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Symmetry reductions, variational symmetries and conservation laws for shallow water and semi-geostrophic PDE systems

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Abstract

The two-dimensional shallow water and semi-geostrophic partial differential equation systems that arise in meteorology and oceanography are analyzed from the point of view of symmetry groups theory, such that their associated classical symmetries, potential symmetries, variational symmetries and conservation laws are found. The MAPLE packages `DESOLV` (authors Carminati and Vu) and `diffgrob2` (author Mansfield) represent essential tools for our study.

1 Introduction

Most natural phenomena are described by a system of nonlinear partial differential equations (PDEs) difficult to solve analytically. While there is no existing general theory for completely solving nonlinear PDEs, the use of group transformations plays a central part. In the last forty years, a remarkable number of mathematical models occurring in various research domains have been studied from the point of view of symmetry groups theory. Sophus Lie (1842-1899), one of the most important mathematicians of the nineteenth century, introduced a new method for studying differential equations by using their associated symmetries. For each PDE (or PDE system) there is a local group of transformations (called *symmetry group*) that acts on the space of its independent and dependent variables, with the property that it maps the set of all analytical solutions to itself, and so leaves the form of the equation (or system) unchanged. The method of finding the symmetry group associated with the a PDE (or PDE system) is called *classical Lie method.* Moreover, the classical Lie method leads to special type of solutions in terms of solutions of lower dimensional equations, such that this is a widely applicable technique

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for finding analytical solutions. Thus, knowledge of the Lie point symmetries allows one to construct group-invariant solutions, namely classes of solutions that invariant under certain subgroups of the full symmetry group. Two solutions are equivalent if there is a symmetry transformation that transforms one into the other. Moreover, new solutions can be constructed from known ones: by applying the symmetry group to a known solution yields a family of new solutions. The classical Lie method is an algorithmic procedure for which many symbolic manipulation programs have became available [2]. For the powerful computer algebra system MAPLE, the packages `DESOLV` ((authors Carminati and Vu [1]) and `diffgrob2` (author Mansfield [5]) represent basic tools for finding and integrating the determining equations of the classical symmetries of a PDE system.

Classical symmetries are related to calculus of variations, such that, if a PDE system represents the Euler-Lagrange equations associated with the a variational problem, then the classical symmetries lead to the variational symmetries. The *variational symmetry group* of a variational problem is a subgroup of the symmetry group associated with the the Euler-Lagrange equations, with the property that it leaves the variational integral unchanged. Knowledge of the variational symmetries lead, by Noether's theorem (Theorem 4.3), to the conservation laws associated with the the studied system: for each one-parameter variational symmetry there is an associated conservation law.

In the past twenty years, the semi-geostrophic equations have become a model for describing atmospheric motions on a synoptic scale, including the presence of fronts [8], [9]. In the two-dimensional shallow water theory [10] the motion of a typical particle can be described by expressing the current horizontal coordinates

$$\begin{aligned} x &= x(a,b,t) \\ y &= y(a,b,t) \end{aligned} \tag{1.1}$$

as functions, on the right, of the particle labels $(a,b) \in \mathbf{R}^2$ and the time $t \in \mathbf{R}_+$. The incompressibility hypothesis requires the current depth $h = h(a,b,t)$ to be a function with the property

$$\frac{h(a,b,0)}{h(a,b,t)} = \frac{\partial(x,y)}{\partial(a,b)}, \tag{1.2}$$

where the jacobian on the right is that of the mapping (1.1). The time derivative of (1.2) following the particle gives the differential equation of continuity. Without loss the generality, one can assume that $h(a,b,0) = 1$, such that the above relation turns into

$$h = \frac{1}{x_a y_b - x_b y_a}. \tag{1.3}$$

The equations of the horizontal momentum balance for the flows over a bed which is rotating with position dependent Coriolis parameter $f = f(y)$ are

$$\left\{ \begin{array}{l} \ddot{x} + gh_x - f\dot{y} = 0 \\ \ddot{y} + gh_y + f\dot{x} = 0, \end{array} \right. \tag{1.4}$$

where g is a nonzero constant (representing the combined effect of the acceleration due to gravity and a centrifugal component due to the Earth's rotation), the superposed dot denotes the time derivative following a particle, and h_x and h_y are given by

$$\begin{array}{rcl} h_x & = & h(y_b h_a - y_a h_b) \\ h_y & = & h(x_a h_b - x_b h_a). \end{array} \tag{1.5}$$

In that follows we consider f to be constant. It is known that, the *shallow-water potential vorticity* defined by

$$\Omega = \frac{1}{h}\left(\frac{\partial \dot{y}}{\partial x} - \frac{\partial \dot{x}}{\partial y} + f\right), \tag{1.6}$$

is conserved on particles.

The semi-geostrophic approximation to equations (1.4) is the replacement of the true acceleration by the time derivative of the vector

$$\left\{ \begin{array}{rcl} u_g & = & -\frac{g}{f} h_y \\ v_g & = & \frac{g}{f} h_x, \end{array} \right. \tag{1.7}$$

following the particle. This vector field is a notional velocity and is called *geostrophic velocity.* Thus, the semi-geostrophic approximation seeks to find motions satisfying

$$\left\{ \begin{array}{l} \dot{u_g} + gh_x - f\dot{y} = 0 \\ \dot{v_g} + gh_y + f\dot{x} = 0. \end{array} \right. \tag{1.8}$$

It is also known that the *semi-geostrophic potential vorticity*, given by

$$\Omega^* = \frac{1}{h}\left(f + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} + \frac{1}{f}\frac{\partial(u_g, v_g)}{\partial(x, y)}\right), \tag{1.9}$$

is conserved on particles.

A complete study of the symmetries and the conservation laws associated with the the two-dimensional shallow water (SW) PDE system (1.4) and with the semi-geostrophic (SG) PDE system (1.8), has not yet been carried out, and hence, the aim of this paper is to apply the groups symmetry theory to them. This theory (Olver [7]) is presented briefly in Sections 2 and 4. In Sections 3 and 4 we determine the Lie point symmetries associated with the the SW and SG PDE systems, written in their classical forms (3.1) and (4.1), or in their potential forms (3.11) and (4.6). We find that the SW and SG PDE systems (3.11) and (4.6) are invariant under translation in t, x, and y, under rotations in the space of their dependent variables, and also under the known particle relabelling symmetry. Moreover, SW PDE system is invariant under helical rotations with respect to t in the (x, y, u, v) - space. In Sections 6 and 7, we determine the conservation laws deriving from the variational symmetries (Tables 1 and 2). The particle relabelling symmetry lead to the conservation of the shallow water potential vorticity Ω (Salmon [11]) and semi-geostrophic potential vorticity (Hydon [3]), the time invariance lead to the conservation of the energy of SW PDE system (Salmon [11]) and of SG PDE system, and the translation invariance to the conservation of linear momentum of the SW and SG PDE systems, respectively. Moreover, the rotation invariance lead to the conservation of the angular momentum of SW PDE system.

2 The classical Lie method

Consider the n-th order PDE system

$$\Delta(x, u^{(n)}) = 0, \tag{2.1}$$

with p independent variables $x = (x^1, ..., x^p)$, and q dependent variables $u = (u^1, ..., u^q)$, where $\Delta = (\Delta_1, ..., \Delta_l)$, and $u^{(n)}$ denote all the partial derivatives of u with respect to x up to order n. The k-th order partial derivative of u^α with respect to x is denoted by

$$u^\alpha_J = \frac{\partial^k u^\alpha}{\partial (x^1)^{j_1} ... \partial (x^n)^{j_k}},$$

where $J = (j_1, ..., j_k)$, $1 \leq j_k \leq n$, $1 \leq k \leq n$, and $u^\alpha_0 = u^\alpha$, $\alpha = 1, ..., q$.

The system (2.1) is a *maximal rank system* if

$$rank\left(\frac{\partial \Delta}{\partial x^i}, \frac{\partial \Delta}{\partial u^\alpha_J}\right) = l \quad \text{whenever} \quad \Delta(x, u^{(n)}) = 0.$$

Denote

$$\mathcal{S} = \{u(x) : \Delta(x, u^{(n)})\} = 0 \tag{2.2}$$

the set of all analytic solutions of the system (2.1) and let us consider

$$\left\{ \begin{array}{rcl} x^{*i} & = & x^i + \epsilon \zeta^i(x, u) + \mathcal{O}(\epsilon^2), \quad i = 1, ..., p \\ u^{*\alpha} & = & u^\alpha + \epsilon \phi^\alpha(x, u) + \mathcal{O}(\epsilon^2), \quad \alpha = 1, ..., q, \end{array} \right. \tag{2.3}$$

an one-parameter group of local (point) transformations acting on an open subset $M \subset \mathcal{X} \times U$ of the space $\mathcal{X}$ of its independent variables and U of its dependent variables, with the associated infintesimal generator

$$X = \sum_{i=1}^{p} \zeta^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi^\alpha(x,u)\frac{\partial}{\partial u^\alpha}. \tag{2.4}$$

The classical Lie method [7] applied to the PDE system (2.1) leads to its associated symmetry group, that consists in one-parameter groups of transformations acting on the space $\mathcal{X} \times U$ and leaving the set $\mathcal{S}$ invariant. These transformations, called *classical symmetries* (or *Lie point symmetries*), map any solution of the system to another solution of it and leave the form of the system unchanged. Knowledge of the symmetry group allows one to construct new solutions from known ones and to classify the set $\mathcal{S}$ in *group-invariant solutions.*

Theorem 2.1. *Let X be the vector field (2.4) defined on an open subset $M \subset \mathcal{X} \times \mathcal{U}$. The n-th order prolongation of X is the vector field*

$$pr^{(n)}X = X + \sum_{\alpha=1}^{q}\sum_{J} \Phi_J^\alpha(x,u^{(n)})\frac{\partial}{\partial u_J^\alpha}, \tag{2.5}$$

defined on the corresponding jet space $M^{(n)} \subset \mathcal{X} \times U^{(n)}$, with

$$\Phi_J^\alpha = D_J\left(\phi^\alpha - \sum_{i=1}^{p}\zeta^i u_i^\alpha\right) + \sum_{i=1}^{p}\zeta^i u_{J,i}^\alpha, \tag{2.6}$$

the summation being over all multi-indices $J=(j_1,...,j_k)$, $1 \le j_k \le p$, $1 \le k \le n$.

The algorithm for finding the classical symmetries associated with the the PDE system (2.1) is based on the following criterion.

Theorem 2.2.(*criterion for infinitesimal invariance*) *If G is a local group of transformations acting on an open set $M \subset \mathcal{X} \times \mathcal{U}$, and*

$$pr^{(n)}X(\Delta)|_{\Delta=0} = 0, \tag{2.7}$$

for every infinitesimal generator X of G, then G is a symmetry group of the system (2.1).

In this case, the vector field X is called *symmetry operator* and its components $\zeta^1,...\zeta^p$, and $\phi^1,...,\phi^q$, are called *infinitesimals.* The infinitesimals are solutions of an over-determined linear PDE system (called *determining equations of the symmetry group*) obtained by eliminating any dependence between the partial derivatives of the functions u^α occurring in (2.7).

3 Classical and potential symmetries for SW PDE system

3.1 Classical symmetries of the shallow water equations

The SW PDE system is given by the equations (1.4), where h satisfies (1.3). Substituting the function h from (1.3) into (1.4), we obtain

(3.1)
$$\begin{cases} \ddot{x} - \frac{g}{(x_a y_b - x_b y_a)^3}\left[y_b^2 x_{aa} - 2y_a y_b x_{ab} + y_a^2 x_{bb} - x_b y_b y_{aa} + (x_a y_b + x_b y_a) y_{ab} - x_a y_a y_{bb}\right] = f\dot{y} \\ \ddot{y} + \frac{g}{(x_a y_b - x_b y_a)^3}\left[x_b y_b x_{aa} - (x_a y_b + x_b y_a) x_{ab} + x_a y_a x_{bb} - x_b^2 y_{aa} + 2x_a x_b y_{ab} - x_a^2 y_{bb}\right] = -f\dot{x}. \end{cases}$$

We can call this second order PDE system system the *classical form* of SW PDE system (1.4), and its associated Lie point symmetries will be called its *classical symmetries.* Consider

$$(3.2) \qquad \begin{cases} a^* &= a + \varepsilon\xi(a,b,t,x,y) + O(\varepsilon^2) \\ b^* &= b + \varepsilon\eta(a,b,t,x,y) + O(\varepsilon^2) \\ t^* &= t + \varepsilon\theta(a,b,t,x,y) + O(\varepsilon^2) \\ x^* &= x + \varepsilon\phi(a,b,t,x,y) + O(\varepsilon^2) \\ y^* &= y + \varepsilon\psi(a,b,t,x,y) + O(\varepsilon^2) \end{cases}$$

a one-parameter group of transformations acting on the space of the independent variables a, b, and t and the dependent variables x,and y for which

$$(3.3) \qquad X = \xi\partial_a + \eta\partial_b + \theta\partial_t + \phi\partial_x + \psi\partial_y$$

is the associated infinitesimal generator (here the infinitesimals ξ, η, θ, ϕ and ψ depend on a, b, t, x, and y). From the classical Lie method it follows that θ must be constant,

$$(3.4) \qquad \begin{aligned} \phi &= Ax - By + C\cos(ft) + D\sin(ft) + E, \\ \psi &= Ay + Bx + D\cos(ft) - C\sin(ft) + F, \end{aligned}$$

where A, B, C, D, E and F are real numbers, and $\xi = \xi(a,b)$ and $\eta = \eta(a,b)$ must verify the relation

$$(3.5) \qquad \xi_a + \eta_b = 4A.$$

Substituting these infinitesimals into (3.3), we get

$$(3.6) \qquad \begin{aligned} X = \; & \xi(a,b)\partial_a + \eta(a,b)\partial_b + \theta\partial_t + (Ax - By + C\cos(ft) + D\sin(ft) + E)\,\partial_x + \\ + \; & (Ay + Bx + D\cos(ft) - C\sin(ft) + F)\,\partial_y. \end{aligned}$$

that represents the general infinitesimal generator associated with the SW PDE system (3.1).

Theorem 3.1. *The symmetry group associated with the SW PDE system (3.1) is generated by the following vector fields*

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y}, \quad X_4 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

$$X_5 = \cos(ft)\frac{\partial}{\partial x} - \sin(ft)\frac{\partial}{\partial y}, \quad X_6 = \sin(ft)\frac{\partial}{\partial x} + \cos(ft)\frac{\partial}{\partial y}, \tag{3.7}$$

$$X_0 = \xi(a,b)\frac{\partial}{\partial a} + \eta(a,b)\frac{\partial}{\partial b} + Ax\frac{\partial}{\partial x} + Ay\frac{\partial}{\partial y},$$

where the functions ξ and η satisfy the condition (3.5). Thus the SW PDE system (3.1) is invariant under translations in t, x and y, respectively, rotations in the (x,y)-space, and helical rotations with respect t in the (x,y)-space. The vector field X_0 corresponds to the particle relabelling symmetry.

Notice that, since each one-parameter group of symmetry transformations

$$\begin{array}{lc} X_1: & (a,b,t+\varepsilon,x,y), \\ X_2: & (a,b,t,x+\varepsilon,y), \\ X_3: & (a,b,t,x,y+\varepsilon), \\ X_4: & (a,b,t,x\cos\varepsilon - y\sin\varepsilon, x\sin\varepsilon + y\cos\varepsilon), \\ X_5: & (a,b,t,x+\varepsilon\cos(ft), y-\varepsilon\sin(ft)), \\ X_6: & (a,b,t,x+\varepsilon\sin(ft), y+\varepsilon\cos(ft)), \end{array} \tag{3.8}$$

is a symmetry group, if $x = F(a,b,t)$, $y = G(a,b,t)$ is a solution of SW PDE system (3.1), so are the functions

$$\left\{\begin{array}{ll} x^{(1)}(a,b,t) & = F(a,b,t-\varepsilon_1) \\ y^{(1)}(a,b,t) & = G(a,b,t-\varepsilon_1), \end{array}\right.$$

$$\left\{\begin{array}{lcl} x^{(2)}(a,b,t) & = & F(a,b,t)+\varepsilon_2 \\ y^{(2)}(a,b,t) & = & G(a,b,t), \end{array}\right.$$

$$\left\{\begin{array}{lcl} x^{(3)}(a,b,t) & = & F(a,b,t) \\ y^{(3)}(a,b,t) & = & F(a,b,t)+\varepsilon_3, \end{array}\right. \tag{3.9}$$

$$\left\{\begin{array}{lcl} x^{(4)}(a,b,t) & = & F(a,b,t)\cos\varepsilon_4 - G(a,b,t)\sin\varepsilon_4 \\ y^{(4)}(a,b,t) & = & F(a,b,t)\sin\varepsilon_4 + G(a,b,t)\cos\varepsilon_4, \end{array}\right.$$

$$\begin{cases} x^{(5)}(a,b,t) &= F(a,b,t) + \varepsilon_5 \cos(ft) \\ y^{(5)}(a,b,t) &= G(a,b,t) - \varepsilon_5 \sin(ft), \end{cases}$$

$$\begin{cases} x^{(6)}(a,b,t) &= F(a,b,t) + \varepsilon_6 \sin(ft) \\ y^{(6)}(a,b,t) &= G(a,b,t) + \varepsilon_6 \cos(ft), \end{cases}$$

where $\varepsilon, \varepsilon_1, ..., \varepsilon_6$ are real numbers). By using the symmetry group transformations, we can express the general solution of the system (3.1) that can be constructed from the known one, namely

$$\text{(3.10)} \qquad \begin{cases} x &= F(a,b,t-\varepsilon_1)\cos\varepsilon_4 - G(a,b,t-\varepsilon_1)\sin\varepsilon_4 + \varepsilon_5\cos(ft) \\ & \quad +\varepsilon_6\sin(ft) + \varepsilon_2\cos\varepsilon_4 - \varepsilon_3\sin\varepsilon_4 \\ y &= F(a,b,t-\varepsilon_1)\sin\varepsilon_4 + G(a,b,t-\varepsilon_1)\cos\varepsilon_4 - \varepsilon_5\sin(ft)+ \\ & \quad +\varepsilon_6\cos(ft) + \varepsilon_2\sin\varepsilon_4 + \varepsilon_3\cos\varepsilon_4. \end{cases}$$

where $\varepsilon_1, ..., \varepsilon_6$ are real constants.

The 6-dimensional Lie algebra associated with the SW PDE system (3.1) is spanned by the solenoidal vector fields $X_1, ..., X_6$ (theorem 3.1), for which Table of the Lie brackets is the following

$[.,.]$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	0	0	$-fX_6$	fX_5
X_2	0	0	0	X_3	0	0
X_3	0	0	0	$-X_2$	0	0
X_4	0	$-X_3$	X_2	0	$-X_6$	X_5
X_5	fX_6	0	0	X_6	0	0
X_6	$-fX_5$	0	0	$-X_5$	0	0

Table 1

3.2 Potential symmetries of the shallow water equations

Moreover, if denote $u = \dot{x}$ and $v = \dot{y}$, respectively, then in SW PDE system (1.4) can ce written as

$$\text{(3.11)} \qquad \begin{cases} \dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &+ gh(y_b h_a - y_a h_b) - fv = 0 \\ \dot{v} &+ gh(x_a h_b - x_b h_a) + fu = 0, \end{cases}$$

here h is given by (1.3). We call this first order PDE system the *potential form* of SW PDE system (1.4), and its associated Lie point symmetries will be called *potential symmetries.* Consider

$$(3.12)\qquad \begin{cases} a^* &= a+\varepsilon\xi(a,b,t,x,y,u,v)+O(\varepsilon^2)\\ b^* &= b+\varepsilon\eta(a,b,t,x,y,u,v)+O(\varepsilon^2)\\ t^* &= t+\varepsilon\theta(a,b,t,x,y,u,v)+O(\varepsilon^2)\\ x^* &= x+\varepsilon\phi(a,b,t,x,y,u,v)+O(\varepsilon^2)\\ y^* &= y+\varepsilon\psi(a,b,t,x,y,u,v)+O(\varepsilon^2)\\ u^* &= u+\varepsilon\alpha(a,b,t,x,y,u,v)+O(\varepsilon^2)\\ v^* &= v+\varepsilon\beta(a,b,t,x,y,u,v)+O(\varepsilon^2)\end{cases}$$

a one-parameter group of transformations on the space of its independent variables a, b, t and dependent variables x, y, u, v, with the infinitesimal generator given by

$$X=\xi\partial_a+\eta\partial_b+\theta\partial_t+\phi\partial_x+\psi\partial_y+\alpha\partial_u+\beta\partial_v,$$

where the infintesimals $\xi,\eta,\theta,\phi,\psi,\alpha$ and β are functions of a, b, t, x, y, u and v. Applying the classical Lie we obtain that θ is a constant,

$$(3.13)\qquad \begin{aligned} \phi &= Ax-By+C\cos(ft)+D\sin(ft)+E,\\ \psi &= Ay+Bx+D\cos(ft)-C\sin(ft)+F,\\ \alpha &= Au-Bv-Cf\sin(ft)+Df\cos(ft),\\ \beta &= Av+Bu-Df\sin(ft)-Cf\cos(ft),\end{aligned}$$

with A, B, C, D, E, F real numbers, and $\xi=\xi(a,b)$ and $\eta=\eta(a,b)$ satisfy (3.5) as before. In this case, the general infinitesimal generator is written as

$$X=\xi(a,b)\partial_a+\eta(a,b)\partial_b+\theta\partial_t+\left(Ax-By+C\cos(ft)+D\sin(ft)+E\right)\partial_x+$$

$$\left(Ay+Bx+D\cos(ft)-C\sin(ft)+F\right)\partial_y+\left(Au-Bv-Cf\sin(ft)+Df\cos(ft)\right)\partial_u+$$

$$(3.14)\qquad \left(Av+Bu-Df\sin(ft)-Cf\cos(ft)\right)\partial_v.$$

Theorem 3.2 *The potential symmetries associated with the SW PDE system (3.11) are generated by the vector fields*

$$X_1=\partial_t,\quad X_2=\partial_x,\quad X_3=\partial_y,\quad X_4=-y\partial_x+x\partial_y-v\partial_u+u\partial_v,$$

$$X_5 = \cos(ft)\partial_x - \sin(ft)\partial_y - f\sin(ft)\partial_u - f\cos(ft)\partial_v, \tag{3.15}$$

$$X_6 = \sin(ft)\partial_x + \cos(ft)\partial_y + f\cos(ft)\partial_u - f\sin(ft)\partial_v,$$

$$X_0 = \xi(a,b)\partial_a + \eta(a,b)\partial_b + A(x\partial_x + y\partial_y + u\partial_u + v\partial_v),$$

where the condition (3.5) is verified. Thus SW PDE system (3.11) is invariant under translations in t, x and y, respectively, rotations in the (x,y,u,v)-space, and helical rotations with respect t in the (x,y,u,v)-space. The vector field X_0 corresponds to the particle relabelling symmetry.

Note that the potential symmetries of SW PDE system are actually, its classical symmetries prolonged to the space of the t-derivatives of the depend variables x and y.

Similarly, each of the vector fields $X_1, ..., X_6$ generates a one-parameter group of symmetry transformations

(3.16)

$$\begin{array}{lcl}
X_1 & : & (a,b,t+\varepsilon_1,x,y,u,v), \\
X_2 & : & (a,b,t,x+\varepsilon_2,y,u,v), \\
X_3 & : & (a,b,t,x,y+\varepsilon_3,u,v), \\
X_4 & : & (a,b,t,x\cos\varepsilon_4 - y\sin\varepsilon_4, x\sin\varepsilon_4 + y\cos\varepsilon_4, u\cos\varepsilon_4 - v\sin\varepsilon_4, u\sin\varepsilon_4 + v\cos\varepsilon_4), \\
X_5 & : & (a,b,t,x+\varepsilon_5\cos(ft), y-\varepsilon_5\sin(ft), u-\varepsilon_5 f\sin(ft), v-\varepsilon_5 f\cos(ft)), \\
X_6 & : & (a,b,t,x+\varepsilon_6\sin(ft), y+\varepsilon_6\cos(ft)u+\varepsilon_6 f\cos(ft), v-\varepsilon_6 f\sin(ft)),
\end{array}$$

here $\varepsilon, \varepsilon_1, ..., \varepsilon_6$ are any real numbers. If $x = F(a,b,t)$, $y = G(a,b,t)$, $u = H(a,b,t)$ and $v = K(a,b,t)$ is a solution of SW PDE system (3.11), then the following are also solutions

$$\left\{\begin{array}{lcl}
x^{(1)}(a,b,t) & = & F(a,b,t-\varepsilon_1) \\
y^{(1)}(a,b,t) & = & G(a,b,t-\varepsilon_1) \\
u^{(1)}(a,b,t) & = & H(a,b,t-\varepsilon_1) \\
v^{(1)}(a,b,t) & = & K(a,b,t-\varepsilon_1),
\end{array}\right.$$

$$\left\{\begin{array}{lcl}
x^{(2)}(a,b,t) & = & F(a,b,t)+\varepsilon_2 \\
y^{(2)}(a,b,t) & = & G(a,b,t) \\
u^{(2)}(a,b,t) & = & H(a,b,t) \\
v^{(2)}(a,b,t) & = & K(a,b,t),
\end{array}\right.$$

$$\left\{\begin{array}{lcl}
x^{(3)}(a,b,t) & = & F(a,b,t) \\
y^{(3)}(a,b,t) & = & G(a,b,t)+\varepsilon_3 \\
u^{(3)}(a,b,t) & = & H(a,b,t) \\
v^{(3)}(a,b,t) & = & K(a,b,t),
\end{array}\right. \tag{3.17}$$

$$\begin{cases} x^{(4)}(a,b,t) &= F(a,b,t)\cos\varepsilon_4 - G(a,b,t)\sin\varepsilon_4 \\ y^{(4)}(a,b,t) &= F(a,b,t)\sin\varepsilon_4 + G(a,b,t)\cos\varepsilon_4 \\ u^{(4)}(a,b,t) &= H(a,b,t)\cos\varepsilon_4 - K(a,b,t)\sin\varepsilon_4 \\ v^{(4)}(a,b,t) &= H(a,b,t)\sin\varepsilon_4 + K(a,b,t)\cos\varepsilon_4, \end{cases}$$

$$\begin{cases} x^{(5)}(a,b,t) &= F(a,b,t) + \varepsilon_5\cos(ft) \\ y^{(5)}(a,b,t) &= G(a,b,t) - \varepsilon_5\sin(ft) \\ u^{(5)}(a,b,t) &= H(a,b,t) - \varepsilon_5 f\sin(ft) \\ v^{(5)}(a,b,t) &= K(a,b,t) - \varepsilon_5 f\cos(ft), \end{cases}$$

$$\begin{cases} x^{(6)}(a,b,t) &= F(a,b,t) + \varepsilon_6\sin(ft) \\ y^{(6)}(a,b,t) &= G(a,b,t) + \varepsilon_6\cos(ft) \\ u^{(6)}(a,b,t) &= H(a,b,t) + \varepsilon_6 f\cos(ft) \\ v^{(6)}(a,b,t) &= K(a,b,t) - \varepsilon_6 f\sin(ft), \end{cases}$$

so that

$$(3.18)\quad \begin{cases} x &= F(a,b,t-\varepsilon_1)\cos\varepsilon_4 - G(a,b,t-\varepsilon_1)\sin\varepsilon_4 + \varepsilon_5\cos(ft) + \varepsilon_6\sin(ft) + \\ &\qquad + \varepsilon_2\cos\varepsilon_4 - \varepsilon_3\sin\varepsilon_4 + \varepsilon_5\cos(ft) + \varepsilon_6\sin(ft) \\ y &= F(a,b,t-\varepsilon_1)\sin\varepsilon_4 + G(a,b,t-\varepsilon_1)\cos\varepsilon_4 - \varepsilon_5\sin(ft) + \varepsilon_6\cos(ft) + \\ &\qquad + \varepsilon_2\sin\varepsilon_4 + \varepsilon_3\cos\varepsilon_4 - \varepsilon_5\sin(ft) + \varepsilon_6\cos(ft) \\ u &= H(a,b,t-\varepsilon_1)\cos\varepsilon_4 - K(a,b,t-\varepsilon_1)\sin\varepsilon_4 - \varepsilon_5 f\sin(ft) + \varepsilon_6 f\cos(ft) \\ v &= H(a,b,t-\varepsilon_1)\sin\varepsilon_4 + K(a,b,t-\varepsilon_1)\cos\varepsilon_4 - \varepsilon_5 f\cos(ft) - \varepsilon_6 f\sin(ft). \end{cases}$$

represents the general solution obtained from the known one.

4 Classical and potential symmetries for SG PDE system

4.1 Classical symmetries of the semi-geostrophic equations

The *classical form* of SG PDE system (1.8)
(4.1)

$$\begin{cases} hh_b\dot{x_a} - hh_a\dot{x_b} + (\dot{h}h_b + h\dot{h_b})x_a - (\dot{h}h_a + h\dot{h_a})x_b + fhh_by_a - fhh_ay_b &= -\frac{f^2}{g}\dot{y} \\ hh_b\dot{y_a} - hh_a\dot{y_b} + (\dot{h}h_b + h\dot{h_b})y_a - (\dot{h}h_a + h\dot{h_a})y_b - fhh_bx_a + fhh_ax_b &= \frac{f^2}{g}\dot{x}, \end{cases}$$

with the function h given by (1.3), is obtained after substituting the functions u_g and v_g given by (1.7) into the system (1.8). The Lie point symmetries of this second order PDE system will be called the *classical symemtries* of the SG PDE system.

In this case, if we consider the one-parameter group of transformations (3.2) with the infinitesimal generator (3.3) and applying the classical Lie method, we obtain that θ real number,

$$\begin{aligned} \phi &= Ax - By + E, \\ \psi &= Ay + Bx + F, \end{aligned} \tag{4.2}$$

A, B, E, F being real numbers, and $\xi = \xi(a, b)$ and $\eta = \eta(a, b)$ must satisfy the condition (3.5). The infinitesimal generator (3.3) becomes

$$X = \xi(a, b)\partial_a + \eta(a, b)\partial_b + \theta\partial_t + (Ax - By + E)\,\partial_x + (Ay + Bx + F)\,\partial_y. \tag{4.3}$$

Theorem 4.1. [6] *The vector fields*

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y}, \quad X_4 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \tag{4.4}$$

$$X_0 = \xi(a, b)\frac{\partial}{\partial a} + \eta(a, b)\frac{\partial}{\partial b} + Ax\frac{\partial}{\partial x} + Ay\frac{\partial}{\partial y},$$

where the condition (3.5) is satisfied, generate the symmetry group associated with the SW PDE system (4.1). Thus, the system is invariant under translations in t, x, y and rotations in the (x, y)-space. The relabelling symmetry corresponds to the vector fields X_0 .

There is a 4-dimensional Lie algebra spanned by the vector fields $X_1, ..., X_4$ associated with the SG PDE system (4.1). This is a subalgebra of the 6-dimensional Lie algebra related to SW PDE system (3.1). If ε_5 and ε_6 are zero in the relation (3.10), then we find the general solution of SW PDE system (4.1) constructed from a known one.

$$\left\{ \begin{aligned} x &= F(a, b, t - \varepsilon_1)\cos\varepsilon_4 - G(a, b, t - \varepsilon_1)\sin\varepsilon_4 + \varepsilon_2\cos\varepsilon_4 - \varepsilon_3\sin\varepsilon_4 \\ y &= F(a, b, t - \varepsilon_1)\sin\varepsilon_4 + G(a, b, t - \varepsilon_1)\cos\varepsilon_4 + \varepsilon_2\sin\varepsilon_4 + \varepsilon_3\cos\varepsilon_4. \end{aligned} \right. \tag{4.5}$$

where $\varepsilon_1, ..., \varepsilon_6$ are real constants.

4.2 Potential symmetries of the semi-geostrophic equations

The potential form of SG PDE system is given by the system (1.7)+(1.8), namely the system

$$\left\{ \begin{aligned} u_g &= -\tfrac{g}{f}h(x_a h_b - x_b h_a) \\ v_g &= \tfrac{g}{f}h(y_b h_a - y_a h_b) \\ \dot{u_g} &+ gh(y_b h_a - y_a h_b) - f\dot{y} = 0 \\ \dot{v_g} &+ gh(x_a h_b - x_b h_a) + f\dot{x} = 0, \end{aligned} \right. \tag{4.6}$$

where f is a nonzero constant and h is given by (1.3), and its Lie point symmetries are called *potential symmetries.*

For the one-parameter group of transformations

$$
\left\{
\begin{array}{lcl}
a^* &=& a + \varepsilon\xi(a,b,t,x,y,u_g,v_g) + O(\varepsilon^2)\\
b^* &=& b + \varepsilon\eta(a,b,t,x,y,u_g,v_g) + O(\varepsilon^2)\\
t^* &=& t + \varepsilon\theta(a,b,t,x,y,u_g,v_g) + O(\varepsilon^2)\\
x^* &=& x + \varepsilon\phi(a,b,t,x,y,u_g,v_g) + O(\varepsilon^2)\\
y^* &=& y + \varepsilon\psi(a,b,t,x,y,u_g,v_g) + O(\varepsilon^2)\\
u_g^* &=& u_g + \varepsilon\alpha(a,b,t,x,y,u_g,v_g) + O(\varepsilon^2)\\
v_g^* &=& v_g + \varepsilon\beta(a,b,t,x,y,u_g,v_g) + O(\varepsilon^2),
\end{array}
\right. \tag{4.7}
$$

acting on the space of the independent variables a, b, t and dependent variables x, y, u_g, v_g, the infinitesimal generator is

$$X = \xi\partial_a + \eta\partial_b + \theta\partial_t + \phi\partial_x + \psi\partial_y + \alpha\partial_{u_g} + \beta\partial_{v_g},$$

with the coefficients $\xi, \eta, \theta, \phi, \psi, \alpha$ and β depending on a, b, t, x, y, u_g, and v_g. The classical Lie method lead to the follow infinitesimals

$$
\begin{array}{lcl}
\phi &=& Ax - By + E,\\
\psi &=& Ay + Bx + F,\\
\alpha &=& Au_g - Bv_g,\\
\beta &=& Av_g + Bu_g,
\end{array} \tag{4.8}
$$

where A, B, E, F, θ being real numbers, and $\xi = \xi(a,b)$ and $\eta = \eta(a,b)$ satisfying (3.5). In this case, the general infintesimal can be written as

$$
\begin{array}{lcl}
X &=& \xi(a,b)\partial_a + \eta(a,b)\partial_b + \theta\partial_t + (Ax - By + E)\,\partial_x + (Ay + Bx + F)\,\partial_y +\\
&& + (Au_g - Bv_g)\,\partial_u + (Av_g + Bu_g)\,\partial_v.
\end{array} \tag{4.9}
$$

Theorem 4.2. *The vector fields*

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = \partial_y, \quad X_4 = -y\partial_x + x\partial_y - v_g\partial_{u_g} + u_g\partial_{v_g},$$

$$X_0 = \xi(a,b)\partial_a + \eta(a,b)\partial_b + A(x\partial_x + y\partial_y + u_g\partial_{u_g} + v_g\partial_{v_g}), \tag{4.10}$$

where (3.5) is verified, generate the potential symmetries associated with the SG PDE system (4.5). Thus the system is invariant under translations in t, x, y and rotations in the (x, y, u_g, v_g)-space. The vector field X_0 corresponds to particle relabelling symmetry.

According to the above considerations, the general solution of the SG PDE constructed from a known one, is given by (3.18), where $\varepsilon_5 = \varepsilon_6 = 0$.

5 Variational symmetries and conservation laws

The classical symmetries have applications to the calculus of variations, that is, one can find conservation laws using some of the classical symmetries associated with the the Euler-Lagrange equations, namely the variational symmetries. In this section we give a brief review of the basic notions, further details can be found in Olver's book [7].

Group methods in the calculus of variations are based on the variational symmetry group of a functional

$$\mathcal{L} = \int_{\Omega_0} L(x, u^{(n)}) dx. \tag{5.1}$$

Consider a local group of transformations G acting on an open set $M \subset \Omega_0 \times U \subset \mathcal{X} \times U$, where $\mathcal{X}$ is the space of the independent variables $x = (x^1, ..., x^p)$, U is the space of the dependent variables $u = (u^1, ..., u^q)$ and $\Omega_0 \subset \mathcal{X}$ is an open connected subset with smooth boundary $\partial\Omega$. If $u = f(x)$ is a smooth function defined over a suitable small subdomain $\Omega \subset \Omega_0$, such that the graph of f lies in M, each transformation g in G sufficiently close to the identity will transform f to another smooth function

$$\tilde{u} = \tilde{f}(\tilde{x}) = g \cdot f(\tilde{x}) \tag{5.2}$$

defined over $\tilde{\Omega} \subset \Omega_0$. A variational symmetry group G will leave the variational integral (5.1) unchanged for all such f.

A local group of transformations G acting on $M \subset \Omega_o \times U$ is a *variational symmetry group* of the functional (5.1) if whenever Ω is a subdomain with closure $\bar{\Omega} \subset \Omega_0$, $u = f(x)$ is a smooth function defined over Ω whose graph lies in M, and $g \in G$ is such that $\tilde{u} = \tilde{f}(\tilde{x}) = g \cdot f(\tilde{x})$ is a single-valued function defined over $\tilde{\Omega}$, then

$$\int_{\tilde{\Omega}} L\left(\tilde{x}, pr^{(n)} \tilde{f}(\tilde{x})\right) d\tilde{x} = \int_{\Omega} L\left(x, pr^{(n)} f(x)\right) dx. \tag{5.3}$$

Theorem 5.1. (*The criterion for the infinitesimal invariance of a variational problem under a group of transformations*) *A connected group of transformations G acting on $M \subset \Omega_0 \times U$ is a variational symmetry group of the functional (5.1) if and only if*

$$pr^{(n)} X(L) + L Div\zeta = 0, \tag{5.4}$$

for all $(x, u^{(n)}) \in M^{(n)}$ and every infinitesimal generator

$$X = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x, u) \frac{\partial}{\partial u^\alpha} \tag{5.5}$$

of G. In (5.4), $Div\zeta$ denotes the total divergence of the p-tuple $\zeta = (\xi^1, ..., \xi^p)$, namely

$$Div\zeta = \sum_{i=1}^{p} D_i(\xi^i).$$

Theorem 5.2. *If G is a variational symmetry group of the functional (5.1), then G is a symmetry group of the Euler-Lagrange equations*

$$\mathcal{E}(L) = 0, \tag{5.6}$$

where $\mathcal{E} = (\mathcal{E}_1, ..., \mathcal{E}_q)$ is the Euler operator with

$$\mathcal{E}_\alpha = \sum_J (-D)_J \frac{\partial}{\partial u_J^\alpha}, \quad \alpha = 1, ..., q. \tag{5.7}$$

Here the sum extending over all multi-indices $J = (j_1, ..., j_k)$ with $1 \le j_k \le p$, $k \ge 0$, and

$$(-D)_J = (-1)^k D_J = (-D_{j_1})(-D_{j_2})...(-D_{j_k}).$$

Notice that not every solution to the Euler-Lagrange equations is an extremal. The other solutions furnish other types of critical points for the functional. If $u = f(x)$ is a smooth extremal of the variational problem (5.1), then it must be a solution of the Euler-Lagrange equations (5.6).

By definition, for a general n-th order PDE system (2.1), *a conservation law* is a divergence expression

$$Div\ \mathcal{P} = 0 \tag{5.8}$$

which vanishes for all solutions $u = f(x)$ of the given system (2.1). Here $\mathcal{P} = (\mathcal{P}_1(x, u^{(n)}), ..., \mathcal{P}_p(x, u^{(n)})$ is a p-tuple of smooth functions of x, u and the derivatives of u, and

$$Div\ \mathcal{P} = D_1(\mathcal{P}_1) + ... + D_p(\mathcal{P}_p) \tag{5.9}$$

is *total divergence.*

In a dynamical problem, one of the independent variables is the time, $x^p = t$, and the remaining $x^1, ..., x^{p-1}$ are spatial variables. In this case, a conservation law can be written as

$$Div\ \mathcal{P} = D_t T + Div\ Z = 0,$$

where T and Z are called *conserved density*, and respectively, *associated flux vector field.* Here Div is the spatial divergence of Z with respect to $x^1, ..., x^{p-1}$. For certain classes of solutions, the conserved density, when integrated, provides us with a constant of the motion of the system.

A conservation law for the PDE system (2.1) can be also written in *characteristic form*

$$Div\ \mathcal{P} = Q \cdot \Delta = \sum_{\alpha=1}^{l} Q_\alpha \Delta_\alpha, \tag{5.10}$$

where the l-tuple $Q = (Q_1, ..., Q_l)$ is called *characteristic of the given conservation law.*

There are two kinds of *trivial conservation laws*: the first if p-tuple $\mathcal{P}$ vanishes for all solutions of the given system (2.1), and the second, if the divergence identity (5.8) holds for all functions $u = f(x)$. If $Div\ \mathcal{P} \equiv 0$, then $\mathcal{P}$ is called *null divergence.* In general, *a trivial conservation law* will be, by definition, a linear combination of conservation laws of the above two kinds. The conservation laws $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are *equivalent* if they differ by a trivial conservation law $\mathcal{R}$:

$$\tilde{\mathcal{P}} = \mathcal{P} + \mathcal{R}. \tag{5.11}$$

The conservation laws are classified up to an equivalence, such that, generally, by *conservation law* one means a *equivalence class of conservation laws.*

A trivial characteristic Q is a characteristic which vanishes for all solutions of the system. Two characteristics Q and $\tilde{Q}$ are *equivalent* if they differ by a trivial characteristic, so

$$Q = \tilde{Q} \tag{5.12}$$

for all solutions $u = f(x)$ of (2.1). Generally, the characteristics are only determined up to an equivalence.

Theorem 5.3 (*Noether*) *Suppose G is a (local) one-parameter group of symmetries of the variational problem (5.1). Let (5.5) be the infinitesimal generator of G, and*

$$Q_\alpha = \phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha, \tag{5.13}$$

the corresponding characteristics of X. Then

$$Q = (Q_1, ..., Q_q) \tag{5.14}$$

is also the characteristic of a conservation law for the Euler-Lagrange equations (5.6); in other words, there is a p-tuple

$$\mathcal{P}(x, u^{(m)}) = \left(\mathcal{P}_1(x, u^{(m)}), ..., \mathcal{P}_p(x, u^{(m)})\right), \tag{5.15}$$

such that

$$\text{(5.16)} \qquad Div\ \mathcal{P} = Q \cdot \mathcal{E}(L) = \sum_{\nu=1}^{q} Q_\nu \mathcal{E}_\nu(L)$$

is a conservation law in characteristic form for the Euler-Lagrange equations (5.6).

From the proof of Noether's theorem ([7], pp. 278) it follows that

$$\text{(5.17)} \qquad \mathcal{P} = -(\mathcal{A} + L\zeta),$$

where $\zeta = (\xi^1, ..., \xi^p)$ and $\mathcal{A} = (\mathcal{A}_1, ..., \mathcal{A}_p)$ are

$$\text{(5.18)} \qquad \mathcal{A}_k = \sum_{\alpha=1}^{q} \sum_{\#I \geq 0} \frac{\tilde{i_k} + 1}{\#I + 1} D_I \left[Q_\alpha \mathcal{E}_\alpha^{I,k}(L) \right], \quad k = 1, ..., p,$$

and

$$\text{(5.19)} \qquad \mathcal{E}_\alpha^J = \sum_{I \supset J} \begin{pmatrix} I \\ J \end{pmatrix} (-D)_{I \setminus J} \frac{\partial}{\partial u_I^\alpha}, \quad \alpha = 1, ..., q, \quad \#J \geq 0$$

are *higher Euler operators* ([7], pp. 356). In the relation (5.18), $\tilde{i_k}$ means the number of the occurrences of the integer k in the multi-index I.

Note that for a variational problem associated with the a first order Lagrangian (i.e., the Lagrangian depends up to the first order partial derivatives of the functions u^α, $\alpha = 1, ..., q$) the higher Euler operators (5.19) applied to this Lagrangian becomes nonzero only if $\#I = 0$ or $\#I = 1$. For $\#I \geq 1$, all terms $\mathcal{E}_\alpha^{I,k}(L)$ in the relation (5.18) are zero, such that the only convenient case is $\#I = 0$. Moreover, the number of occurrences of the integer k in the multi-index I, denoted by $\tilde{i_k}$ in (5.18), becomes zero for $\#I = 0$. It follows that, in this case, the functions given by the relation (5.18) become

$$\text{(5.20)} \qquad \mathcal{A}_k = \sum_{\alpha=1}^{4} Q_\alpha \mathcal{E}_\alpha^{(k)}(L), \quad k = 1, ..., p,$$

and the first order Euler operators given by (5.19) applied to the first order Lagrangian are

$$\text{(5.21)} \qquad \mathcal{E}_\alpha^{(k)} = \frac{\partial L}{u_k^\alpha}, \quad \alpha = 1, ..., q, \ k = 1, ..., p.$$

To summarize, if we know the variational symmetries, then from (5.13) one can determine the characteristics Q_α and form the relations (5.17), (5.20) and (5.21) one can find the associated conservation laws.

According to these previous considerations, for a such variational problem, the expressions (5.17), (5.20) and (5.21) can be used to find the associated conservation laws, where Q_α are already known from (5.13) for each variational symmetry.

6 Variational symmetries and conservation laws for SW PDE system

6.1 Variational symmetries of the shallow water equations

In this section we consider the potential form of SW PDE system (3.11). Salmon [11] proved that there is a first order Lagrangian

$$L = (u - R)\dot{x} + (v + P)\dot{y} - \frac{1}{2}\left(u^2 + v^2 + gh\right), \tag{6.1}$$

where the functions $P = P(x, y)$ and $R = R(x, y)$ satisfy

$$P_x + R_y = f, \tag{6.2}$$

with h given by (1.3) and f constant, for which the Euler-Lagrange equations represent (up to a sign) SW PDE system (3.11).

Theorem 6.1. *The variational symmetry group of the variational problem associated with the the Lagrangian (6.1) is generated by the following vector fields*

$$\begin{array}{rcl} X_1 & = & \partial_t, \\ Y_0 & = & -S_b(a,b)\partial_a + S_a(a,b)\partial_b, \quad S_b = -\xi,\ S_a = \eta. \end{array} \tag{6.3}$$

Moreover, if the functions $P = P(x, y)$ and $R = R(x, y)$ satisfy the PDE system

$$\left\{ \begin{array}{rcl} B(yR_x - xR_y + P) - ER_x - FR_y & = & 0 \\ B(-yP_x + xP_y + R) + EP_x + FP_y & = & 0 \\ P_x + R_y & = & f, \end{array} \right. \tag{6.4}$$

then the vector field

$$Y = (E - By)\partial_x + (F + Bx)\partial_y - Bv\partial_u + Bu\partial_v, \tag{6.5}$$

with B, E, F being real constants, generates a variational symmetry transformation. Particular cases:
Case 1. If $B \neq 0$, then denote $\frac{E}{B} = \lambda$, $\frac{F}{B} = \gamma$. After the change of variables

$$x = -\gamma + r\cos s, \quad y = \lambda + r\sin s$$

we obtain

$$\begin{array}{rcl} P(r,s) & = & -\delta(r)\sin s + \left(\frac{1}{2}fr + \frac{k}{r}\right)\cos s, \\ R(r,s) & = & \delta(r)\cos s + \left(\frac{1}{2}fr + \frac{k}{r}\right)\sin s, \end{array} \tag{6.6}$$

and there is a variational symmetry group generated by

$$Y = Y_1 = -(y-\lambda)\partial_x + (x+\gamma)\partial_y - v\partial_u + u\partial_v. \tag{6.7}$$

Case 2. If $B = 0$ and $E \neq 0$, then denote $\frac{F}{E}$ by λ. In this case it follows

$$\begin{array}{rcl} P(x,y) &=& \delta(y-\lambda x), \\ R(x,y) &=& \lambda\delta(y-\lambda x) - \lambda f x + f y + k, \end{array} \tag{6.8}$$

and the associated variational symmetry corresponds to the vector field

$$Y = Y_2 = \partial_x + \lambda\partial_y. \tag{6.9}$$

3) If $B = 0$ and $E = 0$ then

$$\begin{array}{rcl} P(x,y) &=& f x + k \\ R(x,y) &=& \delta(x), \end{array} \tag{6.10}$$

and

$$Y = Y_3 = \partial_y \tag{6.11}$$

generates a variational symmetry (here k is a real number and δ is an arbitrary function).

Proof. Consider the general infinitesimal generator (3.14) of the symmetry group of SW PDEs system. While the system (3.11) is a first order PDE system, then the criterion for infinitesimal invariance (5.4) becomes

$$pr^{(1)}X(L) + LDiv\,\zeta = 0,$$

where $\zeta = (\xi, \eta, \theta)$. The relation (3.5) implies $Div\,\zeta = \xi_a + \eta_b = 4A$, and thus we have

$$pr^{(1)}X(L) + 4AL = 0. \tag{6.12}$$

The nonzero partial derivatives of the Lagrangian (6.1) are the following

$$\frac{\partial L}{\partial x} = -R_x\dot{x} + P_x\dot{y}, \quad \frac{\partial L}{\partial y} = -R_y\dot{x} + P_y\dot{y}, \quad \frac{\partial L}{\partial u} = \dot{x} - u,$$

$$\frac{\partial L}{\partial v} = \dot{y} - v, \quad \frac{\partial L}{\partial x_a} = \frac{1}{2}gh^2 y_b, \quad \frac{\partial L}{\partial x_b} = -\frac{1}{2}gh^2 y_a, \tag{6.13}$$

$$\frac{\partial L}{\partial \dot{x}} = u - R, \quad \frac{\partial L}{\partial y_a} = -\frac{1}{2}gh^2 x_b, \quad \frac{\partial L}{\partial y_b} = \frac{1}{2}gh^2 x_a \quad \frac{\partial L}{\partial \dot{y}} = v + P.$$

On the other hand,

$$pr^{(1)}X(L) = X(L) + \phi^a \frac{\partial L}{\partial x_a} + \phi^b \frac{\partial L}{\partial u} + \phi^t \frac{\partial L}{\partial \dot{x}} + \psi^a \frac{\partial L}{\partial y_a} + \psi^b \frac{\partial L}{\partial y_b} + \psi^t \frac{\partial L}{\partial \dot{y}} \tag{6.14}$$

represents the first order prolongation of the general infinitesimal generator X given by (3.14) applied to the Lagrangian (6.1), where the infinitesimals ϕ, ψ, α and β are given by (3.13) and moreover,

$$\begin{array}{rcl} \phi^a & = & (A - \xi_a)x_a - By_a - \eta_a x_b, \\ \phi^b & = & -\xi_b x_a + (A - \eta_b)x_b - By_b, \\ \phi^t & = & -Cf\sin(ft) + Df\cos(ft) + A\dot{x} - B\dot{y}, \\ \psi^a & = & Bx_a + (A - \xi_a)y_a - \eta_a y_b, \\ \psi^b & = & Bx_b - \xi_b y_a + (A - \eta_b)y_b, \\ \psi^t & = & -Df\sin(ft) - Cf\cos(ft) + B\dot{x} + A\dot{y}. \end{array} \tag{6.15}$$

Substituting (3.13), (6.15), (6.13) and (6.14) into (6.12), and equalizing to zero the coefficients of the t-derivatives $\dot{x}$, $\dot{y}$ and the free term, it follows $A = C = D = 0$, and the relations (6.4) between the constants B, E, F and the functions $P = P(x, y)$ and $R = R(x, y)$ defining the Lagrangian (6.1). If $A = 0$, then (3.5) implies $\xi_a + \eta_b = 0$, and we can consider a function $S = S(a, b)$ such that $S_b = -\xi$ and $S_a = \eta$. It follows that the variational symmetry are generated by the vector fields X_1 and Y_0 given by (6.3). The compatibility of the system (6.4) implies the form (6.5) for another variational symmetry.

Let us discuss now the particular cases of the system (6.4).

Case 1. 1) If $B \neq 0$, then denote $\frac{E}{B}$ by λ, and $\frac{F}{B}$ by γ. The system (6.4) becomes

$$\left\{ \begin{array}{rcl} (y - \lambda)R_x - (x + \gamma)R_y + P & = & 0 \\ (\lambda - y)P_x + (x + \gamma)P_y + R & = & 0 \\ P_x + R_y & = & f. \end{array} \right. \tag{6.17}$$

Consider the change of variables $x = -\gamma + r\cos s$, $y = \lambda + r\sin s$. Its general solution is given by (6.6) and so the vector field (6.5) is written as (6.7).

For $B = 0$, the system (6.4) is equivalent to

$$\left\{ \begin{array}{rcl} EP_x + FP_y & = & 0 \\ ER_x + FR_y & = & 0 \\ P_x + R_y & = & f. \end{array} \right. \tag{6.18}$$

Case 2. If $B = 0$ and $E \neq 0$, then the system (6.17) is written as

$$\left\{ \begin{array}{l} R_x + \lambda R_y = 0 \\ P_x + \lambda P_y = 0 \\ P_x + R_y = f, \end{array} \right. \tag{6.18}$$

where denote $\frac{F}{E}$ is denoted by λ. Integrating this system, we find that (6.8) represent its general solution, and moreover, there is a variational symmetry generated by the vector field (6.9).

Case 3 If $B = 0$ and $E = 0$, then the system (6.17) turns into

$$\left\{ \begin{array}{r} FR_y = 0 \\ FP_y = 0 \\ P_x + R_y = f. \end{array} \right. \tag{6.19}$$

If $F \neq 0$, then we get $R_y = 0$, $P_y = 0$, and so $P_x = f$. Its general solution is (6.11) and we obtain a variational symmetry generated by the vector field (6.11).

In the case $F = 0$, we get that (6.3) is the only vector field that generates a variational symmetry.

A conservation law for SW PDE system is given by the divergence expression

$$D_a(\mathcal{P}_1) + D_b(\mathcal{P}_2) + D_t(\mathcal{P}_3) = 0, \tag{6.20}$$

where $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$ is a triple of smooth functions of a, b, t, x, y, u, v and the derivatives of x, y, u, v, $\mathcal{P}_3$ is the *conserved density* and the pair $(\mathcal{P}_1, \mathcal{P}_2)$ is the *associated flux vector field.*

Theorem 6.3. *The conservation laws associated with the SW PDE system (3.11), that derive from its variational symmetries, are the following*

variational symmetry generated by	$\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$
$X_1 = \partial_t$	$\mathcal{P}_1 = \frac{1}{2}gh^2(\dot{x}y_b - \dot{y}x_b)$ $\mathcal{P}_2 = \frac{1}{2}gh^2(\dot{y}x_a - \dot{x}y_a)$ $\mathcal{P}_3 = \frac{1}{2}(u^2 + v^2 + gh)$
$Y_0 = -S_b(a,b)\partial_a + S_a(a,b)\partial_b$ $S_b = -\xi,\ S_a = \eta$	$\mathcal{P}_1 = -S_bT_1$ $\mathcal{P}_2 = S_aT_1$ $\mathcal{P}_3 = S_aT_2 - S_bT_3$ $T_1 = \dot{x}(R-u) - \dot{y}(v+P) + \frac{1}{2}(u^2+v^2) + gh$ $T_2 = x_b(u-R) + y_b(v+P)$ $T_3 = x_a(u-R) + y_a(v+P)$
$Y_1 = -(y-\lambda)\partial_x + (x+\gamma)\partial_y - v\partial_u + u\partial_v$	$\mathcal{P}_1 = \frac{1}{2}gh^2\left(x_b(x+\gamma) + y_b(y-\lambda)\right)$ $\mathcal{P}_2 = \frac{1}{2}gh^2\left(y_a(\lambda-y) - x_a(x+\gamma)\right)$ $\mathcal{P}_3 = (y-\lambda)(u-R) - (x+\gamma)(v+P)$
$Y_2 = \partial_x + \lambda\partial_y$	$\mathcal{P}_1 = \frac{1}{2}gh^2(\lambda x_b - y_b)$ $\mathcal{P}_2 = \frac{1}{2}gh^2(y_a - \lambda x_a)$ $\mathcal{P}_3 = R - u - \lambda(v+P)$
$Y_3 = \partial_y$	$\mathcal{P}_1 = \frac{1}{2}gh^2x_b$ $\mathcal{P}_2 = -\frac{1}{2}gh^2x_a$ $\mathcal{P}_3 = -v - P$

Table 2

Moreover, the conservation law deriving from Y_0 is equivalent to $\tilde{\mathcal{P}} = (\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3)$, where

$$\begin{aligned}
\tilde{\mathcal{P}}_1 &= \mathcal{P}_1 + D_t(ST_2), \\
\tilde{\mathcal{P}}_2 &= \mathcal{P}_2 - D_t(ST_3), \\
\tilde{\mathcal{P}}_3 &= -S\Omega = P_3 - D_a(ST_2) + D_b(ST_3),
\end{aligned} \tag{6.21}$$

and Ω is the shallow water potential vorticity (1.6).

Proof. If we apply the higher Euler operators to the first order Lagrangian (6.1), it follows

$$\mathcal{E}_1^{(1)} = \frac{\partial L}{\partial x_a} = \frac{1}{2} g h^2 y_b,\ \mathcal{E}_2^{(1)} = \frac{\partial L}{\partial y_a} = -\frac{1}{2} g h^2 x_b,\ \mathcal{E}_3^{(1)} = \frac{\partial L}{\partial u_a} = 0,\ \mathcal{E}_4^{(1)} = \frac{\partial L}{\partial v_a} = 0,$$

$$\mathcal{E}_1^{(2)} = \frac{\partial L}{\partial x_b} = -\frac{1}{2} g h^2 y_a,\ \mathcal{E}_2^{(2)} = \frac{\partial L}{\partial y_b} = \frac{1}{2} g h^2 x_a,\ \mathcal{E}_3^{(2)} = \frac{\partial L}{\partial u_b} = 0,\ \mathcal{E}_4^{(2)} = \frac{\partial L}{\partial v_b} = 0, \tag{6.22}$$

$$\mathcal{E}_1^{(3)} = \frac{\partial L}{\partial \dot{x}} = u - R, \quad \mathcal{E}_2^{(3)} = \frac{\partial L}{\partial \dot{y}} = v + P, \quad \mathcal{E}_3^{(3)} = \frac{\partial L}{\partial \dot{u}} = 0, \quad \mathcal{E}_4^{(3)} = \frac{\partial L}{\partial \dot{v}} = 0.$$

In that follows, we determine the conservation laws deriving form the variational symmetries generated by the vector fields (6.3), (6.5), and in particular (6.7), (6.9) and (6.11).
1. For the vector field $X_1 = \partial_t$, the corresponding characteristic $Q = (Q_1, ..., Q_4)$ has the components given by (5.13), namely

$$Q_1 = -\dot{x}, \quad Q_2 = -\dot{y}, \quad Q_3 = -\dot{u}, \quad Q_4 = -\dot{v}.$$

According to (5.20), the triple $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ is the following

$$\begin{aligned}
\mathcal{A}_1 &= \tfrac{1}{2} g h^2 (\dot{y} x_b - \dot{x} y_b), \\
\mathcal{A}_2 &= \tfrac{1}{2} g h^2 (\dot{x} y_a - \dot{y} x_a), \\
\mathcal{A}_3 &= \dot{x}(R - u) - \dot{y}(v + P).
\end{aligned}$$

In our case $\zeta = (0, 0, 1)$ and so the triple $\mathcal{P}$ given by (5.17) becomes

$$\begin{aligned}
\mathcal{P}_1 &= \tfrac{1}{2}(\dot{x} y_b - \dot{y} x_b), \\
\mathcal{P}_2 &= \tfrac{1}{2}(\dot{y} x_a - \dot{x} y_a), \\
\mathcal{P}_3 &= \tfrac{1}{2}(u^2 + v^2 + gh),
\end{aligned}$$

Thus, the time invariance lead to the conservation of the energy $\mathcal{P}_3$ of the system (Salmon [11]).

2. The characteristic Q of the vector field

$$Y_0 = -S_b(a,b)\partial_a + S_a(a,b)\partial_b$$

is given by

$$Q_1 = S_b x_a - S_a x_b, \quad Q_2 = S_b y_a - S_a y_b, \quad Q_3 = S_b u_a - S_a u_b, \quad Q_4 = S_b v_a - S_a v_b.$$

In this case, $\zeta = (-S_b, S_a, 0)$ and from (5.20) we get

$$\begin{aligned}\mathcal{A}_1 &= \tfrac{1}{2} gh S_b, \\ \mathcal{A}_2 &= -\tfrac{1}{2} gh S_a, \\ \mathcal{A}_3 &= S_b[x_a(u-R) + y_a(v+P)] - S_a[x_b(u-R) + y_b(v+P)]\end{aligned}$$

Therefore, substituting them into (5.17) one obtains

$$\begin{aligned}\mathcal{P}_1 &= -S_b T_1, \\ \mathcal{P}_2 &= S_a T_1, \\ \mathcal{P}_3 &= S_a T_2 - S_b T_3,\end{aligned}$$

where denote

$$\begin{aligned}T_1 &= \dot{x}(R-u) - \dot{y}(v+P) + \tfrac{1}{2}(u^2+v^2) + gh, \\ T_2 &= x_b(u-R) + y_b(v+P), \\ T_3 &= x_a(u-R) + y_a(v+P).\end{aligned}$$

For the vector field Y_0 we can write another equivalent form of the associated conservation law. Note that the potential vorticity Ω (1.6) satisfies

$$D_a(T_2) - D_b(T_3) = \Omega.$$

Define the triple $\tilde{\mathcal{P}}$ given by (6.21) and let $\mathcal{P}^* = (\mathcal{P}_1^*, \mathcal{P}_2^*, \mathcal{P}_3^*)$ be the triple with the components

$$\mathcal{P}_1^* = D_t(ST_2), \quad \mathcal{P}_2^* = -D_t(ST_3), \quad \mathcal{P}_3^* = -D_a(ST_2) + D_b(ST_3).$$

One verifies that $Div\ P^* = 0$. It follows that the conservation laws P and $\tilde{\mathcal{P}} = \mathcal{P} + \mathcal{P}^*$ are equivalent.

Thus, the particle relabelling symmetry lead to the conservation of the potential vorticity Ω (Salmon [11]).

3. In the case of the vector field (6.5), the characteristic Q is defined by

$$Q_1 = E - By, \quad Q_2 = F + Bx, \quad Q_3 = -Bv, \quad Q_4 = Bu.$$

Because $\zeta = (0,0,0)$, from (5.17) it follows $\mathcal{P}_i = \mathcal{A}_i$, $i = 1,2,3$. From (4.20) we get

$$\begin{aligned}
\mathcal{A}_1 &= \tfrac{1}{2}gh^2[Ey_b - Fx_b - B(xx_b + yy_b)], \\
\mathcal{A}_2 &= \tfrac{1}{2}gh^2[-Ey_a + Fx_a + B(xx_a + yy_a)], \\
\mathcal{A}_3 &= E(u-R) + F(v+P) + B[x(v+P) - y(u-R)],
\end{aligned}$$

and so the triple $\mathcal{P}$ has the components

$$\begin{aligned}
\mathcal{P}_1 &= \tfrac{1}{2}gh^2[-Ey_b + Fx_b + B(xx_b + yy_b)], \\
\mathcal{P}_2 &= \tfrac{1}{2}gh^2[Ey_a - Fx_a - B(xx_a + yy_a)], \\
\mathcal{P}_3 &= E(R-u) - F(v+P) + B[-x(v+P) + y(u-R)].
\end{aligned}$$

In particular, for the vector fields Y_1, Y_2 and Y_3 given by (6.7), (6.9) and (6.11), respectively, if one substitutes $B = 1$, $E = \lambda$, $F = \gamma$, and $B = 0$, $E = 1$, $F = \lambda$, and respectively $B = 0$, $E = 0$, $F = 1$ into the above expressions, it follows the corresponding conservation laws contained in Table 2. The rotation invariance lead to the conservation of the angular momentum and the translation invariance to the conservation of linear momentum.

7 Variational symmetries and conservations laws for SG PDE system

7.1 Variational symmetries for the semi-geostrophic equations

The SW PDE system (4.6) represents the Euler-Lagrange equations associated with the the first order Lagrangian (Hydon [3])

$$L = (u_g - R)\dot{x} + (v_g + P)\dot{y} - \frac{1}{2}(u_g^2 + v_g^2 + gh) - r\dot{u}_g + p\dot{v}_g, \tag{7.1}$$

up to a sign,where $P = P(x,y)$, $R = R(x,y)$, $p = p(u_g, v_g)$ and $r = r(u_g, v_g)$ are arbitrary functions satisfying the equations

$$\begin{cases} P_x + R_y &= f \\ p_{u_g} + r_{v_g} &= \frac{1}{f}, \end{cases} \tag{7.2}$$

where h is given by (1.3).

Theorem 7.1. *The variational symmetry group of the variational problem associated with the the Lagrangian (7.1) is generated by the vector fields (7.4). Moreover, if $P = P(x,y)$ and $R = R(x,y)$ are solutions of the PDE system (6.17), then the vector field*

$$Z = E\partial_x + F\partial_y \tag{7.4}$$

generates also a variational symmetry transformation. Particular cases:
Case 1 If $E \neq 0$, then denote $\frac{F}{E} = \lambda$. It follows that P and R are given by (6.8). In this case, there is an additional variational symmetry generated by the vector field (7.4) denoted by Z_1 and given by (6.9).
Case 2 For $E = 0$, then the functions P and R are given by (6.10). The vector field $Z = Z_2$ defined by (6.11) generates the associated additional variational symmetry.

Proof. Let us consider the infinitesimal generator of the symmetry group associated with the SG PDE system, given by (4.9). The criterion for infinitesimal invariance (Theorem 5.1) implies

$$pr^{(1)}X(L) + LDiv\zeta = 0,$$

where $\zeta = (\xi, \eta, \theta)$. Using $Div\zeta = \xi_a + \eta_b = 4A$, the above condition becomes

$$pr^{(1)}X(L) + 4AL = 0. \tag{7.5}$$

The nonzero partial derivatives of the Lagrangian (7.1) are the following

$$\frac{\partial L}{\partial x} = -R_x\dot{x} + P_x\dot{y}, \quad \frac{\partial L}{\partial y} = -R_y\dot{x} + P_y\dot{y}, \quad \frac{\partial L}{\partial u} = \dot{x} - u_g - r_{u_g}\dot{u_g} + p_{u_g}\dot{v}_g,$$

$$\frac{\partial L}{\partial v} = \dot{y} - v - r_{v_g}\dot{u_g} + p_{v_g}\dot{v}_g, \quad \frac{\partial L}{\partial x_a} = \frac{1}{2}gh^2y_b, \quad \frac{\partial L}{\partial x_b} = -\frac{1}{2}gh^2y_a, \tag{7.6}$$

$$\frac{\partial L}{\partial \dot{x}} = u_g - R, \quad \frac{\partial L}{\partial y_a} = -\frac{1}{2}gh^2x_b, \quad \frac{\partial L}{\partial y_b} = \frac{1}{2}gh^2x_a \quad \frac{\partial L}{\partial \dot{y}} = v_g + P,$$

$$\frac{\partial L}{\partial \dot{u}_g} = -r, \quad \frac{\partial L}{\partial \dot{v}_g} = p.$$

Applying the first order prolongation of the vector field X to the Lagrangian L it follows

$$pr^{(1)}X(L) = \phi\frac{\partial L}{\partial x} + \psi\frac{\partial L}{\partial y} + \alpha\frac{\partial L}{\partial u} + \beta\frac{\partial L}{\partial v} + \phi^a\frac{\partial L}{\partial x_a} + \phi^b\frac{\partial L}{\partial u} + \phi^t\frac{\partial L}{\partial \dot{x}} +$$

$$+\psi^a\frac{\partial L}{\partial y_a} + \psi^b\frac{\partial L}{\partial y_b} + \psi^t\frac{\partial L}{\partial \dot{y}} + \alpha^t\frac{\partial L}{\partial \dot{u_g}} + \beta^t\frac{\partial L}{\partial \dot{v_g}}, \tag{7.7}$$

where the functions ϕ, ψ, α and β are given by (4.8), and moreover

$$
\begin{aligned}
\phi^a &= (A-\xi_a)x_a - By_a - \eta_a x_b, \\
\phi^b &= -\xi_b x_a + (A-\eta_b)x_b - By_b, \\
\phi^t &= A\dot{x} - B\dot{y}, \\
\psi^a &= Bx_a + (A-\xi_a)y_a - \eta_a y_b, \\
\psi^b &= Bx_b - \xi_b y_a + (A-\eta_b)y_b, \\
\psi^t &= B\dot{x} + A\dot{y}, \\
\alpha^t &= A\dot{u_g} - B\dot{v_g}, \\
\beta^t &= B\dot{u_g} + A\dot{v_g}.
\end{aligned} \tag{7.8}
$$

Substituting (4.8), (7.7), (7.8) and (7.6) into the relation (7.5), and equalizing to zero the coefficients of the t-derivatives $\dot{x}, \dot{y}, \dot{u}_g, \dot{v}_g$ and the free term, we get $A = B = 0$, and the conditions (6.17) between the constants E, F and the functions $P = P(x, y)$ and $R = R(x, y)$ from the Lagrangian (7.1). If $A = 0$, then the condition (3.5) turns into $\xi_a + \eta_b = 0$, and so there is a function $S = S(a, b)$ such that $S_a = \eta$, and $S_b = -\xi$. We obtain that the variational symmetries are generated by the vector fields (7.4). Moreover, if the functions P and R satisfy (6.17), then there is an additional variational symmetry corresponding to the vector field Z given by (7.4).

Particular cases for the system (6.17):

Case 1. If $E \neq 0$, then denote $\frac{F}{E}$ by λ. It follows that the system (6.17) can be written as (6.18) with the general solution given by (6.8). The vector field Z denoted by Z_1, becomes (6.9).

Case 2. If $E = 0$, then the system (6.17) is written as (6.19).

- If $F \neq 0$ then its general solution is given by (6.10). The associated vector field Z, denoted it by Z_2, becomes (6.11).
- If $F = 0$, there are only variational symmetry generated by the vector fields (7.4).

Note that the particular cases of the Theorem 7.1 correspond to the particular case 2 of Theorem 6.1.

7.2 Conservation laws for the semi-geostrophic equations

Theorem 7.2. *The conservation laws associated with the SG PDE system, are presented in Table 3, where (P_1, P_2) are the associated flux vector fields and P_3 is the conserved densities.*

$v.f.$	$\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$
$X_1 = \partial_t$	$\mathcal{P}_1 = \frac{1}{2} g h^2 (\dot{x} y_b - \dot{y} x_b)$ $\mathcal{P}_2 = \frac{1}{2} g h^2 (\dot{y} x_a - \dot{x} y_a)$ $\mathcal{P}_3 = \frac{1}{2} (u_g^2 + v_g^2 + gh)$
$Y_0 = -S_b(a,b)\partial_a + S_a(a,b)\partial_b$	$\mathcal{P}_1 = -S_b T_1$ $\mathcal{P}_2 = S_a T_1$ $\mathcal{P}_3 = S_a T_2 - S_b T_3,$ $T_1 = \dot{x}(R - u_g) - \dot{y}(v_g + P) + \frac{1}{2}(u_g^2 + v_g^2) + gh + r\dot{u_g} - p\dot{v_g}$ $T_2 = x_b(u_g - R) + y_b(v_g + P) - r u_{g,b} + p v_{g,b}$ $T_3 = x_a(u_g - R) + y_a(v_g + P) - r u_{g,a} + p v_{g,a}$
$Z_1 = \partial_x + \lambda \partial_y$	$\mathcal{P}_1 = \frac{1}{2} g h^2 (\lambda x_b - y_b)$ $\mathcal{P}_2 = \frac{1}{2} g h^2 (y_a - \lambda x_a)$ $\mathcal{P}_3 = R - u_g - \lambda(v_g + P)$
$Z_2 = \partial_y$	$\mathcal{P}_1 = \frac{1}{2} g h^2 x_b$ $\mathcal{P}_2 = \frac{1}{2} g h^2 x_a$ $\mathcal{P}_3 = -v_g - P$

Table 3

The conservation law deriving from the vector field Y_0 is equivalent to that defines by the triple $\tilde{\mathcal{P}} = (\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3)$ with the components

$$\begin{aligned} \tilde{\mathcal{P}}_1 &= \mathcal{P}_1 + D_t(ST_2), \\ \tilde{\mathcal{P}}_2 &= \mathcal{P}_2 - D_t(ST_3), \\ \tilde{\mathcal{P}}_3 &= -S\Omega^* = \mathcal{P}_3 - D_a(ST_2) + D_b(ST_3), \end{aligned} \tag{7.9}$$

where Ω^* is the potential vorticity (1.9).

Proof. From the relations (5.21) it follows

$$\mathcal{E}_1^{(1)}=\frac{\partial L}{\partial x_a}=\frac{1}{2}gh^2y_b,\ \mathcal{E}_2^{(1)}=\frac{\partial L}{\partial y_a}=-\frac{1}{2}gh^2x_b,\ \mathcal{E}_3^{(1)}=\frac{\partial L}{\partial u_{g,a}}=0,\ \mathcal{E}_4^{(1)}=\frac{\partial L}{\partial v_{g,a}}=0,$$

(7.10)

$$\mathcal{E}_1^{(2)}=\frac{\partial L}{\partial x_b}=-\frac{1}{2}gh^2y_a,\ \mathcal{E}_2^{(2)}=\frac{\partial L}{\partial y_b}=\frac{1}{2}gh^2x_a,\ \mathcal{E}_3^{(2)}=\frac{\partial L}{\partial u_{g,b}}=0,\ \mathcal{E}_4^{(2)}=\frac{\partial L}{\partial v_{g,b}}=0,$$

$$\mathcal{E}_1^{(3)}=\frac{\partial L}{\partial \dot{x}}=u_g-R,\ \mathcal{E}_2^{(3)}=\frac{\partial L}{\partial \dot{y}}=v_g+P,\ \mathcal{E}_3^{(3)}=\frac{\partial L}{\partial \dot{u}_g}=-r,\ \mathcal{E}_4^{(3)}=\frac{\partial L}{\partial \dot{v}_g}=p,$$

that represents the Euler operators applied to the Lagrangian (7.1). For each of the vector fields that defines the associated variational symmetries (Theorem 7.1), one finds the associated conservation laws.

1. For the vector field $Y_1=\partial_t$, the characteristic $Q=(Q_1,...,Q_4)$, has the components given by (5.13), namely,

$$Q_1=-\dot{x},\quad Q_2=-\dot{y},\quad Q_3=-\dot{u},\quad Q_4=-\dot{v}.$$

In this case, the triple $\mathcal{P}=(\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3)$, given by (5.17), is determined by $\zeta=(0,0,1)$ and $\mathcal{A}=(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)$, where

$$\begin{aligned}
\mathcal{A}_1 &= \tfrac{1}{2}gh^2(-\dot{x}y_b+\dot{y}x_b),\\
\mathcal{A}_2 &= \tfrac{1}{2}gh^2(\dot{x}y_a-\dot{y}x_a),\\
\mathcal{A}_3 &= -\dot{x}(u_g-R)-\dot{y}(v_g+P)+\dot{u}_gr-\dot{v}_gp.
\end{aligned}$$

Substituting them into (5.17) it follows that the triple $\mathcal{P}=(\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3)$ with

$$\begin{aligned}
\mathcal{P}_1 &= \tfrac{1}{2}gh^2(\dot{x}y_b-\dot{y}x_b),\\
\mathcal{P}_2 &= \tfrac{1}{2}gh^2(\dot{y}x_a-\dot{x}y_a),\\
\mathcal{P}_3 &= \tfrac{1}{2}(u^2+v^2+gh),
\end{aligned}$$

defines the corresponding conservation law. Thus, the time invariance implies the conservation $\mathcal{P}_3$ of the energy of the SG PDE system.

2. Let us consider the vector field

$$Y_0=-S_b(a,b)\partial_a+S_a(a,b)\partial_b,$$

for which the characteristic is

$$Q_1=S_bx_a-S_ax_b,\quad Q_2=S_by_a-S_ay_b,\quad Q_3=S_bu_{g,a}-S_au_{g,b},\quad Q_4=S_bv_{g,a}-S_av_{g,b}.$$

In the relation (4.17) one substitutes the triples $\zeta = (-S_b, S_a, 0)$ and $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ given by (4.20), namely

$$\begin{aligned}
\mathcal{A}_1 &= \tfrac{1}{2}ghS_b, \quad \mathcal{A}_2 = -\tfrac{1}{2}ghS_a, \\
\mathcal{A}_3 &= S_a[x_b(R-u) - y_b(v+P) + ru_b - pv_b] + S_b[x_a(u-R) + y_a(v+P) - ru_a + pv_a],
\end{aligned}$$

and it follows

$$\begin{aligned}
\mathcal{P}_1 &= -S_bT_1, \\
\mathcal{P}_2 &= S_aT_1, \\
\mathcal{P}_3 &= S_aT_2 - S_bT_3,
\end{aligned}$$

where

$$\begin{aligned}
T_1 &= \dot{x}(R-u_g) - \dot{y}(v_g+P) + \tfrac{1}{2}(u_g^2 + v_g^2) + gh + r\dot{u_g} - p\dot{v_g}, \\
T_2 &= x_b(u_g - R) + y_b(v_g + P) - ru_{g,b} + pv_{g,b}, \\
T_3 &= x_a(u_g - R) + y_a(v_g + P) - ru_{g,a} + pv_{g,a}.
\end{aligned}$$

The conservation law deriving from the vector field Y_0 can be written in an equivalent form. Note that

$$D_a(T_2) - D_b(T_3) = \Omega^*,$$

where Ω^* is the potential vorticity (1.9). Let us consider the triple $\tilde{\mathcal{P}} = (\tilde{\mathcal{P}}_1, \tilde{\mathcal{P}}_2, \tilde{\mathcal{P}}_3)$ defined by

$$\begin{aligned}
\tilde{P}_1 &= -S_bT_1 + D_t(ST_2) = P_1 + D_t(ST_2), \\
\tilde{P}_2 &= S_aT_1 - D_t(ST_3) = P_2 - D_t(ST_3), \\
\tilde{P}_3 &= -S\Omega^* = P_3 - D_a(ST_2) + D_b(ST_3),
\end{aligned}$$

and denote

$$P_1^* = D_t(ST_2), \quad P_2^* = -D_t(ST_3), \quad P_3^* = -D_a(ST_2) + D_b(ST_3).$$

Because $Div\ P^* = 0$, it follows that $\mathcal{P}$ and $\tilde{\mathcal{P}} = \mathcal{P} + \mathcal{P}^*$ are equivalent conservation laws. Thus, the conservation law defines by the triple $\tilde{\mathcal{P}}$, derives from the particle relabelling symmetry, and implies the conservation of the potential vorticity Ω^* (Hydon [3]).
3. In the case of the vector field Z given by (7.4), from (5.13) it follows that its characteristic Q has the components

$$Q_1 = E, \quad Q_2 = F, \quad Q_3 = 0, \quad Q_4 = 0.$$

While $\zeta = (0,0,0)$ it follows $\mathcal{P}_i = \mathcal{A}_i$, $i = 1,2,3$. Using (5.20) we get

$$\begin{aligned}
\mathcal{A}_1 &= \tfrac{1}{2}gh^2(Ey_b - Fx_b), \\
\mathcal{A}_2 &= \tfrac{1}{2}gh^2(Fx_a - Ey_a), \\
\mathcal{A}_3 &= E(u-R) + F(v+P),
\end{aligned}$$

and in this case, the relation (5.17) implies

$$\begin{aligned}
\mathcal{P}_1 &= \tfrac{1}{2} g h^2 (F x_b - E y_b), \\
\mathcal{P}_2 &= \tfrac{1}{2} g h^2 (E y_a - F x_a), \\
\mathcal{P}_3 &= E(R-u) - F(v+P).
\end{aligned}$$

In the particular cases $Z = Z_1$ and $Z = Z_2$, one substitutes $E = 1, F = \lambda$ and respectively, $E = 0, F = 1$ into the above expressions. The corresponding triples $\mathcal{P}$ lead to the conservation of linear momentum (Table 3).

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