ON THE OPTIMALITY OF DOUBLE-BRACKET FLOWS

ANTHONY M. BLOCH AND ARIEH ISERLES

Abstract. In this paper we analyse optimality of the stable fixed point of the double bracket equations. We introduce different types of optimality and prove local and global optimality results with respect to the Schatten p-norms.

1. Introduction

Double-bracket flows were introduced simultaneously by Brockett in the context of control theory and optimization [Bro91] and by Chu and Driessel in connection with the computation of certain problems in numerical linear algebra [CD91] . Let $\mathrm{Sym}(n)$ denote the set of real, $n \times n$ symmetric matrices. A *double-bracket flow (DBF)* is the matrix differential equation

$$\boldsymbol{Y}' = [[\boldsymbol{N}, \boldsymbol{Y}], \boldsymbol{Y}], \quad t \geq 0, \qquad \boldsymbol{Y}(0) = \boldsymbol{Y}_0 \in \mathrm{Sym}(n), \tag{1.1}$$

where $\boldsymbol{N} \in \mathrm{Sym}(n)$ and $[\,\cdot\,,\,\cdot\,]$ is the usual matrix commutator. Without loss of generality, we assume that $\boldsymbol{N}$ is a diagonal matrix, $\boldsymbol{N} = \mathrm{diag}\,\boldsymbol{\lambda}$, say. Otherwise we may reduce (1.1) to this form: $\boldsymbol{Y}(t) = \boldsymbol{P}\tilde{\boldsymbol{Y}}(t)\boldsymbol{P}^\top$, where $\tilde{\boldsymbol{Y}}' = [[\boldsymbol{\Lambda}, \tilde{\boldsymbol{Y}}], \tilde{\boldsymbol{Y}}]$, $\tilde{\boldsymbol{Y}}(0) = \boldsymbol{P}^\top \boldsymbol{Y}_0 \boldsymbol{P}$, and $\boldsymbol{N} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^\top$, with $\boldsymbol{P} \in \mathrm{SO}(n)$ and $\boldsymbol{\Lambda}$ a diagonal matrix.

The Lie algebraic generalization of this flow was analysed in [BBR90], [BBR92].

Equations of the form (1.1) share a number of features that render them of interest both in the context of a number of applications and as mathematical objects in their own right.

Firstly, (1.1) is a special instance of an *isospectral flow*

$$\boldsymbol{Y}' = [\boldsymbol{B}(t, \boldsymbol{Y}), \boldsymbol{Y}], \quad t \geq 0, \qquad \boldsymbol{Y}(0) = \boldsymbol{Y}_0 \in \mathrm{Sym}(n), \tag{1.2}$$

where $\boldsymbol{B} : \mathbb{R}_+ \times \mathrm{Sym}(n) \to \mathfrak{so}(n)$, the set $\mathfrak{so}(n)$ being the Lie algebra of $n \times n$ real, skew-symmetric matrices. It easily follows that

$$\boldsymbol{Y}(t) = \boldsymbol{Q}(t) Y_0 \boldsymbol{Q}^\top(t), \quad t \geq 0, \tag{1.3}$$

where

$$\boldsymbol{Q}' = \boldsymbol{B}(t, \boldsymbol{Q}\boldsymbol{Y}_0\boldsymbol{Q}^\top), \quad t \geq 0, \qquad \boldsymbol{Q}(0) = \boldsymbol{I}.$$

Since the latter is a Lie-group equation, we deduce that $\boldsymbol{Q}$ evolves in $\mathrm{SO}(n)$, therefore, by (1.3), $\boldsymbol{Y}$ is a similarity transformation of the initial value $\boldsymbol{Y}_0$ [IMKNZ00]. In other words, the eigenvalues of $\boldsymbol{Y}(t)$ in (1.2) (and, by implication, in the DBF (1.1)) are invariants of the flow and do not vary as t increases. We note in passing that the equation (1.3) is the key to practical computation of the solution of (1.2), whilst respecting its invariants [CIZ97, Ise02].

The second feature of the DBF (1.1) is that it is a *gradient system,* with a global Lyapunov function, therefore it is assured of convergence to a fixed point of the flow as $t \to \infty$ [Bro91]. More precisely, as shown in [BBR92], [BFR90] it is a gradient

The work of AMB was supported in part by the National Science Foundation.

flow with respect to the so-called *normal metric* on an appropriate adjoint orbit of a compact Lie group. This feature, of critical importance in applications, is in general invalid for arbitrary isospectral flows (1.2): thus, for example, the solution of

$$\boldsymbol{Y}' = [\boldsymbol{B}, \boldsymbol{Y}], \quad t \geq 0, \qquad \boldsymbol{Y}(0) = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \qquad \text{where} \qquad \boldsymbol{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

is

$$\boldsymbol{Y}(t) = \begin{bmatrix} \frac{1}{2}(a+c) + b\sin 2t + \frac{1}{2}(a-c)\cos 2t & b\cos 2t - \frac{1}{2}(a-c)\sin 2t \\ b\cos 2t - \frac{1}{2}(a-c)\sin 2t & \frac{1}{2}(a+c) - b\sin 2t - \frac{1}{2}(a-c)\cos 2t \end{bmatrix},$$

an oscillatory matrix.

It is generally true in fact that integrable Hamiltonian systems are described by isospectral flows. As shown by [Blo90] however, in the case that the matrix $\boldsymbol{Y}$ in the double bracket flow is tridiagonal and the matrix $\boldsymbol{N}$ is the diagonal matrix $\mathrm{diag}(1, 2, \ldots n)$, the double bracket flow is both gradient and Hamiltonian on a level set of its integrals. The flow is the Toda lattice flow and the level set noncompact and diffeomorphic to a product of lines, unlike many Hamiltonian systems where the level set of the integrals is diffeomorphic to a torus. The seminal initial paper on the nonperiodic Toda flow is [Mos74], where its gradient nature is demonstrated. Other early key papers include [Sym80], [Sym82] and [DNT83].

Since the existence of a global Lyapunov function is fundamental to the subject matter of this paper, we herewith reproduce a proof, following that of [Bro91], as well as presenting a second, local Lyapunov function which is (to the best of our knowledge) new and helps to shed light on the asymptotic behaviour of $\boldsymbol{Y}(t)$. We define

$$h(t) = \mathrm{tr}\,[\boldsymbol{Y}(t) - \boldsymbol{N}]^2 = \mathrm{tr}\,\boldsymbol{Y}^2(t) + \mathrm{tr}\,\boldsymbol{N}^2 - \mathrm{tr}\,[\boldsymbol{Y}(t)\boldsymbol{N} + \boldsymbol{N}\boldsymbol{Y}(t)], \quad t \geq 0.$$

Both $\mathrm{tr}\,\boldsymbol{N}^2$ and $\mathrm{tr}\,\boldsymbol{Y}^2(t)$ are independent of t: the first since $\boldsymbol{N}$ itself is a constant matrix, the second because the trace is the sum of the eigenvalues and, since the eigenvalues of $\boldsymbol{Y}(t)$ stay put, so do also the eigenvalues of $\boldsymbol{Y}^2(t)$. We deduce that

$$h'(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}\,[\boldsymbol{Y}(t)\boldsymbol{N} + \boldsymbol{N}\boldsymbol{Y}(t)], \quad t \geq 0.$$

Since $\boldsymbol{N} = \mathrm{diag}\,\boldsymbol{\lambda}$, we have

$$\begin{aligned} (\boldsymbol{Y}\boldsymbol{N} + \boldsymbol{N}\boldsymbol{Y})_{k,l} &= (\lambda_k + \lambda_l) Y_{k,l}, \\ \boldsymbol{Y}'_{k,l} = [[\boldsymbol{N}, \boldsymbol{Y}], \boldsymbol{Y}]_{k,l} &= \sum_{i=1}^{n} (\lambda_k - 2\lambda_i + \lambda_l) Y_{k,i} Y_{i,l}, \end{aligned} \qquad k, l = 1, 2, \ldots, n. \tag{1.4}$$

Substituting the differential equation, (1.1) and the symmetry of $\boldsymbol{Y}$ yield

$$\begin{aligned} h' &= -2\sum_{k=1}^{n} \lambda_k Y'_{k,k} = -4\sum_{k=1}^{n} \lambda_k \sum_{i=1}^{n} (\lambda_k - \lambda_i) Y_{k,i}^2 \\ &= -2\sum_{k=1}^{n}\sum_{i=1}^{n} \lambda_k(\lambda_k - \lambda_i) Y_{k,i}^2 - 2\sum_{i=1}^{n}\sum_{k=1}^{n} \lambda_i(\lambda_i - \lambda_k) Y_{i,k}^2 \\ &= -2\sum_{k=1}^{n}\sum_{i=1}^{n} (\lambda_k - \lambda_i)^2 Y_{k,i}^2. \end{aligned}$$

Therefore, $h'(t) \le 0$ and, unless $\boldsymbol{Y}(t)$ is a diagonal matrix, $h'(t) < 0$. Since $(\boldsymbol{Y} - \boldsymbol{N})^2$ is positive semidefinite, the function h is nonnegative and we deduce that h is a global Lyapunov function and that $\lim_{t\to\infty} h(t)$ exists. This implies $\lim_{t\to\infty} h'(t) = 0$, therefore $\hat{\boldsymbol{Y}} = \lim_{t\to\infty} \boldsymbol{Y}(t)$ exists and, because of our analysis, *is a diagonal matrix.*

Needless to say, the diagonal elements of $\hat{\boldsymbol{Y}}$ are the eigenvalues of $\boldsymbol{Y}_0$ (hence of $\boldsymbol{Y}(t)$, $t \ge 0$), which we denote by $d_1, d_2, \ldots, d_n$. We henceforth assume for the sake of simplicity (and to avoid a welter of special cases of no great intrinsic significance) that the $2n$ real numbers $\{d_1, d_2, \ldots, d_n\}$ and $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ are distinct. Since any diagonal matrix annihilates the vector field in (1.1), all matrices of the from $\operatorname{diag} \boldsymbol{d_\pi}$, where $\boldsymbol{\pi}$ be a permutation of $\{1, 2, \ldots, n\}$, are fixed points of DBF. To enquire of their attractivity, we let $\boldsymbol{Y}(t) = \operatorname{diag} \boldsymbol{d_\pi} + E(t)$, where $\|E(t)\|$ is small. Since diagonal matrices form a commutative algebra, we deduce that

$$\begin{aligned} E' &= Y' = [[N, \operatorname{diag} \boldsymbol{d_\pi} + E], \operatorname{diag} \boldsymbol{d_\pi} + E] = [[N, E], \operatorname{diag} \boldsymbol{d_\pi} + E] \\ &= [[N, E], \operatorname{diag} \boldsymbol{d_\pi}] + \mathcal{O}\big(\|E\|^2\big) . \end{aligned}$$

Therefore, for every $k, l = 1, 2, \ldots, n$,

$$\begin{aligned} E'_{k,l} &= (\lambda_k d_{\pi_l} - \lambda_l d_{\pi_l} - \lambda_k d_{\pi_k} + \lambda_l d_{\pi_k} E_{k,l} + \mathcal{O}\big(\|E\|^2\big) \\ &= -(\lambda_k - \lambda_l)(d_{\pi_k} - d_{\pi_l}) E_{k,l} + \mathcal{O}\big(\|E\|^2\big) . \end{aligned}$$

We deduce that each $E_{k,l}$ decays (at an exponential speed) if and only if

$$(\lambda_k - \lambda_l)(d_{\pi_k} - d_{\pi_l}) > 0, \qquad k, l = 1, 2, \ldots, n, \ k \ne l. \tag{1.5}$$

Therefore there is just one attractive fixed point $\hat{\boldsymbol{Y}}$, given by the permutation of $\boldsymbol{d}$ that arranges the eigenvalues in the same order as the $\boldsymbol{\lambda}$s.

Next, we construct a *local* Lyapunov function,

$$m(t) = \sum_{k=1}^{n} \sum_{\substack{l=1 \\ l \ne k}}^{n} Y_{k,l}^2(t) = \sum_{k=1}^{n} \sum_{l=1}^{n} Y_{k,l}^2(t) - \sum_{k=1}^{n} Y_{k,k}^2(t) = \|\boldsymbol{Y}(t)\|_{\mathsf{F}}^2 - \sum_{k=1}^{n} Y_{k,k}^2(t),$$

where $\|\cdot\|_{\mathsf{F}}$ is the *Frobenius norm.* Since $\|\boldsymbol{A}\|_{\mathsf{F}}^2$ equals the sum of squares of the eigenvalues of a real, square matrix $\boldsymbol{A}$, we use isospectrality to argue that $\|\boldsymbol{Y}(t)\|_{\mathsf{F}}^2$ is constant. Therefore we deduce from (1.4) that

$$\begin{aligned} m'(t) &= -2 \sum_{k=1}^{n} Y_{k,k} Y'_{k,k} = -4 \sum_{k=1}^{n} Y_{k,k} \sum_{i=1}^{n} (\lambda_k - \lambda_i) Y_{k,i}^2 \\ &= -2 \sum_{k=1}^{n} \sum_{i=1}^{n} (\lambda_k - \lambda_i) Y_{k,k} Y_{k,i}^2 - 2 \sum_{i=1}^{n} \sum_{k=1}^{n} (\lambda_i - \lambda_k) Y_{i,i} Y_{k,i}^2 \\ &= -2 \sum_{k=1}^{n} \sum_{i=1}^{n} (\lambda_k - \lambda_i)(Y_{k,k} - Y_{i,i}) Y_{k,i}^2 . \end{aligned}$$

We have already seen that, unless $\boldsymbol{Y}_0$ is a diagonal matrix, the solution $\boldsymbol{Y}$ tends to the diagonal matrix consistently with (1.5). Therefore, there exists $t^* \ge 0$ such that for every $t \ge t^*$ the vector $\operatorname{diag} \boldsymbol{Y}(t)$ lies in the closed, nonempty set

$$\{\boldsymbol{x} \in \mathbb{R}^n \,:\, (\lambda_k - \lambda_i)(x_k - x_i) \ge 0, \ k, l = 1, 2 \ldots, n\}.$$

As we have just demonstrated, within this neighbourhood of the attractive fixed point, $m \geq 0$ is a local Lyapunov function and the Euclidean norm of off-diagonal elements is decaying monotonically.

The third important feature of the DBF (1.1) is that its fixed point $\hat{\boldsymbol{Y}}$, a diagonal matrix with $\boldsymbol{d_\pi}$ along the diagonal (cf. (1.5)) optimizes distance to $\boldsymbol{N}$ in the Frobenius norm.

As a matter of fact, one can distinguish between three types of optimality:

1. *Discrete optimality:* The Frobenius distance from $\boldsymbol{N}$ is minimized at $\hat{\boldsymbol{Y}}$ among all the $n!$ diagonal matrices similar to $\boldsymbol{Y}_0$. This, relatively weak form of optimality is the one considered (and proved) by [Bro91]. Indeed, in Brockett's terminology, the DBF (1.1) is an "analogue computer" for the calculation of eigenvalues, sorting of lists and the solution of linear programming problems, and this makes discrete optimality a natural concept.
2. *Isospectral optimality:* Given an arbitrary matrix norm $\|\cdot\|$, is it true that for all (without loss of generality) diagonal matrices $\boldsymbol{N}$ and all $\boldsymbol{Y}_0 \in \mathrm{Sym}(n)$ the inequality $\|\hat{\boldsymbol{Y}} - \boldsymbol{N}\| \leq \|\boldsymbol{X} - \boldsymbol{N}\|$ holds for all $\boldsymbol{X} \in \mathrm{Sym}(n)$ similar to $\boldsymbol{Y}_0$, such that $\|\boldsymbol{X} - \hat{\boldsymbol{Y}}\|$ is sufficiently small? In this case we say that the norm is isospectrally optimal. Moreover, if the inequality is (for every $\boldsymbol{Y}_0$ and $\boldsymbol{N}$) sharp for $\boldsymbol{X} \neq \hat{\boldsymbol{Y}}$ in an arbitrarily small neighbourhood, except perhaps for a lower-dimensional submanifold, we say that the norm optimality is *strong.* If for some $\boldsymbol{Y}_0$ and $\boldsymbol{N}$ we have equality in a proper neighbourhood of $\hat{\boldsymbol{Y}}$ then optimality is *weak.*
3. *Global optimality:* This is the strongest possible form of optimality,

$$\|\hat{\boldsymbol{Y}} - \boldsymbol{N}\| \leq \|\boldsymbol{X} - \boldsymbol{N}\|, \qquad \boldsymbol{X} \in \mathrm{Sym}(n), \quad \boldsymbol{X} \text{ similar to } \boldsymbol{Y}_0. \tag{1.6}$$

   Intriguingly, although never mentioned in [Bro91], it is implicit in the analysis of that paper that the Frobenius norm is globally optimal. This follows at once from $h'(t) \leq 0$ (since $h(t) = \|Y(t) - \boldsymbol{N}\|_{\mathsf{F}}^2$), the fact that $\hat{Y}$ is the only fixed point for nondiagonal $\boldsymbol{Y}_0$s and discrete optimality. Given that all these facts are already present in [Bro91], it is only fair to attribute the global optimality of the Frobenius norm to Brockett.

In this paper we wish to focus on isospectral optimality for different unitarily-invariant norms: a matrix norm $\|\cdot\|$ is *unitarily invariant* if $\|\boldsymbol{PXQ}\| = \|\boldsymbol{X}\|$ for every $\boldsymbol{P}, \boldsymbol{Q} \in \mathrm{SO}(n)$, $\boldsymbol{X} \in \mathrm{M}_{n,n}$ [HJ91, p. 203]. We commence Section 2 with von Neumann's celebrated characterization of unitarily-invariant norms in terms of symmetric gauges [vN37]. The most important family of unitarily-invariant norms are the so-called *Schatten p-norms,* $1 \leq p \leq \infty$. In the remainder of the section we determine the range of p such that a Schatten p-norm is isospectrally optimal.

We remark that the derivative of the Frobenius norm $h(t) = \|\boldsymbol{Y}(t) - \boldsymbol{N}\|_{\mathsf{F}}^2$ along the flow of the double bracket equations reduces to the derivative of $\mathrm{tr}\,\boldsymbol{YN}$ which is precisely the gradient function for the flow. In this case global optimality follows as in paragraph 3 above. The key difficulty in the current paper is that the more general norms we discuss here involve higher powers of $\boldsymbol{Y}$.

In Section 3 we discuss the case $n = 2$. Although the solution of (1.1) can be easily written down in an explicit fashion, isospectral optimality is far from trivial. We resolve this issue completely and characterize all isospectrally-optimal norms for $n = 2$. We also show that all Schatten p-norms are globally optimal for $n = 2$.

2. Schatten p-norms

A vector norm $|\!|\cdot|\!| : \mathbb{R}^n \to \mathbb{R}_+$ is said to be *symmetric* if $|\!|\boldsymbol{P}\boldsymbol{x}|\!| = |\!|\boldsymbol{x}|\!|$ for every $\boldsymbol{x} \in \mathbb{R}^n$ and every permutation matrix $\boldsymbol{P} \in \mathrm{M}_{n,n}$, and *absolute* if $|\!|\boldsymbol{x}|\!| = |\!||\boldsymbol{x}||\!|$, $\boldsymbol{x} \in \mathbb{R}^n$, where $|\boldsymbol{x}|$ is the vector of the absolute values of the elements of $\boldsymbol{x}$. If $|\!|\cdot|\!|$ is both symmetric and absolute, it is called a *symmetric gauge* [HJ91, LO96]. Thus, for example, every $\ell_p[\mathbb{R}^n]$ norm is a symmetric gauge, but no nontrivial weighted $\ell_p[\mathbb{R}^n]$ norm is symmetric and the norm $|\!|[x_1, x_2]^\top|\!| = [x_1^2 + x_2^2 + (x_1 - x_2)^2]^{1/2}$ is not positive.[1]

Given $\boldsymbol{X} \in \mathrm{M}_{n,n}$, we denote by $\mathrm{sing}(\boldsymbol{X})$ the vector of *singular values* of $\boldsymbol{X}$: to ensure proper definition, we assume that the elements of $\mathrm{sing}(\boldsymbol{X})$ are arranged in increasing order, but this will play no further role in our discussion. Note that for $\boldsymbol{X} \in \mathrm{Sym}(n)$ the map $\mathrm{sing}(\boldsymbol{X})$ produces the absolute values of the eigenvalues of the matrix.

Theorem 1 ([vN37])**.** *A matrix norm $\|\cdot\|$ is unitarily invariant if and only if there exists a symmetric gauge $|\!|\cdot|\!|$ such that*

$$\|\boldsymbol{X}\| = |\!|\mathrm{sing}(\boldsymbol{X})|\!|, \qquad \boldsymbol{X} \in \mathrm{M}_{n,n}. \tag{2.1}$$

We have already mentioned that the standard $\ell_p[\mathbb{R}^n]$ norm, $1 \le p \le \infty$, is a symmetric gauge. Therefore, by Theorem 1, it gives rise to a unitarily-invariant norm, the *Schatten p-norm* $\|\cdot\|_p = |\!|\mathrm{sing}(\,\cdot\,)|\!|_p$ [HJ91, p. 199]. In particular, the special choices $p \in \{1, 2, \infty\}$ are important enough to deserve special names:

$\|\boldsymbol{X}\|_1 = \boldsymbol{1}^\top \mathrm{sing}(\boldsymbol{X})$: the Ky Fan norm [HJ91, p. 195],

$\|\boldsymbol{X}\|_2 = \|\boldsymbol{X}\|_{\mathsf{F}}$: the Frobenius norm,

$\|\boldsymbol{X}\|_\infty = \max \dfrac{|\!|\boldsymbol{X}\boldsymbol{v}|\!|_2}{|\!|\boldsymbol{v}|\!|_2}$: the (operator) Euclidean norm.

Noting in passing that the Schatten 2-norm is isospectrally optimal, we seek to characterize the range of $p \in [1, \infty]$ for which a Schatten p-norm shares this feature.

Assuming that $\boldsymbol{Y}_0$ is not already a diagonal matrix, we have proved in Section 1 that the flow (1.1) converges to $\hat{\boldsymbol{Y}} = \mathrm{diag}\, \boldsymbol{d}_{\boldsymbol{\pi}}$, where the permutation $\boldsymbol{\pi}$ is consistent with (1.5). Let us consider matrices $\boldsymbol{Y}_\varepsilon \in \mathrm{Sym}(n)$ from an isospectral orbit of $\hat{\boldsymbol{Y}}$ (i.e., similar to $\hat{\boldsymbol{Y}}$) within distance $0 < \varepsilon \ll 1$ from $\hat{\boldsymbol{Y}}$. By construction, there exists a nonzero $\boldsymbol{\Omega} \in \mathfrak{so}(n)$ such that

$$\boldsymbol{Y}_\varepsilon = \mathrm{e}^{\varepsilon\boldsymbol{\Omega}} \hat{\boldsymbol{Y}} \mathrm{e}^{-\varepsilon\boldsymbol{\Omega}} = \mathrm{e}^{\varepsilon \mathrm{ad}\,\boldsymbol{\Omega}} \hat{\boldsymbol{Y}} = \hat{\boldsymbol{Y}} + \varepsilon[\boldsymbol{\Omega}, \hat{Y}] + \tfrac{1}{2}\varepsilon^2 [\boldsymbol{\Omega}, [\boldsymbol{\Omega}, \hat{\boldsymbol{Y}}]] + \mathcal{O}\big(\varepsilon^3\big)\,,$$

where $\mathrm{ad}_{\boldsymbol{A}} \boldsymbol{B} = [\boldsymbol{A}, \boldsymbol{B}]$ is the adjoint operator of the Lie algebra $\mathfrak{so}(n)$.

Given $\boldsymbol{A}, \boldsymbol{B} \in \mathrm{Sym}(n)$ and small $\delta > 0$, we denote the eigenvalues of $\boldsymbol{A} + \delta\boldsymbol{B}$ by $\theta_1(\delta), \theta_2(\delta), \ldots, \theta_n(\delta)$. Provided that $\{\theta_1(0), \theta_2(0), \ldots, \theta_n(0)\}$, the spectrum of $\boldsymbol{A}$, is distinct, a classical perturbation result is

$$\theta_k(\delta) = \theta_k(0) + \delta \boldsymbol{v}_k^\top \boldsymbol{B} \boldsymbol{v}_k + \delta^2 \sum_{\substack{i=1 \\ i \ne k}}^{n} \frac{(\boldsymbol{v}_k^\top \boldsymbol{B} \boldsymbol{v}_i)^2}{\theta_k(0) - \theta_i(0)} + \mathcal{O}\big(\delta^3\big)\,, \tag{2.2}$$

---

[1] We leave it as an elementary exercise to the reader to prove that this is indeed a norm. perhaps the easiest proof follows by showing that its unit ball is convex.

where $\boldsymbol{v}_k$ is an eigenvector of $\boldsymbol{A}$ corresponding to the eigenvalue $\theta_k(0)$, such that $\|\boldsymbol{v}_k\|_2 = 1$ [Wil65, p. 70]. We now let

$$\boldsymbol{A} = \hat{\boldsymbol{Y}} - \boldsymbol{N}, \qquad \boldsymbol{B} = \varepsilon^{-1}(\boldsymbol{Y}_\varepsilon - \hat{\boldsymbol{Y}}) = [\boldsymbol{\Omega}, \hat{Y}] + \tfrac{1}{2}\varepsilon[\boldsymbol{\Omega}, [\boldsymbol{\Omega}, \hat{\boldsymbol{Y}}]] + \mathcal{O}\big(\varepsilon^2\big)$$

and $\delta = \varepsilon$. Since the eigenvalues of the diagonal matrix $\hat{\boldsymbol{Y}} - \boldsymbol{N}$ are $\eta_k := d_{\pi_k} - \lambda_k$, $k = 1, 2, \ldots, n$ and the corresponding normalized eigenvectors are the unit vectors $\boldsymbol{e}_k$, (2.2) yields

$$\theta_k(\varepsilon) = \eta_k + \varepsilon \boldsymbol{e}_k^\top [\boldsymbol{\Omega}, \hat{\boldsymbol{Y}}]\boldsymbol{e}_k + \varepsilon^2 \left\{ \tfrac{1}{2}\boldsymbol{e}_k^\top [\boldsymbol{\Omega}, [\boldsymbol{\Omega}, \hat{Y}]]\boldsymbol{e}_k + \sum_{i \neq k} \frac{(d_{\pi_k} - d_i)^2}{\eta_k - \eta_i} \Omega_{k,i}^2 \right\} + \mathcal{O}\big(\varepsilon^3\big) .$$

However, because of the skew-symmetry of $[\boldsymbol{\Omega}, \hat{\boldsymbol{Y}}]$ and diagonality of $\hat{\boldsymbol{Y}}$, it is true that

$$\begin{aligned} \boldsymbol{e}_k^\top [\boldsymbol{\Omega}, \hat{\boldsymbol{Y}}]\boldsymbol{e}_k &= 0, \\ \boldsymbol{e}_k^\top [\boldsymbol{\Omega}, [\boldsymbol{\Omega}, \hat{\boldsymbol{Y}}]]\boldsymbol{e}_k &= -\sum_{i=1}^n (d_{\pi_k} - d_{\pi_i})\Omega_{k,i}^2. \end{aligned}$$

Therefore the $\mathcal{O}(\varepsilon)$ term vanishes and

$$\begin{aligned} \theta_k(\varepsilon) &= \eta_k + \varepsilon^2 \sum_{i \neq k} \frac{d_{\pi_k} - d_{\pi_i}}{\eta_k - \eta_i} (d_{\pi_k} - d_{\pi_i} - \eta_k + \eta_i)\Omega_{k,i}^2 + \mathcal{O}\big(\varepsilon^3\big) \\ &= \eta_k + \varepsilon^2 \sum_{i \neq k} \frac{(d_{\pi_k} - d_{\pi_i})(\lambda_k - \lambda_i)}{\eta_k - \eta_i} \Omega_{k,i}^2 + \mathcal{O}\big(\varepsilon^3\big) = \eta_k + \varepsilon^2 \mu_k + \mathcal{O}\big(\varepsilon^3\big) , \end{aligned}$$

where

$$\mu_k = \sum_{i \neq k} \frac{(d_{\pi_k} - d_{\pi_i})(\lambda_k - \lambda_i)}{d_{\pi_k} - d_{\pi_i} - \lambda_k + \lambda_i} \Omega_{k,i}^2.$$

Letting $\| \cdot \| = |\!|\mathrm{sing}(\,\cdot\,)|\!|$, a unitarily-invariant norm, and assuming that $|\!|\,\cdot\,|\!|$ is (locally) differentiable, we denote by $\boldsymbol{g}$ the gradient of the symmetric gauge,

$$g_k(\boldsymbol{x}) = \frac{\partial |\!|\boldsymbol{x}|\!|}{\partial x_k}, \qquad k = 1, 2, \ldots, n.$$

Since $\boldsymbol{Y}_\varepsilon - \boldsymbol{N} \in \mathrm{Sym}(n)$, its singular values are $|\theta_k(\varepsilon)|$, $k = 1, 2, \ldots, n$. Therefore,

$$\|\boldsymbol{Y}_\varepsilon - \boldsymbol{N}\| = |\!|\boldsymbol{\theta}(\varepsilon)|\!| = |\!|\boldsymbol{\theta}(0) + \varepsilon^2 \boldsymbol{\mu} + \mathcal{O}\big(\varepsilon^3\big)\,|\!| = \|\hat{\boldsymbol{Y}} - \boldsymbol{N}\| + \varepsilon^2 \boldsymbol{\mu}^\top \boldsymbol{g}(\boldsymbol{\eta}) + \mathcal{O}\big(\varepsilon^3\big) .$$

We deduce that the DBF (1.1) is isospectrally optimal if and only if

$$\boldsymbol{\mu}^\top \boldsymbol{g}(\boldsymbol{\eta}) \geq 0, \tag{2.3}$$

and the inequality is sharp for at least one $\boldsymbol{\Omega} \in \mathfrak{so}(n)$.

We specialize our argument to Schatten p-norms, restricting p at the first instance to a finite value, $1 \leq p < \infty$. In that case it is easy to calculate the gradient,

$$g_k(\boldsymbol{x}) = \frac{|x_k|^{p-1}}{|\!|\boldsymbol{x}|\!|_p^{p-1}} \mathrm{sgn} x_k, \qquad k = 1, 2, \ldots, n,$$

with derivative singularities when a component of $\boldsymbol{x}$ is zero. (These derivative discontinuities do not matter since we apply the above formula in the sequel exclusively to vectors with nonvanishing components.) Let

$$U_{k,i} = (d_{\pi_k} - d_{\pi_i})(\lambda_k - \lambda_i)\Omega_{k,i}^2, \qquad k, i = 1, 2, \ldots, n.$$

Since $\hat{\boldsymbol{Y}}$ is the attractive fixed point, it follows from (1.5) that $U_{k,i} \geq 0$ for all $k, i = 1, 2, \ldots, n$ and, since $\boldsymbol{\Omega}$ is nonzero, so is the symmetric matrix $\boldsymbol{U}$. The numbers η_k are all nonzero, since $\boldsymbol{d}$ and $\boldsymbol{\lambda}$ are all distinct, therefore we may partition the set $\{1, 2, \ldots, n\}$ into the union $\mathcal{I}_- \cup \mathcal{I}_+$, where

$$\eta_k < 0 \quad \Rightarrow \quad k \in \mathcal{I}_-, \qquad \eta_k > 0 \quad \Rightarrow \quad k \in \mathcal{I}_+.$$

For the Schatten p-norm we thus have

$$\begin{aligned}
|\boldsymbol{\eta}|_p^{p-1}\boldsymbol{\mu}^\top \boldsymbol{g}(\boldsymbol{\eta}) &= \sum_{k=1}^{n}\sum_{\substack{l=1\\ l\neq k}}^{n} \frac{U_{k,l}}{\eta_k-\eta_l}|\eta_k|^{p-1}\mathrm{sgn}\eta_k \\
&= \sum_{k\in\mathcal{I}_-}\sum_{l\in\mathcal{I}_-\setminus\{k\}} \frac{|\eta_k|^{p-1}U_{k,l}}{|\eta_k|-|\eta_l|} + \sum_{k\in\mathcal{I}_+}\sum_{l\in\mathcal{I}_+\setminus\{k\}} \frac{|\eta_k|^{p-1}U_{k,l}}{|\eta_k|-|\eta_l|} \\
&\quad + \sum_{k\in\mathcal{I}_-}\sum_{l\in\mathcal{I}_+} \frac{|\eta_k|^{p-1}U_{k,l}}{|\eta_k|+|\eta_l|} + \sum_{k\in\mathcal{I}_+}\sum_{l\in\mathcal{I}_-} \frac{|\eta_k|^{p-1}U_{k,l}}{|\eta_k|+|\eta_l|}.
\end{aligned} \tag{2.4}$$

The last two sums are nonnegative. Moreover, $\boldsymbol{U}$ being symmetric, with nonnegative entries,

$$\begin{aligned}
\sum_{k\in\mathcal{I}_\pm}\sum_{l\in\mathcal{I}_\pm\setminus\{k\}} \frac{|\eta_k|^{p-1}U_{k,l}}{|\eta_k|-|\eta_l|} &= \tfrac12 \sum_{k\in\mathcal{I}_\pm}\sum_{l\in\mathcal{I}_\pm\setminus\{k\}} \frac{|\eta_k|^{p-1}U_{k,l}}{|\eta_k|-|\eta_l|} + \tfrac12 \sum_{l\in\mathcal{I}_\pm}\sum_{k\in\mathcal{I}_\pm\setminus\{l\}} \frac{|\eta_l|^{p-1}U_{k,l}}{|\eta_l|-|\eta_k|} \\
&= \sum_{l\in\mathcal{I}_\pm}\sum_{k\in\mathcal{I}_\pm\setminus\{l\}} \frac{|\eta_k|^{p-1}-|\eta_l|^{p-1}}{|\eta_k|-|\eta_l|}U_{k,l} \geq 0.
\end{aligned}$$

We conclude that $\boldsymbol{\mu}^\top \boldsymbol{g}(\boldsymbol{\eta}) \geq 0$. Moreover, examining the expressions above, we observe that, unless $p = 1$ and the first two sums in(2.4) are identically zero, necessarily $\boldsymbol{\mu}^\top \boldsymbol{g}(\boldsymbol{\eta}) > 0$, because $\boldsymbol{U} \neq \boldsymbol{O}$. Therefore, because of (2.3), the Schatten p-norm is strongly isospectrally optimal for $p \in (1, \infty)$

For the Ky Fan norm, $p = 1$, the inner product $\boldsymbol{\mu}^\top \boldsymbol{g}(\boldsymbol{\eta})$ is zero in the special case of all the η_ks being of the same sign. This, of course, need not necessarily imply that strong isospectral optimality fails, since, in principle, it might be assured by higher-order terms. Moreover, it is true that $\|\boldsymbol{Y}(t) - \boldsymbol{N}\|_1 \geq \|\hat{\boldsymbol{Y}} - \boldsymbol{N}\|_1$ by continuity: since for every nonsingular $\boldsymbol{A}$

$$\|\boldsymbol{A}\|_{1+\varepsilon} = \|A\|_1 + \varepsilon \left[\sum_{k=1}^{n} |\sigma_k(\boldsymbol{A})| \log|\sigma_k(\boldsymbol{A})| - \log\sum_{k=1}^{n}|\sigma_k(\boldsymbol{A})|\right] + \mathcal{O}\big(\varepsilon^2\big),$$

where $\sigma_k(\boldsymbol{A})$ are the singular values of $\boldsymbol{A}$, this follows from isospectral optimality for $p > 1$. However, it is perfectly possible that the norm is 'flat' in a nonempty open neighbourhood of $\hat{\boldsymbol{Y}}$, implying weak optimality. Thus, note that $\mathrm{tr}\,[\boldsymbol{Y}(t) - \boldsymbol{N}] = \boldsymbol{1}^\top(\boldsymbol{d} - \boldsymbol{\lambda})$ is independent of t. As long as all the entries along the diagonal of $\boldsymbol{Y}(t) - \boldsymbol{N}$ are nonnegative, it is true that

$$\|\boldsymbol{Y}(t) - \boldsymbol{N}\|_1 = \mathrm{tr}\,[\boldsymbol{Y}(t) - \boldsymbol{N}] \equiv \boldsymbol{1}^\top(\boldsymbol{d} - \boldsymbol{\lambda}).$$

Therefore, there always exists $t^* \geq 0$ such that $\|\boldsymbol{Y}(t) - \boldsymbol{N}\|_1$ is constant for all $t \geq t^*$ and there is no isospectral optimality. If (nonpermuted) $\boldsymbol{d}$ is consistent with (1.5) then it is perfectly possible that we may set $t^* = 0$ and the Ky Fan norm of $\boldsymbol{Y}(t) - \boldsymbol{N}$ is always flat. This is the case if $n = 2$ and, for example, $\min d_k > \max \lambda_k$ (cf. Section 3).

Finally, we consider the Schatten ∞-norm (known also as the operator Euclidean norm). Clearly, it must be at the very least weakly isospectrally optimal, because of a continuity argument similar to the one we have used for the Ky Fan norm. Since, by our analysis,

$$\|\boldsymbol{Y}_\varepsilon - \boldsymbol{N}\|_p^p = \|\hat{\boldsymbol{Y}} - \boldsymbol{N}\|_p^p + \varepsilon^2 p \sum_{k=1}^{n} (\mathrm{sgn}\eta_k)|\eta_k|^{p-1} \sum_{i\neq k} \frac{U_{k,i}}{\eta_k - \eta_i} + \mathcal{O}(\varepsilon^3),$$

taking the pth root and letting $p \to \infty$ results in

$$\|\boldsymbol{Y}_\varepsilon - \boldsymbol{N}\|_\infty = \|\hat{\boldsymbol{Y}} - \boldsymbol{N}\|_\infty + \varepsilon^2 \mathrm{sgn}\eta_r \sum_{\substack{k=1\\k\neq r}}^{n} \frac{U_{r,k}}{\eta_r - \eta_k},$$

where $|\eta_r| = \max_{i=1,2,\ldots,n} |\eta_i|$. We assume without loss of generality that $\eta_r > 0$ (the proof for $\eta_r < 0$ is identical). Then $\eta_r - \eta_k > 0$, $k = 1,2,\ldots,n$ and since $U_{r,k} \geq 0$, $r,k = 1,2,\ldots,n$, we deduce that

$$\|\boldsymbol{Y}_\varepsilon - \boldsymbol{N}\|_\infty \geq \|\hat{\boldsymbol{Y}} - \boldsymbol{N}\|_\infty,$$

with an equality only if $\Omega_{r,k} = \Omega_{k,r} = 0$, $k = 1,2,\ldots,n$. Therefore, except for a lower-dimensional submanifold, the inequality is strict and the norm is strongly isospectrally optimal.

To sum up the results of this section, we have proved the following result.

Theorem 2. *The Schatten p-norm is strongly isospectrally optimal for $1 < p \leq \infty$ and weakly isospectrally optimal for $p = 1$.*

3. Isospectral optimality for $n = 2$

Given (1.1) with $n = 2$, we have just three variables and, because the eigenvalues of $\boldsymbol{Y}(t)$ are invariant, possess just one degree of freedom. No wonder, thus, that the solution can be easily represented explicitly. Further, the equations in this case are just equivalent to the Toda lattice equations since a two-by-two matrix is of course tridiagonal. Letting

$$\boldsymbol{N} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \qquad \boldsymbol{Y}_0 = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix},$$

we have $d_1 = \frac{1}{2}(\alpha + \gamma - \sigma)$, $d_2 = \frac{1}{2}(\alpha + \gamma + \sigma)$,where $\sigma = \sqrt{(\alpha-\gamma)^2 + 4\beta^2}$. Moreover, as can be easily verified by direct differentiation, the solution of (1.1) is

$$\begin{aligned} Y_{1,1}(t) &= \tfrac{1}{2}(\alpha+\gamma) + \tfrac{1}{2}\sigma \tanh\left(\sigma\delta t + \tfrac{1}{2}\log\frac{\sigma+\alpha-\gamma}{\sigma-\alpha+\gamma}\right), \\ Y_{1,2}(y) &= Y_{2,1}(t) = \frac{\sigma\mathrm{sgn}\beta}{2\cosh\left(\sigma\delta t + \frac{1}{2}\log\frac{\sigma+\alpha-\gamma}{\sigma-\alpha+\gamma}\right)}, \\ Y_{2,2} &= \tfrac{1}{2}(\alpha+\gamma) - \tfrac{1}{2}\sigma \tanh\left(\sigma\delta t + \tfrac{1}{2}\log\frac{\sigma+\alpha-\gamma}{\sigma-\alpha+\gamma}\right), \end{aligned} \tag{3.1}$$

where $\delta = \lambda_1 - \lambda_2$.

Alternatively, we may express $\boldsymbol{Y}$ in terms of d_1 and d_2. A general matrix on the isospectral orbit is necessarily of the form

$$\boldsymbol{Y} = \begin{bmatrix} \frac{1}{2}(d_1+d_2)-f & \pm\sqrt{\frac{1}{2}(d_1-d_2)^2-f^2} \\ \pm\sqrt{\frac{1}{2}(d_1-d_2)^2-f^2} & \frac{1}{2}(d_1+d_2)+f \end{bmatrix}, \tag{3.2}$$

where $|f| \leq \frac{\sqrt{2}}{2}|d_1-d_2|$. If $\boldsymbol{Y}$ is a function of t then so is f and substitution in (1.1) yields, regardless of the sign of $Y_{1,2}=Y_{2,1}$, the Riccati equation

$$f' = 2(\lambda_2-\lambda_1)[\tfrac{1}{4}(d_2-d_1)^2-f^2], \quad t\geq 0, \qquad f(0)=f_0, \tag{3.3}$$

where f_0 is given by $\boldsymbol{Y}_0$. Its solution is

$$f(t) = \tfrac{1}{2}|d_1-d_2|\frac{\tilde{f}+\tanh[\sqrt{|\lambda_2-\lambda_1|\cdot|d_2-d_1|}t]}{1+\tilde{f}\tanh[\sqrt{|\lambda_2-\lambda_1|\cdot|d_2-d_1|}t]}, \qquad t\geq 0,$$

where $\tilde{f}=2f_0/|d_1-d_2|$.

We next explore global optimality for $n=2$. We commence by noting from (3.2) that the eigenvalues of $\boldsymbol{Y}(t)-\boldsymbol{N}$ are

$$\kappa_\pm(t) = \tfrac{1}{2}(d_1+d_2-\lambda_1-\lambda_2) \pm \sqrt{(d_2-d_1)^2+(\lambda_2-\lambda_1)^2-4(\lambda_2-\lambda_1)f(t)}.$$

Theorem 3. *In the case $n=2$ all unitarily-invariant norms are globally optimal.*

Proof. Let $q=(d_2-d_1)^2+(\lambda_2-\lambda_1)^2$, $r=d_1+d_2-\lambda_1-\lambda_2$, therefore

$$\kappa_\pm = \tfrac{1}{2}[r\pm\sqrt{q-4(\lambda_2-\lambda_1)f}]. \tag{3.4}$$

We let $\boldsymbol{\kappa}=[\kappa_+,\kappa_-]^\top$. By Theorem 1, any unitarily-invariant norm is a superposition of a symmetric gauge with the singular value map. Therefore, by symmetry of $\boldsymbol{Y}-\boldsymbol{N}$, it is true that

$$\|\boldsymbol{Y}-\boldsymbol{N}\| = |\!|\boldsymbol{\kappa}|\!|$$

for some symmetric gauge $|\!|\cdot|\!|$. Let $w:\mathbb{R}_+^2\to\mathbb{R}_+$ be a function that coincides with the symmetric gauge in the nonnegative quadrant (thus, for the p-norm, $w(\boldsymbol{x}) = \sqrt[p]{x_1^p+x_2^p}$). Assuming that w is differentiable for $\boldsymbol{x}\neq\boldsymbol{0}$ in the positive quadrant, we denote its derivatives by

$$w_1 = \frac{\partial w}{\partial x_1}, \qquad w_2 = \frac{\partial w}{\partial x_2},$$

and note that, being a norm, $|\!|\cdot|\!|$ is a weakly-convex function. Thus, it is necessarily true that

$$w_1(\boldsymbol{x}), w_2(\boldsymbol{x}), \frac{w_1(\boldsymbol{x})-w_2(\boldsymbol{x})}{x_1-x_2} \geq 0, \qquad x_1,x_2\geq 0,\ x_1+x_2>0,\ x_1\neq x_2. \tag{3.5}$$

(In the case when $|\!|\cdot|\!|$ is not differentiable in the open positive quadrant, we may approximate it arbitrarily closely by a differentiable symmetric gauge.) Note that we do not need to bother ourselves with the case $x_1=x_2$, since, by (3.4), away from equilibria the eigenvalues $\kappa_\pm$ are distinct.

Because of the symmetry of $|\!|\cdot|\!|$, it is enough to consider the case $0<x_1<x_2$, when also $w_1\pm w_2\geq 0$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{Y}-\boldsymbol{N}\| = \frac{\mathrm{d}}{\mathrm{d}t}|\!|\boldsymbol{\kappa}|\!| = w_1\frac{\mathrm{d}|\kappa_+|}{\mathrm{d}t} + w_2\frac{\mathrm{d}|\kappa_2|}{\mathrm{d}t}.$$

Therefore, if $\kappa_+\kappa_2 \geq 0$ then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{Y}-\boldsymbol{N}\| = -(w_1-w_2)\frac{(\lambda_2-\lambda_1)f'}{\sqrt{q-4(\lambda_2-\lambda_1)f}},$$

while in the case $\kappa_+\kappa_- < 0$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{Y}-\boldsymbol{N}\| = -(w_1+w_2)\frac{(\lambda_2-\lambda_1)f'}{\sqrt{q-4(\lambda_2-\lambda_1)f}}.$$

Because of (3.3), it is true that $(\lambda_2-\lambda_1)f' \geq 0$. Therefore $\mathrm{d}\|\boldsymbol{Y}-\boldsymbol{N}\|/\mathrm{d}t \leq 0$, hence global optimality. $\square$

We turn our attention to isospectral optimality and distinguish, along the lines of the analysis in Section 2, between weak and strong optimality.

Let $|\!|\cdot|\!|$ be a symmetric gauge. Because $|\!|\boldsymbol{x}|\!| = |\!||\boldsymbol{x}||\!|$, it is not a loss of generality to assume again that $\boldsymbol{x}\in\mathbb{R}^2$ lies in the positive quadrant $\mathbb{R}^2_+$. We assume that it is differentiable for $x_1,x_2>0$ and, resurrecting the terminology of Section 2, let

$$g_k(\boldsymbol{x}) = \frac{\partial|\!|\boldsymbol{x}|\!|}{\partial x_k}, \quad k=1,2, \qquad v(\boldsymbol{x}) = \frac{g_1(\boldsymbol{x})-q_2(\boldsymbol{x})}{x_1-x_2}.$$

Theorem 4. *Every unitarily-invariant norm is weakly isospectrally optimal. If*

$$v(\boldsymbol{x})>0, \qquad x_1,x_2>0, \quad x_1\neq x_2 \tag{3.6}$$

then the the norm is strongly isospectrally optimal.

Proof. It follows from the convexity of the ball of radius $|\!|\boldsymbol{x}|\!|$ that $g_1,g_2,v\geq 0$. The key to isospectral optimality is the inequality (2.3). For $n=2$, though, we have

$$\mu_1 = \frac{U_{1,2}}{\eta_1-\eta_2}, \qquad \mu_2 = -\mu_1,$$

therefore

$$\boldsymbol{\mu}^\top\boldsymbol{g} = U_{1,2}\frac{g_1(|\boldsymbol{\eta}|)-g_2(|\boldsymbol{\eta}|)}{\eta_1-\eta_2}. \tag{3.7}$$

We distinguish between two cases. If $\eta_1\eta_2>0$, we may assume without loss of generality that $\eta_1,\eta_2>0$ (otherwise we multiply both $\boldsymbol{Y}_0$ and $\boldsymbol{N}$ by -1 in (1.1)). In that case (3.7) implies that

$$\boldsymbol{\mu}^\top\boldsymbol{g} = U_{1,2}v(\boldsymbol{\eta})$$

and, $U_{1,2}$ being positive, $\boldsymbol{\mu}^\top\boldsymbol{g}>0$ if and only if $v(\boldsymbol{\eta})>0$. In the second case, $\max\{\eta_k\}>0>\min\{\eta_k\}$, we may assume without loss of generality that $\eta_1>0>\eta_2$, $|\eta_1|\geq|\eta_2|$ (otherwise, again, we change signs in (1.1)). Therefore

$$\boldsymbol{\mu}^\top\boldsymbol{g} = U_{1,2}\frac{|\eta_1|-|\eta_2|}{|\eta_1|+|\eta_2|}v(|\boldsymbol{\eta}|)$$

and, again, $\boldsymbol{\mu}^\top\boldsymbol{g}>0$ is equivalent to $v(|\boldsymbol{\eta}|)>0$. Note that, by the triangle inequality and symmetry,

$$|\!|[\alpha x_2+(1-\alpha)x_1, \alpha x_1+(1-\alpha)x_2]^\top|\!| \leq \alpha|\!|[x_2,x_1]^\top|\!| + (1-\alpha)|\!|[x_1,x_2]^\top|\!| = |\!|\boldsymbol{x}|\!|$$

for every $\alpha\in(0,1)$, therefore

$$\frac{1}{\alpha}[|\!|[\alpha x_2+(1-\alpha)x_1, \alpha x_1+(1-\alpha)x_2]^\top|\!| - |\!|[x_1,x_2]^\top] \geq 0.$$

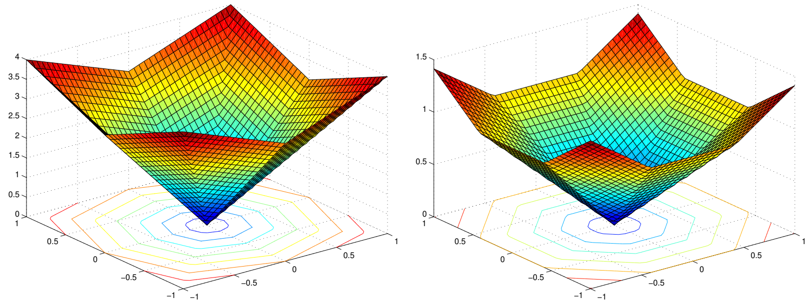

FIGURE 1. The norms (3.8) and (3.9), on the left and the right, respectively.

Letting $\alpha \downarrow 0$, it follows that $v(\boldsymbol{x}) \geq 0$. We thus deduce from (2.3) that, while every unitarily-invariant norm is weakly isospectrally optimal, it is strongly isospectrally optimal if and only if (3.6) holds. □

To illustrate Theorem 4, we commence by considering $|||\cdot|||_p$, $1 \leq p \leq \infty$, corresponding to Schatten p-norms. If $p \in (1, \infty)$, we obtain

$$v(\boldsymbol{x}) = |||\boldsymbol{x}|||_p^{1-p} \frac{x_1^{p-1} - x_2^{p-1}}{x_1 - x_2}.$$

Predictably, $v(\boldsymbol{x}) > 0$ for $1 < p < \infty$. For $p = 1$ we obtain $v(\boldsymbol{x}) \equiv 0$: again, hardly a surprise, considering that we already know from Theorem 2 that the Ky Fan norm is weakly isospectrally optimal. Finally, for $p = \infty$ it is easy to compute

$$v(\boldsymbol{x}) = \frac{1}{|x_1 - x_2|} > 0$$

and again we obtain strong isospectral optimality. Note that v blows up along the line $x_1 = x_2$, but this, as we have already commented, does not interfere with our analysis.

We present three examples of more 'exotic' symmetric gauges (in each case it is easy to prove that the underlying function is, in fact, a symmetric gauge), the first being

$$|||\boldsymbol{x}||| = |x_1| + |x_2| + |x_1 + x_2| + |x_1 - x_2|, \qquad \boldsymbol{x} \in \mathbb{R}^2. \tag{3.8}$$

Therefore $v(\boldsymbol{x}) = 2/|x_1 - x_2|$ and strong isospectral optimality follows from (3.6). (v does not exist for $x_1 = x_2$, where the norm is not differentiable, but this is absolutely no hindrance.)

To illustrate breakdown in strong optimality, our next example is based on the intuitive notion that $\min v(\boldsymbol{x}) = 0$ means that the norm is somehow 'similar' to $|||\cdot|||_1$ near the line $x_1 = x_2$. Thus, we let

$$|||\boldsymbol{x}||| = \begin{cases} |x_1|, & |x_2| \leq (\sqrt{2} - 1)|x_1|, \\ |x_2|, & |x_1| \leq (\sqrt{2} - 1)|x_2|, \\ \frac{\sqrt{2}}{2}(|x_1| + |x_2|), & ||x_1| - |x_2|| \leq (2 - \sqrt{2}) \max\{|x_1|, |x_2|\}. \end{cases} \tag{3.9}$$

Therefore $v \equiv 0$ in a wedge surrounding $x_1 = x_2$ and strong isospectral optimality is lost.

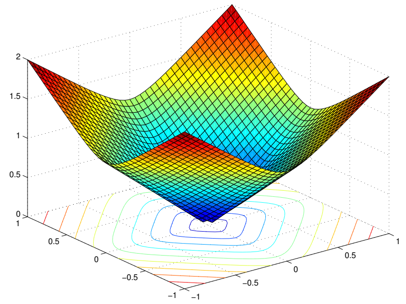

FIGURE 2. The norm (3.10).

Our final example is a symmetric gauge that produces $v = 0$ along the line $x_1 = x_2$, but nowhere else,

$$\|\boldsymbol{x}\| = |x_1| + |x_2| + \tfrac{1}{7}\frac{(|x_1| - |x_2|)^4}{(|x_1| + |x_2|)^3}. \tag{3.10}$$

The constant $\frac{1}{7}$ ensures continuity and it is easy to prove that (3.10) is a symmetric gauge: the only nontrivial item is the satisfaction of the triangle inequality, which follows at once from the observation that the unit ball of (3.10) is convex. We have

$$v(\boldsymbol{x}) = \tfrac{8}{7}\frac{(|x_1| - |x_2|)^2}{(|x_1| + |x_2|)^3} > 0, \qquad x_1, x_2 > 0,\ x_1 \neq x_2,$$

hence strong isospectral optimality. This serves to emphasize the important point that strong isospectral optimality depends on the behaviour of the symmetric gauge *near,* but not *on,* the line $x_1 = x_2$.

4. Further remarks on general $n \geq 2$

The results of Sections 2 and 3, as well as a large volume of numerical computations, are highly suggestive of a conjecture that the attractive fixed point is globally optimal with respect to *every* unitarily-invariant norm.

Depending on the specific norm, numerical experiments highlight two types of behaviour: If the norm in question is strictly isospectrally optimal, $\|\boldsymbol{Y}(t) - \boldsymbol{N}\|$ decreases strictly monotonically until t reaches ∞. However, if we have only weak isospectral optimality, it is perfectly possible for $\|\boldsymbol{Y}(t) - \boldsymbol{N}\|$ to decrease strictly monotonically for a while and then hit a 'plateau': from that moment, $\|\boldsymbol{Y}(t) - \boldsymbol{N}\|$ stays put at its optimal value, although we might be far away from the fixed point.

It is useful and highly illuminating to consider symmetric matrices along an isospectral orbit by examining their *Horn–Schur polytope*. Given a vector $\boldsymbol{\kappa} \in \mathbb{R}^n$, the Horn–Schur polytope $\mathcal{H}$ is the closed convex hull of the $n!$ permutations of $\boldsymbol{\kappa}$ [HJ91]. The following features of the Horn–Schur polytope underlie its relevance to the analysis of DBF:

- Given any $\boldsymbol{A} \in \mathrm{Sym}(n)$ with $\sigma(\boldsymbol{A}) = \boldsymbol{\kappa}$, necessarily $\mathrm{diag}\,\boldsymbol{A} \in \mathcal{H}$. In particular, if $\boldsymbol{Y}(t)$ is the solution of (1.1) then its diagonal forms a trajectory in the Horn–Schur polytope generated by the eigenvalues of $\boldsymbol{Y}_0$.
- Given any $\boldsymbol{x} \in \mathcal{H}$, there exists $\boldsymbol{A} \in \mathrm{Sym}(n)$ such that $\sigma(\boldsymbol{A})$ is given by the vertices of $\mathcal{H}$ and $\mathrm{diag}\,\boldsymbol{A} = \boldsymbol{x}$. (Such a matrix $\boldsymbol{A}$ need not be unique.)

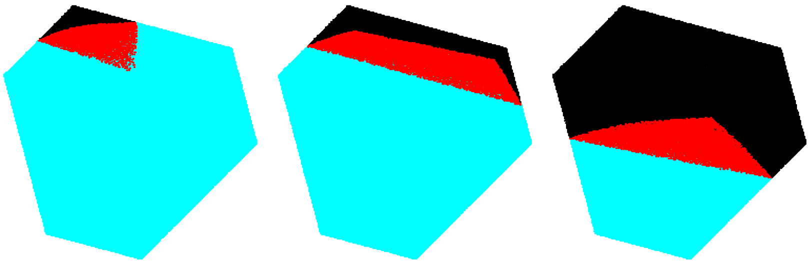

FIGURE 3. The partition of the Horn–Schur polytope for $\boldsymbol{d} = [0, \frac{3}{4}, 2]^\top$ and for (from the left) $\boldsymbol{\lambda} = [\frac{1}{2}, \frac{3}{2}, 3]^\top$, $[\frac{1}{2}, 3]^\top$ and $[\frac{3}{2}, \frac{5}{2}, 3]^\top$.

Therefore, having the freedom to choose $\boldsymbol{Y}_0$ in (1.1), we can commence the trajectory $\operatorname{diag}\boldsymbol{Y}(t)$ at any point in the polytope.

- The diagonal of $\hat{\boldsymbol{Y}}$ is a vertex of $\mathcal{H}$, since the attractive fixed point is a diagonal matrix. Therefore, one privileged vertex of $\mathcal{H}$ attracts all orbits commencing in the interior of the polytope.

Let us focus on the Ky Fan norm which, according to Theorem 2, is weakly isospectrally optimal. Denoting $\boldsymbol{\nu}(t) = \sigma(\boldsymbol{Y}(t) - \boldsymbol{N})$, we observe at once that, as long as all the elements of $\boldsymbol{\nu}$ are of the same sign, i.e. if either $\min \nu_k \geq 0$ or $\max \nu_k \leq 0$, then

$$\|\boldsymbol{Y}(t) - \boldsymbol{N}\|_1 = |\boldsymbol{\nu}|_1 = \mathbf{1}^\top|\boldsymbol{\nu}| = |\mathbf{1}^\top\boldsymbol{\nu}| = |\operatorname{tr}(\boldsymbol{Y}(t) - \boldsymbol{N})| \equiv |\mathbf{1}^\top(\boldsymbol{d} - \boldsymbol{\lambda})|.$$

Hence, in that instance the norm does not vary as t increases. By the same token, if the signs of $\boldsymbol{\nu}$ vary, it follows at once that $\|\boldsymbol{Y}(t) - \boldsymbol{N}\|_1 > |\mathbf{1}^\top(\boldsymbol{d} - \boldsymbol{\lambda})|$. It thus makes sense to partition the polytope $\mathcal{H}$, whose vertices are determined by the eigenvalues of $\boldsymbol{Y}_0$, into three subsets:

1. All points $\boldsymbol{x} \in \mathcal{H}$ such that the eigenvalues of $\boldsymbol{Y} - \boldsymbol{N}$ for every $\boldsymbol{Y} \in \operatorname{Sym}(n)$, $\operatorname{diag}\boldsymbol{Y} = \boldsymbol{x}$, $\sigma(\boldsymbol{Y}) = \sigma(\boldsymbol{Y}_0)$, are all of the same sign (denoted in Fig. 3 by black);
2. All points $\boldsymbol{x} \in \mathcal{H}$ such that the eigenvalues of $\boldsymbol{Y} - \boldsymbol{N}$ for every $\boldsymbol{Y} \in \operatorname{Sym}(n)$, $\operatorname{diag}\boldsymbol{Y} = \boldsymbol{x}$, $\sigma(\boldsymbol{Y}) = \sigma(\boldsymbol{Y}_0)$, change sign (denoted in Fig. 3 by light grey); and
3. All points $\boldsymbol{x} \in \mathcal{H}$ such that there exist $\boldsymbol{Y}_1, \boldsymbol{Y}_2 \in \operatorname{Sym}(n)$ with $\operatorname{diag}\boldsymbol{Y}_1 = \operatorname{diag}\boldsymbol{Y}_2 = \boldsymbol{x}$, $\sigma(\boldsymbol{Y}_1) = \sigma(\boldsymbol{Y}_2) = \sigma(\boldsymbol{Y}_0)$, the eigenvalues of $\boldsymbol{Y}_1 - \boldsymbol{N}$ are all of the same sign and the eigenvalues of $\boldsymbol{Y}_2 - \boldsymbol{N}$ change sign (denoted in Fig. 3 by dark grey).

Since $\mathbf{1}^\top\boldsymbol{x}$ is constant for all $\boldsymbol{x} \in \mathcal{H}$, the Horn–Schur polytope is an $(n-1)$-dimensional object. In particular, given $n = 3$, $\mathcal{H}$ 'lives' on a hyperplane that can be mapped on the plane $\mathbb{R}^2$: a rotation of the hyperplane is probably much more instructive than projection and it has been employed in Fig. 3.

There are three possibilities: sometimes the entire polytope is 'black', sometimes it is 'light grey' and sometimes it includes all three colours. By inspection, everything depends on the relative sizes of $\boldsymbol{\lambda}$ and $\boldsymbol{d}$ (where, without loss of generality, we assume that each vector has monotonically increasing entries). We denote d_k by "$\bullet$" and λ_k by $\circ$, therefore, for example, $\bullet\circ\circ\bullet$ corresponds to the configuration

$d_1 < \lambda_1 < \lambda_2 < d_2$. In $\mathbb{R}^3$, out of all possible configurations, just two, ○ ○ ○ ● ●● and ● ● ● ○ ○○, give a polytope which is entirely 'black' (hence, the Ky Fan norm is constant along all trajectories). Eight further configurations,

○ ○ ● ○ ●●, ○ ○ ● ● ○●, ○ ● ○ ○ ●●, ○ ● ○ ● ○●, ● ○ ● ○ ●○, ● ○ ● ● ○○, ● ● ○ ○ ●○, ● ● ○ ● ○○

result in a 'mixed' polytope, while all remaining choices produce a 'light grey' polytope, where the Ky Fan norm strictly decreases along every trajectory. Fig. 3 displays three instances of 'mixed' polytopes.

The 'exotic norms' (3.9) and (3.10) can be generalized to arbitrary $n \geq 2$. To generalize (3.9), we meld $\| \cdot \|_1$ and $\| \cdot \|_\infty$: the first near the line $x_1 = x_2 = \cdots = x_n$ and its symmetric 'replicas' in other orthants, the other elsewhere, while ensuring continuity and convexity. As long as all the eigenvalues of $\boldsymbol{Y} - \boldsymbol{N}$ lie in the positive orthant, within the n-dimensional wedge where the norm behaves like $\| \cdot \|_1$, we experience weak isospectral optimality.

We generalize (3.10) by setting

$$\|\boldsymbol{x}\| = \|\boldsymbol{x}\|_1 + \frac{1}{7(n-1)} \sum_{k=1}^{n-1} \sum_{l=k+1}^{n} \frac{(|x_k| - |x_l|)^4}{(|x_k| + |x_l|)^3}.$$

This is indeed a symmetric gauge. The only difficulty is in proving that the unit ball with respect to the function above is locally convex along the 'joins' between orthants. Yet, considering tangents therein, it is possible to prove convexity (which implies the triangle inequality) also when $\frac{1}{7}$ is replaced by a smaller positive number. Although we have not proved that this norm is strongly isospectrally optimal, numerical calculations indicate that this is probably the case. Thus, it is not enough for a norm to 'resemble' $\| \cdot \|_1$ just along the line $x_1 = x_2 = \cdots = x_n$ for isospectral optimality to be weak.

5. Conclusions and relationship to the Horn Conjectures

We conclude this paper by making some comparisons to recent work on the Horn conjectures.

A key role is this paper is played by the matrix $-\boldsymbol{N} + \boldsymbol{Y}$. There has been a great deal of interest recently in the the relationship of the eigenvalues of the sum of two symmetric (or Hermitian) matrices to the eigenvalues of the summands. A nice survey of the history and current state of the art of this problem may be found in [Ful00] while the relationship of this problem to combinatorial objects known as 'honeycombs' is discussed in [KT01]. A general list of inequalities between triples (α, β, γ) of n-vectors such that they occur as the eigenvalues of symmetric or Hermitian matrices (A, B, C) with $A + B = C$ was conjectured in [Hor62] and recently proved in a series of papers, as described in [Ful00].

The general case may be clearly reduced to our case of a diagonal matrix summed with a general symmetric matrix, viewed as the conjugation of a diagonal matrix by orthogonal matrices.

Let us consider the Horn conjectures as applied then to the sum $-\boldsymbol{N} + \boldsymbol{Q}\boldsymbol{Y}_0\boldsymbol{Q}^\top$. We do not consider the full set of conjectures but note some special (earlier) necessary conditions that are of interest.

Firstly from [Wey12] one has (using our earlier notation)

$$\nu_{i+j-1} \leq -\lambda_i + d_j, \qquad \text{for } i + j - 1 \leq n\,. \tag{5.1}$$

Fan found the necessary conditions

$$\sum_{i=1}^{r} \nu_i \leq -\sum_{i=1}^{r} \lambda_i + \sum_{i=1}^{r} d_i \qquad \text{for any } r < n \tag{5.2}$$

[Fan49] and Lidskii showed that, viewing the n-tuples of eigenvalues as vectors in $\mathbb{R}^n$, ν must lie in the convex hull of $-\lambda + d_\sigma$ where σ denotes all permutations over the symmetric group [Lid50]. Wielandt in turn showed that this latter condition was equivalent to set of inequalities

$$\sum_{i \in I} \nu_i \leq -\sum_{i \in I} \lambda_i + \sum_{i=1}^{r} d_i \tag{5.3}$$

for every subset I of $\{1, \ldots, n\}$ of cardinality r for all $r < n$ [Wie55].

One observes then that the the spectrum of $-\boldsymbol{N} + \boldsymbol{Q}\boldsymbol{Y}_0\boldsymbol{Q}^\top$ lies in a polytope which is shifted from the Schur–Horn polytope by the vector $\boldsymbol{\lambda}$. That the eigenvalues occur in the convex hull of this shifted polytope rather than diagonals is due to the fact that this polytope arises as the image of a different momentum map. In the Schur–Horn case the relevant map arises from a torus action on an adjoint orbit while the shifted polytope arises from the action of the unitary group on a cross product of orbits.

We note that the inequalities above and indeed the Horn inequalities in general do not imply the inequalities $\nu_i \leq -\lambda_i + d_i$ except for $n = 1$ and thus do not imply the optimality results discussed above. Further it is interesting that the remaining optimality results used in this paper rest on nonlinear functions of the eigenvalues of the relevant matrices and are thus also beyond the scope both of the Horn inequalities and the Schur–Horn inequalities. The latter inequalities are in fact a special case of the former (see [LP03]).

Acknowledgment. We would like to thank Bill Fulton and Sergey Fomin for useful discussions.

References

[BBR90] A. M. Bloch, R. W. Brockett, and T. Ratiu, *A new formulation of the generalized Toda lattice equations and their fixed-point analysis via the moment map*, Bull. AMS **23** (1990), 447–456.

[BBR92] A. M. Bloch, R. W. Brockett, and T. S. Ratiu, *Completely integrable gradient flows*, Comm. Math. Phys. **147** (1992), 57–74.

[BFR90] A. M. Bloch, H. Flaschka, and T. S. Ratiu, *A convexity theorem for isospectral manifolds of jacobi matrices in a compact lie algebra*, Duke Math. Journal **61** (1990), 41–66.

[Blo90] A. M. Bloch, *Steepest descent, linear programming and Hamiltonian flows*, Contemp. Maths AMS **114** (1990), 77–88.

[Bro91] R. W. Brockett, *Dynamical systems that sort lists, diagonalize matrices, and solve linear programming problems*, Linear Algebra Appl. **146** (1991), 79–91.

[CD91] Moody T. Chu and Kenneth R. Driessel, *Constructing symmetric nonnegative matrices with prescribed eigenvalues by differential equations*, SIAM J. Math. Anal. **22** (1991), 1372–1387.

[CIZ97] M. P. Calvo, A. Iserles, and A. Zanna, *Numerical solution of isospectral flows*, Maths Comput. **66** (1997), 1461–1486.

[DNT83] P. Deift, T. Nanda, and C. Tomei, *Differential equations for the symmetric eigenvalue problem*, SIAM J. Num. Anal. **20** (1983), 1–22.

[Fan49] K. Fan, *On a theorem of Weyl concerning linear transformations*, Proc. Nat. Acad. Sci. **35** (1949), 652–655.

[Ful00] W. Fulton, *Eigenvalues, invariant factors, highest weights, and Schubert calculus*, Bull. AMS **37** (2000), 209–249.

[HJ91] Roger A. Horn and Charles R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[Hor62] A. Horn, *Eigenvalues of sums of Hermitian matrices*, Pacific J. Maths **12** (1962), 225–241.

[IMKNZ00] A. Iserles, H. Z. Munthe-Kaas, S. P. Nørsett, and A. Zanna, *Lie-group methods*, Acta Numerica **9** (2000), 215–365.

[Ise02] Arieh Iserles, *On the discretization of double-bracket flows*, Found. Comp. Maths **2** (2002), 305–329.

[KT01] A. Knutson and T. Tao, *Honeycombs and sums of Hermitian matrices*, Notices AMS **48** (2001), 175–186.

[Lid50] V. B. Lidskii, *On the characteristic numbers of the sum and product of symmetric matrices*, Dokl. Akad. Nauk, USSR **75** (1950), 769–772.

[LO96] Adrian S. Lewis and Michael L. Overton, *Eigenvalue optimization*, Acta Numerica **5** (1996), 149–190.

[LP03] C. K. Li and Y.-T. Poon, *Principal submatrices of a Hermitian matrix*, Proc. AMS (2003), to appear.

[Mos74] J. Moser, *Finitely many mass points on the line under the influence of an exponential potential—an integrable system*, Dynamical Systems, Theory and Applications, Lecture Notes in Phys. 38, Springer Verlag, 1974, pp. 467–497.

[Sym80] W. W. Symes, *Hamiltonian group actions and integrable systems*, Physica D **1** (1980), 339–374.

[Sym82] W. W. Symes, *The QR algorithm and scattering for the nonperiodic Toda lattice*, Physica D **4** (1982), 275–280.

[vN37] J. von Neumann, *Some matrix inequalities and metrization of matrix space*, Tomsk University Review **1** (1937), 286–300, In *Collected Works*, Pergamon, Oxford, Volume IV (1962), pp. 205–218.

[Wey12] H. Weyl, *Das asymptotische Verteilungesetz der Eigenwerte lineare partieller Differential-gleichungen*, Math. Ann. **71** (1912), 441–479.

[Wie55] H. Wielandt, *An extremum property of sums of eigenvalues*, Proc. AMS **6** (1955), 106–110.

[Wil65] J. H. Wilkinson, *The algebraic eigenvalue problem*, Oxford University Press, Oxford, 1965.

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA
E-mail address: `abloch@umich.edu`

Department of Applied Mathematics and Theoretical Physics, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK
E-mail address: `A.Iserles@damtp.cam.ac.uk`