|\\ \leq\big(1+\big|O\big(\|z_\mu-z\|\big)\big|\big)\cdot\|v\|\cdot\exp\big\{-t(1-\epsilon)\big\}.\end{aligned}$$

Upon integration this yields the inequality

$$\begin{aligned}\big(1-\big|O\big(\|z_\mu-z\|\big)\big|\big)\cdot\frac{\|v\|}{1+\epsilon}\\ \leq\int_0^{+\infty}\Big\|\frac{\partial}{\partial z}d\big(\varphi(t,z)\big)[v]\Big\|dt\\ \leq\big(1+\big|O\big(\|z_\mu-z\|\big)\big|\big)\cdot\frac{\|v\|}{1-\epsilon}\quad.\end{aligned}$$

Since for a particular z this holds for all $\epsilon>\big|O_2\big(\|z_\mu-z\|\big)\big|$ and since $v\in Z$ was completely arbitrary, it follows that equation (4.22) holds true. Finally, since $\lim_{t\to+\infty}\varphi(t,z)=z_\mu$ for all $z\in D$, arguments similar to those we gave in the last part of Lemma 4.5 show that

$$\int_0^{+\infty}\Big\|\frac{\partial}{\partial z}d\big(\varphi(t,z)\big)[v]\Big\|dt<+\infty$$

for all $z\in D$. □

For the next lemma, note that it follows from Lemma 3.4 that $d''(z_\mu)$ is finite.

Lemma 4.8. *For $t\geq 0$ we have*

$$\Big\|\frac{\partial^2}{\partial z^2}\varphi(t,z)[v,w]\Big\|=\exp\Big\{-t\Big(1-\big|O(\|z_\mu-z\|)\big|\Big)\Big\}\cdot O(1).\tag{4.29}$$

For all $z\in D$ the integral $\int_0^{+\infty}\frac{\partial^2}{\partial z^2}d\big(\varphi(t,z)\big)[\cdot,\cdot]dt$ converges absolutely and

$$\int_0^{+\infty}\Big\|\frac{\partial^2}{\partial z^2}d\big(\varphi(t,z)\big)[\cdot,\cdot]\Big\|dt\leq 2\cdot\big\|d''(z_\mu)\big\|\cdot\Big(1+\big|O(\|z_\mu-z\|)\big|\Big).\tag{4.30}$$

Proof. For all $v,w\in Z$ we have

$$\begin{aligned}\frac{\partial^2}{\partial z^2}d\big(\varphi(t,z)\big)[v,w]\;=\;&d''\big(\varphi(t,z)\big)\Big[\frac{\partial}{\partial z}\varphi(t,z)[v],\frac{\partial}{\partial z}\varphi(t,z)[w]\Big]\\ &+d'\big(\varphi(t,z)\big)\Big[\frac{\partial^2}{\partial z^2}\varphi(t,z)[v,w]\Big],\end{aligned}\tag{4.31}$$

and hence

$$\begin{aligned}
&\Big\|\frac{\partial^2}{\partial z^2} d\big(\varphi(t,z)\big)[v,w]\Big\| \\
\text{(4.32)} \qquad &\leq \big\|d''\big(\varphi(t,z)\big)\big\| \cdot \Big\|\frac{\partial}{\partial z}\varphi(t,z)[v]\Big\| \cdot \Big\|\frac{\partial}{\partial z}\varphi(t,z)[w]\Big\| \\
&\qquad + \big\|d'\big(\varphi(t,z)\big)\big\| \cdot \Big\|\frac{\partial^2}{\partial z^2}\varphi(t,z)[v,w]\Big\|.
\end{aligned}$$

Note also that

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big\|\frac{\partial^2}{\partial z^2}\varphi(t,z)[v,w]\Big\| \\
&\quad = \quad \Big\|\frac{\partial^2}{\partial z^2}\varphi(t,z)[v,w]\Big\|^{-1} \cdot \Big(\frac{\partial^2}{\partial z^2}\varphi(t,z)[v,w]; \frac{\partial}{\partial t}\frac{\partial^2}{\partial z^2}\varphi(t,z)[v,w]\Big) \\
&\quad \overset{\varphi\in\mathcal{C}^3}{=} \quad \Big\|\frac{\partial^2}{\partial z^2}\varphi(t,z)[v,w]\Big\|^{-1} \cdot \Big(\frac{\partial^2}{\partial z^2}\varphi(t,z)[v,w]; \frac{\partial^2}{\partial z^2}\Big(\frac{\partial}{\partial t}\varphi(t,z)\Big)[v,w]\Big) \\
&\quad \overset{3.9}{=} \quad \Big\|\frac{\partial^2}{\partial z^2}\varphi(t,z)[v,w]\Big\|^{-1} \cdot \Big(\frac{\partial^2}{\partial z^2}\varphi(t,z)[v,w]; \frac{\partial^2}{\partial z^2} d\big(\varphi(t,z)\big)[v,w]\Big),
\end{aligned}$$

and equation (4.31) and Lemma 4.6 show that this is equal to

$$\begin{aligned}
&\Big\|\frac{\partial^2}{\partial z^2}\varphi(t,z)[v,w]\Big\|^{-1} \cdot \Big\{\Big(\frac{\partial^2}{\partial z^2}\varphi(t,z)[v,w]; d''\big(\varphi(t,z)\big)\Big[\frac{\partial}{\partial z}\varphi(t,z)[v], \frac{\partial}{\partial z}\varphi(t,z)[w]\Big]\Big) \\
&\qquad + \Big(\frac{\partial^2}{\partial z^2}\varphi(t,z)[v,w]; \Big(-\imath + O\big(\|z_\mu - \varphi(t,z)\|\big)\Big)\Big[\frac{\partial^2}{\partial z^2}\varphi(t,z)[v,w]\Big]\Big)\Big\}.
\end{aligned}$$

If we now apply the Cauchy-Schwarz inequality and Lemma 4.3 to the last expression we get

$$\begin{aligned}
\text{(4.33)} \qquad \frac{\partial}{\partial t}\Big\|\frac{\partial^2}{\partial z^2}\varphi(t,z)[v,w]\Big\| &\leq \big\|d''\big(\varphi(t,z)\big)\big\| \cdot \Big\|\frac{\partial}{\partial z}\varphi(t,z)[v]\Big\| \cdot \Big\|\frac{\partial}{\partial z}\varphi(t,z)[w]\Big\| \\
&\qquad - \Big\|\frac{\partial^2}{\partial z^2}\varphi(t,z)[v,w]\Big\| \cdot \Big(1 - \big|O_3\big(\|z_\mu - z\|\big)\big|\Big).
\end{aligned}$$

Let $\sigma > 1$ and $\delta > 0$ be defined as in Lemma 4.3. Because of Lemma 3.4 there exists a radius $0 < \eta < \delta$ such that

$$\big\|d''(z)\big\| \leq 2 \cdot \big\|d''\big(z_\mu\big)\big\| \qquad \text{for all } z \in B_\eta(z_\mu).$$

Let us refer to the function O introduced in Lemma 4.6 as O_4, and recall the definition of O_2 in Lemma 4.7, as well as the function O_3 introduced above. Let us then define $O_5 := \max\big\{|O_2|, |O_3|, |O_4|\big\}$. For $0 < \epsilon < 1$ let $\eta' > 0$ be a real number such that $O_5(\eta') < \epsilon$ and $\sigma \cdot \eta' \leq \eta$. Then it follows from Lemma 4.3 that for $z \in B_{\eta'}\big(z_\mu\big)$ and $t \geq 0$ we have $\varphi(t,z) \in B_\eta(z_\mu)$. Therefore,

$$\text{(4.34)} \qquad \big\|d''\big(\varphi(t,z)\big)\big\| \leq 2 \cdot \big\|d''\big(z_\mu\big)\big\|.$$

Moreover, since $\big|O_2\big(\|z_\mu - z\|\big)\big| \leq O_5(\eta') < \epsilon$, it follows from inequality (4.28) that

$$\text{(4.35)} \qquad \Big\|\frac{\partial}{\partial z}\varphi(t,z)[v]\Big\| \leq \|v\| \cdot \exp\big\{-t(1-\epsilon)\big\}.$$

Combining equations (4.34), (4.35) and (4.33), we get that for all $z \in B_{\eta'}\big(z_\mu\big)$ and $t \geq 0$

$$\begin{aligned}\frac{\partial}{\partial t}\Big\|\frac{\partial^2}{\partial z^2}\varphi(t,z)[v,w]\Big\| \leq 2\cdot\big\|d''(z_\mu)\big\|\cdot\|v\|\cdot\|w\|\cdot\exp\big\{-2t(1-\epsilon)\big\}\\ -\Big\|\frac{\partial^2}{\partial z^2}\varphi(t,z)[v,w]\Big\|\cdot(1-\epsilon).\end{aligned}\tag{4.36}$$

Note also that

$$\frac{\partial^2}{\partial z^2}\varphi(0,z)[v,w] = \frac{\partial^2}{\partial z^2}z[v,w] = 0.\tag{4.37}$$

Equations (4.36) and (4.37) imply that $\big\|\frac{\partial^2}{\partial z^2}\varphi(t,z)[v,w]\big\|$ is bounded from above by the solution to the initial value problem

$$\begin{aligned}&\dot{y}(t) = 2\cdot\big\|d''(z_\mu)\big\|\cdot\|v\|\cdot\|w\|\cdot\exp\big\{-2t(1-\epsilon)\big\} - y(t)\cdot(1-\epsilon), \qquad t\geq 0,\\ &y(0)=0.\end{aligned}$$

Integrating this equation we get

$$\begin{aligned}&\Big\|\frac{\partial^2}{\partial z^2}\varphi(t,z)[v,w]\Big\|\\ &\quad\leq \exp\big\{-t(1-\epsilon)\big\}\cdot\frac{2\cdot\big\|d''(z_\mu)\big\|}{1-\epsilon}\cdot\|v\|\cdot\|w\|\cdot\Big(1-\exp\big\{-t(1-\epsilon)\big\}\Big)\\ &\quad= \exp\big\{-t(1-\epsilon)\big\}\cdot O(1)\cdot\|v\|\cdot\|w\|.\end{aligned}\tag{4.38}$$

But this inequality implies that equation (4.29) holds true, since $\epsilon > 0$ was arbitrary.

Moreover, since $\varphi(t,z)\in B_\eta(z_\mu)$ for $z\in B_{\eta'}\big(z_\mu\big)$ and $t\geq 0$, Lemma 4.6 implies that

$$\begin{aligned}\big\|d'\big(\varphi(t,z)\big)\big\| &\leq 1 + O_4\big(\|z_\mu - z\|\big)\\ &\leq 1 + O_5(\eta)\\ &\leq 1+\epsilon.\end{aligned}$$

Substituting this inequality and (4.38) into (4.32) we get that

$$\begin{aligned}\Big\|\frac{\partial^2}{\partial z^2}d\big(\varphi(t,z)\big)[v,w]\Big\| &\leq 2\big\|d''(z_\mu)\big\|\cdot\|v\|\cdot\|w\|\cdot\Big(\exp\big\{-2t(1-\epsilon)\big\} +\\ &\quad + \frac{1+\epsilon}{1-\epsilon}\cdot\exp\big\{-t(1-\epsilon)\big\}\cdot\big(1-\exp\{-t(1-\epsilon)\}\big)\Big)\\ &\leq 2\big\|d''(z_\mu)\big\|\cdot\|v\|\cdot\|w\|\cdot\frac{1+\epsilon}{1-\epsilon}\cdot\exp\big\{-t(1-\epsilon)\big\}.\end{aligned}\tag{4.39}$$

Integrating on both sides of this equation yields

$$\begin{aligned}&\int_0^{+\infty}\Big\|\frac{\partial^2}{\partial z^2}d\big(\varphi(t,z)\big)[v,w]\Big\|dt\\ &\qquad\leq 2\big\|d''(z_\mu)\big\|\cdot\|v\|\cdot\|w\|\cdot\frac{1+\epsilon}{(1-\epsilon)^2}\\ &\qquad\qquad= 2\big\|d''(z_\mu)\big\|\cdot\|v\|\cdot\|w\|\cdot\big(1+O(\epsilon)\big).\end{aligned}$$

Since v, w and ϵ were arbitrary it follows that

$$\int_0^{+\infty} \Big\| \frac{\partial^2}{\partial z^2} d\big(\varphi(t,z)\big)[\cdot,\cdot] \Big\| dt \leq 2 \cdot \big\| d''(z_\mu) \big\| \cdot \big(1 + O(\|z_\mu - z\|)\big).$$

This shows that equation (4.30) holds. Again the last statement follows trivially in conjunction with the fact that $\lim_{t\to+\infty} \varphi(t,z) = z_\mu$ for all $z \in D$. $\square$

Lemma 4.9. *For all $z \in D$,*

$$\int_0^{+\infty} \frac{\partial}{\partial z} d\big(\varphi(t,z)\big)[\cdot] dt = -\imath \qquad \textit{and}$$
$$\int_0^{+\infty} \frac{\partial^2}{\partial z^2} d\big(\varphi(t,z)\big)[\cdot,\cdot] dt = 0.$$

Proof. Consider the following trivial identity:

$$\int_0^{+\infty} d\big(\varphi(t,z)\big) dt \overset{3.9}{=} \int_0^{+\infty} \frac{\partial}{\partial t} \varphi(t,z) dt = \lim_{t\to+\infty} \varphi(t,z) - z \overset{(4.2)}{=} z_\mu - z. \tag{4.40}$$

It follows from Lemma 4.7 that the differential operator $\frac{\partial}{\partial z}$ commutes with the integral operator $\int_0^\infty$ on the left-hand side of equation (4.40), and hence we have

$$\int_0^{+\infty} \frac{\partial}{\partial z} d\big(\varphi(t,z)\big)[\cdot] dt = \frac{\partial}{\partial z} \Big(\int_0^{+\infty} d\big(\varphi(t,z)\big) dt \Big)[\cdot] = \frac{\partial}{\partial z} \big(z_\mu - z\big) = -\imath.$$

Likewise, by Lemma 4.8 the operator $\frac{\partial^2}{\partial z^2}$ applied to the left hand side of equation (4.40) commutes with the operator $\int_0^\infty$ and the second identity of the lemma follows also. $\square$

Lemma 4.10. $\lim_{t\to+\infty} e^t \cdot d\big(\varphi(t,z)\big)$ *exists for all $z \in D$.*

Proof. For $0 \neq v \in V$ let $\mathfrak{n}\{v\} := \|v\|^{-1} \cdot v$ denote the normalization of v. Let us start by noting that

$$\begin{aligned} e^t \cdot d\big(\varphi(t,z)\big) &\overset{(1.6)}{=} e^t \cdot M\big(\varphi(t,z)\big)^{-1} \big[\mu e_V - \gamma\big(\varphi(t,z)\big)\big] \\ &= e^t \cdot \big\| \mu e_V - \gamma\big(\varphi(t,z)\big) \big\| \cdot M\big(\varphi(t,z)\big)^{-1} \big[\mathfrak{n}\{\mu e_V - \gamma\big(\varphi(t,z)\big)\}\big] \\ &\overset{(4.1)}{=} \big\| \mu e_V - \gamma(z) \big\| \cdot M\big(\varphi(t,z)\big)^{-1} \big[\mathfrak{n}\{\mu e_V - \gamma\big(\varphi(t,z)\big)\}\big], \end{aligned} \tag{4.41}$$

where M denotes the main part of γ' as in defined in Section 1. Since $M^{-1} \in \mathcal{C}^0$ and $\lim_{t\to+\infty} \varphi(t,z) \overset{(4.2)}{=} z_\mu$, equation (4.41) shows that all we need to prove is that

$$\lim_{t\to+\infty} \mathfrak{n}\big\{\mu e_V - \gamma\big(\varphi(t,z)\big)\big\}$$

exists. In the formulas that follow, let us abbreviate $\varphi(t,z)$ by φ. Note that

$$\begin{aligned}
&\frac{\partial}{\partial t}\,\mathfrak{n}\{\mu e_V - \gamma(\varphi(t,z))\} \\
&\quad = \|\mu e_V - \gamma(\varphi)\|^{-1}\cdot\Big\{-\gamma'(\varphi)\Big[\frac{\partial}{\partial t}\varphi\Big] - \Big(\mu e_V - \gamma(\varphi); -\gamma'(\varphi)\Big[\frac{\partial}{\partial t}\varphi\Big]\Big)\cdot \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot\|\mu e_V - \gamma(\varphi)\|^{-2}\cdot(\mu e_V - \gamma(\varphi))\Big\} \\
&\quad = \|\mu e_V - \gamma(\varphi)\|^{-1}\cdot\Big\{-\gamma'(\varphi)[d(\varphi)] + \Big(\mu e_V - \gamma(\varphi); \gamma'(\varphi)[d(\varphi)]\Big)\cdot \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot\|\mu e_V - \gamma(\varphi)\|^{-2}\cdot(\mu e_V - \gamma(\varphi))\Big\}.
\end{aligned} \tag{4.42}$$

Let $R := \gamma' - M$ as defined in Section 1. Proposition 1.1 *ii)* implies that

$$R(\varphi(t,z))[d(\varphi(t,z))] \perp \mu e_V - \gamma(\varphi(t,z)),$$

and it follows from equation (4.42) that

$$\begin{aligned}
&\frac{\partial}{\partial t}\,\mathfrak{n}\{\mu e_V - \gamma(\varphi)\} \\
&\quad = \quad \|\mu e_V - \gamma(\varphi)\|^{-1}\cdot\Big\{-M(\varphi)[d(\varphi)] - R(\varphi)[d(\varphi)] \\
&\qquad\qquad + \Big(\mu e_V - \gamma(\varphi); M(\varphi)[d(\varphi)]\Big)\cdot\|\mu e_V - \gamma(\varphi)\|^{-2}\cdot(\mu e_V - \gamma(\varphi))\Big\} \\
&\quad \overset{(1.6)}{=} \|\mu e_V - \gamma(\varphi)\|^{-1}\cdot\Big\{-(\mu e_V - \gamma(\varphi)) - R(\varphi)[d(\varphi)] \\
&\qquad\qquad + \Big(\mu e_V - \gamma(\varphi); \mu e_V - \gamma(\varphi)\Big)\cdot\|\mu e_V - \gamma(\varphi)\|^{-2}\cdot(\mu e_V - \gamma(\varphi))\Big\} \\
&\quad = \quad -\|\mu e_V - \gamma(\varphi(t,z))\|^{-1}\cdot R(\varphi(t,z))[d(\varphi(t,z))].
\end{aligned} \tag{4.43}$$

Because of equation (4.2) and the fact that $M^{-1}\in\mathcal{C}^0$ we have

$$\begin{aligned}
d(\varphi(t,z)) &= M(\varphi(t,z))^{-1}[\mu e_V - \gamma(\varphi(t,z))] \\
&= O(\|\mu e_V - \gamma(\varphi(t,z))\|).
\end{aligned} \tag{4.44}$$

Moreover, since $R\in\mathcal{C}^1$, it is true that

$$R(\varphi(t,z)) = R(z_\mu) + O(\|z_\mu - \varphi(t,z)\|). \tag{4.45}$$

Since $\gamma^{-1}\in\mathcal{C}^1$ we have

$$\begin{aligned}
z_\mu - \varphi(t,z) &= \gamma^{-1}(\mu e_V) - \gamma^{-1}\Big(\gamma(\varphi(t,z))\Big) \\
&= \int_0^1 (\gamma^{-1})'\Big(\lambda\cdot\mu e_V + (1-\lambda)\cdot\gamma(\varphi(t,z))\Big)[\mu e_V - \gamma(\varphi(t,z))]\,d\lambda \\
&\overset{(4.2)}{=} (\gamma^{-1})'(\mu e_V)[\mu e_V - \gamma(\varphi(t,z))] + O(\|\mu e_V - \gamma(\varphi(t,z))\|^2).
\end{aligned}$$

Using this in equation (4.45) together with the fact that $R(z_\mu) = 0$ by Proposition 1.1 we get

$$R\big(\varphi(t,z)\big) = O\big(\|\mu e_V - \gamma\big(\varphi(t,z)\big)\|\big). \tag{4.46}$$

Substituting equations (4.44) and (4.46) into (4.43) we get

$$\begin{aligned}\frac{\partial}{\partial t}\mathfrak{n}\big\{\mu e_V - \gamma\big(\varphi(t,z)\big)\big\} &= O\big(\|\mu e_V - \gamma\big(\varphi(t,z)\big)\|\big) \\ &\overset{(4.1)}{=} O(1)\cdot\|\mu e_V - \gamma(z)\|\cdot e^{-t},\end{aligned}$$

and hence

$$\begin{aligned}\Big\|\mathfrak{n}\big\{\mu e_V - \gamma\big(\varphi(t_0,z)\big)\big\} - \mathfrak{n}\big\{\mu e_V - \gamma\big(\varphi(t_0+t,z)\big)\big\}\Big\| &\le \\ &\le \int_{t_0}^{t}\Big\|\frac{\partial}{\partial\theta}\mathfrak{n}\{\mu e_V - \gamma\big(\varphi(\theta,z)\big)\}\Big\|d\theta \\ &\le O(1)\cdot\|\mu e_V - \gamma(z)\|\cdot\int_{t_0}^{t}e^{-\theta}d\theta \\ &= O(1)\cdot\|\mu e_V - \gamma(z)\|\cdot\big(e^{-t_0} - e^{-t}\big).\end{aligned}$$

This implies that the net

$$\Big\{\mathfrak{n}\big\{\mu e_V - \gamma\big(\varphi(t,z)\big)\big\} : t \in I_z\Big\} \tag{4.47}$$

over the directed system $(I_z, \le)$ is Cauchy with values in the sphere $S^{n-1}(V)$, which is a complete set. Therefore, (4.47) converges to a limit point as claimed. □

Lemma 4.11. *For all $z \in D$ it is true that*

$$\lim_{t\to+\infty} e^t\cdot\Big(d\big(\varphi(t,z)\big) - \big(z_\mu - \varphi(t,z)\big)\Big) = 0.$$

Proof. By equation (1.6) we have

$$\begin{aligned}&d\big(\varphi(t,z)\big) - \big(z_\mu - \varphi(t,z)\big) \\ &\qquad = M\big(\varphi(t,z)\big)^{-1}\big[\mu e_V - \gamma\big(\varphi(t,z)\big)\big] + \varphi(t,z) - z_\mu.\end{aligned} \tag{4.48}$$

Moreover, it follows from Propositions 1.1 and 1.1 *iii)* that there exists a convex open neighborhood $N \subseteq \mathcal{F}^\circ(PD)$ of z_μ such that for all $\tilde z \in N$ we have

1. $\gamma'(\tilde z)$ is non-singular and
2. $\|\big(\gamma'(\tilde z)\big)^{-1}\circ R(\tilde z)\| < 1/2$.

Therefore, for $\tilde{z} \in N$ we can develop $M(\tilde{z})$ into the following Neumann series:

$$
\begin{aligned}
M(\tilde{z})^{-1} &= \Big(\gamma'(\tilde{z}) - R(\tilde{z})\Big)^{-1} \\
&= \Big(\gamma'(\tilde{z}) \circ \big(\imath - \gamma'(\tilde{z})^{-1} \circ R(\tilde{z})\big)\Big)^{-1} \\
&= \Big\{\imath + \sum_{k=1}^{\infty} \Big(\gamma'(\tilde{z})^{-1} \circ R(\tilde{z})\Big)^{k}\Big\} \circ \gamma'(\tilde{z})^{-1}.
\end{aligned} \tag{4.49}
$$

Since $\gamma \in \mathcal{C}^1$ and D is convex we have that for all $z \in D$ and $t \in I_z$

$$
\begin{aligned}
\mu e_V - \gamma\big(\varphi(t,z)\big) &= \gamma(z_\mu) - \gamma\big(\varphi(t,z)\big) \\
&= \int_0^1 \gamma'\big(\lambda z_\mu + (1-\lambda)\varphi(t,z)\big)\big[z_\mu - \varphi(t,z)\big]\, d\lambda.
\end{aligned} \tag{4.50}
$$

Equation (4.2) implies that $\varphi(t,z) \in N$ for $t \gg 1$, and by convexity of N we then have

$$
\lambda z_\mu + (1-\lambda)\varphi(t,z) \in N \qquad \forall\, \lambda \in [0,1] \quad \text{and } t \gg 1. \tag{4.51}
$$

Abbreviating $\varphi(t,z)$ by φ, it follows from equations (4.49), (4.50) and (4.51) that for all $t \gg 1$

$$
\begin{aligned}
&M(\varphi)^{-1}\big[\mu e_V - \gamma(\varphi)\big] \\
&\quad = \int_0^1 M(\varphi)^{-1} \circ M\big(\lambda z_\mu + (1-\lambda)\varphi\big) \circ M\big(\lambda z_\mu + (1-\lambda)\varphi\big)^{-1} \circ \\
&\qquad\qquad \circ\, \gamma'\big(\lambda z_\mu + (1-\lambda)\varphi\big)\big[z_\mu - \varphi\big]\, d\lambda \\
&\quad \overset{(4.49)}{=} \int_0^1 M(\varphi)^{-1} \circ M\big(\lambda z_\mu + (1-\lambda)\varphi\big) \circ \\
&\qquad\qquad \circ \Big\{\imath + \sum_{k=1}^{\infty} \Big(\gamma'\big(\lambda z_\mu + (1-\lambda)\varphi\big)^{-1} \circ R\big(\lambda z_\mu + (1-\lambda)\varphi\big)\Big)^{k}\Big\}\big[z_\mu - \varphi\big]\, d\lambda \\
&\quad = \quad z_\mu - \varphi \\
&\qquad - \int_0^1 \Big\{\imath - M(\varphi)^{-1} \circ M\big(\lambda z_\mu + (1-\lambda)\varphi\big)\Big\}\big[z_\mu - \varphi\big]\, d\lambda \\
&\qquad + \int_0^1 M(\varphi)^{-1} \circ M\big(\lambda z_\mu + (1-\lambda)\varphi\big) \circ \\
&\qquad\qquad \circ\, \gamma'\big(\lambda z_\mu + (1-\lambda)\varphi\big)^{-1} \circ R\big(\lambda z_\mu + (1-\lambda)\varphi\big) \circ \\
&\qquad\qquad \circ \Big\{\sum_{k=0}^{\infty} \Big(\gamma'\big(\lambda z_\mu + (1-\lambda)\varphi\big)^{-1} \circ R\big(\lambda z_\mu + (1-\lambda)\varphi\big)\Big)^{k}\Big\}\big[z_\mu - \varphi\big]\, d\lambda.
\end{aligned} \tag{4.52}
$$

It follows from equations (4.48) and (4.52) that for all $t \gg 1$

$$
\begin{aligned}
&e^t\big(d(\varphi) - (z_\mu - \varphi)\big) \\
&= -\int_0^1 \Big\{\imath - M(\varphi)^{-1} \circ M\big(\lambda z_\mu + (1-\lambda)\varphi\big)\Big\}\big[e^t(z_\mu - \varphi)\big]\,d\lambda \\
&\quad + \int_0^1 M(\varphi)^{-1} \circ M\big(\lambda z_\mu + (1-\lambda)\varphi\big) \circ \\
&\qquad \circ \gamma'\big(\lambda z_\mu + (1-\lambda)\varphi\big)^{-1} \circ R\big(\lambda z_\mu + (1-\lambda)\varphi\big) \circ \\
&\qquad \circ \Big\{\sum_{k=0}^{\infty}\Big(\gamma'\big(\lambda z_\mu + (1-\lambda)\varphi\big)^{-1} \circ R\big(\lambda z_\mu + (1-\lambda)\varphi\big)\Big)^k\Big\}\big[e^t(z_\mu - \varphi)\big]\,d\lambda.
\end{aligned} \tag{4.53}
$$

But equation (4.2) implies that the following equation holds true and that the limit converges uniformly for all $\lambda \in [0,1]$:

$$\lim_{t\to+\infty} \lambda z_\mu + (1-\lambda)\varphi(t,z) = \lambda z_\mu + (1-\lambda) z_\mu = z_\mu.$$

Hence, the following identities hold true and the limits converge uniformly for all $\lambda \in [0,1]$:

$$\lim_{t\to+\infty}\Big(\imath - M\big(\varphi(t,z)\big)^{-1} \circ M\big(\lambda z_\mu + (1-\lambda)\varphi(t,z)\big)\Big) = 0, \tag{4.54}$$

$$
\begin{aligned}
&\lim_{t\to+\infty} M\big(\varphi(t,z)\big)^{-1} \circ M\big(\lambda z_\mu + (1-\lambda)\varphi(t,z)\big) \circ \\
&\qquad \circ \gamma'\big(\lambda z_\mu + (1-\lambda)\varphi(t,z)\big)^{-1} = M\big(z_\mu\big)^{-1},
\end{aligned} \tag{4.55}
$$

$$\lim_{t\to+\infty} R\big(\lambda z_\mu + (1-\lambda)\varphi(t,z)\big) = 0, \tag{4.56}$$

$$
\begin{aligned}
&\lim_{t\to+\infty}\sum_{k=0}^{\infty}\Big(\gamma'(\lambda z_\mu + (1-\lambda)\varphi(t,z))^{-1} \circ \\
&\qquad \circ R\big(\lambda z_\mu + (1-\lambda)\varphi(t,z)\big)\Big)^k = \imath.
\end{aligned} \tag{4.57}
$$

Note in particular that for $t \gg 1$ it follows from (4.51) and property 2 above that the k^{th} summand of the series in (4.57) is bounded by $\big(1/2\big)^k$, and hence we can take the limit inside the sum. Since the convergence of expressions (4.54)–(4.57) is uniform in λ, we get that the following limits exist, where the integrals are taken in the normed vector space $\mathrm{End}(Z)$:

$$\lim_{t\to+\infty}\int_0^1 \Big(\imath - M\big(\varphi(t,z)\big)^{-1} \circ M\big(\lambda z_\mu + (1-\lambda)\varphi(t,z)\big)\Big)d\lambda = 0 \tag{4.58}$$

and

$$
\begin{aligned}
&\int_0^1 M(\varphi)^{-1} \circ M\big(\lambda z_\mu + (1-\lambda)\varphi\big) \circ \gamma'\big(\lambda z_\mu + (1-\lambda)\varphi\big)^{-1} \circ \\
&\quad \circ R\big(\lambda z_\mu + (1-\lambda)\varphi\big) \circ \\
&\quad \circ \Big\{\sum_{k=0}^{\infty}\Big(\gamma'\big(\lambda z_\mu + (1-\lambda)\varphi\big)^{-1} \circ R\big(\lambda z_\mu + (1-\lambda)\varphi\big)\Big)^k\Big\}d\lambda = 0
\end{aligned} \tag{4.59}
$$

Now, let $\sigma > 1$ and $\delta > 0$ be defined as in Lemma 4.3, and let $t_0 > 0$ be a positive real number such that $\left\|z_\mu - \varphi(t_0, z)\right\| < \delta$. Then for all $t \geq t_0$ we have

$$\begin{aligned}\left\|e^t \cdot (z_\mu - \varphi)\right\| &\overset{(2.6)}{=} e^t \cdot \left\|z_\mu - \varphi\big(t - t_0, \varphi(t_0, z)\big)\right\| \\ &\overset{4.3}{\leq} e^t \cdot e^{t_0 - t} \cdot \sigma \cdot \left\|z_\mu - \varphi(t_0, z)\right\|,\end{aligned}$$

which shows that $\left\|e^t(z_\mu - \varphi)\right\|$ remains bounded for $t \to +\infty$. Using this fact in conjunction with equations (4.58) and (4.59), the result now follows from equation (4.53). $\square$

Lemma 4.12. *There exists a radius $\tilde{\delta} > 0$ such that for all $z \in B_{\tilde{\delta}}\big(z_\mu\big)$ we have*

$$\left\|z_\mu - \varphi(t, z)\right\| < \|z_\mu - z\| \qquad \forall t > 0.$$

Proof. Clearly, it suffices to prove that $\big(z_\mu - z; d(z)\big) > 0$ in a neighborhood of z_μ. We claim that

$$\lim_{z \to z_\mu} \|z_\mu - z\|^{-1}\big(d(z) - (z_\mu - z)\big) = 0. \tag{4.60}$$

In fact, it follows from equation (4.15) that

$$\begin{aligned}&\|z_\mu - z\|^{-1}\big(d(z) - (z_\mu - z)\big) \\ &\quad = \Big\{M^{-1}(z) \circ \int_0^1 \gamma'\big(\lambda z_\mu + (1-\lambda)z\big)\,d\lambda - \imath\Big\}\big[\mathfrak{n}\{z_\mu - z\}\big].\end{aligned} \tag{4.61}$$

But $M^{-1} \in \mathcal{C}^1$, $\gamma' \in \mathcal{C}^1$, $\gamma'(z_\mu) = M(z_\mu)$ (see Proposition 1.1) and

$$\lambda z_\mu + (1-\lambda)z \overset{z \to z_\mu}{\longrightarrow} z_\mu$$

uniformly for $\lambda \in [0, 1]$, and hence

$$\lim_{z \to z_\mu} M^{-1}(z) \circ \int_0^1 \gamma'\big(\lambda z_\mu + (1-\lambda)z\big)\,d\lambda - \imath = 0. \tag{4.62}$$

Since $\mathfrak{n}\{z_\mu - z\}$ remains bounded, equations (4.61) and (4.62) imply (4.60) at once. But now we have

$$\begin{aligned}\big(z_\mu - z; d(z)\big) &= \big(z_\mu - z; z_\mu - z\big) + \Big(z_\mu - z; d(z) - (z_\mu - z)\Big) \\ &\overset{C.S.}{\geq} \|z_\mu - z\|^2 \cdot \Big(1 - \|z_\mu - z\|^{-1} \cdot \left\|d(z) - (z_\mu - z)\right\|\Big).\end{aligned}$$

Together with equation (4.60) this implies that $\big(z_\mu - z; d(z)\big) > 0$ must hold in a neighborhood of z_μ. $\square$

Lemma 4.13. *For all $z \in D$ the following limit exists:*

$$\lim_{t \to +\infty} e^t \cdot \frac{\partial}{\partial t}\Big(e^t \cdot d\big(\varphi(t,z)\big)\Big).$$

Proof. First note that

$$\begin{aligned}
\frac{\partial}{\partial t} d\big(\varphi(t,z)\big) &\overset{4.1}{=} \frac{\partial}{\partial t}\Big(\big\|\mu e_V - \gamma(z)\big\| \cdot e^{-t} \cdot M(\varphi)^{-1}\big[\mathfrak{n}\big\{\mu e_V - \gamma(\varphi)\big\}\big]\Big) \\
&\overset{(4.43)}{=} -\big\|\mu e_V - \gamma(z)\big\| \cdot e^{-t} \cdot M(\varphi)^{-1}\big[\mathfrak{n}\big\{\mu e_V - \gamma(\varphi)\big\}\big] + \\
&\quad + \big\|\mu e_V - \gamma(z)\big\| \cdot e^{-t} \cdot \Big(\big(M^{-1}\big)'(\varphi)\big[d(\varphi)\big]\Big)\big[\mathfrak{n}\big\{\mu e_V - \gamma(\varphi)\big\}\big] \\
&\quad + \big\|\mu e_V - \gamma(z)\big\| \cdot e^{-t} \cdot M(\varphi)^{-1}\big[-\|\mu e_V - \gamma(\varphi)\|^{-1} \cdot R(\varphi)[d(\varphi)]\big] \\
&= -d(\varphi) \\
&\quad + \big\|\mu e_V - \gamma(z)\big\| \cdot e^{-2t} \cdot \Big(\big(M^{-1}\big)'(\varphi)\big[e^t \cdot d(\varphi)\big]\Big)\big[\mathfrak{n}\big\{\mu e_V - \gamma(\varphi)\big\}\big] \\
&\quad + \big\|\mu e_V - \gamma(z)\big\| \cdot e^{-2t} \cdot M(\varphi)^{-1}\big[-\|\mu e_V - \gamma(\varphi)\|^{-1} \cdot R(\varphi)[e^t \cdot d(\varphi)]\big].
\end{aligned} \tag{4.63}$$

Now observe that since $\gamma' \in \mathcal{C}^1$ and $M \in \mathcal{C}^1$, we have

$$\begin{aligned}
&\big\|\mu e_V - \gamma\big(\varphi(t,z)\big)\big\|^{-1} \cdot R(\varphi) \overset{4.1,1.1}{=} \\
&\quad = \big\|\mu e_V - \gamma(z)\big\|^{-1} \cdot e^t \cdot \Big(\gamma'(\varphi) - \gamma'\big(z_\mu\big) + M\big(z_\mu\big) - M(\varphi)\Big) \\
&\quad = \big\|\mu e_V - \gamma(z)\big\|^{-1} \cdot \Big(\int_0^1 \big\{\gamma'' - M'\big\}\big(\lambda z_\mu + (1-\lambda)\varphi\big)\, d\lambda\Big)\big[e^t \cdot (\varphi - z_\mu)\big]
\end{aligned} \tag{4.64}$$

But it follows from (4.2) that

$$\lim_{t \to \infty} \lambda z_\mu + (1-\lambda)\varphi = z_\mu$$

converges uniformly in $\lambda \in [0,1]$, and hence

$$\lim_{t \to \infty} \int_0^1 \big\{\gamma'' - M'\big\}\big(\lambda z_\mu + (1-\lambda)\varphi\big)\, d\lambda = \gamma''(z_\mu) - M'(z_\mu) \qquad \text{exists.} \tag{4.65}$$

Moreover, it follows from Lemmas 4.10 and 4.11 that

$$\lim_{t \to +\infty} e^t \cdot \big(\varphi - z_\mu\big) \qquad \text{exists.} \tag{4.66}$$

Using (4.65) and (4.66) in (4.64) we get that

$$\lim_{t \to +\infty} \big\|\mu e_V - \gamma\big(\varphi(t,z)\big)\big\|^{-1} \cdot R(\varphi) \qquad \text{exists.} \tag{4.67}$$

Moreover, clearly it follows from equation (4.2) that

$$\lim_{t \to +\infty} \big(M^{-1}\big)'(\varphi) = \big(M^{-1}\big)'(z_\mu) \tag{4.68}$$

and

$$\lim_{t\to+\infty} M(\varphi)^{-1} = M(z_\mu)^{-1}. \tag{4.69}$$

Finally, it follows from Lemma 4.10 and equations (4.2), (4.47) and (4.67) that all of the following limits exist:

$$\begin{aligned}
&\lim_{t\to+\infty} e^t \cdot \frac{\partial}{\partial t}\Big(e^t \cdot d\big(\varphi(t,z)\big)\Big)\\
&\quad= \lim_{t\to+\infty} e^t \cdot \Big(e^t \cdot d(\varphi) + e^t \cdot \frac{\partial}{\partial t} d\big(\varphi(t,z)\big)\Big)\\
&\overset{(4.63)}{=} \big\|\mu e_V - \gamma(z)\big\| \cdot\\
&\qquad\qquad \cdot \Big\{\lim_{t\to+\infty}\big(M^{-1}\big)'(\varphi)\Big[\lim_{t\to+\infty} e^t d(\varphi)\Big]\Big\}\Big[\lim_{t\to+\infty} \mathfrak{n}\big\{\mu e_V - \gamma(\varphi)\big\}\Big]\\
&\quad + \big\|\mu e_V - \gamma(z)\big\| \cdot\\
&\qquad\qquad \cdot \Big(\lim_{t\to+\infty} M(\varphi)^{-1}\Big)\Big[-\lim_{t\to+\infty}\Big(\big\|\mu e_V - \gamma(\varphi)\big\|^{-1} \cdot R(\varphi)\Big)\big[\lim_{t\to+\infty} e^t \cdot d(\varphi)\big]\Big]\\
&= \big\|\mu e_V - \gamma(z)\big\| \cdot \Big\{\big(M^{-1}\big)'(z_\mu)\Big[\lim_{t\to+\infty} e^t d(\varphi)\Big]\Big\}\Big[\lim_{t\to+\infty} \mathfrak{n}\big\{\mu e_V - \gamma(\varphi)\big\}\Big]\\
&\quad + \big\|\mu e_V - \gamma(z)\big\| \cdot M(z_\mu)^{-1}\Big[-\lim_{t\to+\infty}\Big(\big\|\mu e_V - \gamma(\varphi)\big\|^{-1} \cdot R(\varphi)\Big)\big[\lim_{t\to+\infty} e^t \cdot d(\varphi)\big]\Big].
\end{aligned}$$

□

Lemma 4.14. *The following identities hold true:*

$$\begin{aligned}
&i) \quad \frac{\partial}{\partial z} d\big(\varphi(t,z)\big) = e^{-t} \cdot \big(-\imath + O\big(\|z_\mu - z\|\big)\big),\\
&ii) \quad \frac{\partial}{\partial z}\varphi(t,z) = e^{-t} \cdot \big(\imath + O\big(\|z_\mu - z\|\big)\big).
\end{aligned}$$

Proof. *i)* For all $w \in Z$ we have

$$\begin{aligned}
\frac{\partial}{\partial z} d\big(\varphi(t,z)\big)[w] &= d'\big(\varphi(t,z)\big)\Big[\frac{\partial}{\partial z}\varphi(t,z)[w]\Big]\\
&\overset{4.6}{=} -\imath\Big[\frac{\partial}{\partial z}\varphi(t,z)[w]\Big] + O\big(\|z_\mu - \varphi(t,z)\|\big)\Big[\frac{\partial}{\partial z}\varphi(t,z)[w]\Big]\\
&\overset{4.3}{=} -\frac{\partial}{\partial z}\varphi(t,z)[w] + e^{-t} \cdot O\big(\|z_\mu - z\|\big)\Big[\frac{\partial}{\partial z}\varphi(t,z)[w]\Big].
\end{aligned} \tag{4.70}$$

Hence, part *i)* of the lemma will follow at once from part *ii)*.

ii) For all $w \in Z$ we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big\|\frac{\partial}{\partial z}\varphi(t,z)[w] - e^{-t} \cdot w\Big\|\\
&\quad= \Big\|\frac{\partial}{\partial z}\varphi[w] - e^{-t} \cdot w\Big\|^{-1} \cdot \Big(\frac{\partial}{\partial z}\varphi[w] - e^{-t} \cdot w; \frac{\partial^2}{\partial t\partial z}\varphi(t,z)[w] + e^{-t} \cdot w\Big).
\end{aligned} \tag{4.71}$$

But since $\varphi \in \mathcal{C}^2$ we have

$$\begin{aligned}\frac{\partial^2}{\partial t \partial z}\varphi(t,z)[w] &= \frac{\partial}{\partial z}\Big(\frac{\partial}{\partial t}\varphi(t,z)\Big)[w] \\ &= \frac{\partial}{\partial z} d\big(\varphi(t,z)\big)[w] \\ &\overset{(4.70)}{=} -\frac{\partial}{\partial z}\varphi(t,z)[w] + e^{-t}\cdot O\big(\|z_\mu - z\|\big)\Big[\frac{\partial}{\partial z}\varphi(t,z)[w]\Big].\end{aligned}$$

Substituting this result into (4.71) yields the following upper and lower bounds:

$$\begin{aligned}(4.72)\qquad & -e^{-t}\cdot\Big|O\big(\|z_\mu - z\|\big)\Big|\cdot\Big\|\frac{\partial}{\partial z}\varphi[w]\Big\| \\ &\le \frac{\partial}{\partial t}\Big\|\frac{\partial}{\partial z}\varphi[w]-e^{-t}\cdot w\Big\| + \Big\|\frac{\partial}{\partial z}\varphi[w] - e^{-t}\cdot w\Big\| \\ &\le +e^{-t}\cdot\Big|O\big(\|z_\mu - z\|\big)\Big|\cdot\Big\|\frac{\partial}{\partial z}\varphi[w]\Big\|.\end{aligned}$$

Let $c > 0$ be a real number chosen such that for $\|z_\mu - z\|$ sufficiently small and $t > 0$ the inequality

$$(4.73)\qquad \Big|O\big(\|z_\mu - z\|\big)\Big| \le c\cdot\|z_\mu - z\|$$

holds for the functions O from equation (4.72) and

$$(4.74)\qquad \Big|O\big(\|z_\mu - \varphi(t,z)\|\big)\Big| \le c\cdot e^{-t}\cdot\|z_\mu - z\|$$

for the function O from equation (4.25). It follows from Lemma 4.3 that such a constant c exists indeed. Equations (4.25) and (4.74) now imply that for all $t > 0$ and for $\|z_\mu - z\|$ sufficiently small the following inequalities hold:

$$\begin{aligned}&-\Big\|\frac{\partial}{\partial z}\varphi(t,z)[w]\Big\|\cdot\Big(1+c\cdot e^{-t}\cdot\|z_\mu - z\|\Big) \\ &\le \frac{\partial}{\partial t}\Big\|\frac{\partial}{\partial z}\varphi(t,z)[w]\Big\| \\ &\le -\Big\|\frac{\partial}{\partial z}\varphi(t,z)[w]\Big\|\cdot\Big(1 - c\cdot e^{-t}\cdot\|z_\mu - z\|\Big).\end{aligned}$$

Moreover, we have $\big\|\frac{\partial}{\partial z}\varphi(0,z)[w]\big\| = \|w\|$, and it follows that an upper bound on $\|\frac{\partial}{\partial z}\varphi(t,z)[w]\|$ can be derived as the solution to the initial value problem

$$\begin{aligned}\dot{y}(t) &= -y(t)\cdot\Big(1 - c\cdot e^{-t}\cdot\|z_\mu - z\|\Big), \\ y(0) &= \|w\|.\end{aligned}$$

Integrating this equation we get

$$y(t) = \|w\|\cdot\exp\big\{c\cdot\|z_\mu - z\|\cdot\big(1-e^{-t}\big) - t\big\}.$$

Substituting this bound into equation (4.72) and using equation (4.73) we get that for all $t > 0$ and for $\|z_\mu - z\|$ sufficiently small

$$
\begin{aligned}
-c \cdot \|z_\mu - z\| \cdot \exp\Big\{-2t + c \cdot \|z_\mu - z\| \cdot \big(1 - e^{-t}\big)\Big\} & \\
\leq \frac{\partial}{\partial t}\Big\|\frac{\partial}{\partial z}\varphi(t, z) - e^{-t} \cdot \imath\Big\| + \Big\|\frac{\partial}{\partial z}\varphi(t, z) - e^{-t} \cdot \imath\Big\| & \\
\leq +c \cdot \|z_\mu - z\| \cdot \exp\Big\{-2t + c \cdot \|z_\mu - z\| \cdot \big(1 - e^{-t}\big)\Big\}. &
\end{aligned}
$$

Moreover,

$$\Big\|\frac{\partial}{\partial z}\varphi(0, z) - e^{-0} \cdot \imath\Big\| = \Big\|\frac{\partial}{\partial z}z - \imath\Big\| = 0.$$

It follows that for $t > 0$ and for $\|z_\mu - z\|$ sufficiently small an upper bound on $\big\|\frac{\partial}{\partial z}\varphi(t, z) - e^{-t} \cdot \imath\big\|$ is given by the solution to the initial value problem

$$
\begin{aligned}
\dot{y}(t) + y(t) &= c \cdot \big\|z_\mu - z\big\| \cdot \exp\big\{-2t + c \cdot \|z_\mu - z\| \cdot \big(1 - e^{-t}\big)\big\}, \\
y(0) &= 0.
\end{aligned}
$$

Integration of this equation yields

$$y(t) = e^{-t} \cdot \Big(\exp\Big\{c \cdot \|z_\mu - z\| \cdot \big(1 - e^{-t}\big)\Big\} - 1\Big) = e^{-t} \cdot O\big(\|z_\mu - z\|\big).$$

Part *ii)* of the lemma now follows readily. □

A Proof of Lemma 3.13

Lemma A.1. *Let $Q, \varrho > 0$ be real numbers that satisfy conditions $1-12$ stated in Section 3 prior to Lemma 3.13, and let $k \in \mathbb{N} \cup \{0\}$. Then, for $z \in B_\varrho(z_\mu) \setminus \{z_\mu\}$ and for $\tau \in \mathcal{T}_{3k+1}$ it is true that*

$$\begin{aligned}
&i) && \int_0^\infty \big\| L_\tau\big(\varphi(t,z)\big)\big\| dt &\leq\ & \|z_\mu - z\|^{k+1} Q^{2k+2} 2^k, \\
&ii) && \int_0^\infty \Big\| \frac{\partial}{\partial z} L_\tau\big(\varphi(t,z)\big)\Big\| dt &\leq\ & \|z_\mu - z\|^{k} Q^{2k+1} 2^{2k} \\
&&&&& + \|z_\mu - z\|^{k+1} Q^{2k+2} 3^{2k+1}, \\
&iii) && \int_0^\infty \Big\| \frac{\partial^2}{\partial z^2} L_\tau\big(\varphi(t,z)\big)\Big\| dt &\leq\ & \|z_\mu - z\|^{k-1} Q^{2k} \chi(k) \\
&&&&& + \|z_\mu - z\|^{k} Q^{2k+1} \big(3^{2k+2} + Q 3^{2k}\big) \\
&&&&& + \|z_\mu - z\|^{k+1} Q^{2k+2} \big(3^{2k+3} + Q 3^{2k+1}\big),
\end{aligned}$$

where $$\chi(k) := \begin{cases} 0 & \text{if } k = 0, \\ 2^{2(k+1)} & \text{if } k \geq 1. \end{cases}$$

Proof. Induction over k. Un-bracketed references written over inequality signs refer to the twelve conditions on Q and ϱ listed in Section 3 prior to Lemma 3.13.

Let $k = 0$. Then $\tau_0 = \bullet$. Therefore, we have

$$\begin{aligned}
i)\ \int_0^\infty \big\| L_{\tau_0}\big(\varphi(t,z)\big)\big\| dt &= \int_0^\infty \big\| \tilde{\Omega}'(\varphi)\big[\pi d(\varphi)\big] \circ \tilde{\Omega}^*(\varphi)\big\| dt \\
&\overset{1,8}{\leq} \int_0^\infty Q \big\| d\big(\varphi(t,z)\big)\big\| dt \\
&\overset{7}{\leq} Q \int_0^\infty \|z_\mu - z\| \cdot Q \cdot e^{-\xi} d\xi \\
&= 2^0 Q^{0+2} \cdot \|z_\mu - z\|^{0+1}.
\end{aligned}$$

Next,

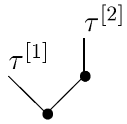
$$
\begin{aligned}
ii)\ \int_0^\infty \Big\| \frac{\partial}{\partial z} L_{\tau_0}\big(\varphi(t,z)\big)\Big\| dt
&\leq \int_0^\infty \Big\| \tilde{\Omega}''\Big[\pi d, \frac{\partial}{\partial z}\varphi\Big] \circ \tilde{\Omega}^* \Big\| dt + \int_0^\infty \Big\| \tilde{\Omega}'\Big[\pi \frac{\partial}{\partial z} d\Big] \circ \tilde{\Omega}^* \Big\| dt \\
&\qquad + \int_0^\infty \Big\| \tilde{\Omega}'[\pi d] \circ \tilde{\Omega}^{*\prime}\Big[\frac{\partial}{\partial z}\varphi\Big] \Big\| dt \\
&\overset{1,2,3,8}{\leq} Q \int_0^\infty \big\| d\big(\varphi(t,z)\big)\big\| \cdot \Big\| \frac{\partial}{\partial z}\varphi(t,z)\Big\| dt + Q \int_0^\infty \Big\| \frac{\partial}{\partial z} d\big(\varphi(t,z)\big)\Big\| dt \\
&\qquad + Q \int_0^\infty \big\| d\big(\varphi(t,z)\big)\big\| \cdot \Big\| \frac{\partial}{\partial z}\varphi(t,z)\Big\| dt \\
&\overset{7,9,11}{\leq} 2Q^2 \cdot \|z_\mu - z\| + Q\big(1 + Q\|z_\mu - z\|\big) \\
&= \|z_\mu - z\|^0 \cdot Q^{0+1} 2^0 + \|z_\mu - z\|^1 \cdot Q^{0+2} 3^{0+1}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
iii)\ &\int_0^\infty \Big\| \frac{\partial^2}{\partial z^2} L_{\tau_0}\big(\varphi(t,z)\big) \Big\| dt \\
&\le \int_0^\infty \Big\| \tilde{\Omega}''' \Big[\pi d, \frac{\partial}{\partial z}\varphi, \frac{\partial}{\partial z}\varphi\Big] \circ \tilde{\Omega}^* \Big\| dt + \int_0^\infty \Big\| \tilde{\Omega}'' \Big[\pi \frac{\partial}{\partial z} d, \frac{\partial}{\partial z}\varphi\Big] \circ \tilde{\Omega}^* \Big\| dt \\
&\quad + \int_0^\infty \Big\| \tilde{\Omega}'' \Big[\pi d, \frac{\partial^2}{\partial z^2}\varphi\Big] \circ \tilde{\Omega}^* \Big\| dt + \int_0^\infty \Big\| \tilde{\Omega}'' \Big[\pi d, \frac{\partial}{\partial z}\varphi\Big] \circ \tilde{\Omega}^{*\prime} \Big[\frac{\partial}{\partial z}\varphi\Big] \Big\| dt \\
&\quad + \int_0^\infty \Big\| \tilde{\Omega}'' \Big[\pi \frac{\partial}{\partial z} d, \frac{\partial}{\partial z}\varphi\Big] \circ \tilde{\Omega}^* \Big\| dt + \int_0^\infty \Big\| \tilde{\Omega}' \Big[\pi \frac{\partial^2}{\partial z^2} d\Big] \circ \tilde{\Omega}^* \Big\| dt \\
&\quad + \int_0^\infty \Big\| \tilde{\Omega}' \Big[\pi \frac{\partial}{\partial z} d\Big] \circ \tilde{\Omega}^{*\prime} \Big[\frac{\partial}{\partial z}\varphi\Big] \Big\| dt + \int_0^\infty \Big\| \tilde{\Omega}'' \Big[\pi d, \frac{\partial}{\partial z}\varphi\Big] \circ \tilde{\Omega}^{*\prime} \big[\frac{\partial}{\partial z}\varphi\big] \Big\| dt \\
&\quad + \int_0^\infty \Big\| \tilde{\Omega}' \Big[\pi \frac{\partial}{\partial z} d\Big] \circ \tilde{\Omega}^{*\prime} \Big[\frac{\partial}{\partial z}\varphi\Big] \Big\| dt + \int_0^\infty \Big\| \tilde{\Omega}'[\pi d] \circ \tilde{\Omega}^{*\prime\prime} \Big[\frac{\partial}{\partial z}\varphi, \frac{\partial}{\partial z}\varphi\Big] \Big\| dt \\
&\quad + \int_0^\infty \Big\| \tilde{\Omega}'[\pi d] \circ \tilde{\Omega}^{*\prime} \Big[\frac{\partial^2}{\partial z^2}\varphi\Big] \Big\| dt \\
&\overset{1-6,8}{\le} Q \int_0^\infty \|d\| \cdot \Big\| \frac{\partial}{\partial z}\varphi \Big\|^2 dt + Q \int_0^\infty \Big\| \frac{\partial}{\partial z} d \Big\| \cdot \Big\| \frac{\partial}{\partial z}\varphi \Big\| dt \\
&\quad + Q \int_0^\infty \|d\| \cdot \Big\| \frac{\partial^2}{\partial z^2}\varphi \Big\| dt + Q \int_0^\infty \|d\| \cdot \Big\| \frac{\partial}{\partial z}\varphi \Big\|^2 dt \\
&\quad + Q \int_0^\infty \Big\| \frac{\partial}{\partial z} d \Big\| \cdot \Big\| \frac{\partial}{\partial z}\varphi \Big\| dt + Q \int_0^\infty \Big\| \frac{\partial^2}{\partial z^2} d \Big\| dt \\
&\quad + Q \int_0^\infty \Big\| \frac{\partial}{\partial z} d \Big\| \cdot \Big\| \frac{\partial}{\partial z}\varphi \Big\| dt + Q \int_0^\infty \|d\| \cdot \Big\| \frac{\partial}{\partial z}\varphi \Big\|^2 dt \\
&\quad + Q \int_0^\infty \Big\| \frac{\partial}{\partial z} d \Big\| \cdot \Big\| \frac{\partial}{\partial z}\varphi \Big\| dt + Q \int_0^\infty \|d\| \cdot \Big\| \frac{\partial}{\partial z}\varphi \Big\|^2 dt \\
&\quad + Q \int_0^\infty \|d\| \cdot \Big\| \frac{\partial^2}{\partial z^2}\varphi \Big\| dt \\
&\overset{9,10}{\le} (4+2Q)Q \int_0^\infty \|d\| dt + 4Q \int_0^\infty \Big\| \frac{\partial}{\partial z} d \Big\| dt + Q \int_0^\infty \Big\| \frac{\partial^2}{\partial z^2} d \Big\| dt \\
&\overset{7,11,12}{\le} (4+2Q)Q^2 \|z_\mu - z\| + 4Q\big(1 + Q\|z_\mu - z\|\big) + Q^2\big(1 + Q\|z_\mu - z\|\big) \\
&\le \|z_\mu - z\|^{-1} Q^0 0 + \|z_\mu - z\|^0 Q^1 \big(3^2 + Q3^0\big) + \|z_\mu - z\|^1 Q^2 \big(3^3 + Q3^1\big).
\end{aligned}
$$

This completes the base case.

Now, to prove the inductive step, let $k \geq 1$ and suppose the lemma holds true for all $\tau \in \mathcal{T}_{3i+1}$, $(i = 0, \ldots, k-1)$. Let $\tilde{\tau} \in \mathcal{T}_{3k+1}$. Because of the recursive nature of the definition of $\mathcal{T}$ there exist an integer $l < k$ and two oriented rooted trees $\tau^{[1]} \in \mathcal{T}_{3l+1}$ and $\tau^{[2]} \in \mathcal{T}_{3(k-l-1)+1}$ such that

$\tilde{\tau} =$ 

Therefore, we have

$$
\begin{aligned}
i)\ \int_0^\infty \left\|L_{\tilde{\tau}}\right\| dt &= \int_0^\infty \left\| \left[L_{\tau^{[1]}}, \int_t^\infty L_{\tau^{[2]}}\big(\varphi(\xi,z)\big)\,d\xi\right] \right\| dt \\
&\leq 2\int_0^\infty \left\|L_{\tau^{[1]}}\right\| \cdot \int_t^\infty \left\|L_{\tau^{[2]}}\big(\varphi(\xi,z)\big)\right\| d\xi\, dt \\
&\overset{(2.3)}{=} 2\int_0^\infty \left\|L_{\tau^{[1]}}\right\| \cdot \int_0^\infty \left\|L_{\tau^{[2]}}\Big(\varphi\big(\xi,\varphi(t,z)\big)\Big)\right\| d\xi\, dt \\
&\overset{i)}{\leq} \int_0^\infty \left\|L_{\tau^{[1]}}\right\| \cdot 2^{k-l} Q^{2k-2l} \cdot \left\|z_\mu - \varphi(t,z)\right\|^{k-l} dt \\
&\overset{8}{\leq} 2^{k-l} Q^{2k-2l} \cdot \|z_\mu - z\|^{k-l} \cdot \int_0^\infty \left\|L_{\tau^{[1]}}\right\| dt \\
&\overset{i)}{\leq} 2^{k-l} Q^{2k-2l} \cdot \|z_\mu - z\|^{k-l} 2^l Q^{2l+2} \cdot \|z_\mu - z\|^{l+1} \\
&= 2^k Q^{2k+2} \cdot \|z_\mu - z\|^{k+1}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
ii)\ \int_0^\infty \Big\| \frac{\partial}{\partial z} L_{\tilde\tau} \Big\| dt
&\le \int_0^\infty \Big\| \Big[\frac{\partial}{\partial z} L_{\tau[1]}, \int_t^\infty L_{\tau[2]}\big(\varphi(\xi, z)\big) d\xi \Big] \Big\| dt \\
&\quad + \int_0^\infty \Big\| \Big[L_{\tau[1]}, \int_t^\infty \frac{\partial}{\partial z} L_{\tau[2]}\big(\varphi(\xi, z)\big) d\xi \Big] \Big\| dt \\
&\le 2 \int_0^\infty \Big\| \frac{\partial}{\partial z} L_{\tau[1]} \Big\| \cdot \int_t^\infty \| L_{\tau[2]} \| d\xi\, dt \\
&\quad + 2 \int_0^\infty \| L_{\tau[1]} \| \cdot \int_t^\infty \Big\| \frac{\partial}{\partial z} L_{\tau[2]} \Big\| d\xi\, dt \\
&\overset{(2.3)}{=} 2 \int_0^\infty \Big\| \frac{\partial}{\partial z} L_{\tau[1]} \Big\| \cdot \int_0^\infty \big\| L_{\tau[2]}\big(\varphi\big(\xi, \varphi(t, z)\big)\big) \big\| d\xi\, dt \\
&\quad + 2 \int_0^\infty \| L_{\tau[1]} \| \cdot \int_t^\infty \Big\| \frac{\partial}{\partial \tilde z} L_{\tau[2]}\big(\varphi(\xi, \tilde z)\big) \Big|_{\tilde z = \varphi(t,z)} \Big[\frac{\partial}{\partial z} \varphi(t, z) \Big] \Big\| d\xi\, dt \\
&\overset{i),ii)}{\le} 2^{k-l} Q^{2k-2l} \int_0^\infty \Big\| \frac{\partial}{\partial z} L_{\tau[1]} \Big\| \cdot \big\| z_\mu - \varphi(t, z) \big\|^{k-l} dt \\
&\quad + 2 \cdot 2^{2k-2l-2} Q^{2k-2l-1} \cdot \int_0^\infty \| L_{\tau[1]} \| \cdot \big\| z_\mu - \varphi(t, z) \big\|^{k-l-1} \cdot \Big\| \frac{\partial}{\partial z} \varphi(t, z) \Big\| dt \\
&\quad + 2 \cdot 3^{2k-2l-1} Q^{2k-2l} \cdot \int_0^\infty \| L_{\tau[1]} \| \cdot \big\| z_\mu - \varphi(t, z) \big\|^{k-l} \cdot \Big\| \frac{\partial}{\partial z} \varphi(t, z) \Big\| dt \\
&\overset{i),ii),8,9}{\le} 2^{k-l} Q^{2k-2l} \| z_\mu - z \|^{k-l} \cdot \Big\{ 2^{2l} Q^{2l+1} \cdot \| z_\mu - z \|^{l} + 3^{2l+1} Q^{2l+2} \| z_\mu - z \|^{l+1} \Big\} \\
&\quad + 2 \cdot \Big\{ 2^{2k-2l-2} Q^{2k-2l-1} \cdot \| z_\mu - z \|^{k-l-1} + 3^{2k-2l-1} Q^{2k-2l} \cdot \| z_\mu - z \|^{k-l} \Big\} \cdot \\
&\qquad\qquad \cdot 2^{l} Q^{2l+2} \cdot \| z_\mu - z \|^{l+1} \\
&= \| z_\mu - z \|^{k} \cdot Q^{2k+1} \big(2^{k-l} 2^{2l} + 2^{l+1} 2^{2k-2l-2} \big) \\
&\quad + \| z_\mu - z \|^{k+1} Q^{2k+2} \big(2^{k-l} 3^{2l+1} + 2^{l+1} 3^{2k-2l-1} \big) \\
&\le \| z_\mu - z \|^{k} Q^{2k+1} 2^{2k} + \| z_\mu - z \|^{k+1} Q^{2k+2} 3^{2k+1}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
iii)\ &\int_0^\infty \Big\| \frac{\partial^2}{\partial z^2} L_{\tilde{\tau}} \Big\| dt \\
&= \int_0^\infty \Big\| \frac{\partial^2}{\partial z^2} \Big[L_{\tau[1]}, \int_t^\infty L_{\tau[2]}\big(\varphi(\xi,z)\big) d\xi \Big] \Big\| dt \\
&\leq 2\int_0^\infty \Big\| \frac{\partial^2}{\partial z^2} L_{\tau[1]} \Big\| \cdot \int_t^\infty \big\| L_{\tau[2]}\big(\varphi(\xi,z)\big) \big\| d\xi\, dt \\
&\quad + 4\int_0^\infty \Big\| \frac{\partial}{\partial z} L_{\tau[1]} \Big\| \cdot \int_t^\infty \Big\| \frac{\partial}{\partial z} L_{\tau[2]}\big(\varphi(\xi,z)\big) \Big\| d\xi\, dt \\
&\quad + 2\int_0^\infty \big\| L_{\tau[1]} \big\| \cdot \int_t^\infty \Big\| \frac{\partial^2}{\partial z^2} L_{\tau[2]}\big(\varphi(\xi,z)\big) \Big\| d\xi\, dt \\
&\overset{(2.3)}{\leq} 2\int_0^\infty \Big\| \frac{\partial^2}{\partial z^2} L_{\tau[1]} \Big\| \cdot \int_0^\infty \big\| L_{\tau[2]}\big(\varphi\big(\xi,\varphi(t,z)\big)\big) \big\| d\xi\, dt \\
&\quad + 4\int_0^\infty \Big\| \frac{\partial}{\partial z} L_{\tau[1]} \Big\| \cdot \int_0^\infty \Big\| \frac{\partial}{\partial \tilde{z}} L_{\tau[2]}\big(\varphi(\xi,\tilde{z})\big)\Big|_{\tilde{z}=\varphi(t,z)} \Big\| \cdot \Big\| \frac{\partial}{\partial z}\varphi(t,z) \Big\| d\xi\, dt \\
&\quad + 2\int_0^\infty \big\| L_{\tau[1]} \big\| \cdot \int_0^\infty \Big\| \frac{\partial^2}{\partial \tilde{z}^2} L_{\tau[2]}\big(\varphi(\xi,\tilde{z})\big)\Big|_{\tilde{z}=\varphi(t,z)} \Big\| \cdot \Big\| \frac{\partial}{\partial z}\varphi(t,z) \Big\|^2 d\xi\, dt \\
&\quad + 2\int_0^\infty \big\| L_{\tau[1]} \big\| \cdot \int_0^\infty \Big\| \frac{\partial}{\partial \tilde{z}} L_{\tau[2]}\big(\varphi(\xi,\tilde{z})\big)\Big|_{\tilde{z}=\varphi(t,z)} \Big\| \cdot \Big\| \frac{\partial^2}{\partial z^2}\varphi(t,z) \Big\| d\xi\, dt \\
&\overset{i),ii),iii)}{\leq} 2\int_0^\infty \Big\| \frac{\partial^2}{\partial z^2} L_{\tau[1]} \Big\| \cdot 2^{k-l-1} Q^{2k-2l} \cdot \big\| z_\mu - \varphi(t,z) \big\|^{k-l} dt \\
&\quad + 4\int_0^\infty \Big\| \frac{\partial}{\partial z} L_{\tau[1]} \Big\| \cdot \Big\{ 2^{2k-2l-2} Q^{2k-2l-1} \cdot \big\| z_\mu - \varphi(t,z) \big\|^{k-l-1} \\
&\qquad + 3^{2k-2l-1} Q^{2k-2l} \cdot \big\| z_\mu - \varphi(t,z) \big\|^{k-l} \Big\} \cdot \Big\| \frac{\partial}{\partial z}\varphi(t,z) \Big\| dt \\
&\quad + 2\int_0^\infty \big\| L_{\tau[1]} \big\| \cdot \Big\{ \chi(k-l-1) Q^{2k-2l-2} \big\| z_\mu - \varphi(t,z) \big\|^{k-l-2} \\
&\qquad + \big(3^{2k-2l} + Q 3^{2k-2l-2}\big) Q^{2k-2l-1} \cdot \big\| z_\mu - \varphi(t,z) \big\|^{k-l-1} \\
&\qquad + \big(3^{2k-2l+1} + Q 3^{2k-2l-1}\big) Q^{2k-2l} \cdot \big\| z_\mu - \varphi(t,z) \big\|^{k-l} \Big\} \cdot \Big\| \frac{\partial}{\partial z}\varphi(t,z) \Big\|^2 dt \\
&\quad + 2\int_0^\infty \big\| L_{\tau[1]} \big\| \cdot \Big\{ 2^{2k-2l-2} Q^{2k-2l-1} \big\| z_\mu - \varphi(t,z) \big\|^{k-l-1} \\
&\qquad + 3^{2k-2l-1} Q^{2k-2l} \big\| z_\mu - \varphi(t,z) \big\|^{k-l} \Big\} \cdot \Big\| \frac{\partial^2}{\partial z^2}\varphi(t,z) \Big\| dt \overset{8-10,i),ii),iii)}{\leq}
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & 2\cdot\Big\{\chi(l)Q^{2l}\cdot\|z_\mu-z\|^{l-1}+\big(3^{2l+2}+Q3^{2l}\big)Q^{2l+1}\cdot\|z_\mu-z\|^{l}\\
&\qquad+\big(3^{2l+3}+Q3^{2l+1}\big)Q^{2l+2}\cdot\|z_\mu-z\|^{l+1}\Big\}\cdot 2^{k-l-1}Q^{2k-2l}\cdot\|z_\mu-z\|^{k-l}\\
&+4\cdot\Big\{2^{2l}Q^{2l+1}\cdot\|z_\mu-z\|^{l}+3^{2l+1}Q^{2l+2}\cdot\|z_\mu-z\|^{l+1}\Big\}\cdot\\
&\qquad\cdot\Big\{2^{2k-2l-2}Q^{2k-2l-1}\cdot\|z_\mu-z\|^{k-l-1}+3^{2k-2l-1}Q^{2k-2l}\cdot\|z_\mu-z\|^{k-l}\Big\}\cdot 1\\
&+2^{l+1}Q^{2l+2}\cdot\|z_\mu-z\|^{l+1}\cdot\Big\{\chi(k-l-1)Q^{2k-2l-2}\cdot\|z_\mu-z\|^{k-l-2}\\
&\qquad+\big(3^{2k-2l}+Q3^{2k-2l-2}\big)Q^{2k-2l-1}\cdot\|z_\mu-z\|^{k-l-1}\\
&\qquad+\big(3^{2k-2l+1}+Q3^{2k-2l-1}\big)Q^{2k-2l}\cdot\|z_\mu-z\|^{k-l}\Big\}\cdot 1\\
&+2^{l+1}Q^{2l+2}\cdot\|z_\mu-z\|^{l+1}\cdot\Big\{2^{2k-2l-2}Q^{2k-2l-1}\cdot\|z_\mu-z\|^{k-l-1}\\
&\qquad+3^{2k-2l-1}Q^{2k-2l}\cdot\|z_\mu-z\|^{k-l}\Big\}\cdot Q\\
=\ & \|z_\mu-z\|^{k-1}Q^{2k}\Big\{2^{k-l}\chi(l)+2^{2k}+2^{l+1}\chi(k-l-1)\Big\}\\
&+\|z_\mu-z\|^{k}Q^{2k+1}\cdot\Big\{2^{k-l}3^{2l+2}+2^{2l+2}3^{2k-2l-1}+2^{2k-2l}3^{2l+1}\\
&\qquad+2^{l+1}3^{2k-2l}+Q(2^{k-l}3^{2l}+2^{l+1}3^{2k-2l-2}+2^{2k-l-1})\Big\}\\
&+\|z_\mu-z\|^{k+1}Q^{2k+2}\cdot\Big\{2^{k-l}3^{2l+3}+2^{2}3^{2k}+2^{l+1}3^{2k-2l+1}\\
&\qquad+Q\big(2^{k-l}3^{2l+1}+2^{l+1}3^{2k-2l-1}+2^{l+1}3^{2k-2l-1}\big)\Big\}.
\end{aligned}
$$

But

$$
\begin{aligned}
&2^{k-l}\chi(l)+2^{2k}+2^{l+1}\chi(k-l-1)\\
&\qquad\leq\begin{cases}2^{2k}+2\cdot 2^{2k} & \text{if } l=0,\\ 2^{2l+2}2^{k-l}+2^{2k}+2^{l+1}2^{2k-2l} & \text{if } l\geq 1,\end{cases}\\
&\qquad\leq\begin{cases}2^{2k+2} & \text{if } l=0,\\ 2^{2k+1}+2^{2k}+2^{2k} & \text{if } l\geq 1,\end{cases}\\
&\qquad\leq\begin{cases}2^{2k+2} & \text{if } l=0,\\ 2^{2k+2} & \text{if } l\geq 1,\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
2^{k-l}3^{2l+2} &+ 2^{2l+2}3^{2k-2l-1} + 2^{2k-2l}3^{2l+1} + 2^{l+1}3^{2k-2l} \\
&\leq \begin{cases} 3^{2k}\{\left(2/9\right)^k \cdot 9 + 2 + \left(2/3\right)^k \cdot 3 + 2\} & \text{if } l = 0, \\ 3^{2k+1} + 3^{2k+1} + 2^2 3^{2k-2l-2} 3^{2l+1} + 2^2 3^{l-1} 3^{2k-2l} & \text{if } l \geq 1, \end{cases} \\
&\leq \begin{cases} 3^{2k} \cdot 8 & \text{if } l = 0, \\ 3^{2k+1} + 3^{2k+1} + 8 \cdot 3^{2k-1} & \text{if } l \geq 1, \end{cases} \\
&\leq \begin{cases} 3^{2k+2} & \text{if } l = 0, \\ 3^{2k+2} & \text{if } l \geq 1, \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&2^{k-l}3^{2l} + 2^{l+1}3^{2k-2l-2} + 2^{2k-l-1} \leq 3 \cdot 3^{2k-1} = 3^{2k}, \\
&2^{k-l}3^{2l+3} + 2^2 3^{2k} + 2^{l+1}3^{2k-2l+1} \leq 3 \cdot 3^{2k+2} = 3^{2k+3}, \\
&2^{k-l}3^{2l+1} + 2^{l+1}3^{2k-2l-1} + 2^{l+1}3^{2k-2l-1} \leq 3 \cdot 3^{2k} = 3^{2k+1}.
\end{aligned}
$$

This concludes therefore the proof. □

Acknowledgements

The author would like to thank Mike Todd for having introduced him to the open problem that led to this work, and for the many fruitful discussions they had on this subject.

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