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Abstract

In this paper we introduce a new technique for deriving the determining equations of the nonclassical symmetries associated with a nonlinear partial differential equation system, such that, all the symbolic manipulation programs designated to calculate classical Lie symmetries can also be used for finding nonclassical symmetries, without any adaptation of the program. This algorithm was implemented as the MAPLE routine `gendefnc`, and uses the MAPLE package DESOLV (authors Carminati and Vu). The nonlinear system of the determining equations for nonclassical symmetries can now be integrated (interactively) using the MAPLE packages `diffgrob2` (author Mansfield) or RIF (authors Reid and Wittkopf). The two-dimensional semigeostrophic partial differential equation system arising from meteorology was considered as an application of this procedure.

Keywords: classical symmetries, nonclassical symmetries, nonlinear partial differential equation systems.

AMS Subject Classification: 58J70, 35A30, 70G65, 68W30.

1 Introduction

The modern approach for studying systems of nonlinear partial differential equations (PDEs) from the point of view of group analysis theory was pioneered by Sophus Lie at the beginning of this century [10]. While there is no existing general theory for completely solving of nonlinear PDEs, such in the case of linear ones, the application of group of transformations to

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their study has to be one of the powerful methods that answer this difficult problem. In the past years, a variety of methods have been developed in order to find special classes of solutions of nonlinear differential equations. [23] showed that the one common theme of all these methods has been the appearance of some form of group invariance. Further remarks on the historical development of this subject and its applications to nonlinear models occurring in different research domains can be found for example in [1], [4], [7], [8], [9], [20], [14], [25], [30], [31].

In this paper, we restrict ourselves to the *classical Lie method* (due originally to [17], popularized by [25], and presented in a modern form using the jet space theory by [22]) and to the *nonclassical method* (due to [4]). In order to apply these algorithmic methods, the use of symbolic manipulation programs has became imperative, such that a variety of packages have been developed for many computer algebra systems (a survey of these programs can be found in Hereman's paper [12]). For the computer system MAPLE, the programs DESOLV [5], SYMMETRY [13] and RIF [26] contain routines for generating classical symmetries. At the moment, for the computer system MAPLE, there are no symbolic manipulation programs available for finding nonclassical symmetries, but for other computer systems there are available several packages: the MACSYMA program SYMMGRP.MAX [6] that can be adapted for this purpose [9], the interactive program NUSY in REDUCE [21]), and the MUMATH program LIE [11]. The last one can be used for analysis of differential equations on IBM type PCs but this program is restricted due to the system memory up to 256kB.

The classical Lie method leads us to those one-parameter group of transformations called *classical symmetries* or *Lie point symmetries* that leave a PDE system unchanged, and hence, they map the set $\mathcal{S}$ of all analytical solutions to itself. Moreover, the *nonclassical symmetries* are those one-parameter groups of transformations that leave only a subset $\mathcal{S}_\psi$ of the set $\mathcal{S}$ invariant. The classical symmetries are solutions to an over-determined system of linear PDEs, while the nonclassical symmetries are solutions to an over-determined system of nonlinear PDEs. Any classical symmetry is a nonclassical symmetry, but not conversely. It has been showed that there are PDEs for which these kind of symmetries are the same, but we still do not know the answer to the question: "How to determine *a priori* which PDEs posses symmetry reductions that are not obtainable using the classical Lie approach?" [8]. However, the nonclassical method has been successfully applied to a large number of nonlinear PDEs, and so, new classes of symmetry reductions, and new classes of analytical solutions have been found.

Introduced by Bluman and Cole in 1969, in their study of new solutions to linear heat equation that are not obtained from the classical Lie method, the nonclassical method, also known as the *method of conditional symmetries* [16] or as the *method of partial symmetries of the first type* [31], has been applied using two techniques: the first is the standard procedure [16], and the second is based on the theory of differential Gröbner basis [9]. Due to the infinite loops that can occur in the reduction process, Clarkson and Mansfield introduced a new algorithm (that was implemented using the program SYMMGRP.MAX) based on the reducing of the studied PDE system (1) to its normal form (13), by using the associated invariant surface conditions (8) and its differential consequences. By *normal form* is meant some well-defined reduction of the system such that no further eliminations from the invariant surface conditions are possible. The next step is to apply the classical Lie method to the normal form system, where the symmetry operator related to the invariance surface condition. It should be pointed out that the invariant surface conditions represent the characteristics of the symmetry operator of the normal system, that we are looking for. The resulting nonlinear PDE system represents the determining equations of nonclassical symmetries. This can be integrated (interactively) using the MAPLE packages `diffgrob2` [20] or RIF [26]. Following this idea, all the available programs designated to find nonclassical symmetries, automatically add the invariant surface conditions and their differential consequences to the studied PDE system, so those appear in the input of the program as initial data.

The aim of this paper is to introduce an easier procedure for deriving the determining equations of nonclassical symmetries, namely, we prove that the determining equations of the nonclassical symmetries of a PDE system derive from the determining equations of the symmetry group associated with its normal form. Contrary to the standard procedure, we are looking for a symmetry operator of the system (13) that is can be viewed as not related to the surface invariant conditions anymore, such that, we apply the classical Lie method to the system (13) with an arbitrary vector field.

The algorithm introduced in this paper is a three-step procedure: at Step 1 the studied PDE system is reduced to its normal form (13); at Step 2 the coefficients of the remaining partial derivatives may be denoted by arbitrary functions $\mathcal{A}_\beta$, so we get the system (18), and the determining equations of the symmetry group associated with this new system (18) are found by using any package design for this purpose; finally, at Step 3, the determining equations of the nonclassical symmetries of the studied PDE system are obtained from the determining equations of the symmetry group associated with the system (18).

As a main consequence of this algorithm, one need not design new programs for finding nonclassical symmetries, such that any symbolic manipulation program designated to calculate the classical Lie symmetries can be used, without any change, for the finding of nonclassical symmetries. To avoid running into memory problems, as that happens in the case of large PDE systems, we restrict ourselves to the computer system MAPLE. Our algorithm has been implemented as the MAPLE routine `gendefnc` and uses the DESOLV program, that can handle now both the finding of the determining equations of the classical and nonclassical symmetries.

The structure of this paper is as follows. In Section 2 we recall briefly the classical Lie method and the nonclassical method (more details on this subject can be found for example in [22], [15] and [14]). The new procedure is shown in Section 3, and the MAPLE routine `gendefnc` is explained in Section 4. In Section 5, an application of this procedure to the two-dimensional semi-geostrophic PDE system is given. This large PDE system arises from meteorology, namely, in the semi-geostrophic theory [27], [28], [29].

2 Classical and nonclassical symmetries

Consider a n-th order PDE system

$$\Delta(x, u^{(n)}) = 0, \tag{1}$$

with p independent variables $x = (x^1, ..., x^p) \in \mathcal{X} \subset \mathbf{R}^p$, and q dependent variables $u = (u^1, ..., u^q) \in U \subset \mathbf{R}^q$. Here $\mathcal{X}$ is the space of the independent variables and U is the space of the dependent variables associated with the system. Denote by $\Delta = (\Delta_1, ..., \Delta_l)$ the function whose components define the equations of the system (1). Recall that by $u^{(n)}$ we denote all the partial derivatives of u with respect to x up to order n, where the k-th order partial derivative of u^α with respect to x is

$$u^\alpha_J = \frac{\partial^k u^\alpha}{\partial x^{j_1} ... \partial x^{j_k}},$$

for $J = (j_1, ..., j_k)$, $1 \leq j_k \leq p$, $1 \leq k \leq n$, and $u^\alpha_0 = u^\alpha$, $\alpha = 1, ..., q$. The system (1) is said to be a *maximal rank system* if

$$rank \left(\frac{\partial \Delta}{\partial x^i}, \frac{\partial \Delta}{\partial u^\alpha_J} \right) = l \quad \text{whenever} \quad \Delta(x, u^{(n)}) = 0.$$

Denote the set of all analytic solutions of this system (1) by

$$\mathcal{S} = \{u(x) : \Delta(x, u^{(n)}) = 0\} \tag{2}$$

and let G be a (connected) local group of transformations

$$\begin{cases} x^{*i} &= x^i + \epsilon\zeta^i(x,u) + \mathcal{O}(\epsilon^2), \quad i = 1, ..., p \\ u^{*\alpha} &= u^\alpha + \epsilon\phi^\alpha(x,u) + \mathcal{O}(\epsilon^2), \quad \alpha = 1, ..., q, \end{cases} \tag{3}$$

acting on an open subset $M \subset \mathcal{X} \times U \subset \mathbf{R}^p \times \mathbf{R}^q$, with the infinitesimal generator given by the vector field

$$X = \sum_{i=1}^{p} \zeta^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi^\alpha(x,u)\frac{\partial}{\partial u^\alpha}. \tag{4}$$

2.1 The classical Lie method

The classical Lie method [22] applied to a maximal rank PDE system (1) leads us to its associated symmetry group G, that consists of one-parameter groups of transformations acting on an open set of the space $\mathcal{X} \times U$ and leaving the set $\mathcal{S}$ invariant. Thus, these transformations, called *classical symmetries* or *Lie point symmetries*, map any solution of the system to another solution of it and leave the form of the system unchanged. The vector field X is called a *symmetry operator* associated with the system (1), and its components $\zeta^1, ..., \zeta^p$, and $\phi^1, ..., \phi^q$ are named *infinitesimals*. Knowledge of the symmetry group G, allows one to construct and to classify the *group-invariant solutions* of the studied PDE system.

The method of finding the symmetry group of a maximal rank PDE system is based on the following results.

Theorem 1 1 *Let X be a vector field defined on an open set $M \subset \mathcal{X} \times \mathcal{U}$, given by (4). Then its associated n-th order prolongation is the vector field*

$$pr^{(n)}X = X + \sum_{\alpha=1}^{q}\sum_{J} \Phi_J^\alpha(x, u^{(n)})\frac{\partial}{\partial u_J^\alpha}, \tag{5}$$

defined on the corresponding jet space $M^{(n)} \subset \mathcal{X} \times U^{(n)}$, where

$$\Phi_J^\alpha(x, u^{(n)}) = D_J\left(\phi^\alpha - \sum_{i=1}^{p}\zeta^i u_i^\alpha\right) + \sum_{i=1}^{p}\zeta^i u_{J,i}^\alpha, \tag{6}$$

the summation being over all multi-indices $J = (j_1, ..., j_k)$, $1 \le j_k \le p$, $1 \le k \le n$.

Theorem 2 1 *(Criterion for infinitesimal invariance) Let (1) be a maximal rank PDE system. A connected group of (local) transformations G is a symmetry group of the PDE system (1) if and only if the condition*

$$pr^{(n)}X(\Delta)|_{\Delta=0} = 0 \tag{7}$$

is identically satisfied for every infinitesimal generator X of G.

Substituting the n-th order prolongation of the vector field X given by the relations (5) and (6) into the condition (7), after that eliminating any dependencies (caused by the system itself) by substituting the higher order partial derivative occurring in (1) into (7), and finally, equalizing to zero the coefficients of the remaining partial derivatives, we get an over-determined linear PDE system for the infinitesimals ζ^i and ϕ^α of the symmetry operator X. This system is called the *determining equations of the symmetry group* of the PDE system (1).

2.2 The nonclassical method

The nonclassical method [4] yields the one-parameter groups of transformations called *nonclassical* or *conditional* symmetries, that leave the subset

$$\mathcal{S}_\psi = \{u(x) : \Delta = 0, \psi = 0\} \subset \mathcal{S}$$

invariant, where $\psi = (\psi^1, ..., \psi^q)$ is defined by

$$\psi^\alpha \equiv \sum_{i=1}^{p} \zeta^i(x,u)u_i^\alpha - \phi^\alpha(x,u) = 0, \ \alpha = 1, ..., q. \tag{8}$$

The relations (8) represent the characteristics of the vector field X given by (4) and they are called the *invariant surface conditions* associated with the PDE system (1). These transformations do not leave the set of all analytical solutions of the system (1) unchanged, but they map the set of the solutions of the system (1)+(8) to itself.

The algorithm of finding nonclassical symmetries associated with the maximal rank PDE system (1) is based on the same criterion for infinitesimal invariance (see Theorem 2) but in this case, this is rewritten as

$$\begin{aligned} pr^{(n)}X(\Delta)|_{\Delta=0,\psi=0} &= 0 \\ pr^{(1)}X(\psi)|_{\Delta=0,\psi=0} &= 0. \end{aligned} \tag{9}$$

Note that the second condition in (9) is identically satisfied, because

$$pr^{(1)}X(\psi^{\alpha}) = -\psi^{\alpha}\psi_u^{\alpha}, \quad \alpha = 1, ..., q. \tag{10}$$

In this case, a vector field X satisfying (9) is called *nonclassical operators* associated with the PDE system (1). Notice that the nonclassical operators do not form a vector space, still less a Lie algebra as the symmetry operators do.

So far, the nonclassical method has been applied using two techniques:

1. The standard procedure [16] can create difficulties in the implementation of symbolic manipulation programs, because it requires the use of certain derivatives of u^{α} obtained by differentiating the invariant surface conditions (8) (infinite loops can occur in the reduction process).

2. In order to eliminate the difficulties appearing in this first technique, Clarkson and Mansfield introduced a new algorithm based on the theory of differential Gröbner basis [9]. The system (1) is reduced to its normal form using the invariant surface conditions (8) and its differential consequences, and after that one applies the classical Lie method to the resulting system, but with the symmetry operator X related to (8).

Let us discuss in the following the procedure in the nontrivial case when $\zeta^p \neq 0$. Without loss the generality, one can assume that $\zeta^p = 1$ (if X is a nonclassical operator, then so is λX, for any function $\lambda = \lambda(x, u)$). In this case, let us denote by Y the new vector field X,

$$Y = \sum_{j=1}^{p-1} \zeta^j(x, u)\frac{\partial}{\partial x^j} + \frac{\partial}{\partial x^p} + \sum_{\alpha=1}^{q} \phi^{\alpha}(x, u)\frac{\partial}{\partial u^{\alpha}}. \tag{11}$$

Its associated invariant surface conditions (8) turn into

$$\psi_0 \equiv u_p^{\alpha} + \sum_{j=1}^{p-1} \zeta^j(x, u)u_j^{\alpha} - \phi^{\alpha}(x, u) = 0, \quad \alpha = 1, ..., q. \tag{12}$$

The relations (12) divide the set of all the partial derivatives of the unknown functions u^{α} into two subsets $\{u_I^{\alpha}\}$ and $\{u_p^{\alpha}, u_{p,K}^{\alpha}\}$. The first one does not contain the partial derivatives of u^{α} with respect to the independent variable x^p, so $I = (i_1, ..., i_s)$, $1 \leq i_j \leq p-1$, $1 \leq j \leq s$, $1 \leq s \leq n$. In the second one u^α_{pK} represent all the partial derivatives that appear in the PDE system (1), where $K = (k_1, ..., k_r)$, $1 \le k_j \le p$, $1 \le j \le r$, $1 \le r \le n-1$. These can be expressed in the terms of the partial derivatives of the first subset $\{u^\alpha_I\}$ (and of the infinitesimals ζ^i and ϕ^α also) by using the relations (12). These notations for the multi-indices K and I are kept through all the paper.

The determining equations of the nonclassical symmetries are found by the following two-step procedure.
Step 1. Determine all the partial derivatives u^α_{pK} that appear in the PDE system (1), in terms of the partial derivatives of of u^α_I by using the invariant surface conditions (12). Substitute u^α_p and u^α_{pK} into the PDE system (1) and reduce it to its normal form, say

$$\Omega(x, u^{[n]}) = 0. \tag{13}$$

This is a PDE system for the unknown functions $u^\alpha = u^\alpha(x^1, ..., x^{p-1}; x^p)$ of x^1,...,x^{p-1}, with $\alpha = 1, ..., q$ (here x^p is considered as a parameter). Note that the function $\Omega = (\Omega_1, ..., \Omega_l)$ defining the system (13) depends on x and $u^{[n]}$, where by $u^{[n]}$ we denote all the partial derivatives u^α's up to order n (as functions only of $x^1, ..., x^{p-1}$).
Step 2. If the PDE system (13) is of maximal rank, then apply the classical Lie method to this, with Y given by (11) as a symmetry operator. The n-th order prolongation of the vector field Y is

$$pr^{(n)}Y = Y + \sum_{\alpha=1}^{q} \sum_{J} \Psi^\alpha_J(x, u^{[n]}) \frac{\partial}{\partial u^\alpha_J}, \tag{14}$$

with

$$\Psi^\alpha_J(x, u^{[n]}) = D_J \left(\phi^\alpha - \sum_{j=1}^{p-1} \zeta^j u^\alpha_j \right) + \sum_{j=1}^{p-1} \zeta^j u^\alpha_{J,j}. \tag{15}$$

In this case, the criterion for infinitesimal invariance is rewritten as

$$pr^{(n)}Y(\Omega)|_{\Omega=0} = 0. \tag{16}$$

Substituting the n-th order prolongation of the vector field Y given by (14) into (16), we get the relation

$$\sum_{j=1}^{p-1} \zeta^j \frac{\partial \Omega_k}{\partial x^j} + \frac{\partial \Omega_k}{\partial x^p} + \sum_{\alpha=1}^{q} \phi^\alpha \frac{\partial \Omega_k}{\partial u^\alpha} + \tag{17}$$

$$+\sum_{\alpha=1}^{q}\sum_{I}\left[D_I\left(\phi^\alpha-\sum_{j=1}^{p-1}\zeta^j u_j^\alpha\right)+\sum_{j=1}^{p-1}\zeta^j u_{I,j}^\alpha\right]\frac{\partial\Omega_k}{\partial u_I^\alpha}=0,\quad k=1,...,l,$$

that must be satisfied on the set of the solutions of the system (13).

By eliminating any dependence between the partial derivatives u_I^α occurring in (17)+(13) - this means the higher order partial derivative from (13) is substituted into (17) - we get an over-determined nonlinear PDE system for the infinitesimals ζ^i and ϕ^α. This system is called the *determining equations of the nonclassical symmetries* associated with the system (1).

3 A new procedure for finding nonclassical symmetries

In this section it is shown that the determining equations of the nonclassical symmetries of the system (1) derive from the determining equations of the symmetry group of the system (13). Let us start by making the following remarks on the procedure for finding nonclassical symmetries [3].

Remark 1. The coefficients of the partial derivatives u_I^α that appear into the normal form (13) of the system (1), depend on the infinitesimals ζ^i and ϕ^α, so these coefficients may be denoted by some arbitrary functions $\mathcal{A}_\beta$.

Remark 2. Since the partial derivatives u_p^α and u_{pK}^α do not occur both into the system (13) and the coefficients Ψ_J^α, then the invariance surface conditions (12) are not used any more at Step 2 (see also the relation (10)).

Remark 3. The available packages for finding the determining equations of the nonclassical symmetries are designed to use the invariant surface conditions (12) and their differential consequences as an input data.

Remark 4. The nonclassical symmetries of the system (1) are found by seeking the classical symmetries of the system (13) but with the symmetry operator we are looking for related to the invariant surface conditions (12) (these relations representing its associated characteristics).

Let us consider the normal form (13) of the system (1). Denoting the coefficients of the partial derivatives $u_{\mathcal{I}}^\alpha$ by some arbitrary functions $\mathcal{A}_\beta$, we get the following system

$$\tilde{\Omega}(x,u^{[n]})=0. \tag{18}$$

Consider a symmetry operator

$$W=\sum_{i=1}^{p}L^i(x,u)\frac{\partial}{\partial x^i}+\sum_{\alpha=1}^{q}M^\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{19}$$

of this system, whose coefficients are not related to the invariant surface conditions (12) any more. We prove that in the particular case when the vector field W is a nonclassical operator of the system (1), then the determining equations of the nonclassical symmetries of the system (1) derive from the determining equations of the symmetry group of the system (18) (or associated with the system (13) if we do not use the notations $\mathcal{A}_\beta$). Indeed, in this case, the criterion for infinitesimal invariance is written as

$$pr^{(n)}W(\tilde{\Omega})|_{\tilde{\Omega}=0} = 0, \tag{20}$$

where, the n-th order prolongation of the vector field W is given by

$$pr^{(n)}W = W + \sum_{\alpha=1}^{q}\sum_{J}\mathcal{M}_J^\alpha(x,u^{(n)})\frac{\partial}{\partial u_J^\alpha}, \tag{21}$$

where

$$\mathcal{M}_J^\alpha(x,u^{(n)}) = D_J\left(M^\alpha - \sum_{j=1}^{p-1}L^j u_j^\alpha\right) + \sum_{j=1}^{p-1}L^j u_{J,i}^\alpha + L^p u_{J,p}^\alpha - D_J\left(L^p u_p^\alpha\right). \tag{22}$$

After substituting (21) and (22) into the condition (20), we get the relation

$$\sum_{j=1}^{p-1}L^j\frac{\partial\tilde{\Omega}_k}{\partial x^j} + L^p\frac{\partial\tilde{\Omega}_k}{\partial x^p} + \sum_{\alpha=1}^{q}M^\alpha\frac{\partial\tilde{\Omega}_k}{\partial u^\alpha} + \tag{23}$$

$$+\sum_{\alpha=1}^{q}\sum_{I}\left[D_I\left(M^\alpha - \sum_{j=1}^{p-1}L^j u_j^\alpha\right) + \sum_{j=1}^{p-1}L^j u_{I,j}^\alpha\right]\frac{\partial\tilde{\Omega}_k}{\partial u_I^\alpha} =$$

$$= \sum_{\alpha=1}^{q}\sum_{I}\left[D_I\left(L^p u_p^\alpha\right) - L^p u_{I,p}^\alpha\right]\frac{\partial\tilde{\Omega}_k}{\partial u_I^\alpha}, \quad k = 1,...,l,$$

which must be identically satisfied on the set of the solutions of the system (18). To find the determining equations of the symmetry group of the system (18), first we have to eliminate any dependence between the partial derivatives of u^α occurring in (18) and (23), by substituting the higher order partial derivative from (18) into (23).

Recall that u_p^α, u_{pK}^α do not appear in the system (18) anymore, but they do appear now in the n-th order prolongation of the vector field W. The right hand side of the relation (23) contains the partial derivatives u_p^α and $u_{p,K}^\alpha$.

Remark that if L^p is constant, then the coefficients of the partial derivatives u^α_p, u^α_{pK} are zero. Substitute the higher order partial derivative from (18) into (23), and keep these partial derivatives with respect the variable x^p only in the right side of the resulting relation. Since now u^α_p, u^α_{pK} and other remained partial derivatives of u's are linearly independent, then by equalizing to zero their coefficients we get a PDE system divided into two subsets of equations, obtained from each hand side respectively: one comes from the coefficients of u^α_p, u^α_{pK}, and another from the remaining partial derivatives of u^α.

To find the determining equations of the nonclassical symmetries of the system (1), we have to substitute $L^p = 1$, $L^j = \zeta^j$, $j = 1, ..., p-1$, and $M^\alpha = \phi^\alpha$, $\alpha = 1, ..., q$ into the above system. It follows that the subset of PDEs occurring from the right hand side are identically zero and the subset of PDEs occuring from the left hand side represents the determining equations of the nonclassical symmetries of the system (1). In this particular case, the symmetry operator W becomes a nonclassical operator Y of the system (1).

The above procedure can be summarize as follows:

Step 1. Find the normal form (13) of the PDE system (1) by reducing the initial system using the invariant surface conditions (8) and its differential consequences.

Step 2. Denote the coefficients of the partial derivatives u^α_I that appear in the system (13) by some arbitrary functions $\mathcal{A}_\beta$ and get the system (18). Use any package designated for finding classical symmetries to generate the determining equations of the symmetry group associated with the system (18).

Step 3. Substitute the coefficients of the symmetry operator W by $L^p = 1$, $L^j = \zeta^j$, and $M^\alpha = \phi^\alpha$, respectively, into the PDE system obtained at Step 2, and get the determining equations of the nonclassical symmetries of the initial PDE system.

4 The MAPLE routine `gendefnc`

The procedure for finding nonclassical symmetries described above has been implemented as the MAPLE routine `gendefnc`. As may be apparent from its name, it is designed with the `gendef` routine in the DESOLV package [5] which calculates *classical* symmetries. Hence, the user interface of `gendefnc` is similar to that of `gendef`. Furthermore, the `gendefnc` routine by default calls **gendef** to finish the work.

Let us now describe the **gendefnc** routine in detail. It takes five arguments. The last two are optional, and the interface reduces to that of **gendef** if they are left out. The arguments are:

- the PDE system (1);
- the independent variables $x = (x^1, ..., x^p)$;
- the dependent variables $u = (u^1, ..., u^q)$;
- the independent variable x^k corresponding to the infinitesimal ζ^k that is supposed to be one (by default, the routine chooses the last variable, namely x^p);
- the fifth argument, which defaults to 0, can be set to skip various parts of the computation. This is explained below.

Given this input, the routine goes through the following procedure:

1. Given the general vector field (4), the infinitesimal ζ^k is set identically equal to 1, so that we get the vector field (11).
2. The associated invariant surface conditions (12) are formed.
3. The original system (1) is reduced relative to these surface conditions to determine the normal form (13). It is not necessary to program the whole machinery of differential Gröbner bases (see [9]) to achieve this reduction because of the simple form that the invariant surface conditions take.
4. If the fifth argument equals 1, the rest of the steps are skipped and the routine terminates here.
5. The coefficients of the partial derivatives appearing in the normal form (13) are denoted by $\mathcal{A}_\beta$, and so, the PDE system (18) is written. This step can be skipped by setting the fifth argument to 2.
6. The resulting system (18) is passed to **gendef**. This routine returns the determining equations of the *classical* symmetries associated with the PDE system (18).

7. In the determining equations found in the previous step, the coefficient of $\frac{\partial}{\partial x^k}$, denoted by L^k in (19), is set to 1, and also $L^i = \zeta^i$, $i = 1, ..., p$, $i \neq k$, and $M^\alpha = \phi^\alpha$, $\alpha = 1, ..., q$.

8. If the coefficients were replaced by $\mathcal{A}_\beta$ in step 5, then that substitution is reversed.

9. The resulting system, which form the determining equations for the nonclassical symmetries of the original system (1), are returned to the user.

The fifth argument to the **`gendef`** routine allows the user to intervene at two places. The substitutions by $\mathcal{A}_\beta$ in steps 5 and 8 save time and memory in most cases as they prevent expression swell, but this may not always be so. Therefore, the user has the possibilities to skip these substitutions by setting the fifth argument to 2.

Setting the fifth argument to 1 causes steps 5–9 to be skipped. In this case, the result of the computation is the normal form (13). The user can use any method to determine classical symmetries of (13).

The last feature enables one to use the **`gendefnc`** routine for reducing a PDE system (1) to its normal form (13) by using the invariant surface conditions (12), without requiring the DESOLV package. An example of the use of the **`gendefnc`** routine can be found in the next section.

5 Application to the two-dimensional semi-geostrophic PDE system

In the past years, the semi-geostrophic theory has become a model for describing atmospheric motions on a synoptic scale, including the presence of fronts [27], [28], [29]. This theory is one example of so-called *balanced models* meaning a model constructed so as to eliminate the fast 'inertia waves' or oscillations that can occur in numerical weather prediction or other 'primitive-equations' models.

In the Lagrangian description of the fluid motion, a typical particle (more precisely, fluid column) has Cartesian coordinates

$$\begin{aligned} x &= x(a,b,t), \\ y &= y(a,b,t) \end{aligned} \tag{24}$$

expressed as functions, on the right, of the particle labels $(a,b) \in \mathbf{R}^2$ and the time $t \in \mathbf{R}_+$. Each particle is labelled by its position at a reference time $t=0$, namely

$$x(a,b,0) = a, \quad y(a,b,0) = b.$$

The incompressibility hypothesis requires the layer depth h to satisfy the relation

$$\frac{h(a,b,0)}{h(a,b,t)} = \frac{\partial(x,y)}{\partial(a,b)}, \tag{25}$$

where the jacobian on the right is that of the mapping (24). Assuming that the mapping (24) is invertible,

$$\begin{array}{rcl} a & = & a(x,y,t) \\ b & = & b(x,y,t) \end{array} \tag{26}$$

and substituting (26) into (25), then it follows that the function h (now considered as a function of x and y) satisfies the relation

$$h(x,y,t) = h(x,y,0)\frac{\partial(a,b)}{\partial(x,y)},$$

that gives the Eulerian description of the fluid motion. Supposing $h(a,b,0) = 1$, the incompressibility hypothesis turns into

$$h(x,y,t) = \frac{1}{x_a y_b - x_b y_a}. \tag{27}$$

Since the transformation (24) is invertible, we can express the partial derivatives of h as follows

$$\begin{array}{rcl} h_x & = & h(y_b h_a - y_a h_b) \\ h_y & = & h(x_a h_b - x_b h_a), \end{array} \tag{28}$$

where, from (27), we have

$$\begin{array}{rcl} h_a & = & -h^2\left(x_{aa}y_b - x_{ab}y_a - x_b y_{aa} + x_a y_{ab}\right) \\ h_b & = & -h^2\left(x_{ab}y_b - x_{bb}y_a - x_b y_{ab} + x_a y_{bb}\right). \end{array} \tag{29}$$

The semi-geostrophic approximation to the shallow water equations [29] is the replacement of the true acceleration $(\dot{x}, \dot{y})$ of the particle by the time derivative of the vector

$$\left\{ \begin{array}{rcl} u_g & = & -\frac{g}{f}h_y \\ v_g & = & \frac{g}{f}h_x, \end{array} \right. \tag{30}$$

following the particle. Here f is a constant representing the Coriolis parameter and g is a constant gave by the combined effect of the acceleration due to gravity and a centrifugal component due to the Earth's rotation. This vector field is a notional velocity and is called *geostrophic velocity.* The resulting equations

$$\left\{\begin{array}{rcl} \dot{u}_g - f(\dot{y} - v) & = & 0 \\ \dot{v}_g + f(\dot{x} - u) & = & 0, \end{array}\right. \tag{31}$$

together with (30) are called the *semi-geostrophic PDE system.* The superposed dot denotes the time derivative following a particle.

In this paper we deal with the Lagrangian description of the semi-geostrophic PDE system, so that x, y, u_g and v_g are the independent variables, considered as functions of the particle labelling (a,b) and the time t. The function h depending on a, b and t, is considered as a notation of the inverse of the jacobian written on the right hand side of the relation (27). Notice that the semi-geostrophic PDE system might be rewritten in a simplified form by substituting the components of the geostrophic velocity u_g and v_g given by (30), into the system (31). This gives the third order PDE system

$$\left\{\begin{array}{l} hh_b\dot{x}_a - hh_a\dot{x}_b + (\dot{h}h_b + h\dot{h}_b)x_a - (\dot{h}h_a + h\dot{h}_a)x_b + fhh_by_a - fhh_ay_b + \frac{f^2}{g}\dot{y} = 0 \\ hh_b\dot{y}_a - hh_a\dot{y}_b + (\dot{h}h_b + h\dot{h}_b)y_a - (\dot{h}h_a + h\dot{h}_a)y_b - fhh_bx_a + fhh_ax_b - \frac{f^2}{g}\dot{x} = 0, \end{array}\right. \tag{32}$$

for the unknown function $x = x(a,b,t)$ and $y = y(a,b,t)$, where h is given by the relation (27) (the substitution of the function h given by (27) into (32) lead us to a very large PDE system, and unfortunately, because of its large form, we cannot give its explicit form in this paper). Note that in this resulting third order PDE system, the higher order partial derivatives $\dot{x}_{aa}$, $\dot{x}_{ab}$, $\dot{x}_{bb}$, $\dot{y}_{aa}$, $\dot{y}_{ab}$ and $\dot{y}_{bb}$ are contained in the partial derivatives $\dot{h}_a$ and $\dot{h}_b$ in each of the equation of the system.

The classical symmetries associated with the PDE system (32) are generated by the following vector fields [2]

$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = \frac{\partial}{\partial y},\ X_4 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \tag{33}$$

$$X_0 = \xi(a,b)\frac{\partial}{\partial a} + \eta(a,b)\frac{\partial}{\partial b} + A\frac{\partial}{\partial x} + A\frac{\partial}{\partial y},\ \xi_a + \eta_b = 4A,$$

that corresponds to translations in the time-space, x-space, y-space, and rotation in the (x,y)-space, respectively. The vector field X_0 is related to the so-called *particle-relabelling symmetry.*

As an application of our procedure introduced in Section 3, we consider the nonclassical symmetry analysis of the semi-geostrophic PDE system (32), in the special case when the time-infinitesimal of the nonclassical operator is nonzero, such that, without loss the generality, we can assume that equals to 1. Using the notations introduced in the above sections, we now have $x^1 = a$, $x^2 = b$, $x^3 = t$, $u^1 = x$ and $u^2 = y$.

Consider the one-parameter group of transformations

$$\begin{array}{rcl} a^* & = & a + \varepsilon\zeta(a,b,t,x,y) + \mathcal{O}(\varepsilon^2) \\ b^* & = & b + \varepsilon\tau(a,b,t,x,y) + \mathcal{O}(\varepsilon^2) \\ t^* & = & t + \varepsilon + \mathcal{O}(\varepsilon^2) \\ x^* & = & x + \varepsilon\phi(a,b,t,x,y) + \mathcal{O}(\varepsilon^2) \\ y^* & = & y + \varepsilon\psi(a,b,t,x,y) + \mathcal{O}(\varepsilon^2) \end{array} \tag{34}$$

acting on an open set of the (a,b,t,x,y)-space of the independent and dependent variables of the PDE system (32) and let

$$X = \zeta\frac{\partial}{\partial a} + \tau\frac{\partial}{\partial b} + \frac{\partial}{\partial t} + \phi\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial y}, \tag{35}$$

be its infinitesimal generator, where ζ, τ, ϕ and ψ are functions of a, b, t, x and y. In this case, the invariant surface conditions (12) are written as

$$\begin{aligned} \dot{x} &= \phi - \zeta x_a - \tau x_b \\ \dot{y} &= \psi - \zeta y_a - \tau y_b. \end{aligned} \tag{36}$$

¿From these relations we obtain the second order partial derivatives

$$\begin{array}{rcl} \dot{x}_a & = & \phi_a + (\phi_x - \zeta_a)x_a - \tau_a x_b + \phi_y y_a - \zeta_x x_a^2 - \tau_x x_a x_b - \zeta_y x_a y_a - \\ & & -\tau_y x_b y_a - \zeta x_{aa} - \tau x_{ab}, \\ \dot{x}_b & = & \phi_b - \zeta_b x_a + (\phi_x - \tau_b)x_b + \phi_y y_b + \zeta_x x_a x_b - \eta_x x_b^2 + \zeta_y x_a y_b - \\ & & -\tau_y x_b y_b - \zeta x_{ab} - \tau x_{bb}, \\ \dot{y}_a & = & \psi_a + \psi_x x_a + (\psi_y - \zeta_a)y_a - \tau_a y_b - \zeta_x x_a y_a - \tau_x x_a y_b - \zeta_y y_a^2 - \\ & & -\tau_y y_a y_b - \zeta y_{aa} - \tau y_{ab}, \\ \dot{y}_b & = & \psi_b + \psi_x x_b - \zeta_b y_a + (\psi_y - \tau_b)y_b - \zeta_x x_b y_a - \tau_x x_b y_b - \zeta_y y_a y_b - \\ & & -\tau_y y_b^2 - \zeta y_{ab} - \tau y_{bb}, \end{array} \tag{37}$$

and subsequently the third order partial derivatives $\dot{x}_{aa}$, $\dot{x}_{ab}$, $\dot{x}_{bb}$, $\dot{y}_{aa}$, $\dot{y}_{ab}$ and $\dot{y}_{bb}$, which appear in the semi-geostrophic PDE system (32).

Step 1. Substituting the partial derivatives with respect to time into the system (32), we get a very large PDE system for the unknown functions $x = x(a, b; t)$ and $y = y(a, b; t)$ as functions of the variables a and b (where the variable t is considered as parameter), that represents the normal form of the semi-geostrophic PDE system in this case.
Step 2. Running the routine `gendfnc`, we find automatically the determining equations of the nonclassical symmetries of this last system, which is a large nonlinear PDE system. The CPU time is decreased if we denote by some arbitrary functions $\mathcal{A}_\beta$ the coefficients of the remaining partial derivatives. Using the MAPLE package `diffgrob2` we can integrate this system interactively and find that its general solution is given by

$$
\begin{aligned}
\zeta(a, b, t, x, y) &= \xi(a, b), \\
\tau(a, b, t, x, y) &= \eta(a, b), \\
\theta(a, b, t, x, y) &= 1, \\
\phi(a, b, t, x, y) &= Ax - By + C, \\
\psi(a, b, t, x, y) &= Bx + Ay + D,
\end{aligned}
\tag{38}
$$

where A, B, C and D are real numbers and ξ and η satisfy the condition

$$
\frac{\partial \xi}{\partial a}(a, b) + \frac{\partial \eta}{\partial b}(a, b) = 4A. \tag{39}
$$

In this case, the nonclassical method does not yield any symmetry reductions additional to those obtained using the classical Lie method. Hence the classical and the nonclassical symmetries are the same in this case.

6 Conclusions

There are large PDE systems for which the finding of the nonclassical symmetries become a difficult problem due to the memory of the system on which the symbolic manipulation program runs. The algorithm introduced in this paper is based on a remark on the application of the nonclassical method. This procedure that we have presented can be applied to finding nonclassical symmetries by a different approach than classical ones. This prove that any symbolic manipulation programs designed to determine classical symmetries can handle the finding of the nonclassical symmetries, without any change of the program. The MAPLE routine `gendefnc` reduces a PDE system to its normal form by using the invariant surface conditions. The normal form of the studied system is passed to the MAPLE routine `gendef` of the package `DESOLV`, such that the determining equations of the nonclassical symmetries can be found by using this routine designated to find classical symmetries.

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