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Abstract:
Radial basis function interpolation has attracted a lot of interest in recent years. For popular choices, for example thin plate splines, this problem has a variational formulation, i.e. the interpolant minimizes a semi-norm on a certain space of radial functions. This gives rise to a function space, called the native space. Every function in this space has the property that the semi-norm of an arbitrary interpolant to this function is uniformly bounded. In applications it is of interest whether a sufficiently smooth function belongs to the native space. In this paper we give sufficient conditions on the differentiability of a function with compact support, in the case of cubic, linear and thin plate splines. In the case of multiquadrics and Gaussian functions, it is shown that the only compactly supported function that satisfies these conditions is identically zero.

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1 Introduction

Let n pairwise different points $x_1, \dots, x_n \in I\!R^d$ be given, where n and d are any positive integers. Further, let $\lambda_1, \dots, \lambda_n$ be real numbers, p be in Π_m, the space of polynomials of degree at most m, and denote the Euclidean norm by $\|.\|$. A radial basis function is of the form

$$s(x) = \sum_{i=1}^{n} \lambda_i \phi(\|x - x_i\|) + p(x), \quad x \in I\!R^d. \tag{1.1}$$

In this paper we consider the following choices of ϕ:

$$\left.\begin{array}{lcll} \phi(r) & = & r & \text{(linear)}, \\ \phi(r) & = & r^3 & \text{(cubic)}, \\ \phi(r) & = & r^2 \log r & \text{(thin plate spline)}, \\ \phi(r) & = & \sqrt{r^2 + c^2} & \text{(multiquadric)}, \\ \phi(r) & = & e^{-r^2} & \text{(Gaussian)}, \end{array}\right\} \quad r \geq 0. \tag{1.2}$$

Let the matrix $\Phi \in I\!R^{n \times n}$ be defined by

$$(\Phi)_{ij} := \phi(\|x_i - x_j\|), \quad i, j = 1, \dots, n. \tag{1.3}$$

In all the cases in (1.2), Φ is conditionally definite. Specifically, letting m_0 be 1 in the cubic and thin plate spline case, 0 in the linear and multiquadric case and -1 in the Gaussian case, we obtain (see e.g. [6])

$$(-1)^{m_0+1} \lambda^T \Phi \lambda > 0, \tag{1.4}$$

if $\lambda \in I\!R^n$ is any nonzero vector that satisfies

$$\sum_{i=1}^{n} \lambda_i q(x_i) = 0 \quad \forall\, q \in \Pi_m, \tag{1.5}$$

where Π_{-1} is the space that contains only the zero function. We choose m to be an integer that is not less than m_0.

Now fix $n \in I\!N$ and the centres $x_1, \dots, x_n \in I\!R^d$. The interpolation problem can be posed in the following way. Find a radial basis function s of the form (1.1) that satisfies

$$\left.\begin{array}{rcl} s(x_i) & = & f_i, \quad i = 1, \dots, n \\ \sum_{i=1}^{n} \lambda_i q(x_i) & = & 0, \quad q \in \Pi_m \end{array}\right\}. \tag{1.6}$$

Let $\hat{m}$ be the dimension of Π_m. Further, let $p_1, \dots, p_{\hat{m}}$ be a basis of this linear space. The matrix P is defined by

$$P := \begin{pmatrix} p_1(x_1) & \cdots & p_{\hat{m}}(x_1) \\ \vdots & & \vdots \\ p_1(x_n) & \cdots & p_{\hat{m}}(x_n) \end{pmatrix}.$$

In the Gaussian case with $m = -1$, P is omitted. Further, let $F \in I\!R^n$ be the vector whose entries are the data values $f_1, \ldots, f_n$. Therefore the system (1.6) can be written as

$$\begin{pmatrix} \Phi & P \\ P^T & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ c \end{pmatrix} = \begin{pmatrix} F \\ 0_{\hat{m}} \end{pmatrix}, \tag{1.7}$$

where $\lambda \in I\!R^n$ has the components λ_i, where $c \in I\!R^{\hat{m}}$ and where $0_{\hat{m}}$ is the zero in $I\!R^{\hat{m}}$. The components of c are the coefficients of the polynomial p with respect to the basis $p_1, \ldots, p_{\hat{m}}$. Powell [6] shows that the interpolation matrix

$$A = \begin{pmatrix} \Phi & P \\ P^T & 0 \end{pmatrix} \in I\!R^{(n+\hat{m})\times(n+\hat{m})} \tag{1.8}$$

is nonsingular for any prescribed $m \geq m_0$, if and only if $x_1, \ldots, x_n$ satisfy

$$q \in \Pi_m \quad \text{and} \quad q(x_i) = 0, \quad i = 1, \ldots, n \quad \Longrightarrow \quad q \equiv 0. \tag{1.9}$$

An interesting and useful property is that the radial basis function that solves the interpolation problem (1.6) is the solution of a minimization problem. For any choice of ϕ in (1.2) and $m \geq m_0$, let us define the linear space $\mathcal{A}_{\phi,m}$ as the space of functions of the form

$$\sum_{i=1}^{N} \lambda_i \phi(\|x - y_i\|) + p(x), \quad x \in I\!R^d,$$

where N is any positive integer, where $y_1, \ldots, y_N$ are any pairwise different points in $I\!R^d$, where p is any polynomial in Π_m and where $\lambda = (\lambda_1, \ldots, \lambda_N)^T$ satisfies

$$\sum_{i=1}^{N} \lambda_i q(y_i) = 0 \quad \forall\, q \in \Pi_m. \tag{1.10}$$

On this space a semi-inner product and a semi-norm can be defined. Let s and u be any functions in $\mathcal{A}_{\phi,m}$, i.e.

$$s(x) = \sum_{i=1}^{N(s)} \lambda_i \phi(\|x - y_i\|) + p(x) \quad \text{and} \quad u(x) = \sum_{j=1}^{N(u)} \mu_j \phi(\|x - z_j\|) + q(x).$$

The semi-inner product is the expression

$$\langle s, u \rangle := (-1)^{m_0+1} \sum_{i=1}^{N(s)} \lambda_i u(y_i). \tag{1.11}$$

Clearly, it is bilinear. To show symmetry, by employing

$$\sum_{i=1}^{N(s)} \lambda_i q(y_i) = 0 \quad \text{and} \quad \sum_{j=1}^{N(u)} \mu_j p(z_j) = 0,$$

we deduce

$$\begin{aligned}
\langle s,u\rangle &= (-1)^{m_0+1}\sum_{i=1}^{N(s)}\lambda_i\left(\sum_{j=1}^{N(u)}\mu_j\phi(\|y_i-z_j\|)+q(y_i)\right)\\
&= (-1)^{m_0+1}\sum_{i=1}^{N(s)}\sum_{j=1}^{N(u)}\lambda_i\mu_j\phi(\|y_i-z_j\|)\\
&= (-1)^{m_0+1}\sum_{j=1}^{N(u)}\mu_j\left(\sum_{i=1}^{N(s)}\lambda_i\phi(\|z_j-y_i\|)+p(z_j)\right)\\
&= (-1)^{m_0+1}\sum_{j=1}^{N(u)}\mu_j s(z_j)=\langle u,s\rangle\,.
\end{aligned}$$

By (1.4), the expression

$$\langle s,s\rangle=(-1)^{m_0+1}\sum_{i=1}^{N(s)}\lambda_i s(y_i)=(-1)^{m_0+1}\sum_{i,j=1}^{N(s)}\lambda_i\lambda_j\phi(\|y_i-y_j\|) \tag{1.12}$$

is strictly positive, if λ satisfies (1.5) for $n=N(s)$ and any $m\ge m_0$. Thus (1.11) is indeed a semi-inner product. Then $(\langle s,s\rangle)^{1/2}$ is a semi-norm on the space $\mathcal{A}_{\phi,m}$ with null space Π_m.

Schaback [8] shows that the solution of (1.6) can be characterized as follows.

Theorem 1 *Let ϕ be any radial basis function from (1.2), and let m be chosen such that $m\ge m_0$. Given are points $x_1,\dots,x_n$ in $\mathbb{R}^d$ that satisfy (1.9) and values $f_1,\dots,f_n$ in $\mathbb{R}$. Let s be the radial function of the form (1.1) that solves the system (1.6). Then s minimizes the semi-norm $\langle g,g\rangle^{1/2}$ on the set of functions $g\in\mathcal{A}_{\phi,m}$ that satisfy*

$$g(x_i)=f_i,\quad i=1,\dots,n. \tag{1.13}$$

Proof:
Let $g(x):=\sum_{j=1}^{n(g)}\mu_j\phi(\|x-y_j\|)+q(x)\in\mathcal{A}_{\phi,m}$ satisfy (1.13). We consider the semi-norm of $g-s$. Since $s(x_i)=g(x_i)$, $i=1,\dots,n$, it has the value

$$\begin{aligned}
\langle g-s,g-s\rangle &= \langle g,g\rangle-2\langle g,s\rangle+\langle s,s\rangle\\
&= \langle g,g\rangle+\langle s,s\rangle-2(-1)^{m_0+1}\sum_{i=1}^{n}\lambda_i g(x_i)\\
&= \langle g,g\rangle+\langle s,s\rangle-2(-1)^{m_0+1}\sum_{i=1}^{n}\lambda_i s(x_i)\\
&= \langle g,g\rangle+\langle s,s\rangle-2\langle s,s\rangle\\
&= \langle g,g\rangle-\langle s,s\rangle\,.
\end{aligned}\tag{1.14}$$

Thus, we deduce the required condition

$$\langle s, s\rangle = \langle g, g\rangle - \langle g-s, g-s\rangle \le \langle g, g\rangle . \qquad \square$$

One can associate with a particular ϕ in (1.2) and $m \ge m_0$ an interesting function space, the so-called native space $\mathcal{N}_{\phi,m}$ (see Schaback [9]). Schaback and Wendland [10] show that it can be characterized as the space of all functions $F : \mathbb{R}^d \to \mathbb{R}$ for which the following condition holds.

Condition 2 *There exists a real number C, that only depends on F, such that, for any choice of interpolation points $x_1, \ldots, x_n \in \mathbb{R}^d$ for which (1.9) holds, the interpolant s_n to F at these points satisfies*

$$\langle s_n, s_n\rangle \le C.$$

For many types of radial basis functions, it is unknown what functions F belong to $\mathcal{N}_{\phi,m}$. It is the subject of this paper to prove Condition 2 for sufficiently differentiable functions F with bounded support. A useful application of the results presented here is given by a method for global optimization (Gutmann [3]). Here the global minimum of a continuous function $f : \mathcal{D} \to \mathbb{R}$ is sought, where $\mathcal{D} \subset \mathbb{R}^d$ is compact. If $x_1, \ldots, x_n \in \mathcal{D}$ and their function values have been calculated, the next point x_{n+1} is determined as follows. Choose an estimate of the global minimum, f^* say. For each $y \in \mathcal{D} \setminus \{x_1, \ldots, x_n\}$ there exists a radial basis function $s_y \in \mathcal{A}_{\phi,m}$ that interpolates $(x_1, f(x_1)), \ldots, (x_n, f(x_n))$ and (y, f^*). We take the view that the "least bumpy" of these interpolants yields the most promising location for evaluating the objective function. The "bumpiness" of s_y is measure by its semi-norm. This means that x_{n+1} minimizes $\langle s_y, s_y\rangle$, $y \in \mathcal{D} \setminus \{x_1, \ldots, x_n\}$. A crucial part of the proof of convergence of this method is to show that suitable functions with bounded support satisfy Condition 2. Another use is that the assumptions that guarantee convergence can be weakened if the objective function f itself satisfies Condition 2.

We propose a new technique that relates the semi-norm introduced above to a semi-norm defined for radial basis function interpolants on an infinite regular grid. For that purpose we generalize the semi-norm by allowing infinite sums in (1.1), and show in Section 2 that a semi-inner product and a semi-norm are well-defined. In Section 3, the extended semi-norm is applied to interpolants s_h on the infinite regular grid $h\mathbb{Z}^d$ with the mesh-size $h > 0$, where we restrict our attention to interpolation of functions with bounded support. Proposition 8 shows that, if the interpolants on $h\mathbb{Z}^d$ have uniformly bounded semi-norms, as $h \to 0$, then Condition 2 holds. A condition on f that ensures the uniform boundedness of these semi-norms is given in Theorem 12.

Multi-index notation will be used in the following sections. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)^T \in I\!N^d$, the order $|\alpha|$ and the factorial $\alpha!$ are defined as

$$|\alpha| := \sum_{i=1}^{d} \alpha_i \quad \text{and} \quad \alpha! := \prod_{i=1}^{d} \alpha_i!.$$

The power x^α, $x \in I\!R^d$, and the derivative $D^\alpha u(x)$ for a sufficiently smooth function $u : I\!R^d \to I\!R$ are

$$x^\alpha := \prod_{i=1}^{d} x_i^{\alpha_i} \quad \text{and} \quad D^\alpha u(x) := \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}},$$

respectively.

2 The semi-norm for radial basis functions with infinitely many centres

We let the linear space $\tilde{\mathcal{A}}_{\phi,m}$ be the space of functions of the form

$$s(x) = \sum_{i=1}^{\infty} \lambda_i \phi(\|x - x_i\|) + p(x), \quad x \in I\!R^d, \tag{2.1}$$

where

$$\sum_{i=1}^{\infty} \lambda_i q(x_i) = 0 \quad \forall\, q \in \Pi_m. \tag{2.2}$$

Here ϕ is a radial basis function from (1.2), $(x_i)_{i \in I\!N}$ is a sequence in $I\!R^d$, the x_i being pairwise different, $(\lambda_i)_{i \in I\!N}$ is a sequence in $I\!R$ and p is a polynomial in Π_m where $m \geq m_0$, m_0 being defined in Section 1. We require $(\lambda_i)_{i \in I\!N}$ and $(x_i)_{i \in I\!N}$ to satisfy

$$\sum_{i=1}^{\infty} |\lambda_i| < \infty, \quad \sum_{i=1}^{\infty} |\lambda_i \phi(\|x_i\|)| < \infty, \tag{2.3}$$

and

$$\sum_{i=1}^{\infty} |\lambda_i||q(x_i)| < \infty, \quad q \in \Pi_m. \tag{2.4}$$

Every choice of ϕ in (1.2) has the property

$$|\phi(\|x - y\|)| \leq \Lambda[1 + |\phi(\|x\|)| + |\phi(\|y\|)|], \quad x, y \in I\!R^d, \tag{2.5}$$

where Λ is a constant. Thus (2.3) implies that the sum in expression (2.1) is absolutely convergent for each $x \in I\!R^d$. Note that, if m is increased, then fewer functions of the form (2.1) satisfy (2.2).

Let

$$s(x)=\sum_{i=1}^{\infty}\lambda_i\phi(\|x-x_i\|)+p(x) \quad \text{and} \quad u(x)=\sum_{j=1}^{\infty}\mu_j\phi(\|x-y_j\|)+q(x)$$

be elements of $\tilde{\mathcal{A}}_{\phi,m}$, and let $\mathcal{P}=\{z_k : k\in I\!N\}$ be the union of $\{x_1,x_2,\ldots\}$ and $\{y_1,y_2,\ldots\}$, where any point that is in both of the last sets is included only once in $\mathcal{P}$. New coefficients $\tilde{\lambda}_k$ and $\tilde{\mu}_k$ are defined by

$$\tilde{\lambda}_k := \begin{cases} \lambda_i, & \text{if } z_k=x_i \quad \text{for an } i\in I\!N \\ 0, & \text{otherwise} \end{cases}$$

and

$$\tilde{\mu}_k := \begin{cases} \mu_j, & \text{if } z_k=y_j \quad \text{for a } j\in I\!N \\ 0, & \text{otherwise} \end{cases},$$

which gives the expressions

$$s(x)=\sum_{k=1}^{\infty}\tilde{\lambda}_k\phi(\|x-z_k\|)+p(x) \quad \text{and} \quad u(x)=\sum_{k=1}^{\infty}\tilde{\mu}_k\phi(\|x-z_k\|)+q(x).$$

Then the sum $s+u$ is

$$(s+u)(x)=\sum_{k=1}^{\infty}(\tilde{\lambda}_k+\tilde{\mu}_k)\phi(\|x-z_k\|)+(p+q)(x), \quad x\in I\!R^d,$$

and the product of s with a scalar α is

$$(\alpha s)(x)=\sum_{i=1}^{\infty}(\alpha\lambda_i)\phi(\|x-x_i\|)+\alpha p(x).$$

Therefore $\tilde{\mathcal{A}}_{\phi,m}$ is a linear space. It contains $\mathcal{A}_{\phi,m}$ as defined in Section 1.

Next, by analogy with expression (1.11), we introduce the bilinear form

$$\langle s,u\rangle := (-1)^{m_0+1}\sum_{i=1}^{\infty}\lambda_i u(x_i), \tag{2.6}$$

where s and u are as above. We will show that it is a well-defined semi-inner product.

We consider the formula

$$\begin{aligned}\langle s,u\rangle &= (-1)^{m_0+1}\sum_{i=1}^{\infty}\lambda_i u(x_i)\\ &= (-1)^{m_0+1}\sum_{i=1}^{\infty}\lambda_i\left[\sum_{j=1}^{\infty}\mu_j\phi(\|x_i-y_j\|)+q(x_i)\right].\end{aligned}$$

Equation (2.2) shows that $\sum_i \lambda_i q(x_i)$ vanishes. Further, inequalities (2.3) and (2.5) provide the absolute convergence condition

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\lambda_i \mu_j \phi(\|x_i - y_j\|)| < \infty, \tag{2.7}$$

the functions s and u being in $\tilde{\mathcal{A}}_{\phi,m}$. Therefore $\langle s, u\rangle$ and $\langle s, s\rangle$ are well-defined. Symmetry can be derived easily now, as the sums in the double series can be exchanged. Thus using (2.2) again,

$$\begin{aligned} \langle s, u\rangle &= (-1)^{m_0+1} \sum_{j=1}^{\infty} \mu_j \left[\sum_{i=1}^{\infty} \lambda_i \phi(\|y_j - x_i\|) + p(y_j) \right] \\ &= (-1)^{m_0+1} \sum_{j=1}^{\infty} \mu_j s(y_j) \ = \ \langle u, s\rangle \,. \end{aligned}$$

It remains to prove that $\langle s, s\rangle$ is nonnegative. Definition (2.6) gives the value

$$\langle s, s\rangle = (-1)^{m_0+1} \sum_{i,j=1}^{\infty} \lambda_i \lambda_j \phi(\|x_i - x_j\|).$$

If only finitely many λ_i are nonzero, then s is of the form (1.1) and the λ_i satisfy (1.10). Hence $\langle s, s\rangle$ is a finite sum, and the assertion follows from (1.4).

However, if infinitely many λ_i are nonzero, we have to argue in a different way. As in the previous section, let $\hat{m}$ be the dimension of Π_m, and let $p_1, \ldots, p_{\hat{m}}$ be a basis of Π_m. There exist $\hat{m}$ pairwise different points $y_1, \ldots, y_{\hat{m}}$ anywhere in $I\!R^d$ that satisfy (1.9) for $n = \hat{m}$, if $x_i = y_i$, $i = 1, \ldots, \hat{m}$. Thus the matrix

$$P := \begin{pmatrix} p_1(y_1) & \cdots & p_1(y_{\hat{m}}) \\ \vdots & & \vdots \\ p_{\hat{m}}(y_1) & \cdots & p_{\hat{m}}(y_{\hat{m}}) \end{pmatrix}$$

is nonsingular. For large n, let $\sigma^{(n)}$ be the solution of the system

$$P\sigma^{(n)} = b^{(n)}, \tag{2.8}$$

where the right-hand side vector has the components

$$b_k^{(n)} = -\sum_{i=1}^{n} \lambda_i p_k(x_i), \quad k = 1, \ldots, \hat{m}.$$

When n tends to infinity, each component of $b^{(n)}$ tends to zero by (2.2). Therefore, $\sigma^{(n)} = P^{-1} b^{(n)}$ also tends to zero.

The choices of $b^{(n)}$ and $\sigma^{(n)}$ provide the equations

$$\sum_{i=1}^{n} \lambda_i q(x_i) + \sum_{j=1}^{\hat{m}} \sigma_j^{(n)} q(y_j) = 0, \quad q \in \Pi_m.$$

Therefore the function

$$\sum_{i=1}^{n} \lambda_i \phi(\|x - x_i\|) + \sum_{j=1}^{\hat{m}} \sigma_j^{(n)} \phi(\|x - y_j\|), \quad x \in I\!R^d,$$

is in the linear space $\mathcal{A}_{\phi,m}$ of Section 1. Thus the conditional positive or negative definiteness of Φ yields the inequality

$$\begin{aligned}(-1)^{m_0+1} \sum_{i,j=1}^{n} \lambda_i \lambda_j \phi(\|x_i - x_j\|) \;&+\; (-1)^{m_0+1} 2 \sum_{j=1}^{\hat{m}} \sigma_j^{(n)} \sum_{i=1}^{n} \lambda_i \phi(\|x_i - y_j\|) \\ &+\; (-1)^{m_0+1} \sum_{i,j=1}^{\hat{m}} \sigma_i^{(n)} \sigma_j^{(n)} \phi(\|y_i - y_j\|) \geq 0.\end{aligned}$$

The first term in the sum on the left-hand side tends to $\langle s, s\rangle$, as n tends to infinity, whereas the other two terms tend to zero, since $\sigma^{(n)}$ tends to zero and since the definition (2.1) gives the limit

$$\lim_{n\to\infty} \sum_{j=1}^{n} \lambda_j \phi(\|y_i - x_j\|) = s(y_i) - p(y_i), \quad i = 1, \ldots, \hat{m}.$$

Thus the left-hand side, as $n \to \infty$, provides a sequence of nonnegative numbers that tend to $\langle s, s\rangle$. It follows that $\langle s, s\rangle$ is nonnegative as required. We state the result formally.

Theorem 3 *$\langle s, u\rangle$ defined in (2.6) is a semi-inner product on $\tilde{\mathcal{A}}_{\phi,m}$, and therefore $\langle s, s\rangle^{1/2}$ is a semi-norm on this space.*

Now one can extend the variational principle in Theorem 1 to $\tilde{\mathcal{A}}_{\phi,m}$.

Theorem 4 *Let ϕ be any radial basis function from (1.2), let m be chosen such that $m \geq m_0$, where m_0 is defined in the previous section, and let one of the two following statements hold:*

(i) *The data are finitely many points $x_1, \ldots, x_n$ in $I\!R^d$, that satisfy (1.9), and the values $f_1, \ldots, f_n$ in $I\!R$. Let s be the radial function of the form (1.1) that solves the system (1.6).*

(ii) *The data are a sequence of pairwise different points* $(x_i)_{i\in\mathbb{N}}$ *in* $\mathbb{R}^d$ *and a sequence of values* $(f_i)_{i\in\mathbb{N}}$ *in* $\mathbb{R}$*. Assume there is a radial basis function* $s \in \tilde{\mathcal{A}}_{\phi,m}$ *of the form (2.1) that interpolates* (x_i, f_i), $i \in \mathbb{N}$.

In both cases s *minimizes the semi-norm* $\langle g, g\rangle^{1/2}$ *on the set of functions* $g \in \tilde{\mathcal{A}}_{\phi,m}$ *that satisfy*

$$g(x_i) = f_i, \quad i = 1, \ldots, n, \tag{2.9}$$

or

$$g(x_i) = f_i, \quad i \in \mathbb{N}, \tag{2.10}$$

respectively.

Proof:
Let $g(x) := \sum_{j=1}^{\infty} \mu_j \phi(\|x - y_j\|) + q(x) \in \tilde{\mathcal{A}}_{\phi,m}$ satisfy (2.9) or (2.10). We consider the semi-norm of $g - s$. Equation (1.14) is valid in case (i), and is the identity

$$\langle g - s, g - s\rangle = \langle g, g\rangle - \langle s, s\rangle .$$

For case (ii), the finite sums of equation (1.14) are replaced by the infinite series

$$\sum_{i=1}^{\infty} \lambda_i g(x_i) \quad \text{and} \quad \sum_{i=1}^{\infty} \lambda_i s(x_i),$$

and the same argument applies, since $s(x_i) = g(x_i)$, $i \in \mathbb{N}$. Therefore, in both cases,

$$\langle s, s\rangle = \langle g, g\rangle - \langle g - s, g - s\rangle \le \langle g, g\rangle ,$$

which completes the proof. □

3 Application to interpolation on the infinite regular grid

An important special case of interpolation at infinitely many points is interpolation on an infinite regular grid $h\mathbb{Z}^d$, where $h > 0$ is the mesh-size. It is known [6] that an interpolant to a function $F : \mathbb{R}^d \to \mathbb{R}$ can be written in the form

$$u_h(x) = \sum_{k\in\mathbb{Z}^d} F(hk)\, \chi\left(\frac{x}{h} - k\right), \quad x \in \mathbb{R}^d, \tag{3.1}$$

provided that the sum (3.1) is absolutely convergent for all $x \in \mathbb{R}^d$, where χ is the unique cardinal spline

$$\chi(x) = \sum_{j\in\mathbb{Z}^d} \gamma_j \phi(\|x - j\|), \quad x \in \mathbb{R}^d, \tag{3.2}$$

that satisfies $\chi(k) = \delta_{k0}$, $k \in \mathbb{Z}^d$. Because of the rapid decrease of $|\chi(x)|$ when $\|x\|$ tends to infinity, the absolute convergence condition is achieved for many functions F, including all bounded functions.

In this section we restrict our attention to continuous functions F that have bounded support, i.e. there exists a bounded subset $U \subset \mathbb{R}^d$ such that $F(x) = 0$ for all $x \in \mathbb{R}^d \backslash U$. It follows that every F is bounded. First, we want to show that the interpolant to F on $h\mathbb{Z}^d$ is in $\tilde{\mathcal{A}}_{\phi,m}$. Then we apply the results of the previous section to show that, if the semi-norm of this interpolant is uniformly bounded, as $h \to 0$, then the semi-norm of an interpolant at finitely many arbitrary points is also uniformly bounded. Conditions on the differentiability of F that are sufficient for the uniform boundedness of the semi-norm of the interpolant on $h\mathbb{Z}^d$ will be investigated in the next section.

Since the argument of χ in (3.1) is $x/h - k$, and not $x - hk$ as is needed for the form (2.1), we define $\phi_h(r) := \phi(hr)$, $r \geq 0$, and we let the coefficients $\gamma_j^{(h)}$, $j \in \mathbb{Z}^d$, have the values that cause the function

$$\chi_h(x) = \sum_{j \in \mathbb{Z}^d} \gamma_j^{(h)} \phi_h(\|x - j\|), \quad x \in \mathbb{R}^d, \tag{3.3}$$

to satisfy the Lagrange conditions $\chi_h(k) = \delta_{k0}$, $k \in \mathbb{Z}^d$. Then, instead of u_h in (3.1), we consider the interpolant

$$s_h(x) = \sum_{k \in \mathbb{Z}^d} F(hk)\, \chi_h\left(\frac{x}{h} - k\right), \quad x \in \mathbb{R}^d. \tag{3.4}$$

The coefficients of the cardinal functions (3.2) and (3.3) can be written explicitly as (see [6])

$$\gamma_j = \left(\frac{1}{2\pi}\right)^d \int_{[0,2\pi]^d} \frac{e^{ij^Tt}}{\sum_{\ell \in \mathbb{Z}^d} \hat{\phi}(\|t + 2\pi\ell\|)}\, dt, \quad j \in \mathbb{Z}^d, \tag{3.5}$$

and

$$\gamma_j^{(h)} = \left(\frac{1}{2\pi}\right)^d \int_{[0,2\pi]^d} \frac{e^{ij^Tt}}{\sum_{\ell \in \mathbb{Z}^d} \hat{\phi}_h(\|t + 2\pi\ell\|)}\, dt, \quad j \in \mathbb{Z}^d, \tag{3.6}$$

where $\hat{\phi}(\|.\|)$ and $\hat{\phi}_h(\|.\|)$ are the Fourier transforms of $\phi(\|.\|)$ and $\phi_h(\|.\|)$, respectively. The Fourier transform of an absolutely integrable function $\psi : \mathbb{R}^d \to \mathbb{R}$ is the integral

$$\hat{\psi}(t) = \int_{\mathbb{R}^d} e^{-i\,x^Tt} \psi(x)\, dx, \quad t \in \mathbb{R}^d.$$

The generalized transforms $\hat{\phi}$ are known for all choices of ϕ in (1.2) and can be found in [6]. They take the values

$$\left.\begin{array}{lcll}\hat{\phi}(r) & = & C\,r^{-d-1} & \text{(linear)},\\ \hat{\phi}(r) & = & C\,r^{-d-3} & \text{(cubic)},\\ \hat{\phi}(r) & = & C\,r^{-d-2} & \text{(thin plate spline)},\\ \hat{\phi}(r) & = & C\,(c/r)^{(d+1)/2}K_{(d+1)/2}(cr) & \text{(multiquadric)},\\ \hat{\phi}(r) & = & C\,e^{-r^2/4} & \text{(Gaussian)},\end{array}\right\} r>0. \tag{3.7}$$

In each case C is a constant that does not depend on t and c, and $r = \|t\|,\ t \in I\!R^d$. $K_{(d+1)/2}$ is a modified Bessel function that is positive for $r > 0$, and that decays exponentially as r tends to infinity (Abramowitz and Stegun [1]). The definition $\phi_h(r) = \phi(hr)$, $r \geq 0$, implies

$$\hat{\phi}_h(\|t\|) = h^{-d}\hat{\phi}\left(\frac{\|t\|}{h}\right), \quad t \in I\!R^d \setminus \{0\}. \tag{3.8}$$

The coefficients $\gamma_j^{(h)}$ decrease rapidly (cf. [6]). Specifically, they have the property

$$|\gamma_j^{(h)}| \leq C_h \|j\|^{-\ell},\ \ \|j\| \to \infty, \tag{3.9}$$

where C_h depends only on h, and ℓ is $2d+1$ in the linear and the multiquadric cases, $2d+2$ in the thin plate spline case and $2d+3$ in the cubic case. In the Gaussian case, any integer ℓ is admissible. There is a difference between even and odd dimensions, i.e. the decay is faster, often exponential, in either even or odd dimensions, but the bound (3.9) is sufficient for our purposes.

For any $h > 0$, let N_h be the finite set $\{k \in Z\!\!Z^d : hk \in U\}$. Thus (3.4) can be expressed in the form

$$\begin{aligned} s_h(x) &= \sum_{k\in N_h} F(hk)\chi_h\left(\frac{x}{h} - k\right) \\ &= \sum_{k\in N_h} F(hk) \sum_{j\in Z\!\!Z^d} \gamma_j^{(h)}\phi_h\left(\left\|\frac{x}{h} - k - j\right\|\right) \\ &= \sum_{k\in N_h} F(hk) \sum_{j\in Z\!\!Z^d} \gamma_j^{(h)}\phi(\|x - hk - hj\|) \\ &= \sum_{k\in N_h} F(hk) \sum_{j\in Z\!\!Z^d} \gamma_{j-k}^{(h)}\,\phi(\|x - hj\|), \quad x \in I\!R^d. \end{aligned}$$

By (2.5) and (3.9) the sum $\sum_{j\in Z\!\!Z^d} \gamma_{j-k}^{(h)}\,\phi(\|x-hj\|)$ is absolutely convergent for each $k \in N_h$ and each $x \in I\!R^d$. It follows from the finiteness of N_h that we can write

$$s_h(x) = \sum_{j\in Z\!\!Z^d}\left[\sum_{k\in N_h} F(hk)\gamma_{j-k}^{(h)}\right]\phi(\|x-hj\|) = \sum_{j\in Z\!\!Z^d}\lambda_j^{(h)}\phi(\|x-hj\|),$$

where

$$\lambda_j^{(h)} := \sum_{k \in N_h} F(hk)\gamma_{j-k}^{(h)}, \quad j \in \mathbb{Z}^d. \tag{3.10}$$

Because of (3.9), the coefficients have the property

$$\left|\lambda_j^{(h)}\right| = \left|\sum_{k \in N_h} F(hk)\gamma_{j-k}^{(h)}\right| \le \hat{C}_h \|j\|^{-\ell}, \ \|j\| \to \infty, \tag{3.11}$$

with the values of ℓ specified after (3.9) and $\hat{C}_h$ depending on h and F. From (3.10) we can deduce

$$\begin{aligned} \sum_{j \in \mathbb{Z}^d} \left|\lambda_j^{(h)}\right| &= \sum_{j \in \mathbb{Z}^d} \left|\sum_{k \in N_h} F(hk)\gamma_{j-k}^{(h)}\right| \\ &\le \|F\|_\infty \sum_{k \in N_h} \sum_{j \in \mathbb{Z}^d} \left|\gamma_{j-k}^{(h)}\right| < \infty, \end{aligned}$$

and

$$\sum_{j \in \mathbb{Z}^d} \left|\lambda_j^{(h)} \phi(\|hj\|)\right| \le \|F\|_\infty \sum_{k \in N_h} \sum_{j \in \mathbb{Z}^d} \left|\gamma_{j-k}^{(h)}\right| |\phi(\|hj\|)| < \infty,$$

so the $\lambda_j^{(h)}$ satisfy the required conditions (2.3). The remaining question for deciding if s_h is in $\tilde{\mathcal{A}}_{\phi,m}$ is whether they also satisfy equation (2.2). If the coefficients of the cardinal function (3.3) have the properties

$$\sum_{j \in \mathbb{Z}^d} \left|\gamma_j^{(h)} q(j)\right| < \infty \quad \text{and} \quad \sum_{j \in \mathbb{Z}^d} \gamma_j^{(h)} q(j) = 0 \quad \forall\, q \in \Pi_m, \tag{3.12}$$

then, because $q(x+k)$, $x \in \mathbb{R}^d$, is in Π_m for every $k \in \mathbb{Z}^d$, we deduce the identities

$$\sum_{j \in \mathbb{Z}^d} \gamma_{j-k}^{(h)}\, q(j) = \sum_{j \in \mathbb{Z}^d} \gamma_j^{(h)} q(j+k) = 0 \quad \forall\, q \in \Pi_m, \quad \forall\, k \in \mathbb{Z}^d. \tag{3.13}$$

Thus the definition (3.10) implies

$$\begin{aligned} \sum_{j \in \mathbb{Z}^d} \lambda_j^{(h)} q(j) &= \sum_{j \in \mathbb{Z}^d} \sum_{k \in N_h} F(hk)\gamma_{j-k}^{(h)}\, q(j) \\ &= \sum_{k \in N_h} F(hk) \sum_{j \in \mathbb{Z}^d} \gamma_{j-k}^{(h)}\, q(j) \ = \ 0, \quad q \in \Pi_m, \end{aligned}$$

which is the required condition (2.2). Therefore the interpolant (3.4) is in the linear space $\tilde{\mathcal{A}}_{\phi,m}$ if the conditions (3.12) are achieved for the choice of m. The following remark shows that one cannot expect any $m \ge m_0$ to be admissible.

Remark 5 *We consider linear splines in one dimension, i.e.* $\phi(r) = r$*. Then the cardinal function (3.3) takes the form*

$$\chi_h(x) = \frac{1}{2h}|x+h| - \frac{1}{h}|x| + \frac{1}{2h}|x-h|, \quad x \in I\!R,$$

so just three of the coefficients $\{\gamma_j^{(h)} : j \in Z\!\!Z^d\}$ *are nonzero. Thus the first of the conditions (3.12) is achieved for every* m*. Further, for* $q(x) \equiv 1$ *and* $q(x) = x$*, the sum* $\sum_{j \in Z\!\!Z^d} \gamma_j^{(h)} q(j)$ *takes the values*

$$\begin{aligned} \frac{1}{2h} - \frac{1}{h} + \frac{1}{2h} &= 0 \quad \textit{and} \\ \frac{1}{2h} \cdot (-1) - \frac{1}{h} \cdot 0 + \frac{1}{2h} \cdot 1 &= 0. \end{aligned}$$

On the other hand, for $q(x) = x^2$ *the sum takes the nonzero value*

$$\frac{1}{2h} \cdot 1 - \frac{1}{h} \cdot 0 + \frac{1}{2h} \cdot 1 = \frac{1}{h}.$$

Thus the coefficients fail to satisfy the constraints (3.12) for $m = 2$*.*

Proposition 6 gives values of m that satisfy the constraints (3.12), but the Gaussian case is excluded.

Proposition 6 *Let* ϕ *be any of the radial basis functions in (1.2), except the Gaussian function. Let* m *take the value* $d, d+2, d+1$ *and* d *in the linear, cubic, thin plate spline and multiquadric case respectively. Then the coefficients of the cardinal function (3.3) satisfy*

$$\sum_{j \in Z\!\!Z^d} \gamma_j^{(h)} q(j) = 0 \quad \forall\, q \in \Pi_m,$$

and this sum is absolutely convergent.

Proof:
See Buhmann [2], chapter 5. □

We summarize the results obtained so far in the following theorem.

Theorem 7 *Let* ϕ *be any of the radial basis functions in (1.2). Let the polynomial space* Π_m *be chosen such that* m *takes an integer value in* $[0, d]$, $[1, d+2]$, $[1, d+1]$ *and* $[0, d]$ *in the linear, cubic, thin plate spline and multiquadric case respectively. In the Gaussian case, the polynomial space is omitted, i.e.* $m = m_0 = -1$*. Let* $F : I\!R^d \to I\!R$ *be a continuous function with bounded support. Then the interpolant* *to F on $h\mathbb{Z}^d$ of the form (3.4) lies in the space $\tilde{\mathcal{A}}_{\phi,m}$, and its semi-norm is the square root of*

$$\langle s_h, s_h \rangle = (-1)^{m_0+1} \sum_{j \in \mathbb{Z}^d} \lambda_j^{(h)} F(hj). \tag{3.14}$$

Now we turn to the second problem posed in the second paragraph of this section. We use the results of the previous section to study the behaviour of the semi-norm of a function (1.1) that interpolates a continuous function F with bounded support at finitely many points. It turns out that this semi-norm can be bounded by a constant that does not depend on the particular choice of the interpolation points, if the semi-norm of the interpolant on $h\mathbb{Z}^d$ is uniformly bounded, as $h \to 0$.

Proposition 8 *Let ϕ be any of the radial basis functions in (1.2), and let m take one of the values stated in Theorem 7. $F : \mathbb{R}^d \to \mathbb{R}$ is a continuous function that has bounded support. For $h > 0$ let s_h be the interpolant (3.4) to F on $h\mathbb{Z}^d$. Further, let pairwise different points $x_1, \ldots, x_n \in \mathbb{R}^d$ satisfy (1.9), and let $s_n \in \mathcal{A}_{\phi,m}$ be the unique function (1.1) that interpolates F at $x_1, \ldots, x_n$. If $\langle s_h, s_h \rangle$ is uniformly bounded by a constant $K > 0$, as $h \to 0$, then*

$$\langle s_n, s_n \rangle \le K.$$

Proof:
We regard the data points $x_1, \ldots, x_n$ as fixed. For a sufficiently small $h > 0$ there exist pairwise different points $x_1(h), \ldots, x_n(h) \in h\mathbb{Z}^d$ such that

$$\|x_i - x_i(h)\|_\infty < h, \ i = 1, \ldots, n, \tag{3.15}$$

where $\|.\|_\infty$ denotes the maximum norm in $\mathbb{R}^d$. The entries of the interpolation matrix (1.8) are continuous with respect to the data points. Thus there exists $h_0 > 0$ such that, for all $h < h_0$, (3.15) holds and the nonsingularity of the interpolation matrix is preserved, if each x_i is replaced by $x_i(h)$. For any $h < h_0$, let $s_n^{(h)}$ be the radial basis function interpolant $\sum_{i=1}^n \lambda_i^{(h)} \phi(\|x - x_i(h)\|) + p^{(h)}(x)$ that satisfies

$$s_n^{(h)}(x_i(h)) \;=\; F(x_i(h)), \qquad i = 1, \ldots, n.$$

The continuity of F yields the continuity of the right-hand side vector of the system (1.7), and the nonsingularity of (1.8) for all $h < h_0$ implies the continuity of the solution vector (1.7), considered as a function of the data points, in

$(x_1, \ldots, x_n)$. Formula (1.12) then yields that the semi-norm as a function of the data points is continuous in $(x_1, \ldots, x_n)$. Thus, as $\lim_{h\to 0} x_i(h) = x_i$, $i = 1, \ldots, n$,

$$\langle s_n, s_n \rangle = \lim_{h\to 0} \left\langle s_n^{(h)}, s_n^{(h)} \right\rangle . \tag{3.16}$$

Since $x_i(h) \in h\mathbb{Z}^d$, $i = 1, \ldots, n$, Theorem 4 implies

$$\left\langle s_n^{(h)}, s_n^{(h)} \right\rangle \le \langle s_h, s_h \rangle .$$

By assumption, the right-hand side is bounded above by a constant K that does not depend on h. Therefore (3.16) proves the assertion. $\square$

4 Sufficient conditions for a uniform bound on the semi-norm

Proposition 8 shows that F is in $\mathcal{N}_{\phi,m}$ if the semi-norm of the interpolant s_h on $h\mathbb{Z}^d$ is uniformly bounded. In this section we investigate conditions on F that guarantee this property. We will find that it is sufficient if F is in the function space $\tilde{\mathcal{N}}_\phi$ defined below. The definition requires the remark that $(-1)^{m_0+1}\hat{\phi}_h(\|t\|)$ is nonnegative for $h > 0$, which is proved as follows. Using (3.6) and (3.10), we write the nonnegative expression (3.14) as

$$\langle s_h, s_h \rangle = (-1)^{m_0+1} \left(\frac{1}{2\pi}\right)^d \int_{[0,2\pi]^d} \left| \sum_{k\in\mathbb{Z}^d} F(hk) e^{-ik^T t} \right|^2 \frac{1}{\sum_{\ell\in\mathbb{Z}^d} \hat{\phi}_h(\|t + 2\pi\ell\|)} \, dt. \tag{4.1}$$

In all the cases in (3.7), $\hat{\phi}$ does not change sign. Therefore formula (3.8) implies that $(-1)^{m_0+1}\hat{\phi}_h(\|t\|)$ is nonnegative.

Definition 9 *The linear space $\tilde{\mathcal{N}}_\phi$ is the space of continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ that are absolutely integrable and whose Fourier transform $\hat{f}$ satisfies*

$$\int_{\mathbb{R}^d} \left| \hat{f}(t) \right|^2 \frac{1}{\left| \hat{\phi}(\|t\|) \right|} \, dt < \infty.$$

It is endowed with the inner product

$$(f, g) := \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \hat{f}(t)\overline{\hat{g}(t)} \frac{1}{\left| \hat{\phi}(\|t\|) \right|} \, dt$$

that induces the norm

$$(f,f)^{1/2} = \left[\left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \left|\hat{f}(t)\right|^2 \frac{1}{\left|\hat{\phi}(\|t\|)\right|} \, dt \right]^{1/2}.$$

Remark 10 *The Fourier transform of a function $f \in \tilde{\mathcal{N}}_\phi$ is absolutely integrable which can be shown as follows. If we denote the closed unit ball by B and observe that $\hat{\phi}(\|.\|)$ is in $L^1(\mathbb{R}^d \setminus B)$ in all the cases in (3.7), we find*

$$\begin{aligned}
\int_{\mathbb{R}^d} \left|\hat{f}(t)\right| dt &= \int_B \left|\hat{f}(t)\right| dt + \int_{\mathbb{R}^d \setminus B} \frac{\left|\hat{f}(t)\right|}{\sqrt{\left|\hat{\phi}(\|t\|)\right|}} \sqrt{\left|\hat{\phi}(\|t\|)\right|} \, dt \\
&\leq \int_B \left|\hat{f}(t)\right| dt + \left(\int_{\mathbb{R}^d \setminus B} \left|\hat{f}(t)\right|^2 \frac{1}{\left|\hat{\phi}(\|t\|)\right|} \, dt \right)^{1/2} \left(\int_{\mathbb{R}^d \setminus B} \left|\hat{\phi}(\|t\|)\right| \, dt \right)^{1/2} < \infty.
\end{aligned}$$

Thus the inverse Fourier transform theorem holds, i.e.

$$f(x) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \hat{f}(t) e^{itx} dt, \quad x \in \mathbb{R}^d.$$

The following lemma shows that the radial basis function interpolant s_h to a function F with bounded support is in $\tilde{\mathcal{N}}_\phi$.

Lemma 11 *Let $F : \mathbb{R}^d \to \mathbb{R}$ be a continuous function with bounded support, let s_h be the interpolant (3.4) to F on $h\mathbb{Z}^d$, and let g be any function in $\tilde{\mathcal{N}}_\phi$. Then*

(i) *s_h is in $\tilde{\mathcal{N}}_\phi$, and $\langle s_h, s_h \rangle = (s_h, s_h)$.*

(ii) *$(g, s_h) = (-1)^{m_0+1} \sum_{j \in \mathbb{Z}^d} \lambda_j^{(h)} g(hj)$, where $\lambda_j^{(h)}$ has the value (3.10).*

Proof of (i):
First we obtain the Fourier transform of s_h from expression (3.4). Because of the rapid decrease of the cardinal spline χ_h, s_h is absolutely integrable. For each $k \in \mathbb{Z}^d$, the Fourier transform of $\chi_h(./h - k)$ is

$$\widehat{\chi_h\left(\frac{\cdot}{h} - k\right)}(t) = h^d e^{-ihk^T t} \hat{\chi}_h(ht), \quad t \in \mathbb{R}^d.$$

Moreover, $\hat{\chi}_h$ is related to $\hat{\phi}_h$ (see [6]) by the equation

$$\hat{\chi}_h(ht) = \frac{\hat{\phi}_h(h\|t\|)}{\sum_{\ell\in\mathbb{Z}^d}\hat{\phi}_h(h\|t+2\pi\ell/h\|)}, \quad t\in\mathbb{R}^d.$$

and (3.8) provides $\hat{\phi}_h(h\|t\|) = h^{-d}\hat{\phi}(\|t\|)$, $t\in\mathbb{R}^d\setminus\{0\}$. Thus the Fourier transform of s_h becomes

$$\hat{s}_h(t) = h^d\sum_{k\in\mathbb{Z}^d}F(hk)e^{-ihk^Tt}\frac{\hat{\phi}(\|t\|)}{\sum_{\ell\in\mathbb{Z}^d}\hat{\phi}(\|t+2\pi\ell/h\|)}, \quad t\in\mathbb{R}^d. \tag{4.2}$$

Further, for every $j\in\mathbb{Z}^d$,

$$\hat{s}_h\left(t+\frac{2\pi j}{h}\right) = h^d\sum_{k\in\mathbb{Z}^d}F(hk)e^{-ihk^Tt}\frac{\hat{\phi}(\|t+2\pi j/h\|)}{\sum_{\ell\in\mathbb{Z}^d}\hat{\phi}(\|t+2\pi\ell/h\|)}, \quad t\in\mathbb{R}^d, \tag{4.3}$$

which implies

$$\sum_{j\in\mathbb{Z}^d}\hat{s}_h\left(t+\frac{2\pi j}{h}\right) = h^d\sum_{k\in\mathbb{Z}^d}F(hk)e^{-ihk^Tt}, \quad t\in\mathbb{R}^d. \tag{4.4}$$

Therefore expression (4.1) and application of (4.3) and (4.4) yield

$$\begin{aligned}\langle s_h, s_h\rangle &= \left(\frac{h^2}{2\pi}\right)^d\int_{[0,2\pi/h]^d}\left|\sum_{k\in\mathbb{Z}^d}F(hk)e^{-ihk^Tt}\right|^2\frac{dt}{\sum_{\ell\in\mathbb{Z}^d}\left|\hat{\phi}(\|t+2\pi\ell/h\|)\right|}\\ &= \left(\frac{1}{2\pi}\right)^d\int_{[0,2\pi/h]^d}\sum_{j\in\mathbb{Z}^d}\left|\hat{s}_h\left(t+\frac{2\pi j}{h}\right)\right|^2\frac{1}{\left|\hat{\phi}(\|t+2\pi j/h\|)\right|}\,dt.\end{aligned}$$

All the terms in the sum are nonnegative, and $\langle s_h, s_h\rangle$ exists. Therefore the monotone convergence theorem provides

$$\begin{aligned}\langle s_h, s_h\rangle &= (-1)^{m_0+1}\left(\frac{1}{2\pi}\right)^d\sum_{j\in\mathbb{Z}^d}\int_{[0,2\pi/h]^d}\left|\hat{s}_h\left(t+\frac{2\pi j}{h}\right)\right|^2\frac{1}{\hat{\phi}(\|t+2\pi j/h\|)}\,dt\\ &= (-1)^{m_0+1}\left(\frac{1}{2\pi}\right)^d\int_{\mathbb{R}^d}|\hat{s}_h(t)|^2\frac{1}{\hat{\phi}(\|t\|)}\,dt.\end{aligned}$$

If follows that s_h is in $\tilde{\mathcal{N}}_\phi$, and $\langle s_h, s_h\rangle = (s_h, s_h)$.

Proof of (ii):
For $j \in \mathbb{Z}^d$, (3.6) and (3.10) show that $\lambda_j^{(h)}$ can be expressed as

$$\lambda_j^{(h)} = \left(\frac{1}{2\pi}\right)^d \int_{[0,2\pi]^d} \left(\sum_{k\in\mathbb{Z}^d} F(hk)e^{-ik^Tt}\right) \frac{e^{ij^Tt}}{\sum_{\ell\in\mathbb{Z}^d} \hat{\phi}_h(\|t+2\pi\ell\|)}\, dt.$$

Thus $\lambda_j^{(h)},\ j \in \mathbb{Z}^d$, are the Fourier coefficients of the periodic function

$$\left(\sum_{k\in\mathbb{Z}^d} F(hk)e^{-ik^Tt}\right) \frac{1}{\sum_{\ell\in\mathbb{Z}^d} \hat{\phi}_h(\|t+2\pi\ell\|)}, \quad t \in \mathbb{R}^d,$$

and (3.11) shows a decay of the coefficients that provides

$$\sum_{j\in\mathbb{Z}^d} \lambda_j^{(h)} e^{-ij^Tt} = \left(\sum_{k\in\mathbb{Z}^d} F(hk)e^{-ik^Tt}\right) \frac{1}{\sum_{\ell\in\mathbb{Z}^d} \hat{\phi}_h(\|t+2\pi\ell\|)}, \quad t \in \mathbb{R}^d. \tag{4.5}$$

Expressions (4.2) and (4.5) imply

$$\sum_{j\in\mathbb{Z}^d} \lambda_j^{(h)} e^{-ihj^Tt} = \frac{\hat{s}_h(t)}{\hat{\phi}(\|t\|)}, \quad t \in \mathbb{R}^d.$$

Thus from Definition 9 and $|\hat{\phi}| = (-1)^{m_0+1}\hat{\phi}$ we obtain

$$(g, s_h) = (-1)^{m_0+1} \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \hat{g}(t) \sum_{j\in\mathbb{Z}^d} \lambda_j^{(h)} e^{ihj^Tt}\, dt. \tag{4.6}$$

For $n \in \mathbb{N}$, define $S_n : \mathbb{R}^d \to \mathbb{R}$ by $S_n(t) := \sum_{\|j\|_\infty \le n} \lambda_j^{(h)} e^{ihj^Tt},\ t \in \mathbb{R}^d$. Then the function $\hat{g}(.)S_n(.)$ converges to the integrand of (4.6) as $n \to \infty$, and satisfies the bound

$$|\hat{g}(t)S_n(t)| \le \left(\sum_{j\in\mathbb{Z}^d} \left|\lambda_j^{(h)}\right|\right) |\hat{g}(t)|, \quad t \in \mathbb{R}^d.$$

The function on the right hand side is integrable, because $g \in \tilde{\mathcal{N}}_\phi$, so by the dominated convergence theorem

$$(g, s_h) = (-1)^{m_0+1} \sum_{j\in\mathbb{Z}^d} \lambda_j^{(h)} \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \hat{g}(t) e^{ihj^Tt}\, dt.$$

Thus the inverse Fourier transform formula gives the required property

$$(g, s_h) = (-1)^{m_0+1} \sum_{j\in\mathbb{Z}^d} \lambda_j^{(h)} g(hj). \qquad \square$$

Now we are ready to state and prove the theorem.

Theorem 12 *Let ϕ be any of the functions in (1.2), and choose m according to Theorem 7. Let $F : I\!R^d \to I\!R$ be in $\tilde{\mathcal{N}}_\phi$ and have bounded support. For $h > 0$ denote the interpolant (3.4) to F on $h\mathbb{Z}^d$ by s_h. Then $\langle s_h, s_h \rangle$ is bounded above by a number that does not depend on h.*

Proof:
Lemma 11(i) shows that $s_h \in \tilde{\mathcal{N}}_\phi$ and $(s_h, s_h) = \langle s_h, s_h \rangle$. By analogy to the proof of Theorem 4, using Lemma 11(ii), one obtains

$$\begin{aligned} 0 &\le (F - s_h, F - s_h) = (F, F) - 2(F, s_h) + (s_h, s_h) \\ &= (F, F) - 2(-1)^{m_0+1} \sum_{j \in \mathbb{Z}^d} \lambda_j^{(h)} F(hj) + (s_h, s_h) \\ &= (F, F) - (s_h, s_h). \end{aligned}$$

Thus

$$\langle s_h, s_h \rangle = (s_h, s_h) \le (F, F).$$

Because (F, F) does not depend on h, it serves as the required number. □

In the linear, cubic and thin plate spline cases, a bound on $\langle s_h, s_h \rangle$ can be obtained for a class of functions F. Specifically, let $\kappa = 1$ in the linear case, $\kappa = 2$ in the thin plate spline case and $\kappa = 3$ in the cubic case. Then the following result holds.

Corollary 13 *Let $\phi(r) = r$, $\phi(r) = r^2 \log r$ or $\phi(r) = r^3$, and let m be as in Theorem 7. Let $F \in C^{(d+\kappa)/2}(I\!R^d)$ if $d+\kappa$ is even, or $F \in C^{(d+\kappa+1)/2}(I\!R^d)$ if $d+\kappa$ is odd, where κ is defined above, and let F have bounded support. For $h > 0$ let s_h be the interpolant (3.4) on $h\mathbb{Z}^d$. Then $\langle s_h, s_h \rangle$ is uniformly bounded above, so F satisfies Condition 2.*

Proof:
It will be proved that F is in the linear space $\tilde{\mathcal{N}}_\phi$. Because F has bounded support, it is absolutely integrable. Further, let ν be $(d+\kappa)/2$ if $d+\kappa$ is even, or $(d+\kappa+1)/2$ if $d + \kappa$ is odd. Integration by parts shows that the Fourier transform $\hat{F}$ satisfies

$$\widehat{D^\alpha F}(t) = (it)^\alpha \hat{F}(t), \quad t \in I\!R^d, \tag{4.7}$$

for any multi-index α with order $|\alpha| = \nu$. Also, as $D^\alpha F$ has bounded support and thus is in $L_2(I\!R^d)$, the Plancherel formula gives

$$\left(\frac{1}{2\pi}\right)^d \int_{I\!R^d} \left|\widehat{D^\alpha F}(t)\right|^2 \, dt = \int_{I\!R^d} |D^\alpha F(x)|^2 \, dx < \infty, \quad |\alpha| = \nu. \tag{4.8}$$

Moreover, expression (3.7) states that $\hat{\phi}$ has the form

$$\hat{\phi}(\|t\|) = \frac{C}{\|t\|^{d+\kappa}}, \quad t \in I\!R^d \setminus \{0\},$$

for a constant C, and a generalization of the binomial formula provides

$$\|t\|^{2\nu} = \sum_{|\alpha|=\nu} \frac{\nu!}{\alpha!} t^{2\alpha}, \quad t \in I\!R^d.$$

Thus, if $d+\kappa$ is even, the term (F,F) of Definition 9 has the value

$$\begin{aligned}
(F,F) &= \left(\frac{1}{2\pi}\right)^d |C|^{-1} \int_{I\!R^d} \left|\hat{F}(t)\right|^2 \|t\|^{2\nu}\, dt \\
&= \left(\frac{1}{2\pi}\right)^d |C|^{-1} \sum_{|\alpha|=\nu} \frac{\nu!}{\alpha!} \int_{I\!R^d} \left|\hat{F}(t)\right|^2 |t^\alpha|^2\, dt \\
&= \left(\frac{1}{2\pi}\right)^d |C|^{-1} \sum_{|\alpha|=\nu} \frac{\nu!}{\alpha!} \int_{I\!R^d} \left|\widehat{D^\alpha F}(t)\right|^2\, dt \;<\; \infty.
\end{aligned}$$

Analogously, if $d+\kappa$ is odd, one obtains

$$(F,F) \;=\; \left(\frac{1}{2\pi}\right)^d |C|^{-1} \sum_{|\alpha|=\nu} \frac{\nu!}{\alpha!} \int_{I\!R^d} \left|\widehat{D^\alpha F}(t)\right|^2 \frac{1}{\|t\|}\, dt.$$

For each multi-index α with order ν, (4.7) shows that $\lim_{t\to 0} |\widehat{D^\alpha F}(t)|^2 \|t\|^{-1} = 0$. It follows from (4.8) that (F,F) is finite. Thus, in both cases, $F \in \tilde{\mathcal{N}}_\phi$. Therefore Theorem 12 provides a uniform bound on the semi-norm of interpolants to F on $h\mathbb{Z}^d$. Proposition 8 then completes the proof that F satisfies Condition 2. □

This corollary includes some well-known special cases including the following one.

Remark 14 *Consider natural cubic splines in one dimension, i.e.* $d=1$, $\phi(r)=r^3$ *and* $m=1$. *The variational principle (e.g. Powell [5]) states that for* $F \in C^2(I\!R)$ *with bounded support, the cubic spline interpolant* s *at finitely many points satisfies*

$$\int_{I\!R} s''(x)^2\, dx \le \int_{I\!R} F''(x)^2\, dx.$$

It can be shown easily that the left hand side equals $12\,\langle s,s\rangle$. *Thus the constant* $(1/12)\int F''(x)^2\, dx$ *is a uniform bound on* $\langle s,s\rangle$. *The existence of such a bound is included in Corollary 13, and the corollary also holds for* $m=2$ *and* $m=3$.

In the multiquadric and Gaussian cases, however, there is no useful class of functions with bounded support. In these cases, $\hat{\phi}(\|t\|)$ decays exponentially, as $\|t\| \to \infty$. One can show that the only function in $\tilde{\mathcal{N}}_\phi$ with bounded support is the zero function. This follows from a Paley-Wiener theorem (see Katznelson [4] for a one-dimensional version).

Theorem 15 *Let $F : I\!R^d \to I\!R$ be continuous and compactly supported, and let $e^{\beta\|\cdot\|}\hat{F} \in L^2(I\!R^d)$ for some $\beta > 0$. Then $F \equiv 0$.*

Proof:
The assumption $e^{\beta\|\cdot\|}\hat{F} \in L^2(I\!R^d)$ shows that $\hat{F}$ is absolutely integrable. Therefore the inverse Fourier transform formula

$$F(x) = \left(\frac{1}{2\pi}\right)^d \int_{I\!R^d} \hat{F}(t)e^{itx}\, dt, \quad x \in I\!R^d, \tag{4.9}$$

holds.

Let $D := \{z \in C : |\mathrm{Im}\, z| < \beta\}$. For any arbitrary $v \in I\!R^d$ with $\|v\| = 1$ define $g : D \to C$ as

$$g(z) = \left(\frac{1}{2\pi}\right)^d \int_{I\!R^d} \hat{F}(t)e^{i(zv)^T t}\, dt, \quad z \in D.$$

We show that g is well-defined and continuous on D. For $z \in D$

$$\begin{aligned}
\int_{I\!R^d} \left|\hat{F}(t)e^{iz(v^T t)}\right| dt &= \int_{I\!R^d} \left|\hat{F}(t)e^{-\mathrm{Im}\, z(v^T t)}\right| dt \\
&= \int_{I\!R^d} \left|\hat{F}(t)e^{\beta\|t\|}\right| \left|e^{-\beta\|t\|}e^{-\mathrm{Im}\, z(v^T t)}\right| dt \\
&\le \left\|\hat{F}e^{\beta\|\cdot\|}\right\|_{L^2(I\!R^d)} \left(\int_{I\!R^d} e^{-2(\mathrm{Im}\, z+\beta)\|t\|}\, dt\right)^{1/2}.
\end{aligned}$$

The right hand side is finite, so g is well-defined on D. The continuity of g can be deduced as follows. Fix $z \in D$. Then there is $\alpha \in I\!R$ such that $|\mathrm{Im}\, z| < \alpha < \beta$. For $w \in D$ sufficiently close to z, $|\mathrm{Im}\, w| < \alpha$, which provides the bound

$$\left|\hat{F}(t)e^{iw(v^T t)}\right| = \left|\hat{F}(t)\right| e^{-(v^T t)\mathrm{Im}\, w} \le \left|\hat{F}(t)\right| e^{\alpha\|t\|}, \quad t \in I\!R^d.$$

The function on the right hand side is integrable, because a similar argument as above gives

$$\begin{aligned}
\int_{I\!R^d} \left|\hat{F}(t)\right| e^{\alpha\|t\|}\, dt &= \int_{I\!R^d} \left|\hat{F}(t)e^{\beta\|t\|}\right| e^{-(\beta-\alpha)\|t\|}\, dt \\
&\le \left\|\hat{F}e^{\beta\|\cdot\|}\right\|_{L^2(I\!R^d)} \left(\int_{I\!R^d} e^{-2(\beta-\alpha)\|t\|}\, dt\right)^{1/2} < \infty.
\end{aligned}$$

Therefore, as w tends to z, the dominated convergence theorem implies

$$\begin{aligned}
g(z) &= \left(\frac{1}{2\pi}\right)^d \int_{I\!R^d} \lim_{w\to z} \hat{F}(t)e^{iw(v^T t)}\, dt \\
&= \lim_{w\to z} \left(\frac{1}{2\pi}\right)^d \int_{I\!R^d} \hat{F}(t)e^{iw(v^T t)}\, dt \;=\; \lim_{w\to z} g(w).
\end{aligned}$$

Thus g is continuous on D.

Further, Morera's Theorem (see e.g. Rudin [7]) shows that g is analytic on D. Let T be any triangle in the interior of D, and denote its boundary by ∂T. Then, by Fubini's Theorem,

$$\begin{aligned} \int_{\partial T} g(z)dz &= \left(\frac{1}{2\pi}\right)^d \int_{\partial T} \int_{I\!R^d} \hat{F}(t) e^{i(zv)^T t}\, dt\, dz \\ &= \left(\frac{1}{2\pi}\right)^d \int_{I\!R^d} \hat{F}(t) \int_{\partial T} e^{i(v^T t)z}\, dz\, dt = 0, \end{aligned}$$

because $e^{i(v^T t)z}$ is analytic for every $t \in I\!R^d$. As T is arbitrary, g is analytic.

Now choose $\tau^* \in I\!R$ such that $\tau^* v$ is outside the support of F. Because g and $F(.\,v)$ coincide on the real line, g and all its derivatives vanish at τ^*. Therefore the Taylor expansion of g around τ^* yields that $g \equiv 0$ on D. Thus $F(\tau v) = 0$ for all $\tau \in I\!R$. Because v has been chosen arbitrarily, this proves the theorem. □

It follows that in the multiquadric and Gaussian cases $\tilde{\mathcal{N}}_\phi$ contains no nonzero function with compact support.

Corollary 16 *Let $\phi(r) = \sqrt{r^2 + c^2}$ or $\phi(r) = e^{-r^2}$. If $F \in \tilde{\mathcal{N}}_\phi$ has bounded support, then $F \equiv 0$.*

Proof:
In both cases considered here, $\hat{\phi}(\|.\|)$ decays exponentially, i.e. there exist $\beta > 0$ and $K > 0$ such that

$$\left|\hat{\phi}(\|t\|)\right| \leq K e^{-2\beta\|t\|}, \quad t \in I\!R^d \setminus B, \tag{4.10}$$

where B is the closed unit ball. Now $F \in \tilde{\mathcal{N}}_\phi$ provides

$$\int_{I\!R^d \setminus B} \left|\hat{F}(t)\right|^2 e^{2\beta\|t\|}\, dt \leq K \int_{I\!R^d \setminus B} \left|\hat{F}(t)\right|^2 \frac{1}{\left|\hat{\phi}(\|t\|)\right|}\, dt < \infty.$$

It follows that $e^{\beta\|\cdot\|}\hat{F} \in L^2(I\!R^d)$. Theorem 15 now implies the required result. □

5 Conclusions

We have introduced a new technique to prove Condition 2 for sufficiently smooth functions with bounded support. It provides an extension of well-known results in the linear, cubic and thin plate spline cases. In particular, the choice of m is less restricted than before.

The situation in the multiquadric and Gaussian cases, however, is disappointing. The exponential decay of $\hat{\phi}$ prevents a uniform bound on the semi-norms when F has bounded support. As mentioned in the introduction, the application to global optimization we have in mind requires Condition 2 for such functions. Thus the presented approach is not useful in these cases. In the other cases the question arises whether there exist functions with bounded support that are not in $\tilde{\mathcal{N}}_\phi$ but still satisfy Condition 2. In particular, it is interesting to investigate how the semi-norm behaves when F is less differentiable than required by Corollary 13.

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