Polyharmonic spline interpolation on a semi-space lattice

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Abstract

We consider the problem of semi-cardinal interpolation for polyharmonic splines. For absolutely summable data sequences, we construct a solution to this problem using a Lagrange series representation. The corresponding Lagrange functions are defined using Fourier transforms and the technique of Wiener-Hopf factorizations for semi-space lattices.

1 Introduction

§ Let m and d be positive integers such that $2m \geq d+1$. A tempered distribution s is called an *m-harmonic cardinal spline* if it satisfies the following two properties:

$$s \in C^{2m-d-1}(\mathbb{R}^d), \tag{1.1}$$

$$\Delta^m s = 0 \text{ on } \mathbb{R}^d \setminus \mathbb{Z}^d, \tag{1.2}$$

where $\Delta = \sum_{q=1}^{d} \partial^2/\partial x_q^2$ is the Laplace operator and Δ^m is defined iteratively by $\Delta^m s = \Delta(\Delta^{m-1}s)$ for $m \geq 2$. Denote by $SH_m(\mathbb{R}^d)$ the space of all m-harmonic cardinal splines in $\mathbb{R}^d$. The associated problem of *cardinal interpolation* for m-harmonic splines is the following:

(A) Given a sequence $\{f_j\}_{j\in\mathbb{Z}^d} \subset \mathbb{R}$, find an element $s \in SH_m(\mathbb{R}^d)$ such that $s(j) = f_j$ for all $j \in \mathbb{Z}^d$.

For $d \geq 2$, this problem was introduced and studied by Madych and Nelson [13], extending the Fourier transform approach employed by Schoenberg [19] for $d = 1$. Madych and Nelson proved that, if $\{f_j\}_{j\in\mathbb{Z}^d}$ is a sequence of polynomial growth, then problem (A) admits a unique solution.

Consider now the set $SH_m^+(\mathbb{R}^d)$ of all tempered distributions s which verify condition (1.1) together with the property

$$\Delta^m s = 0 \text{ on } \mathbb{R}^d \setminus (\mathbb{Z}^{d-1}\times\mathbb{Z}_+). \tag{1.3}$$

Any element of $SH_m^+(\mathbb{R}^d)$ will be called an *m-harmonic semi-cardinal spline*. Note that $SH_m^+(\mathbb{R}^d)$ is a subspace of $SH_m(\mathbb{R}^d)$. The corresponding problem of *semi-cardinal interpolation* for m-harmonic splines can be formulated as follows:

(B) Given a sequence $\{f_j\}_{j\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} \subset \mathbb{R}$, find an element $s \in SH_m^+(\mathbb{R}^d)$ such that $s(j) = f_j$ for all $j \in \mathbb{Z}^{d-1}\times\mathbb{Z}_+$.

In the case $d = 1$ treated by Schoenberg [20], problem (B) does not have a unique solution if s is only required to be in $SH_m^+(\mathbb{R}^d)$. What guarantees unicity in this case is the additional string of *natural* end-conditions

$$s^{(m)}(0) = s^{(m+1)}(0) = \ldots = s^{(2m-2)}(0) = 0. \tag{1.4}$$

These conditions also ensure, for a suitable class of data sequences, that s is the unique function that minimizes $\int_{\mathbb{R}} |s^{(m)}(x)|^2 dx$ among all functions interpolating the data and having a square-integrable derivative of order m (cf. Schoenberg [20, Theorem 5]). For a finite number of interpolation points, the analogous variational problem in the multivariable case was investigated by Duchon [7] in the framework of the so-called Beppo-Levi space

$$L_m^2(\mathbb{R}^d) = \{u \in \mathcal{S}' : \partial^\alpha u \in L^2(\mathbb{R}^d),\ \forall\, |\alpha| = m\}, \tag{1.5}$$

on which a semi-norm to be minimized is defined. Note that in [14], Madych and Nelson showed that their solution to problem (A) also minimizes the corresponding semi-norm, among all functions in $L_m^2(\mathbb{R}^d)$ which interpolate the data sequence on $\mathbb{Z}^d$.

In this paper we construct a solution $s \in SH_m^+(\mathbb{R}^d) \cap L_m^2(\mathbb{R}^d)$ to problem (B), in the case when the data sequence is absolutely summable, i.e. $\sum_{j\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} |f_j| < \infty$ (cf. Theorem 3.4). The conjecture that this solution is uniquely determined by the "natural" condition $s \in L_m^2(\mathbb{R}^d)$, and that it solves the variational problem associated with semi-cardinal interpolation in the sense of Duchon, remains open at present.

§ For a data sequence $\{f_j\}_{j\in\mathbb{Z}^d}$ of polynomial growth, the solution s of problem (A) is given by Madych and Nelson in the "Lagrange form"

$$s(x) = \sum_{j\in\mathbb{Z}^d} f_j\, L(x-j), \qquad x \in \mathbb{R}^d, \tag{1.6}$$

where L is the unique function in $SH_m(\mathbb{R}^d)$ that satisfies

$$L(k) = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } k \in \mathbb{Z}^d\backslash\{0\}. \end{cases} \tag{1.7}$$

The above series is absolutely and uniformly convergent on compact subsets of $\mathbb{R}^d$ due to the exponential rate of decay of $|L(y)|$ for $\|y\| \to \infty$. Here, as throughout the paper, $\|\cdot\|$ denotes the Euclidean norm. The Lagrange function L is defined as the inverse Fourier transform of the integrable product $\|\cdot\|^{-2m}\omega$, for a continuous and even function $\omega : \mathbb{R}^d \to \mathbb{R}$ which is 2π-periodic in each of its variables and which has a zero of order $2m$ at the origin.

Note that, up to a constant factor, $\|\cdot\|^{-2m}$ coincides on $\mathbb{R}^d\backslash\{0\}$ with the distributional Fourier transform of the radial function

$$\phi(x) = \begin{cases} \|x\|^{2m-d}, & \text{if } d \text{ is odd}, \\ \|x\|^{2m-d}\log\|x\|, & \text{if } d \text{ is even}, \end{cases} \qquad x \in \mathbb{R}^d, \tag{1.8}$$

which is the fundamental solution of the iterated Laplacian Δ^m in $\mathbb{R}^d$. On the other hand, Madych and Nelson proved that ω has a Fourier-Laurent series representation valid in a tube in $\mathbb{C}^d$,

$$\omega(t) = \sum_{k\in\mathbb{Z}^d} a_k e^{ikt}, \tag{1.9}$$

with coefficients a_k, $k \in \mathbb{Z}^d$, decaying exponentially as $\|k\| \to \infty$. Consequently, the Lagrange function L admits the following *radial basis* representation in terms of ϕ:

$$L(x) = \sum_{k\in\mathbb{Z}^d} a_k\, \phi(x-k), \qquad x \in \mathbb{R}^d, \tag{1.10}$$

the series being absolutely and uniformly convergent on compact subsets of $\mathbb{R}^d$. (For the extension of the radial basis approach for cardinal interpolation to a large class of functions ϕ, see Buhmann [4, 5].)

§ In Section 3, for each $j \in \mathbb{Z}^{d-1}\times\mathbb{Z}_+$, we define a Lagrange function $L_j \in SH_m^+(\mathbb{R}^d) \cap L_m^2(\mathbb{R}^d)$ which satisfies

$$L_j(k) = \begin{cases} 1, & \text{if } k = j, \\ 0, & \text{if } k \in (\mathbb{Z}^{d-1}\times\mathbb{Z}_+) \setminus \{j\}. \end{cases} \tag{1.11}$$

The solution to problem (B) for an absolutely summable data sequence is then given by

$$s(x) = \sum_{j\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} f_j\, L_j(x), \qquad x \in \mathbb{R}^d, \tag{1.12}$$

as a locally and uniformly convergent series.

The construction of the symbols ω_j, $j \in \mathbb{Z}^{d-1}\times\mathbb{Z}_+$, which are replacing ω in the semi-cardinal case, is performed in Section 2, using the technique of multi-dimensional Wiener-Hopf factorizations (this technique seems not to have been used before in the radial basis literature for the case of several variables). As a result, for each $j \in \mathbb{Z}^{d-1}\times\mathbb{Z}_+$, we obtain an absolutely convergent Fourier representation

$$\omega_j(t) = \sum_{k\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} b_{jk} e^{ikt}, \qquad t \in \mathbb{R}^d, \tag{1.13}$$

and ω_j has a zero of order only m for $t = 0$. An interesting new feature is that, due to the halving in the order of the zero of ω_j as compared to that of ω, for $m \geq d$ we have to use modified Fourier transforms of $\|\cdot\|^{-2m}\omega_j$ in order to define L_j (cf. (3.2)).

The approach of the present article is modelled on that of [3], where we treated the one-dimensional case from a radial basis and Fourier transform perspective (Schoenberg's work [20] depends on the piecewise polynomial form of the univariate splines). In [3], we also indicated the relevance of the semi-cardinal construction for explaining the deterioration in accuracy that occurs near the end-points of an interval, when interpolating with natural splines. From this point of view, the present multivariable construction may prove to be a useful step in explaining the boundary effects that were noticed for interpolatory schemes such as the thin plate splines. Another potential application is to the problem of bounding the Lebesgue constants of surface spline interpolation, as formulated in [1, 2]. Thus it becomes important for further research to establish precise asymptotic properties of the Lagrange functions L_j and of the Fourier coefficients of ω_j, as well as to develop similar constructions for quarter-plane lattices or for other radial basis functions.

§ Throughout the paper, we work with $d \geq 2$ and $\mathbb{Z}_+ := \{0, 1, \ldots\}$. Thus $\mathbb{Z}_+^d = \mathbb{Z}_+ \times \ldots \times \mathbb{Z}_+$ (d times). A d-dimensional multi-index α is an element of $\mathbb{Z}_+^d$; if $\alpha = (\alpha_1, \ldots, \alpha_d)$, then $|\alpha| := \alpha_1 + \ldots + \alpha_d$, $\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}$, and $D^\alpha = i^{-|\alpha|}\partial^\alpha$, where i will always denote the usual root of -1. The dot product between two vectors x and t is denoted by xt. For any d-dimensional vector x, we denote by x' the vector of its first $d-1$ coordinates, so $x = (x', x_d)$. For a countable set $\mathcal{M}$, we denote by $\ell^1(\mathcal{M})$ the set of absolutely summable sequences indexed on $\mathcal{M}$. By δ_{ab} we always mean the Kronecker delta, having value 1 if $a = b$, and 0 if $a \neq b$.

Any function which is 2π-periodic in each of its variables will be called a *symbol* in the following. The space of polynomials of total degree up to n in $\mathbb{R}^d$ is denoted by Π_n^d.

2 Semi-space symbol functions

§ Our approach to construct the Lagrange functions for m-harmonic spline interpolation on the semi-space lattice $\mathbb{Z}^{d-1}\times\mathbb{Z}_+$ is based on the technique for solving multi-dimensional discrete Wiener-Hopf equations. Let $Q = [-\pi, \pi]$. Denote by P_+ the orthogonal projection operator

$$\mathrm{P}_+ \left\{ \sum_{k\in\mathbb{Z}^d} \mu_k e^{ikt} \right\} = \sum_{k\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} \mu_k e^{ikt}, \qquad t \in Q^d, \tag{2.1}$$

for any $\{\mu_k\}_{k\in\mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d)$.

Given the sequence $\{\sigma_k\}_{k\in\mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d)$, define the symbol $\sigma : \mathbb{R}^d \to \mathbb{C}$,

$$\sigma(t) = \sum_{k\in\mathbb{Z}^d} \sigma_k e^{ikt}, \qquad t \in Q^d. \tag{2.2}$$

For each $t' \in Q^{d-1}$, let the winding number of $\sigma(t', \cdot)$ with respect to the origin be $\kappa(t') = \frac{1}{2\pi}[\arg \sigma(t', t_d)]_{t_d=-\pi}^{t_d=\pi}$. To describe the Wiener-Hopf factorization technique, assume that the symbol σ is "normal", *i.e.* it satisfies

$$\sigma(t) \neq 0, \qquad \forall\, t \in Q^d, \tag{2.3}$$

$$\kappa(t') = 0, \qquad \forall\, t' \in Q^{d-1}. \tag{2.4}$$

These conditions ensure the existence of two sequences $\{\sigma_k^\pm\}_{k\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} \in \ell^1(\mathbb{Z}^{d-1}\times\mathbb{Z}_+)$ such that the symbols

$$\sigma_\pm(t) = \sum_{k\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} \sigma_k^\pm e^{\pm ikt}, \qquad t \in Q^d, \tag{2.5}$$

provide the factorization

$$[\sigma(t)]^{-1} = \sigma_+(t)\sigma_-(t), \qquad t \in Q^d, \tag{2.6}$$

and their inverses, σ_+^{-1} and σ_-^{-1}, have similar Fourier series expansions in exponentials e^{ikt} and, respectively, e^{-ikt}, for $k \in \mathbb{Z}^{d-1}\times\mathbb{Z}_+$. In turn, (2.6) implies, for each $j \in \mathbb{Z}^{d-1}\times\mathbb{Z}_+$, the existence of a unique solution $\{\xi_{jk}\}_{k\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} \in \ell^1(\mathbb{Z}^{d-1}\times\mathbb{Z}_+)$ of the infinite "semi-cardinal" system

$$\sum_{k\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} \sigma_{l-k}\xi_{jk} = \delta_{jl}, \qquad l \in \mathbb{Z}^{d-1}\times\mathbb{Z}_+. \tag{2.7}$$

Specifically, letting $\xi_j(t) = \sum_{k\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} \xi_{jk}e^{ikt}$, the Fourier coefficients ξ_{jk} are determined from the formula

$$\xi_j(t) = \sigma_+(t)\mathrm{P}_+\left\{e^{ijt}\sigma_-(t)\right\}, \qquad t \in Q^d. \tag{2.8}$$

These results are due to Goldenstein and Gohberg [11] and Goldenstein [10].

§ Returning to m-harmonic spline interpolation, let the parameters $m, d \in \{2, 3, \ldots\}$ satisfy $2m \geq d+1$. The cardinal "inverse symbol" ω that was mentioned in Section 1 is defined by

$$\omega(t) = \frac{1}{\sum_{j\in\mathbb{Z}^d} \|t + 2\pi j\|^{-2m}}, \qquad t \in \mathbb{R}^d. \tag{2.9}$$

In [13], Madych and Nelson showed that ω extends to a periodic and analytic function on the tube $\mathbb{R}^d_\epsilon \subset \mathbb{C}^d$, for some $\epsilon > 0$ where $\mathbb{R}_\epsilon = \{z = u + iv \in \mathbb{C} : |v| < \epsilon\}$. Letting $\zeta_q = e^{it_q}$, $q \in \{1, 2, \ldots, d\}$, this property is equivalent to the fact that, as a function of $\zeta_1, \ldots, \zeta_d$, the symbol ω is analytic in a neighbourhood of the torus $T^d = \{(\zeta_1, \ldots, \zeta_d) \in \mathbb{C}^d : |\zeta_1| = \ldots = |\zeta_d| = 1\}$. In particular, the usual Cauchy estimates imply the exponential decay of the Fourier coefficients $\{a_k\}_{k\in\mathbb{Z}^d}$ of ω, as $\|k\| \to \infty$, and the Fourier-Laurent expansion (1.9) of ω is absolutely convergent for $t \in \mathbb{R}^d_\epsilon$.

Our aim is to obtain a factorization similar to (2.6) for the m-harmonic cardinal spline inverse symbol ω. The first apparent obstruction is that ω does not satisfy the "normal" condition (2.3), since (2.9) shows $\omega(0) = 0$. In the one-dimensional situation (see [3]), we can divide the inverse symbol by a factor containing its zero, such that the quotient has no zeros and is analytic in a neighbourhood of the unit circle as a function of e^{it}, but this approach fails for $d \geq 2$. Indeed, ω being an analytic function of at least two complex variables, its zero at $(\zeta_1, \ldots, \zeta_d) = (1, \ldots, 1)$ is not isolated and cannot be divided away globally on the analyticity domain of ω.

In the present multi-dimensional case we take a different approach, based on the possibility of dividing ω by its zero on the torus T^d, at the expense of loosing the analyticity of the quotient. Specifically, for all $t \in Q^d$, we have

$$\begin{aligned} \omega(t) &= \left\{4\textstyle\sum_{q=1}^d [\sin(t_q/2)]^2\right\}^m \omega_0(t) \\ &= \left\{\textstyle\sum_{q=1}^d \left(2 - e^{-it_q} - e^{it_q}\right)\right\}^m \omega_0(t), \end{aligned} \tag{2.10}$$

where

$$\omega_0(t) = \left\{\frac{\|t\|^2}{4\sum_{q=1}^d [\sin(t_q/2)]^2}\right\}^m \frac{1}{1 + \|t\|^{2m}\sum_{j\in\mathbb{Z}^d\setminus 0} \|t + 2\pi j\|^{-2m}}, \quad t \in Q^d. \tag{2.11}$$

We point out that (2.10) appears in the work of Rabut [17, Section II.3], [18, Eq. (4.40)], in connection with the problem of finding a polyharmonic multivariable analog of the one-dimensional B-spline basis (for this idea, see also Dyn and Levin [8], Chui, Jetter and Ward [6]).

Due to its first factor, the periodic quotient ω_0 is no longer analytic, but admits integrable partial derivatives of total order up to $d+1$. Furthermore, the asymptotic expansion of ω_0 as $t \to 0$ provides the following result, whose proof will be given in Section 4.

Lemma 2.1 *If $\{\widetilde{a}_k\}_{k\in\mathbb{Z}^d}$ denotes the sequence of Fourier coefficients of ω_0, there exists a constant $A > 0$ such that*

$$|\widetilde{a}_k| \le \frac{A}{(1+\|k\|)^{d+2}}, \qquad \forall\ k \in \mathbb{Z}^d. \tag{2.12}$$

In particular, this result guarantees an absolutely convergent Fourier series representation for ω_0. Moreover, from (2.11) we see that ω_0 takes only strictly positive values, which shows that the two "normal" conditions (2.3) and (2.4) hold for ω_0 replacing σ. Therefore we can apply the classical theory to obtain a factorization as (2.6) for ω_0, but more information is provided in the explicit construction below.

§ We proceed by considering the continuous symbol $\log\omega_0$, whose asymptotic expansion as $t \to 0$ is similar to that of ω_0. The proof of the next result is postponed to Section 4.

Lemma 2.2 *If $\{\lambda_k\}_{k\in\mathbb{Z}^d}$ is the sequence of Fourier coefficients of $\log\omega_0$, there exists a constant $A' > 0$ such that*

$$|\lambda_k| \le \frac{A'}{(1+\|k\|)^{d+2}}, \qquad \forall\ k \in \mathbb{Z}^d. \tag{2.13}$$

Consequently, $\log\omega_0$ has an absolutely convergent Fourier series

$$\log\omega_0(t) = \sum_{k\in\mathbb{Z}^d} \lambda_k e^{ikt}, \qquad t \in Q^d. \tag{2.14}$$

Using the procedure of Goldenstein and Gohberg [11], we define the "semi-space" symbols

$$\Lambda_\pm(t) := \frac{1}{2}\sum_{k'\in\mathbb{Z}^{d-1}} \lambda_{(k',0)} e^{ik't'} + \sum_{k\in\mathbb{Z}^{d-1}\times\{1,2,\ldots\}} \lambda_k e^{\pm ikt}, \qquad t \in Q^d, \tag{2.15}$$

$$\Omega_\pm(t) := \exp[\Lambda_\pm(t)], \qquad t \in Q^d, \tag{2.16}$$

and obtain the factorization

$$\omega_0(t) = \Omega_+(t)\Omega_-(t), \qquad t \in Q^d. \tag{2.17}$$

Since ω_0 and $\log\omega_0$ are even functions and $\Lambda_-(t) = \overline{\Lambda_+(t)} = \Lambda_+(-t)$, we also have

$$\Omega_-(t) = \overline{\Omega_+(t)} = \Omega_+(-t). \tag{2.18}$$

Thus, using the power series expansion of the exponential function and the fact that $\mathbb{Z}^{d-1}\times\mathbb{Z}_+$ is a semigroup with respect to addition, it follows that the factors $\Omega_\pm$ admit a Fourier series representation

$$\Omega_\pm(t) = \sum_{k\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} \alpha_k e^{\pm ikt}, \qquad t\in Q^d, \tag{2.19}$$

where $\{\alpha_k\}_{k\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} \in \ell^1(\mathbb{Z}^{d-1}\times\mathbb{Z}_+)$.

§ Having obtained a factorization of the type (2.6) for ω_0, we turn to the remaining factor of ω in (2.10), which is the symbol associated to the d-dimensional discrete Laplacian of order m. We treat this factor as a function of e^{it_d} which depends on the parameter $t' = (t_1,\ldots,t_{d-1})$, noting that

$$\begin{aligned}\sum_{q=1}^{d}(2-e^{-it_q}-e^{it_q}) &= -e^{-it_d}\left\{1-2\left(1+2\textstyle\sum_{q=1}^{d-1}[\sin(t_q/2)]^2\right)e^{it_d}+e^{2it_d}\right\} \\ &= -e^{-it_d}(e^{it_d}-\rho)(e^{it_d}-\rho^{-1}).\end{aligned} \tag{2.20}$$

Here $\rho=\rho(t')$ and $\rho^{-1}=\rho^{-1}(t')$ are the binomial roots

$$\rho^{\pm1}(t') := 1+2\sum_{q=1}^{d-1}[\sin(t_q/2)]^2 \mp 2\Theta(t'), \tag{2.21}$$

where

$$\Theta(t') := \left\{\textstyle\sum_{q=1}^{d-1}[\sin(t_q/2)]^2\right\}^{1/2}\left\{1+\textstyle\sum_{q=1}^{d-1}[\sin(t_q/2)]^2\right\}^{1/2}, \qquad t'\in Q^{d-1}. \tag{2.22}$$

Note that

$$0 < 2d-1-2\sqrt{d(d-1)} = \min_{t'\in Q^{d-1}}\rho(t') < 1 = \rho(0) = \max_{t'\in Q^{d-1}}\rho(t'). \tag{2.23}$$

Our task is now to express (2.20) as the product of two factors like in (2.6), the first being a polynomial in e^{it_d}, and the second a polynomial in e^{-it_d}, but we are faced with the problem of choosing between several such factorizations. Therefore, adopting a criterion that was systematically used by Krein in his fundamental work [12] on univariate Wiener-Hopf factorizations, we select the so-called "canonical" factorization

$$\sum_{q=1}^{d}\left(2-e^{-it_q}-e^{it_q}\right) = (\rho^{-1}-e^{it_d})(1-\rho e^{-it_d}), \tag{2.24}$$

which, up to a factor not depending on t_d, is the unique factorization of (2.20) of the form (2.6) such that, for each $t'\in Q^{d-1}$, the first (respectively second) factor has no zero for $|\zeta_d|<1$ (respectively $|\zeta_d|>1$) as a function of the complex variable $\zeta_d = e^{it_d}$. As will be seen from the proof of Proposition 3.2, it is the canonical factorization which ensures that our definition for L_j satisfies the semi-cardinal Lagrange conditions (1.11).

Lemma 2.3 *If $\{\theta_k\}_{k\in\mathbb{Z}^{d-1}}$ is the sequence of Fourier coefficients of the function Θ, then there exists a constant $A''>0$ such that*

$$|\theta_k| \le \frac{A''}{(1+\|k\|)^d}, \qquad \forall\ k\in\mathbb{Z}^{d-1}. \tag{2.25}$$

(The proof of this lemma is given in Section 4). Therefore the symbols ρ and ρ^{-1} possess the absolutely convergent Fourier series

$$\rho^{\pm 1}(t') = 1 + \sum_{q=1}^{d-1}\left(1 - \frac{1}{2}e^{-it_q} - \frac{1}{2}e^{it_q}\right) \mp 2 \sum_{k\in\mathbb{Z}^{d-1}} \theta_k\, e^{ikt'}, \qquad t' \in Q^{d-1}. \tag{2.26}$$

§ Using (2.10), (2.17) and (2.24), we obtain the factorization of ω in the form

$$\omega(t) = \omega_+(t)\omega_-(t), \qquad t \in Q^d, \tag{2.27}$$

where

$$\omega_+(t) := [\rho^{-1}(t') - e^{it_d}]^m \Omega_+(t), \quad \omega_-(t) := [1 - \rho(t')e^{-it_d}]^m \Omega_-(t). \tag{2.28}$$

By (2.19) and (2.26), ω_+ and ω_- admit absolutely convergent Fourier series representations. For further use, we let $\{\gamma_k\}_{k\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+}$ be the Fourier coefficients of ω_-, so

$$\omega_-(t) = \sum_{k\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} \gamma_k\, e^{-ikt} = \sum_{n=0}^{\infty} \Gamma_n(t')e^{-int_d}, \qquad t \in Q^d, \tag{2.29}$$

where

$$\Gamma_n(t') := \sum_{k'\in\mathbb{Z}^{d-1}} \gamma_{(k',n)} e^{-ik't'}, \qquad n \in \mathbb{Z}_+. \tag{2.30}$$

Note also that (2.18) implies the relation

$$\omega_-(t) := [\rho(t')]^m \omega_+(-t), \qquad t \in Q^d. \tag{2.31}$$

Guided by (2.8), we define, for each $j \in \mathbb{Z}^{d-1}\times\mathbb{Z}_+$, the "semi-space" symbol

$$\begin{aligned} \omega_j(t) &:= \omega_+(t)\, \mathrm{P}_+\left\{e^{ijt}\omega_-(t)\right\} \\ &= \sum_{k\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} b_{jk}e^{ikt}, \qquad t \in Q^d, \end{aligned} \tag{2.32}$$

which will be used in the construction of L_j. The above considerations ensure that the last series is absolutely convergent, where the Fourier coefficients b_{jk}, $k \in \mathbb{Z}^{d-1}\times\mathbb{Z}_+$, can be computed explicitly in terms of those of ω_+ and ω_-.

Lemma 2.4 a) *For all $j = (j', j_d) \in \mathbb{Z}^{d-1}\times\mathbb{Z}_+$, the following identity holds:*

$$\omega_{(0,j_d)}(t) = e^{-ij't'}\omega_j(t), \qquad t \in Q^d. \tag{2.33}$$

b) *There exist positive constants M_1, M_2, which depend only on m and d, such that, for all $j \in \mathbb{Z}^{d-1}\times\mathbb{Z}_+$ and all $t \in Q^d$, we have the inequalities:*

$$|\omega_j(t)| \le M_1, \tag{2.34}$$

$$|\omega_j(t)| \le M_2\|t\|^m. \tag{2.35}$$

c) *There exists a constant* $B>0$*, such that the Fourier coefficients of the symbols* ω_j*,* $j\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+$*, satisfy*

$$\sup_{j\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+}\ \textstyle\sum_{k\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+}|b_{jk}|\le B. \tag{2.36}$$

Proof. a) This is a direct consequence of the fact that, for any $q\in\{1,\dots,d-1\}$, multiplication by e^{itq} commutes with the semi-space projection P_+.

b) For all $t\in Q^d$, we have the estimates

$$|\rho^{-1}-e^{it_d}|\le|\rho^{-1}|+|e^{it_d}|\le 2d+2\sqrt{d(d-1)}, \tag{2.37}$$

$$\begin{aligned}|\rho^{-1}-e^{it_d}| &\le |\rho^{-1}-1|+|1-e^{it_d}|\\ &\le 2(\sqrt{d-1}+\sqrt{d})\left\{\sum_{q=1}^{d-1}[\sin(t_q/2)]^2\right\}^{1/2}+2|\sin(t_d/2)|\\ &\le (\sqrt{d-1}+\sqrt{d})\,\|t'\|+|t_d| \quad\le\quad \left(1+\sqrt{d-1}+\sqrt{d}\right)\,\|t\|,\end{aligned} \tag{2.38}$$

$$\begin{aligned}\left|\mathrm{P}_+\left\{e^{ijt}\omega_-(t)\right\}\right| &= \left|\mathrm{P}_+\left\{e^{ijt}\sum_{k\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+}\gamma_k e^{-ikt}\right\}\right|\\ &\le \sum_{k\in\mathbb{Z}^{d-1}\times\{0,\dots,j_2\}}|\gamma_k| \quad\le\quad \Gamma,\end{aligned} \tag{2.39}$$

where $\Gamma:=\sum_{k\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+}|\gamma_k|$. Setting $M_0:=\max\{|\Omega_+(t)|:t\in Q^d\}$, we obtain (2.34) and (2.35) from the above estimates and (2.32), where $M_1:=\left(2d+2\sqrt{d(d-1)}\right)^m M_0\Gamma$ and $M_2:=\left(1+\sqrt{d-1}+\sqrt{d}\right)^m M_0\Gamma$.

c) Using the absolutely convergent Fourier representation of ω_+,

$$\omega_+(t)=:\sum_{k\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+}\beta_k\,e^{ikt},\qquad t\in Q^d, \tag{2.40}$$

from (2.32) we have

$$\omega_j(t)=\left(\sum_{k\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+}\beta_k\,e^{ikt}\right)\left(\sum_{k_d=0}^{j_d}\sum_{k'\in\mathbb{Z}^{d-1}}\gamma_k e^{i(j-k)t}\right). \tag{2.41}$$

Thus a standard convolution argument implies

$$\begin{aligned}\sum_{k\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+}|b_{jk}| &\le \left(\sum_{k\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+}|\beta_k|\right)\left(\sum_{k_d=0}^{j_d}\sum_{k'\in\mathbb{Z}^{d-1}}|\gamma_k|\right)\\ &\le \left(\sum_{k\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+}|\beta_k|\right)\left(\sum_{k\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+}|\gamma_k|\right) \quad=:\ B,\end{aligned} \tag{2.42}$$

which proves (2.36). □

Remark 2.5 By (2.32) and (2.33), we have $b_{jk}=b_{kj}$, $\forall\ j,k\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+$.

3 Lagrange functions for polyharmonic semi-cardinal interpolation

§ In order to define, for each $j \in \mathbb{Z}^{d-1}\times\mathbb{Z}_+$, an m-harmonic semi-cardinal spline L_j satisfying the Lagrange conditions (1.11), we distinguish two cases.

First, assume that the parameters $d, m \in \{2, 3, \ldots\}$ are such that $m < d$. Then the estimates of Lemma 2.4 b) guarantee the integrability of $\|\cdot\|^{-2m}\omega_j$ over $\mathbb{R}^d$ and we define $L_j : \mathbb{R}^d \to \mathbb{R}$ as the classical inverse Fourier transform

$$L_j(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ixt}\,\frac{\omega_j(-t)}{\|t\|^{2m}}\,dt, \qquad x \in \mathbb{R}^d. \tag{3.1}$$

In the second case, assume $m \geq d$. We start with the auxiliary function $F_j : \mathbb{R}^d \to \mathbb{R}$, defined as the "modified" inverse Fourier transform

$$F_j(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \left\{ e^{ixt} - \sum_{q=0}^{m-d} \frac{(ixt)^q}{q!} \right\} \frac{\omega_j(-t)}{\|t\|^{2m}}\,dt, \qquad x \in \mathbb{R}^d. \tag{3.2}$$

Note that, for each x and j, the estimates of Lemma 2.4 b) together with

$$\left| e^{ixt} - \textstyle\sum_{q=0}^{m-d}(ixt)^q/(q!) \right| = \begin{cases} O(\|t\|^{m-d+1}), & \text{for } \|t\| \to 0, \\ O(\|t\|^{m-d}), & \text{for } \|t\| \to \infty, \end{cases} \tag{3.3}$$

show that $F_j(x)$ is defined by an absolutely convergent integral. Further, any partial derivative with respect to x of total order up to $2m-d-1$ of the integrand remains absolutely integrable on $\mathbb{R}^d$, so $F_j \in C^{2m-d-1}(\mathbb{R}^d)$. Next, for each $n \in \mathbb{Z}_+$, consider the set of lattice points

$$\mathcal{G}(n) := \{k = (k_1, \ldots, k_d) \in \mathbb{Z}_+^d : k_1 + \ldots + k_d \leq n\}. \tag{3.4}$$

A basic result in multivariable polynomial interpolation (Nicolaides [15]) asserts that $\mathcal{G}(n)$ is Π_n^d-unisolvent, i.e. given any set of scalars $\{y_k\}_{k\in\mathcal{G}(n)}$, there exists a unique polynomial $p \in \Pi_n^d$ such that $p(k) = y_k$, for all $k \in \mathcal{G}(n)$. We thus select the unique polynomial $p_j \in \Pi_{m-d}^d$ which satisfies

$$p_j(k) = \delta_{jk} - F_j(k), \qquad \forall\, k \in \mathcal{G}(m-d), \tag{3.5}$$

and define the Lagrange function L_j by the formula

$$L_j(x) := p_j(x) + F_j(x), \qquad x \in \mathbb{R}^d. \tag{3.6}$$

Remark 3.1 The above definitions guarantee the property $L_j \in C^{2m-d-1}(\mathbb{R}^d)$ in both cases. Note that, by making the conventions $p_j := 0$ (or $\Pi_{m-d}^d := \emptyset$) and $\sum_{q=0}^{m-d} := 0$ for $m < d$, we can treat (3.1) as a special case of (3.6).

A useful property that follows from the above definition of L_j and (2.33) is the identity:

$$L_{(0,j_d)}(x - (j',0)) = L_j(x), \qquad \forall\, j = (j', j_d) \in \mathbb{Z}^{d-1}\times\mathbb{Z}_+,\ \forall\, x \in \mathbb{R}^d. \tag{3.7}$$

§ The proof of the next result will make use of a class of finite difference operators, described as follows. For $m \geq d$, let

$$\mathcal{H}(m,d) := \{\nu = (\nu_1, \dots, \nu_d) \in \mathbb{Z}_+^d : \nu_1 + \dots + \nu_d = m - d + 1\}. \tag{3.8}$$

For each $\nu \in \mathcal{H}(m,d)$, let $\mathcal{I}_\nu = \{\mu \in \mathbb{Z}_+^d : \mu_q \leq \nu_q,\ q = 1, \dots, d\}$ and let $\{c_\mu\}_{\mu \in \mathcal{I}_\nu}$ be the sequence of coefficients generated by the formal monomial expansion

$$\prod_{q=1}^{d} (\zeta_q - 1)^{\nu_q} = \sum_{\mu \in \mathcal{I}_\nu} c_\mu \zeta_1^{\mu_1} \cdots \zeta_d^{\mu_d}. \tag{3.9}$$

Further, define the finite difference operator T^ν by

$$(T^\nu y)(k) = \sum_{\mu \in \mathcal{I}_\nu} c_\mu y(k + \mu), \qquad \forall\ k \in \mathbb{Z}^{d-1} \times \mathbb{Z}_+, \tag{3.10}$$

for any sequence $y = \{y(k)\}_{k \in \mathbb{Z}^{d-1} \times \mathbb{Z}_+}$. Note that

$$T^\nu y = 0, \qquad \text{if } y = \{p(k)\}_{k \in \mathbb{Z}^{d-1} \times \mathbb{Z}_+},\ p \in \Pi_{m-d}^d. \tag{3.11}$$

Also, if $m < d$, set $\mathcal{H}(m,d) := \{0\}$ and let T^0 be the identity operator on the set of sequences indexed by $\mathbb{Z}^{d-1} \times \mathbb{Z}_+$.

Proposition 3.2 *For each $j \in \mathbb{Z}^{d-1} \times \mathbb{Z}_+$, the function L_j defined above satisfies the "semi-space cardinal conditions":*

$$L_j(k) = \delta_{jk}, \qquad \forall\ k \in \mathbb{Z}^{d-1} \times \mathbb{Z}_+. \tag{3.12}$$

Proof. Due to (3.7), we can assume, without loss of generality, that $j = (0, j_d)$, where $j_d \in \mathbb{Z}_+$. From the definition of L_j and (3.11) it follows that, for any $k \in \mathbb{Z}^{d-1} \times \mathbb{Z}_+$ and $\nu \in \mathcal{H}(m,d)$, we have

$$(T^\nu L_j)(k) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ikt} \prod_{q=1}^{d} (e^{it_q} - 1)^{\nu_q} \frac{\omega_j(-t)}{\|t\|^{2m}}\, dt. \tag{3.13}$$

The last integral can be transformed as follows:

$$\begin{aligned} &\int_{\mathbb{R}^d} e^{ikt} \prod_{q=1}^{d} (e^{it_q} - 1)^{\nu_q} \frac{\omega_j(-t)}{\|t\|^{2m}}\, dt \\ &= \sum_{l \in \mathbb{Z}^2} \int_{Q^d} e^{ikt} \prod_{q=1}^{d} (e^{it_q} - 1)^{\nu_q} \frac{\omega_j(-t)}{\|t - 2\pi l\|^{2m}}\, dt \\ &= \int_{Q^d} e^{ikt} \prod_{q=1}^{d} (e^{it_q} - 1)^{\nu_q} \omega_j(-t) \sum_{l \in \mathbb{Z}^2} \|t - 2\pi l\|^{-2m}\, dt \\ &= \int_{Q^d} e^{-ikt} \prod_{q=1}^{d} (e^{-it_q} - 1)^{\nu_q} \frac{\omega_j(t)}{\omega(t)}\, dt \\ &= \int_{Q^d} e^{-ikt} \prod_{q=1}^{d} (e^{-it_q} - 1)^{\nu_q} \frac{\mathrm{P}_+\{e^{ijt}\omega_-(t)\}}{\omega_-(t)}\, dt. \end{aligned} \tag{3.14}$$

The second of the above equalities is an application of Lebesgue's dominated convergence theorem (since $2m > d$), while the last two equalities follow from (2.9), (2.27) and (2.32). Using the expansion (2.29), we have

$$\text{(3.15)}\qquad \begin{aligned} \mathrm{P}_+\{e^{ijt}\omega_-(t)\} &= \mathrm{P}_+\left\{e^{ijt}\sum_{n=0}^{\infty}\Gamma_n(t')e^{-int_d}\right\} \\ &= e^{ijt}\omega_-(t) - \sum_{n=1}^{\infty}\Gamma_{n+j_d}(t')e^{-int_d}, \end{aligned}$$

so (3.13)–(3.15) imply

$$\text{(3.16)}\quad \begin{aligned} (T^\nu L_j)(k) &= (T^\nu \delta_j)(k) \\ &\quad - \frac{1}{(2\pi)^d}\int_{Q^d} e^{-ikt}\prod_{q=1}^{d}(e^{-it_q}-1)^{\nu_q}\,\frac{\sum_{n=1}^{\infty}\Gamma_{n+j_d}(t')e^{-int_d}}{[1-\rho(t')e^{-it_d}]^m\Omega_-(t)}\,dt, \end{aligned}$$

where δ_j is the sequence given by $\delta_j(j) = 1$, $\delta_j(l) = 0$, $\forall\, l \in (\mathbb{Z}^{d-1}\times\mathbb{Z}_+)\setminus\{j\}$, and we have used the definition (2.28) for $\omega_-(t)$.

We treat the last integrand as a function of t_d that depends on the parameter t'. Since, by (2.16), $[\Omega_-(t)]^{-1} = \exp[-\Lambda_-(t)]$, it follows , as for (2.19), that $[\Omega_-(t)]^{-1}$ has an absolutely convergent expansion in powers e^{-ikt}, $k \in \mathbb{Z}^{d-1}\times\mathbb{Z}_+$. Thus we can arrange this expansion as an absolutely convergent series in non-negative powers of e^{-it_d}. Further, for each $t' \neq 0$, definition (2.21) provides $0 < \rho(t') < 1$, which ensures the expansion

$$\text{(3.17)}\qquad [1-\rho(t')e^{-it_d}]^{-m} = \sum_{n=0}^{\infty}\frac{(n+m-1)!}{n!(m-1)!}[\rho(t')]^n e^{-int_d}.$$

Consequently, for each $t' \neq 0$, the fractional factor of the integrand of (3.16) has an absolutely convergent expansion in strictly positive powers of e^{-it_d}:

$$\text{(3.18)}\qquad [1-\rho(t')e^{-it_d}]^{-m}[\Omega_-(t)]^{-1}\sum_{n=1}^{\infty}\Gamma_{n+j_d}(t')e^{-int_d} =: \sum_{n=1}^{\infty}\Upsilon_n(t')e^{-int_d},$$

where each Υ_n also depends on j. Since $\nu_d \geq 0$ and $k_d \geq 0$, for each $t' \neq 0$ we have

$$\text{(3.19)}\qquad \begin{aligned} &\int_{Q^{d-1}} e^{-ik_d t_d}\,(e^{-it_d}-1)^{\nu_d}\sum_{n=1}^{\infty}\Upsilon_n(t')e^{-int_d}\,dt_d \\ &\quad= \sum_{n=1}^{\infty}\Upsilon_n(t')\int_{Q^{d-1}}(e^{-it_d}-1)^{\nu_d}e^{-i(k_d+n)t_d}\,dt_d \;=\; 0, \end{aligned}$$

so (3.16), (3.18), (3.19) and Fubini's theorem for Lebesgue integrals imply

$$\text{(3.20)}\qquad (T^\nu L_j)(k) = (T^\nu\delta_j)(k), \qquad \forall\, k \in \mathbb{Z}^{d-1}\times\mathbb{Z}_+.$$

If $m < d$, then (3.20) coincides with (3.12).

In the remaining case, $m \geq d$, we first prove that relations (3.12) hold for $k \in \mathbb{Z}_+^d$, using induction over the sum of the components of k. For $n \geq m-d$, consider the assertion $A(n)$: "$L_j(k) = \delta_{jk}$ holds for all lattice points $k \in \mathcal{G}(n)$". Since (3.5) provides $L_j(k) = \delta_{jk}$ for $k \in \mathcal{G}(m-d)$, it follows that $A(m-d)$ is true. Next, assume $A(n)$ is true and let $k \in \mathbb{Z}_+^d$ such that $|k| = n+1$. Then there exists at least one element $\nu_0 \in \mathcal{H}(m,d)$ such that $k - \nu_0 \in \mathbb{Z}_+^d$. Replacing k by $k-\nu_0$ and ν by ν_0 in (3.20) and taking account of (3.10), we obtain

$$0 = [T^{\nu_0}(L_j - \delta_j)](k-\nu_0) = \sum_{\mu \in \mathcal{I}_{\nu_0}} c_\mu (L_j - \delta_j)(k - \nu_0 + \mu). \tag{3.21}$$

On the other hand, for any $\mu \in \mathcal{I}_{\nu_0} \setminus \{\nu_0\}$, we have $|k - \nu_0 + \mu| \le n$, and therefore $(L_j - \delta_j)(k - \nu_0 + \mu) = 0$ by the inductive hypothesis. Then (3.21) implies $0 = (L_j - \delta_j)(k)$, and we deduce that $A(n+1)$ is true. Thus the statement at the beginning of this paragraph is proved.

The complete proof of (3.12) when $m \ge d$ now follows from the result of the previous paragraph and the observation that (3.20) remains true if we modify the definition of the finite difference T^ν by replacing e^{it_q} with e^{-it_q} for any $q \in \{1, \ldots, d-1\}$. $\square$

Proposition 3.3 *For any $j \in \mathbb{Z}^{d-1} \times \mathbb{Z}_+$, we have $L_j \in SH_m^+(\mathbb{R}^d) \cap L_m^2(\mathbb{R}^d)$.*

Proof. We first show that the continuous function L_j is of polynomial growth and thus belongs to the space of tempered distributions. For this, note that differentiation with respect to x under the integral sign in (3.1) and (3.2) gives

$$(D^\mu L_j)(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ixt} \, \frac{t^\mu \omega_j(-t)}{\|t\|^{2m}} \, dt, \qquad x \in \mathbb{R}^d, \tag{3.22}$$

for any multi-index μ such that $\max\{0, m-d+1\} \le |\mu| \le 2m-d-1$. Using the estimates of Lemma 2.4 b), we deduce that there exists a constant $M > 0$ such that

$$|(D^\mu L_j)(x)| \le M, \qquad \forall\, x \in \mathbb{R}^d, \ \ \forall\, j \in \mathbb{Z}^{d-1} \times \mathbb{Z}_+, \tag{3.23}$$

for any multi-index μ as above. In particular, if $m < d$, L_j is a tempered distribution, being bounded on $\mathbb{R}^d$.

If $m \ge d$, we start by establishing the existence of a constant $\widetilde{M} > 0$ such that

$$\max_{\alpha \in \mathcal{G}(m-d)} |(D^\alpha L_j)(0)| \le \widetilde{M}, \qquad \forall\, j \in \mathbb{Z}^{d-1} \times \mathbb{Z}_+. \tag{3.24}$$

Since (3.2) provides $(D^\alpha F_j)(0) = 0$, $\forall\, \alpha \in \mathcal{G}(m-d)$, it follows that we need to prove

$$\max_{\alpha \in \mathcal{G}(m-d)} |(D^\alpha p_j)(0)| \le \widetilde{M}, \qquad \forall\, j \in \mathbb{Z}^{d-1} \times \mathbb{Z}_+, \tag{3.25}$$

where p_j is the polynomial from (3.6). To this aim, note that (3.2) and the estimates of Lemma 2.4 b) imply the existence of a constant $M' > 0$ such that

$$|F_j(k)| \le M', \qquad \forall\, k \in \mathcal{G}(m-d), \ \forall\, j \in \mathbb{Z}^{d-1} \times \mathbb{Z}_+, \tag{3.26}$$

so the defining conditions (3.5) lead to

$$|p_j(k)| \le |\delta_{jk}| + |F_j(k)| \le 1 + M', \qquad \forall\, k \in \mathcal{G}(m-d), \ \forall\, j \in \mathbb{Z}^{d-1} \times \mathbb{Z}_+. \tag{3.27}$$

Thus, writing p_j in monomial form, $p_j(x) =: \sum_{\alpha\in\mathcal{G}(m-d)} N_{j\alpha}x^\alpha$, it is straightforward to deduce the uniform estimate

$$\max_{\alpha\in\mathcal{G}(m-d)} |N_{j\alpha}| \leq M'', \qquad \forall\, j \in \mathbb{Z}^{d-1}\times\mathbb{Z}_+, \tag{3.28}$$

for some constant $M'' > 0$. Since $(D^\alpha p_j)(0) = \alpha! N_{j\alpha}$, we obtain (3.25) and hence (3.24). Next, if μ is a multi-index with $|\mu| = m - d$, we use (3.24) and the Newton-Leibniz formula to obtain

$$\begin{aligned} |(D^\mu L_j)(x)| &= \left| (D^\mu L_j)(0) + \int_0^1 x\,[\nabla(D^\mu L_j)](\tau x)\,d\tau \right| \\ &\leq \widetilde{M} + M\|x\|, \quad \forall\, x \in \mathbb{R}^d,\ \forall\, j \in \mathbb{Z}^{d-1}\times\mathbb{Z}_+, \end{aligned} \tag{3.29}$$

since any component of the gradient $\nabla(D^\mu L_j)$ is a partial derivative of L_j of total order $m - d + 1$, to which (3.23) applies.

Arguing successively as in (3.29), there exists a constant $M > 0$ (the same as in (3.23)) such that

$$|(D^\mu L_j)(x)| \leq M(1 + \|x\|)^{m-d+1-|\mu|}, \qquad \forall\, x \in \mathbb{R}^d,\ \forall\, j \in \mathbb{Z}^{d-1}\times\mathbb{Z}_+, \tag{3.30}$$

for any multi-index $\mu \in \mathcal{G}(m - d)$. For $\mu = 0$, this shows that each L_j is of polynomial growth on $\mathbb{R}^d$, hence it belongs to $\mathcal{S}'$.

We now make use the Fourier transform of the tempered distribution L_j in order to show that $L_j \in SH_m^+(\mathbb{R}^d) \cap L_m^2(\mathbb{R}^d)$.

If $m < d$, then, due to the definition (3.1), $\widehat{L_j}$ coincides with the L^1-function $\omega_j(-t)/\|t\|^{2m}$ (defined everywhere on $\mathbb{R}^d$ except 0). Thus, for any multi-index α with $|\alpha| = m$, the following identity holds in $\mathcal{S}'$:

$$(D^\alpha L_j)\hat{} = t^\alpha \widehat{L_j} = \frac{t^\alpha \omega_j(-t)}{\|t\|^{2m}}. \tag{3.31}$$

Since the estimates of Lemma 2.4 b) show that the above right-hand side belongs to $L^2(\mathbb{R}^d)$ if $|\alpha| = m$, by Plancherel's theorem we have $L_j \in L_m^2(\mathbb{R}^d)$. Further, we also have the distributional identity

$$(\Delta^m L_j)\hat{} = \|t\|^{2m}\widehat{L_j} = \omega_j(-t), \tag{3.32}$$

so taking the inverse Fourier transform of the periodic distribution in the right-hand side, we obtain

$$\Delta^m L_j = \sum_{k\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} b_{jk}\,\delta_k, \tag{3.33}$$

where the last series converges in $\mathcal{S}'$ and δ_k is the translated Dirac distribution defined by $\langle\delta_k, \eta\rangle = \eta(k)$ for any test function η. We deduce that condition (1.3) is satisfied by L_j, which is thus an m-harmonic cardinal spline, as claimed.

If $m \geq d$, then (3.22) implies that the distributional Fourier transform $(D^\mu L_j)\hat{}$ coincides with the L^1-function $t^\mu\omega_j(-t)/\|t\|^{2m}$ whenever $|\mu| = m - d + 1$. Given any multi-index α with $|\alpha| = m$, we can write $D^\alpha = D^\nu D^\mu$ for two multi-indices ν, μ such that $|\nu| = d-1$, $|\mu| = m-d+1$, and compute

$$(D^\alpha L_j)\hat{} = t^\nu (D^\mu L_j)\hat{} = \frac{t^\alpha \omega_j(-t)}{\|t\|^{2m}}. \tag{3.34}$$

Thus $L_j \in L_m^2(\mathbb{R}^d)$. Also,

$$(\Delta^m L_j)\hat{} = \sum_{q=1}^{d} t_q^{m+d-1} (D_q^{m-d+1} L_j)\hat{} = \sum_{q=1}^{d} t_q^{m+d-1} \frac{t_q^{m-d+1}\omega(-t)}{\|t\|^{2m}} = \omega_j(-t), \tag{3.35}$$

from which, as in the previous paragraph, $L_j \in SH_m^+(\mathbb{R}^d)$. This proves Proposition 3.3. □

§ The following theorem provides a solution to the semi-cardinal interpolation problem for polyharmonic splines, when the data sequence is absolutely summable.

Theorem 3.4 *Assume that* $\{f_j\}_{j\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} \subset \mathbb{R}$ *satisfies* $\sum_{j\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} |f_j| < \infty$. *Then the series*

$$s(x) = \sum_{j\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} f_j L_j(x), \qquad x \in \mathbb{R}^d, \tag{3.36}$$

is absolutely and uniformly convergent over compact subsets of $\mathbb{R}^d$, *we have* $s(k) = f_k$, $\forall\, k \in \mathbb{Z}^{d-1}\times\mathbb{Z}_+$, *and* $s \in SH_m^+(\mathbb{R}^d) \cap L_m^2(\mathbb{R}^d)$.

Proof. Since the data sequence $\{f_j\}_{j\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+}$ is absolutely summable and the constant M of estimates (3.23) and (3.30) is independent of j, we deduce: first, that the series (3.36) is absolutely and uniformly convergent over compact subsets of $\mathbb{R}^d$, and hence, by permitted termwise differentiation, $s \in C^{2m-d-1}$; second, that the series (3.36) converges in $\mathcal{S}'$, so s is a tempered distribution. The property $s(k) = f_k$, $\forall\, k \in \mathbb{Z}^{d-1}\times\mathbb{Z}_+$, is now a consequence of (3.12).

On the other hand, from (2.34) and the absolute summability of the data sequence, it follows that the series

$$\sigma_f(t) := \sum_{j\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} f_j \omega_j(-t), \qquad t \in \mathbb{R}^d, \tag{3.37}$$

is convergent in $\mathcal{S}'$, as well as being absolutely and uniformly convergent for $t \in \mathbb{R}^d$. Letting α be any multi-index such that $|\alpha| = m$, we obtain in $\mathcal{S}'$:

$$(D^\alpha s)\hat{} = \sum_{j\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} f_j (D^\alpha L_j)\hat{} = \sum_{j\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} f_j \frac{t^\alpha \omega_j(-t)}{\|t\|^{2m}} = \frac{t^\alpha}{\|t\|^{2m}} \sigma_f(t). \tag{3.38}$$

Since (2.35) implies

$$|\sigma_f(t)| \le M_2 \|t\|^m \sum_{j\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} |f_j|, \qquad t \in Q^d, \tag{3.39}$$

we deduce $(D^\alpha s)\hat{} \in L^2(\mathbb{R}^d)$ and, by Plancherel's theorem, $s \in L_m^2(\mathbb{R}^d)$. Similarly,

$$(\Delta^m s)\hat{} = \sigma_f. \tag{3.40}$$

Note that (2.32) implies the following Fourier series representation for σ_f:

$$\sigma_f(t) = \sum_{j\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} f_j \left(\sum_{k\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} b_{jk} e^{-ikt} \right) = \sum_{k\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} c_k e^{-ikt}, \tag{3.41}$$

where $c_k := \sum_{j\in\mathbb{Z}^{d-1}\times\mathbb{Z}_+} f_j b_{jk}$ and the two repeated series are absolutely convergent and equal due to Lemma 2.4 c). As in the proof of the previous result (for L_j), we now deduce $s \in SH_m^+(\mathbb{R}^d)$, which completes the proof of the theorem. □

4 Proofs for Lemmas 2.1, 2.2 and 2.3

§ In order to prove Lemmas 2.1–2.3, we apply a well-known result in asymptotic analysis, concerning the decay rate of the d-dimensional inverse Fourier transform

$$\widetilde{f}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ixt} f(t)\, dt, \qquad x \in \mathbb{R}^d, \tag{4.1}$$

as $\|x\| \to \infty$. In each case, f is locally integrable in $\mathbb{R}^d$, but its partial derivatives of a certain order admit singularities.

The following theorem is taken from the textbook [22], in which f needs not belong to $L^1(\mathbb{R}^d)$, being assumed only that the Cauchy limit

$$\lim_{\epsilon\to\infty} \int_{\|t\|<\epsilon} e^{ixt} f(t)\, dt \tag{4.2}$$

exists for every $x \in \mathbb{R}^d$. Further, let f satisfy the following three conditions.

(C_1) *$f \in C^{2n}(\mathbb{R}^d \setminus \{0\})$, for some $n \in \mathbb{Z}_+$. Also, f is expressible in the form*

$$f(t) = \sum c_{pq} t^p \|t\|^q + \varphi(t), \tag{4.3}$$

where the q's are real numbers, the p's are multi-indices and the sum is finite. For all p and q under the summation, we require $q + p > -d$, and if $\kappa = \max\{q + |p|\}$, then $2n - 1 \le \kappa + d < 2n + 1$.

(C_2) *As $\|t\| \to 0$, the remainder φ in (4.3) satisfies*

$$(\Delta^j \varphi)(t) = O(\|t\|^{\kappa-2j+1}), \qquad j = 0, 1, \ldots, n, \tag{4.4}$$

if $\kappa + d \ne 2n - 1$, and

$$(\Delta^j \varphi)(t) = O(\|t\|^{\kappa-2j+2}), \qquad j = 0, 1, \ldots, n, \tag{4.5}$$

if $\kappa + d = 2n - 1$, where Δ is the Laplacian operator.

(C_3) *For some $\epsilon > 0$, each of the integrals*

$$\int_{\|t\|\ge\epsilon} e^{ixt} (\Delta^j f)(t)\, dt, \qquad j = 0, 1, \ldots, n, \tag{4.6}$$

converges uniformly for all sufficiently large $\|x\|$.

The above conditions ensure that the Fourier integral (4.1) converges uniformly for all sufficiently large values of $\|x\|$.

Theorem 4.1 (Wong [22, p. 503]) *Under the conditions* (C_1), (C_2) *and* (C_3), *there exist real constants A_q and B_q which depend on d, such that*

$$(4.7) \qquad \begin{aligned} \widetilde{f}(x) &= {\sum}' c_{pq} A_q D^p(\|x\|^{-q-d}) \\ &\quad + {\sum}'' c_{pq} B_q D^p(\|x\|^{2l} \log \|x\|) + o(\|x\|^{-2n}), \qquad \text{as } \|x\| \to \infty, \end{aligned}$$

where $\sum'$ excludes those q's for which $q+d$ is a negative even integer and $\sum''$ includes only those q's for which $q+d=-2l$, a negative even integer.

For the rigorous proof of this theorem using the theory of distributions, see Wong [21, 22] (a similar result is used by Buhmann [5, Lemma 5] in his work on cardinal interpolation).

§ In Lemmas 2.1 and 2.2, we have to estimate the magnitude of the Fourier coefficients $\{\sigma_k\}_{k\in\mathbb{Z}^d}$ of a certain continuous symbol $\sigma : Q^d \to \mathbb{R}$ (Lemma 2.3 deals with a $(d-1)$-dimensional symbol). Since the Fourier coefficients are given by the formula

$$(4.8) \qquad \sigma_k = \frac{1}{(2\pi)^d} \int_{Q^d} e^{ikt} \sigma(t)\, dt, \qquad k \in \mathbb{Z}^d,$$

in order to apply Theorem 4.1 we have to write the last integral as an integral over $\mathbb{R}^d$ of a function that coincides with σ near 0. This well-known procedure uses a partition of unity subordinated to the covering $\{Q^d + 2\pi j : j \in \mathbb{Z}^d\}$ of $\mathbb{R}^d$, as described for example by Friedlander [9, Section 8.5] or Powell [16, Lemma 6.3]. Specifically, there exists $\psi \in C_0^\infty(\mathbb{R}^d)$ such that $\operatorname{supp}(\psi) \subset \operatorname{int}(Q^d)$, $\psi|_{\delta Q^d} \equiv 1$ for some constant $\delta > 0$, and $\sum_{j\in\mathbb{Z}^d} \psi(t - 2\pi j) = 1$ for all $t \in \mathbb{R}^d$. For any symbol σ, the integral in (4.8) is then written

$$(4.9) \qquad \int_{Q^d} e^{ikt}\sigma(t)\, dt = \int_{Q^d} e^{ikt}\sigma(t) \sum_{j\in\mathbb{Z}^d} \psi(t-2\pi j)\, dt = \int_{\mathbb{R}^d} e^{ikt}\sigma(t)\psi(t)\, dt,$$

where we have used the periodicity of σ.

Proof of Lemma 2.1. Define $\vartheta : Q^d \to \mathbb{R}$ by

$$(4.10) \qquad \vartheta(t) = \left(1 + \|t\|^{2m} \sum_{j\in\mathbb{Z}^d\setminus\{0\}} \|t + 2\pi j\|^{-2m}\right)^{-1}, \qquad t \in Q^d,$$

and note that ϑ is a factor of ω_0 in (2.11). Since ϑ is a C^∞ function, any singularity in the derivatives of ω_0 is due to the remaining factor, which admits the following asymptotic expansion as $t \to 0$:

$$(4.11) \qquad \begin{aligned} \omega_0(t)/\vartheta(t) &= \left\{ \frac{\|t/2\|^2}{\sum_{l=1}^d [\sin(t_l/2)]^2} \right\}^m \\ &= \left\{ \frac{\|t/2\|^2}{\sum_{l=1}^d [(t_l/2) - \frac{1}{3!}(t_l/2)^3 + \frac{1}{5!}(t_l/2)^5 - \ldots]^2} \right\}^m \end{aligned}$$

$$= \left\{1 - \|t\|^{-2}\left[\frac{1}{12}\sum_{l=1}^{d} t_l^4 - \frac{1}{360}\sum_{l=1}^{d} t_l^6 + \dots\right]\right\}^{-m}$$

$$= 1 + \left[\frac{m\sum_{l=1}^{d} t_l^4}{12\|t\|^2}\right] - \left[\frac{m\sum_{l=1}^{d} t_l^6}{360\|t\|^2} - \frac{m(m+1)(\sum_{l=1}^{d} t_l^4)^2}{288\|t\|^4}\right] + \dots .$$

The r-th term of the last expansion is of magnitude $O(\|t\|^{2r-2})$ and the first three terms are shown explicitly. Guided by this expansion, we check the conditions of Theorem 4.1 for $f := (\omega_0 - 1)\psi$, where ψ is the auxiliary function described in the previous paragraph. To this aim, we define

$$\varphi(t) := f(t) - \frac{m}{12}\|t\|^{-2}\sum_{l=1}^{d} t_l^4, \tag{4.12}$$

as well as $q := -2$, and $p_l := 4u_l$ for $l = 1, \dots, d$, where u_l is the unit l-th coordinate vector. Thus condition (C_1) is satisfied by letting n be the integer part of $(d+3)/2$. Further, multiplying (4.11) with the asymptotic expansion of ϑ as $t \to 0$, namely

$$\begin{aligned} \vartheta(t) \quad = \quad & 1 - \left[\|t\|^{2m}\sum_{j\in\mathbb{Z}^d\setminus\{0\}}\|2\pi j\|^{-2m}\right] + \left[m\|t\|^{2m+2}\sum_{j\in\mathbb{Z}^d\setminus\{0\}}\|2\pi j\|^{-2m-2}\right. \\ & \left. - \frac{m(m+1)}{2\pi^2}\|t\|^{2m}\sum_{j\in\mathbb{Z}^d\setminus\{0\}}\frac{(jt)^2}{(2\pi)^{2m}\|j\|^{2m+4}}\right] + \dots , \end{aligned} \tag{4.13}$$

and applying termwise differentiation to the ensuing expansion, we obtain the property

$$(\Delta^j\varphi)(t) = O(\|t\|^{4-2j}), \qquad \forall\, j \in \mathbb{Z}_+ . \tag{4.14}$$

This shows that (C_2) is also verified, since $\kappa = 2$ in the present case. Condition (C_3) is ensured by the compact support of the auxiliary function ψ.

Applying Theorem 4.1 we obtain, as $\|x\| \to \infty$,

$$\widetilde{f}(x) = C\frac{m}{12}\begin{cases} (D_1^4 + D_2^4)(\log\|x\|), & \text{if } d = 2 \\ (\sum_{l=1}^{d} D_l^4)(\|x\|^{2-d}), & \text{if } d \geq 3 \end{cases} + o(\|x\|^{-d-3}), \tag{4.15}$$

for some constant $C = C(d)$. Letting $x := k$ and recalling (4.9), it follows that the Fourier coefficient of index k of ω_0 is of magnitude $\|k\|^{-d-2}$ as $\|k\| \to \infty$, for any $d \geq 2$. This proves Lemma 2.1. □

Proof of Lemma 2.2. The asymptotic expansion of $\log\omega_0(t)$ as $t \to 0$ shows that ω_0 and $\log\omega_0$ have the same leading singularity in their partial derivatives of total order $d + 2$. Therefore Lemma 2.2 is obtained by letting $f := \psi\log\omega_0$ in Theorem 4.1 and arguing as in the preceding proof. □

Proof of Lemma 2.3. As $t \to 0$, the first factor of the symbol Θ in (2.22) has the asymptotic expansion

$$\left\{\sum_{l=1}^{d-1}[\sin(t_l/2)]^2\right\}^{1/2} \tag{4.16}$$

$$= \left\{\sum_{l=1}^{d-1}\left[\frac{t_l^2}{4}-\frac{t_l^4}{48}+\frac{t_l^6}{1440}-\ldots\right]\right\}^{1/2}$$
$$= \frac{\|t'\|}{2}\left\{1-\|t'\|^{-2}\left[\frac{1}{12}\sum_{l=1}^{d-1}t_l^4-\frac{1}{360}\sum_{l=1}^{d-1}t_l^6+\ldots\right]\right\}^{1/2}$$
$$= \frac{\|t'\|}{2}\left\{1-\left[\frac{\sum_{l=1}^{d-1}t_l^4}{24\|t'\|^2}\right]+\left[\frac{\sum_{l=1}^{d-1}t_l^6}{720\|t'\|^2}-\frac{(\sum_{l=1}^{d-1}t_l^4)^2}{1152\|t'\|^4}\right]+\ldots\right\}.$$

The second factor of Θ is a C^∞ function whose corresponding expansion as $t \to 0$ is

$$\left\{1+\sum_{l=1}^{d-1}[\sin(t_l/2)]^2\right\}^{1/2} \tag{4.17}$$
$$= \left\{1+\frac{1}{4}\sum_{l=1}^{d-1}t_l^2-\frac{1}{48}\sum_{l=1}^{d-1}t_l^4+\ldots\right\}^{1/2}$$
$$= 1+\frac{1}{8}\sum_{l=1}^{d-1}t_l^2-\frac{1}{32}\left[\frac{1}{3}\sum_{l=1}^{d-1}t_l^4+\frac{1}{4}\left(\sum_{l=1}^{d-1}t_l^2\right)^2\right]+\ldots.$$

Now replace d by $d-1$ and let $f := \Theta\psi$ in Theorem 4.1, where ψ is the corresponding auxiliary function of $d-1$ variables (t and x are replaced by t' and x', respectively). Also, define

$$\varphi(t') := f(t') - \frac{1}{2}\|t'\|, \tag{4.18}$$

$q := 1$, $p := 0$ (thus $\kappa = 1$) and let n be the integer part of $(d+1)/2$, so that condition (C_1) is satisfied. Further, from (4.16) and (4.17) we deduce after multiplication and termwise differentiation the property

$$(\Delta^j\varphi)(t') = \mathcal{O}(\|t'\|^{3-2j}), \qquad \forall\, j \in \mathbb{Z}_+, \tag{4.19}$$

so (C_2) holds as well. As (C_3) is ensured by the compact support of ψ, we apply Theorem 4.1 with d replaced by $d-1$, which shows (after letting $x' := k \in \mathbb{Z}^{d-1}$) that the Fourier coefficient θ_k of Θ is of magnitude $\|k\|^{-d}$ for large $\|k\|$. Thus Lemma 2.3 is proved. □

Remark 4.2 If $d = 2$, then Θ is a one-dimensional symbol, so Lemma 2.3 admits a simpler proof using integration by parts twice in the formula $\theta_k = \pi^{-1}\int_0^\pi \cos(kt)\Theta(t)dt$.

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