A new approach to semi-cardinal spline interpolation

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Abstract

The problem of semi-cardinal spline interpolation was solved by Schoenberg exploiting the piecewise polynomial form of the splines. In the present paper, we propose a new construction for the Lagrange functions of semi-cardinal spline interpolation, based on a radial basis and Fourier transform approach. This approach suggests a way of extending semi-cardinal interpolation to polyharmonic splines in the multivariable case, as considered in a sequel to this article.

1 Introduction

§ For an integer $m \geq 2$, let $\mathcal{S}^+_{2m-1}$ be the space of functions S such that

(1.1) $$S \in C^{2m-2}(\mathbb{R}),$$

(1.2) $$S|_{[k,k+1]} \in \Pi_{2m-1}, \quad k = 0, 1, \dots,$$

(1.3) $$S|_{(-\infty,0]} \in \Pi_{m-1}.$$

Here, as throughout the paper, Π_n denotes the space of polynomials of degree at most n. The *semi-cardinal interpolation* problem for the spline-space $\mathcal{S}^+_{2m-1}$ is the following

(1.4) Given a sequence $\{f_j\}_{j=0}^{\infty} \subset \mathbb{R}$, find an element $s \in \mathcal{S}^+_{2m-1}$ such that $s(j) = f_j$ for all $j \in \mathbb{Z}_+$.

This problem was introduced and studied in [17] by Schoenberg, who constructed solutions to (1.4) in two cases: a) when $f_j = O(j^\gamma)$ as $j \to \infty$, for some $\gamma \geq 0$; b) when the sequence $\{(\Delta^m f)_j\}_{j=0}^{\infty}$ of forward divided differences of order m of $\{f_j\}_{j=0}^{\infty}$, satisfies $\{(\Delta^m f)_j\}_{j=0}^{\infty} \in \ell^p(\mathbb{Z}_+)$, for some $p \in [1, \infty]$.

In case a), Schoenberg obtains his solution as a locally and uniformly convergent Lagrange series

(1.5) $$S(x) = \sum_{j=0}^{\infty} f_j \, L_j(x), \qquad x \in [0, \infty),$$

where L_j is the unique function in $\mathcal{S}^+_{2m-1}$, bounded on $[0, \infty)$, that satisfies

(1.6) $$L_j(k) = \begin{cases} 1, & \text{if } k = j, \\ 0, & \text{if } k \in \mathbb{Z}_+ \backslash \{j\}. \end{cases}$$

Further, L_j is constructed by means of the so-called eigensplines (special cases of the exponential Euler splines).

In case b), Schoenberg uses the B-spline representation of $S^{(m)}$ to arrive at a system of discrete Wiener-Hopf equations. This is solved using Krein's theory of Wiener-Hopf factorizations for periodic functions (or symbols) which have no zeros on the unit circle and whose winding number vanishes.

In the present paper, we propose a new construction for the Lagrange functions L_j, based on Wiener-Hopf factorizations for symbols which possess zeros on the unit circle. Our construction provides a Fourier transform definition for L_j, parallelling the well-known radial-basis approach for multivariable cardinal interpolation.

§ The Fourier transform approach for cardinal spline interpolation in one dimension appears in the celebrated paper [16] of Schoenberg. Specifically, he constructs the unique bounded Lagrange spline function $L \in C^{2m-2}(\mathbb{R})$, satifying $L|_{[k,k+1]} \in \Pi_{2m-1}$, $\forall\ k \in \mathbb{Z}$, and

$$L(k) = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } k \in \mathbb{Z}\backslash\{0\}, \end{cases} \tag{1.7}$$

by means of the inverse Fourier transform

$$L(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixt} \frac{\psi(t)}{\sigma(t)}\, dt, \tag{1.8}$$

where

$$\psi(t) = \left[\frac{2\sin(t/2)}{t}\right]^{2m}, \tag{1.9}$$

and σ is the 2π-periodization of ψ,

$$\begin{aligned} \sigma(t) &= \sum_{k=-\infty}^{\infty} \psi(t + 2\pi k) \\ &= [2\sin(t/2)]^{2m} \sum_{k=-\infty}^{\infty} (t + 2\pi k)^{-2m}. \end{aligned} \tag{1.10}$$

As well-known, ψ is precisely the Fourier transform of a B-spline, so using the Fourier series expansion of σ^{-1}, one can derive from (1.8) the B-spline representation of L (for the extension to box spline series, see the monograph by de Boor, Höllig and Riemenschneider [3]; for the complementary extension to polyharmonic B-splines in several dimensions, see Rabut [14, 15]).

The simplification of the fraction of (1.8) by the factor $[2\sin(t/2)]^{2m}$ gives

$$L(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixt} \frac{\omega(t)}{t^{2m}}\, dt, \tag{1.11}$$

where the 2π-periodic function

$$\omega(t) = \frac{1}{\sum_{k=-\infty}^{\infty} (t + 2\pi k)^{-2m}}, \qquad t \in \mathbb{R}, \tag{1.12}$$

has a zero of order $2m$ for $t = 0$. It is formula (1.11) which leads to the *radial basis* representation of L as a series of shifts of the basis function $\phi(x) = |x|^{2m-1}$ (for the extension of this representation to several dimensions and to a large class of basis functions ϕ, see Madych and Nelson [12] and Buhmann [4, 5]).

§ In Section 2, we perform a "canonical" Wiener-Hopf factorization of the symbol ω, defining, for each $j \in \mathbb{Z}_+$, a symbol ω_j that has a zero of order m only. Thus a "modified" Fourier transform of $t^{-2m}\omega_j(t)$ is proposed for defining L_j in Section 3. This approach offers new proofs of the properties that characterize the Lagrange functions L_j.

The motivation for the present article comes from the problem of lifting the construction of semi-cardinal spline interpolation to the case of several variables. Such a project is pursued in our subsequent paper [2] on polyharmonic splines. The constructions for semi-cardinal interpolation may serve to explain the boundary effects that are present in multivariable schemes like the thin plate splines. The convergence order of Section 4 is a first step in this direction, showing that the accuracy of the scaled semi-cardinal spline scheme is half of the maximum accuracy of the similar cardinal scheme. For a different analogy between radial basis functions and univariate splines in connection with convergence orders, see Schaback [18].

In the radial basis literature dealing with the one-dimensional case, the Wiener-Hopf factorization technique has been used before by Buhmann [6, Theorem 7], [7, Theorem 5.4], for a class of basis functions ϕ that decay at infinity, including the inverse multi-quadric $\phi(x) = (x^2+1)^{-1/2}$. The technique is also applied in the work of Goodman, Micchelli, Rodriguez and Seatzu [9], in connection with exponentially decaying univariate basis functions that generate zero-free symbols. By contrast, the basis functions $\phi(x) = |x|^{2m-1}$ of our spline setup are growing at infinity.

2 Semi-cardinal symbol functions

§ In this section, we use Wiener-Hopf factorizations to construct the semi-cardinal symbols ω_j, $j \in \mathbb{Z}_+$. Denote $Q = [-\pi, \pi]$, and let P_+ be the orthogonal projection operator

$$\mathrm{P}_+\left(\sum_{k=-\infty}^{\infty} \mu_k e^{ikt}\right) = \sum_{k=0}^{\infty} \mu_k e^{ikt}, \qquad t \in Q, \tag{2.1}$$

for $\{\mu_k\}_{k=-\infty}^{\infty} \in \ell^1(\mathbb{Z})$.

Given $\{\sigma_k\}_{k=-\infty}^{\infty} \in \ell^1(\mathbb{Z})$, define the symbol $\sigma : \mathbb{R} \to \mathbb{C}$,

$$\sigma(t) = \sum_{k=-\infty}^{\infty} \sigma_k e^{ikt}, \qquad t \in Q. \tag{2.2}$$

The winding number of σ with respect to the origin is $\mathrm{ind}(\sigma) = \frac{1}{2\pi}[\arg \sigma(t)]_{t=-\pi}^{t=\pi}$. To describe the Wiener-Hopf factorization technique, assume that the symbol σ is "normal", i.e. it satisfies

$$\sigma(t) \neq 0, \qquad \forall\, t \in Q, \tag{2.3}$$

$$\mathrm{ind}(\sigma) = 0. \tag{2.4}$$

These conditions ensure the existence of two sequences $\{\sigma_k^{\pm}\}_{k=0}^{\infty} \in \ell^1(\mathbb{Z}_+)$ such that the symbols

$$\sigma_{\pm}(t) = \sum_{k=0}^{\infty} \sigma_k^{\pm} e^{\pm ikt}, \qquad t \in Q, \tag{2.5}$$

satisfy $\operatorname{ind}(\sigma_+) = \operatorname{ind}(\sigma_-) = 0$ and

$$[\sigma(t)]^{-1} = \sigma_+(t)\sigma_-(t), \qquad t \in Q. \tag{2.6}$$

It is this representation that provides, for each $j \in \mathbb{Z}_+$, an explicit construction for the solution $\{\xi_{jk}\}_{k=0}^{\infty} \in \ell^1(\mathbb{Z}_+)$ of the infinite "semi-cardinal" Wiener-Hopf system

$$\sum_{k=0}^{\infty} \sigma_{l-k}\xi_{jk} = \delta_{lj}, \qquad l \in \mathbb{Z}_+\,. \tag{2.7}$$

Specifically, letting $\xi_j(t) = \sum_{k=0}^{\infty} \xi_{jk} e^{ikt}$, the Fourier coefficients ξ_{jk} are determined from the formula

$$\xi_j(t) = \sigma_+(t)\mathrm{P}_+\left\{e^{ijt}\sigma_-(t)\right\}, \qquad t \in Q. \tag{2.8}$$

For the theory of discrete Wiener-Hopf equations, see Krein [11, Sections 5 and 13] and Calderón, Spitzer and Widom [8]. The factorization (2.6) is called *canonical* by Krein, in the sense that, as a function of $\zeta = e^{it}$, the symbol σ_+ (respectively σ_-) is analytic and has no zeros for $|\zeta| < 1$ (respectively $|\zeta| > 1$).

§ Returning to spline interpolation, let $m \geq 2$ be fixed. The analiticity of the cardinal spline "inverse symbol" ω defined by (1.12) is an implicit property in the treatment of Schoenberg [16] (see Madych and Nelson [12] for the extension of this property to the multivarible case). Specifically, there exists $\epsilon > 0$ such that ω is a periodic and analytic function in the strip $\mathbb{R}_\epsilon = \{t = u + iv \in \mathbb{C} : |v| < \epsilon\}$. Letting $\zeta = e^{it}$, this is equivalent to the fact that, as a function of ζ, the symbol ω is analytic in the annulus $\mathcal{A}_\epsilon = \{\zeta \in \mathbb{C} : e^{-\epsilon} < |\zeta| < e^{\epsilon}\}$, which contains the unit circle. In particular, if $\{a_k\}_{k=-\infty}^{\infty}$ is the sequence of Fourier coefficients of ω, then

$$|a_k| \leq A\exp(-a|k|)\ , \qquad \forall\ k \in \mathbb{Z}\,, \tag{2.9}$$

for some positive constants a, A, and ω has an absolutely convergent Fourier-Laurent expansion

$$\omega(t) = \sum_{k=-\infty}^{\infty} a_k e^{ikt}, \qquad t \in \mathbb{R}_\epsilon. \tag{2.10}$$

Our aim is to obtain a factorization similar to (2.6) for ω, but note that the "normal" condition (2.3) is not satisfied, as $\omega(0) = 0$. However, since $\omega(t) = O(t^{2m})$ for $t \to 0$ (i.e. ω has a zero of order $2m$ at $t = 0$), there exists a function $\widetilde{\omega} = \widetilde{\omega}(\zeta)$ analytic and zero-free for $\zeta \in \mathcal{A}_\epsilon$, such that

$$\omega(t) = (\zeta - 1)^{2m}\widetilde{\omega}(\zeta), \qquad \zeta = e^{it} \in \mathcal{A}_\epsilon. \tag{2.11}$$

Using $\zeta - 1 = 2ie^{it/2}\sin(t/2)$ and formula (1.12), it follows that

$$\widetilde{\omega}(\zeta) = e^{-mit}\omega_0(t), \qquad t \in Q, \tag{2.12}$$

where ω_0 is the periodic and analytic function in $Q_\epsilon = \{t = u+iv \in \mathbb{C} : u \in Q, |v| < \epsilon\}$,

$$\omega_0(t) = \left\{\frac{t/2}{\sin(t/2)}\right\}^{2m} \frac{1}{1 + t^{2m} \sum_{j\in\mathbb{Z}\setminus\{0\}} (t+2\pi j)^{-2m}}\ ,\ t \in Q_\epsilon. \tag{2.13}$$

Note that (2.11) and (2.12) imply

$$\omega(t) = [2\sin(t/2)]^{2m}\omega_0(t) = \{2 - e^{-it} - e^{it}\}^m \omega_0(t), \qquad t \in Q. \tag{2.14}$$

By its analyticity property as a function of $\zeta = e^{it}$, the symbol ω_0 has an absolutely convergent Fourier-Laurent expansion for $\zeta \in \mathcal{A}_\epsilon$, its Fourier coefficients decaying exponentially. Moreover, from (2.13) we see that ω_0 takes only strictly positive values for $t \in Q$, so the two "normal" conditions (2.3) and (2.4) hold for ω_0 in place of σ and we can apply the classical theory to obtain a canonical factorization of ω_0. However, in order to exploit the analyticity of ω_0, we incorporate this property in the construction given in the next paragraph.

§ By reducing ϵ if necessary, we can ensure that $\omega_0(t)$ has a strictly positive real part for all $t \in Q_\epsilon$. Consequently, we can define the periodic function $\log\omega_0$ to be analytic for $t \in Q_\epsilon$ (analytic in the annulus $\mathcal{A}_\epsilon$ as a function of ζ). Letting $\{\lambda_k\}_{k=-\infty}^{\infty}$ be the sequence of exponentially decaying Fourier coefficients of $\log\omega_0$, we have the Fourier-Laurent expansion

$$\log\omega_0(t) = \sum_{k=-\infty}^{\infty} \lambda_k e^{ikt}, \qquad t \in Q_\epsilon. \tag{2.15}$$

Using the procedure of Krein [11], we now define the "semi-line" symbols

$$\Lambda_\pm(t) := \frac{1}{2}\lambda_0 + \sum_{k=1}^{\infty} \lambda_k e^{\pm ikt}, \qquad t \in Q, \tag{2.16}$$

$$\Omega_\pm(t) := \exp[\Lambda_\pm(t)], \qquad t \in Q, \tag{2.17}$$

and obtain the canonical factorization

$$\omega_0(t) = \Omega_+(t)\Omega_-(t), \qquad t \in Q. \tag{2.18}$$

Since ω_0 and $\log\omega_0$ are even functions and $\Lambda_-(t) = \overline{\Lambda_+(t)} = \Lambda_+(-t)$, we also have

$$\Omega_-(t) = \overline{\Omega_+(t)} = \Omega_+(-t). \tag{2.19}$$

In addition, as functions of $\zeta = e^{it}$, Λ_+ and Ω_+ (respectively Λ_- and Ω_-) are analytic for $|\zeta| < e^\epsilon$ (respectively for $|\zeta| > e^{-\epsilon}$). Therefore, if $\{\beta_k\}_{k=0}^{\infty}$ is the sequence of coefficients in the power series expansion of $\Omega_+(\zeta)$ (respectively $\Omega_-(\zeta)$) with respect to 0 (respectively ∞), we have the Fourier series representation

$$\Omega_\pm(t) = \sum_{k=0}^{\infty} \beta_k e^{\pm ikt}, \qquad t \in Q, \tag{2.20}$$

and there exist positive constants $\widetilde{A}$, $\widetilde{a}$, such that

$$|\beta_k| \le \widetilde{A}\exp(-\widetilde{a}|k|)\ , \qquad \forall\ k \in \mathbb{Z}_+\,. \tag{2.21}$$

Remark 1 Note that ω_0 coincides, up to a constant factor, with the reciprocal of the trigonometric polynomial defined by Schoenberg in [17, Eq. (7.9)] as the symbol associated to a B-spline. The canonical factorization of ω_0 is constructed by Schoenberg using the decomposition of the so-called Euler-Frobenius polynomial into linear factors.

We turn to the remaining factor of ω in (2.14), which is the symbol associated to a centred finite difference. Note that $2-\zeta^{-1}-\zeta$ admits, up to a constant factor, the unique *canonical* factorization

$$2-\zeta^{-1}-\zeta=(1-\zeta)(1-\zeta^{-1}), \tag{2.22}$$

in the sense that the first (respectively second) factor of the above product has no zero for $|\zeta|<1$ (respectively $|\zeta|>1$). Using (2.14), (2.18), (2.19) and (2.22), we obtain the canonical factorization of ω in the form

$$\omega(t)=\omega_+(t)\omega_-(t), \qquad t\in Q, \tag{2.23}$$

where

$$\omega_+(t):=(1-e^{it})^m\Omega_+(t)\,, \quad \omega_-(t):=\omega_+(-t)=\overline{\omega_+(t)}. \tag{2.24}$$

The sequence $\{\gamma_k\}_{k=0}^\infty$ of Fourier coefficients of ω_+ can be computed explicitly in terms of the Fourier sequence $\{\beta_k\}_{k=0}^\infty$ of Ω_+. In particular, there exist positive constants C, c, such that

$$|\gamma_k|\le C\exp(-c|k|)\;, \qquad \forall\; k\in\mathbb{Z}_+\,, \tag{2.25}$$

and we have the Fourier series representation

$$\omega_+(t)=\sum_{k=0}^\infty\gamma_k e^{ikt}, \qquad t\in Q. \tag{2.26}$$

§ By analogy to (2.8), for each $j\in\mathbb{Z}_+$ we define the "semi-cardinal" symbol

$$\begin{aligned}\omega_j(t) \;&:=\; \omega_+(t)\,\mathrm{P}_+\left\{e^{ijt}\omega_-(t)\right\}\\ &=\; \sum_{k=0}^\infty\gamma_k e^{ikt}\sum_{r=0}^j\gamma_r e^{i(j-r)t}\\ &=\; \sum_{k=0}^\infty b_{jk}e^{ikt}, \qquad t\in Q,\end{aligned} \tag{2.27}$$

where

$$b_{jk}=\sum_{r=0}^{\min\{j,k\}}\gamma_{j-r}\gamma_{k-r}, \qquad k\in\mathbb{Z}_+. \tag{2.28}$$

Proposition 2 *Let $m\ge 2$ be fixed.*

a) *For all $j\in\mathbb{Z}_+$ and all $t\in Q$, we have the inequalities:*

$$|\omega_j(t)|\le 2^m M_0\Gamma, \tag{2.29}$$

$$|\omega_j(t)|\le M_0\Gamma|t|^m, \tag{2.30}$$

where $M_0:=\max\{|\Omega_+(t)|\colon t\in Q\}$ and $\Gamma:=\sum_{k=0}^\infty|\gamma_k|$.

b) *There exist positive constants B and b, depending only on m, such that*

$$|b_{jk}| \le B \exp(-b|k-j|)\ , \qquad \forall\ j,k \in \mathbb{Z}_+\,. \tag{2.31}$$

c) *The following "moment conditions" hold:*

$$\sum_{k=0}^{\infty} p(k) b_{jk} = 0, \qquad \forall\ p \in \Pi_{m-1}. \tag{2.32}$$

Proof. a) The required inequalities follow from (2.27) and the estimates $|1-e^{it}| \le 2$, $|1-e^{it}| \le 2|\sin(t/2)| \le |t|$, and $|\,\mathrm{P}_+\{e^{ijt}\omega_-(t)\}| \le \sum_{k=0}^{j} |\gamma_k| \le \Gamma$, which are valid for all $t \in Q$.

b) Since $b_{jk} = b_{kj}$, it is sufficient to consider the case $j \le k$. Then (2.28) and (2.25) imply the estimate

$$\begin{aligned} |b_{jk}| &\le \sum_{r=0}^{j} |\gamma_{j-r}|\,|\gamma_{k-r}| \\ &\le C \sum_{r=0}^{j} |\gamma_{j-r}|\, e^{-c(k-r)} \\ &\le C\, e^{-c(k-j)} \sum_{r=0}^{j} |\gamma_r| \quad \le \quad C\Gamma\, e^{-c(k-j)}, \end{aligned} \tag{2.33}$$

which is (2.31), with $B := C\Gamma$ and $b := c$.

c) The moment conditions are due to the zero of order m of ω_j. □

3 Lagrange functions for semi-cardinal spline interpolation

Let $m \ge 2$. Having constructed the "inverse" symbol ω_j for each $j \in \mathbb{Z}_+$, we now proceed to the definition of a semi-cardinal spline $L_j \in \mathcal{S}^+_{2m-1}$ satisfying the Lagrange conditions (1.6).

Consider the auxiliary function $F_j : \mathbb{R} \to \mathbb{R}$,

$$F_j(x) := \frac{1}{2\pi} \int_{\mathbb{R}} \left\{ e^{ixt} - \sum_{q=0}^{m-1} \frac{(ixt)^q}{q!} \right\} \frac{\omega_j(-t)}{t^{2m}}\, dt, \qquad x \in \mathbb{R}. \tag{3.1}$$

Note that, for each x and j, $F_j(x)$ is defined by an absolutely convergent integral, due to (2.29), (2.30) and the estimates

$$\left| e^{ixt} - \sum_{q=0}^{m-1} \frac{(ixt)^q}{q!} \right| = \begin{cases} O(|t|^m), & \text{for } |t| \to 0, \\ O(|t|^{m-1}), & \text{for } |t| \to \infty. \end{cases} \tag{3.2}$$

Further, by differentiation with respect to x under the integral , it follows that $F_j \in C^{2m-2}(\mathbb{R})$. Selecting the unique polynomial $p_j \in \Pi_{m-1}$ which satisfies

$$p_j(k) = \delta_{jk} - F_j(k), \qquad k = 0, 1, \ldots, m-1, \tag{3.3}$$

we define the Lagrange function L_j by

$$L_j(x) := p_j(x) + F_j(x), \qquad x \in \mathbb{R}. \tag{3.4}$$

Theorem 3 *Let $m \geq 2$ be given.*

a) *For each $j \in \mathbb{Z}_+$, L_j satisfies the "semi-cardinal conditions":*

$$L_j(k) = \delta_{jk}, \qquad \forall\, k \in \mathbb{Z}_+. \tag{3.5}$$

b) *There exist positive constants C_m, c_m such that*

$$|L_j(x)| \leq C_m \exp(-c_m|x-j|)\,, \qquad \forall\, x \in [0,\infty),\ \forall\, j \in \mathbb{Z}_+. \tag{3.6}$$

c) *For each $j \in \mathbb{Z}_+$, we have $L_j \in \mathcal{S}^+_{2m-1}$ and*

$$L_j^{(m)}(x) = 0, \qquad \forall\, x \in (-\infty, 0]. \tag{3.7}$$

Proof. a) Denote by Δ^m the forward difference operator that associates to each sequence $\{y(k)\}_{k=0}^\infty$ the sequence $\{(\Delta^m y)(k)\}_{k=0}^\infty$ such that

$$(\Delta^m y)(k) = \sum_{\nu=0}^{m} (-1)^\nu \frac{m!}{\nu!(m-\nu)!} y(k+\nu). \tag{3.8}$$

Letting $x = k \in \mathbb{Z}_+$ in (3.4) and (3.1), we compute

$$(\Delta^m L_j)(k) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikt}\,(1-e^{it})^m\, \frac{\omega_j(-t)}{t^{2m}}\, dt. \tag{3.9}$$

The last integral can be transformed as follows:

$$\begin{aligned}
&\int_{\mathbb{R}} e^{ikt}\,(1-e^{it})^m\, \frac{\omega_j(-t)}{t^4}\, dt \\
&= \sum_{l=-\infty}^{\infty} \int_{-\pi}^{\pi} e^{ikt}\,(1-e^{it})^m\, \frac{\omega_j(-t)}{(t-2\pi l)^{2m}}\, dt \\
&= \int_{-\pi}^{\pi} e^{ikt}\,(1-e^{it})^m\, \omega_j(-t) \sum_{l=-\infty}^{\infty} (t-2\pi l)^{-2m}\, dt \\
&= \int_{-\pi}^{\pi} e^{-ikt}\,(1-e^{-it})^m\, \frac{\omega_j(t)}{\omega(t)}\, dt \\
&= \int_{-\pi}^{\pi} e^{-ikt}\,(1-e^{-it})^m\, \frac{\mathrm{P}_+\{e^{ijt}\omega_-(t)\}}{\omega_-(t)}\, dt.
\end{aligned} \tag{3.10}$$

The second of the above equalities is an application of Lebesgue's dominated convergence theorem, while the last two equalities follow from (1.12), (2.23) and (2.27). Using the expansion (2.26), we have

$$\begin{aligned}
\mathrm{P}_+\{e^{ijt}\omega_-(t)\} &= \mathrm{P}_+\left\{e^{ijt}\sum_{n=0}^{\infty}\gamma_n e^{-int}\right\} \\
&= e^{ijt}\omega_-(t) - \sum_{n=1}^{\infty}\gamma_{n+j}e^{-int}.
\end{aligned} \tag{3.11}$$

From (3.9)–(3.11), we obtain

$$(\Delta^m L_j)(k) = (\Delta^m \delta_j)(k) - \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}\,\frac{\sum_{n=1}^{\infty}\gamma_{n+j}e^{-int}}{\Omega_-(t)}\,dt. \tag{3.12}$$

where $\delta_j(j) = 1$, $\delta_j(l) = 0$, $\forall\, l \in \mathbb{Z}_+ \setminus \{j\}$, and we have used the definition (2.24).

Since, by (2.17), $[\Omega_-(t)]^{-1} = \exp(-\Lambda_-(t))$ has an absolutely convergent expansion in non-negative powers of e^{-it}, it follows that the fraction under the integral (3.12) has an absolutely convergent expansion in strictly positive powers of e^{-it}:

$$[\Omega_-(t)]^{-1}\sum_{n=1}^{\infty}\gamma_{n+j}e^{-int} =: \sum_{n=1}^{\infty}\Upsilon_n e^{-int}, \tag{3.13}$$

where each Υ_n also depends on j. Thus we obtain

$$\begin{aligned}(\Delta^m L_j)(k) &= (\Delta^m \delta_j)(k) - \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}\sum_{n=1}^{\infty}\Upsilon_n e^{-int}dt \\ &= (\Delta^m \delta_j)(k) - \frac{1}{2\pi}\sum_{n=1}^{\infty}\Upsilon_n\int_{-\pi}^{\pi} e^{-i(n+k)t}dt \\ &= (\Delta^m \delta_j)(k), \qquad \text{since } k \geq 0.\end{aligned} \tag{3.14}$$

Since (3.3) provides $L_j(k) = \delta_j(k)$ for $k \in \{0, 1, \ldots, m-1\}$, from (3.14) we deduce (3.5).

b) We start from the derivative of order m of L_j. Differentiating with respect to x under the integral in (3.1), we obtain

$$L_j^{(m)}(x) := \frac{i^m}{2\pi}\int_{\mathbb{R}} e^{ixt}\,\frac{\omega_j(-t)}{t^m}\,dt, \qquad x \in \mathbb{R}, \tag{3.15}$$

the integral being absolutely convergent due to $m \geq 2$ and the estimates (2.29), (2.30). Recalling the definitions (2.24) and (2.27), it follows that $L_j^{(m)}$ is, up to a constant factor, the inverse Fourier transform of the function

$$\frac{\omega_j(-t)}{t^m} = \left(\frac{1-e^{-it}}{t}\right)^m \Omega_-(t)e^{-ijt}\sum_{r=0}^{j}\gamma_r e^{irt} \qquad t \in \mathbb{R}. \tag{3.16}$$

The factor $[(1-e^{-it})/t]^m$ of the above product is an entire function (i.e. analytic for $t \in \mathbb{C}$). We also know that $\Omega_-(t)$ is analytic on the lower semi-plane $\mathcal{L}_\epsilon := \{t = u + iv \in \mathbb{C} : v < \epsilon\}$. Thus, we deduce that the function $\omega_j(-t)/t^m$ is analytic in the lower semi-plane $\mathcal{L}_\epsilon$.

Choose $\eta \in \{-\epsilon/2, \epsilon/2\}$ such that $(x-j)\eta \geq 0$, i.e. $(x-j)\eta = |x-j|\epsilon/2$. Due to the above analyticity property, we can integrate on $\mathbb{R} + i\eta$ instead of $\mathbb{R}$ in (3.15), in order to obtain the estimate

$$\begin{aligned}&|L_j^{(m)}(x)| \\ &= \frac{1}{2\pi}\left|\int_{\mathbb{R}} \frac{e^{i(x-j)(t+i\eta)}}{(t+i\eta)^m}\sum_{k=0}^{\infty}\gamma_k e^{-ik(t+i\eta)}\sum_{r=0}^{j}\gamma_r e^{ir(t+i\eta)}\,dt\right| \\ &\leq \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-(x-j)\eta}}{(t^2+\eta^2)^{m/2}}\sum_{k=0}^{\infty}|\gamma_k e^{k\eta}|\sum_{r=0}^{j}|\gamma_r e^{-r\eta}|\,dt \\ &\leq \widetilde{C}e^{-\widetilde{c}|x-j|}, \qquad \forall\, x \in \mathbb{R},\ \forall\, j \in \mathbb{Z}_+.\end{aligned} \tag{3.17}$$

Here $\widetilde{c} := \epsilon/2$ and, with c, C from (2.25),

$$\widetilde{C} := \frac{C}{2\pi} \sum_{k=0}^{\infty} |e^{-(c-\epsilon/2)k}| \sum_{r=0}^{\infty} |\gamma_r| \int_{\mathbb{R}} \frac{dt}{(t^2+\epsilon^2/4)^{m/2}}. \tag{3.18}$$

Note that ϵ is reduced if necessary, in order to ensure $\epsilon < 2c$ in the above estimate.

Next, for each $q \in \{0, 1, \ldots, m-1\}$ and $j \in \mathbb{Z}_+$ we define a strictly increasing unbounded sequence $\mathcal{Z}(q,j) \subset \mathbb{R}_+$ such that

$$L_j^{(q)}(z) = 0, \qquad \forall\ z \in \mathcal{Z}(q,j). \tag{3.19}$$

Specifically, for $q = 0$, let $\mathcal{Z}(0,j) := \mathbb{Z}_+ \setminus \{j\}$. Further, assume that we have already defined $\mathcal{Z}(q,j)$ for a value $q \le m-2$ and for all $j \in \mathbb{Z}_+$. Then, for any two consecutive elements $z' < z''$ of $\mathcal{Z}(q,j)$, we have $L_j^{(q)}(z') = L_j^{(q)}(z'') = 0$, and, by Rolle's theorem, we may select $z \in (z', z'')$ satisfying $L_j^{(q+1)}(z) = 0$. We define $\mathcal{Z}(q+1,j)$ to be the increasing sequence of all such selected z. Note that this construction provides the property

$$|z' - z''| < q + 2 \le m + 1, \tag{3.20}$$

for any two consecutive elements $z', z'' \in \mathcal{Z}(q,j)$, and, in addition, the first element of any such sequence $\mathcal{Z}(q,j)$ is smaller than $m+1$.

Consider now any $x > 0$. Then there exists $z \in \mathcal{Z}(m-1,j)$ such that $|x-z| \le m+1$, so the Newton-Leibniz formula and the estimate (3.17) imply

$$\begin{aligned} |L_j^{(m-1)}(x)| &= \left| \int_z^x L_j^{(m)}(\tau)\, d\tau \right| \\ &\le |x-z| \max_{\tau \in [z,x]} \left| L_j^{(m)}(\tau) \right| \\ &\le (m+1)\widetilde{C} \max\{e^{-\widetilde{c}|x-j|}, e^{-\widetilde{c}|z-j|}\} \\ &\le (m+1)\widetilde{C} e^{(m+1)\widetilde{c}} e^{-\widetilde{c}|x-j|}. \end{aligned} \tag{3.21}$$

Applying the above argument successively, we obtain (3.6).

c) We first prove (3.7). Fix any $x < 0$. The analyticity of the integrand of (3.15) on the lower semi-plane $\mathcal{L}_\epsilon$ allows changing the integration path from $\mathbb{R}$ to $\mathbb{R} + i\eta$, for any $\eta < -1$. Thus we obtain the estimate

$$\begin{aligned} &|L_j^{(m)}(x)| \\ &\le \frac{1}{2\pi} e^{-x\eta} \sum_{k=0}^{\infty} |\gamma_k| e^{k\eta} \sum_{r=0}^{j} |\gamma_r| e^{(j-r)\eta} \int_{\mathbb{R}} \frac{dt}{(t^2+\eta^2)^{m/2}} \\ &\le \frac{1}{2\pi} e^{-x\eta} \left(\sum_{k=0}^{\infty} |\gamma_k| \right)^2 \int_{\mathbb{R}} \frac{dt}{(t^2+1)^{m/2}}. \end{aligned} \tag{3.22}$$

Letting $\eta \to -\infty$, it follows that $L_j^{(m)}(x) = 0$, hence (3.7) is true, as x was an arbitrary negative number.

Since L_j is a polynomial of degree $m-1$ on $(-\infty,0]$ and it is bounded on $[0,\infty)$, it follows that L_j is a tempered distribution. By (3.15), the Fourier transform of $L_j^{(m)}$ is the square integrable function $i^m\omega_j(-t)/t^m$, so we have the distributional equalities

$$(L_j^{(2m)})\hat{}=t^m(L_j^{(m)})\hat{}=\omega_j(-t). \tag{3.23}$$

Using the representation (2.27), it is now straightforward to deduce

$$L_j^{(2m)}=\sum_{k=0}^{\infty}b_{jk}\delta_k, \tag{3.24}$$

in the sense of distributions, where δ_k is the translated Dirac distribution associating to each test function its value in k. From (3.7) and (3.24) we deduce $L_j\in\mathcal{S}_{2m-1}^{+}$, which finishes the proof of the theorem. ☐

The assertions of the following corollary can be deduced from the above construction using distributional arguments, which we omit for brevity.

Corollary 4 a) *For any function f having a square-integrable derivative of order m, we have the "fundamental identity"*

$$\int_{\mathbb{R}}L_j^{(m)}(x)f^{(m)}(x)\,dx=\sum_{k=0}^{\infty}b_{jk}f(k). \tag{3.25}$$

By a standard variational argument, it follows that, among all functions f as above which satisfy the semi-cardinal Lagrange interpolation conditions, L_j is the unique minimizer of $\int_{\mathbb{R}}|f^{(m)}(x)|^2dx$.

b) *For any $j\in\mathbb{Z}_+$, there exists $\widetilde{p}_j\in\Pi_{m-1}$ such that the following radial basis representation holds:*

$$L_j(x)=\widetilde{p}_j+\sum_{k=0}^{\infty}b_{jk}|x-k|^{2m-1},\qquad x\in\mathbb{R}. \tag{3.26}$$

c) *We have the polynomial reproduction property*

$$p(x)=\sum_{j=0}^{\infty}p(j)L_j(x),\qquad\forall\ x\in[0,\infty),\ \forall\ p\in\Pi_{m-1}. \tag{3.27}$$

4 A convergence order

Given $m\geq 2$ and a continuous data function $f:\mathbb{R}_+\to\mathbb{R}$ of polynomial growth at ∞, consider, for each $h>0$, the semi-cardinal scaled scheme

$$S_{f,h}(x)=\sum_{j=0}^{\infty}f(hj)L_j(h^{-1}x),\qquad x\in[0,\infty), \tag{4.1}$$

which is absolutely and uniformly convergent over compact subsets of $[0,\infty)$ by (3.6).

Corollary 5 *If $f \in C^m(\mathbb{R}_+)$ has a bounded m-order derivative on $\mathbb{R}_+$, there exists a constant $c_f > 0$ such that*

$$\max_{x \in \mathbb{R}_+} |f(x) - S_{f,h}(x)| \le c_f h^m, \tag{4.2}$$

for all sufficiently small h.

Proof. Since $f^{(m)}$ is bounded on $\mathbb{R}_+$, it follows that f is of polynomial growth at ∞, so $S_{f,h}$ is well-defined. We use a well-known technique for proving convergence estimates in the radial basis function literature (see Jackson [10], Buhmann [4], Powell [13, Theorem 8.2]). So far, that technique was applied to stationary Lagrange schemes similar to (4.1), but in which $L_j(h^{-1}x)$ is replaced by the translate $\widetilde{\psi}(h^{-1}x - j)$ of a single function $\widetilde{\psi}$, and the summation index j ranges through the full set of integers $\mathbb{Z}$. Two ingredients are required: the first is the condition

$$|\widetilde{\psi}(x)| < c\,|x|^{-m-1-\epsilon}, \qquad \forall x \in \mathbb{R}, \tag{4.3}$$

for some $c > 0$, $\epsilon > 0$, and the second is the polynomial reproduction property

$$p(x) = \sum_{j=-\infty}^{\infty} p(j)\widetilde{\psi}(x-j), \qquad \forall\, x \in \mathbb{R},\ \forall\, p \in \Pi_{m-1}. \tag{4.4}$$

The technique only uses the bound (4.3) for x replaced by $h^{-1}x - j$, so in our case the exponential rate of decay (3.6) provides the analogous bound for $L_j(h^{-1}x)$, with a vengeance. The other changes in order to adapt the mentioned technique are to replace (4.4) by (3.27) and to work only with $x \in \mathbb{R}_+$ and $j \in \mathbb{Z}_+$. Then the corresponding arguments imply the error estimate (4.2), as required. $\square$

Since x^m is not reproduced by the semi-cardinal scheme, it follows that (4.2) cannot be improved without requiring end-conditions for f, even if the order of differentiability of f is increased. Thus the approximation order of the semi-cardinal scheme is half of that of the cardinal scheme for general sufficiently smooth data functions.

It is now straightforward to modify the arguments of [1] in order to deduce the same order of convergence uniformly on a finite interval, when interpolating with natural splines at equally spaced points, assuming no end-conditions for the data function.

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