On the numerical quadrature of highly-oscillating integrals II: Irregular oscillators

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Abstract

In this paper we set out to understand Filon-type quadrature of highly-oscillating integrals of the form $\int_0^1 f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x$, where g is a real-valued function and $\omega \gg 1$. Employing *ad hoc* analysis, as well as perturbation theory, we demonstrate that for most functions g of interest the moments behave asymptotically according to a specific model that allows for an optimal choice of quadrature nodes. Filon-type methods that employ such quadrature nodes exhibit significantly faster decay of the error for high frequencies ω. Perhaps counterintuitively, as long as optimal quadrature nodes are used, rapid oscillation leads to significantly more precise and more affordable quadrature.

1 Introduction

Highly-oscillating integrals occur in a wide range of practical problems, ranging from electromagnetics and nonlinear optics to fluid dynamics, plasma transport, computerized tomography, celestial mechanics, computation of Schrödinger spectra, Bose–Einstein condensates, ... Moreover, they feature extensively in some new methods for the discretization of highly-oscillatory differential equations (Iserles 2003). Their quadrature is thus a numerical challenge of abiding importance and relevance.

A received numerical 'wisdom' and the lore of application areas is that the computation of highly-oscillatory integrals is an inherently difficult task and that, in general, high oscillation is inimical to computation. In (Iserles 2004) we have set out to promote an opposing point of view, namely that, once the right quadrature methods are employed, high oscillation is a most welcome phenomenon that renders affordable and precise computation much easier. In the present paper we extend this paradigm to considerably more general, irregular oscillators.

To set the stage for our analysis, we need to review very briefly the main result of (Iserles 2004). That paper concerns itself with the computation of the Fourier transform

$$I[f] = \int_0^1 f(x)\mathrm{e}^{\mathrm{i}\omega x}\mathrm{d}x, \tag{1.1}$$

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where $\omega \gg 1$ and the function f is $\mathrm{C}^\infty[0,1]$. The familiar *interpolatory quadrature* is

$$Q^{\mathrm{GC}}[f] = \sum_{l=1}^{\nu} b_l f(c_l) \mathrm{e}^{\mathrm{i}\omega c_l},$$

where $0 \le c_1 < c_2 < \cdots < c_\nu \le 1$ and the weights are selected so that $Q^{\mathrm{GC}}[x^k \mathrm{e}^{-\mathrm{i}\omega x}] = I[x^k \mathrm{e}^{-\mathrm{i}\omega x}]$ for $k = 0, 1, \ldots, p-1$ and some $\nu \le p \le 2\nu$ (Davis & Rabinowitz 1984, Gautschi 1981). Equivalently, the integrand $f(x)\mathrm{e}^{\mathrm{i}\omega x}$ is replaced by an interpolating polynomial at the nodes $c_1, c_2, \ldots, c_\nu$, which is integrated exactly. The most useful interpolatory method is the familiar Gaussian quadrature, $p = 2\nu$, but it is beneficial to treat the more general case in the present setting. As demonstrated in (Iserles 2004), interpolatory quadrature is of little use in the presence of high oscillation, indeed

$$Q^{\mathrm{GC}}[f] - I[f] \sim \mathcal{O}(1)\,, \qquad \omega \to \infty.$$

In this narrow sense, high oscillation is indeed an enemy of computation. Yet, there are numerical methods that deliver incomparably better results and that thrive on high oscillation! None of these methods is particularly new, the main one dates back to 1928 but, unfortunately, with the single exception of Levin (1997), their analysis has either received practically no attention or been performed in an unsatisfactory manner. As a consequence, the wider computational community is broadly unaware of their efficacy and remains wedded to a view that deems highly oscillatory problems intractable or, at best, difficult. (Iserles 2004) focuses on three algorithms: the Zamfirescu method (Zamfirescu 1963), the Levin method (Levin 1982, Levin 1996) and, in particular, the Filon method (Filon 1928, Flinn 1960). They all share the same behaviour when applied to the oscillator (1.1). Specifically,

$$Q[f] - I[f] \sim \mathcal{O}\big(\omega^{-1}\big)\,, \qquad \omega \to \infty$$

for general nodes $c_1 < c_2 < \cdots < c_\nu$, while the choice $c_1 = 0$, $c_\nu = 1$ results in an improved behaviour,

$$Q[f] - I[f] \sim \mathcal{O}\big(\omega^{-2}\big)\,, \qquad \omega \to \infty.$$

Here, $Q[f]$ corresponds to either of the three quadrature methods: Filon, Levin or Zamfirescu. In other words, for a judicious choice of quadrature points, the error decays in an inverse proportion to the square of the frequency.

The purpose of the present paper is to delve deeper into highly-oscillatory quadrature and analyse the behaviour of the Filon quadrature for *irregular oscillators* of the form

$$I_g[f] = \int_0^1 f(x) \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x, \tag{1.2}$$

where g is a real, nonzero, sufficiently smooth function. In the present context, a (generalised) Filon method is

$$Q^{\mathrm{F}}[f] = \sum_{l=1}^{\nu} b_l(\omega) f(c_l), \tag{1.3}$$

where the weights $b_l(\omega)$ are chosen so that $Q^{\mathrm{F}}[x^k] = I_g[x^k]$, $k = 0, 1, \ldots, \nu - 1$. The latter conditions yield a set of ν linear algebraic equations with a nonsingular Vandermonde matrix, hence such $b_l(\omega)$s always exist. An alternative interpretation of (1.3) is that we replace the function f (rather than the entire integrand, *á la* interpolatory quadrature) by an interpolating polynomial, which we subsequently integrate to produce $Q^{\mathrm{F}}[f]$.

The key observation in (Iserles 2004), fundamental to the analysis of Filon's method, is that in the Fourier case (1.1) (corresponding to (1.2) with $g(x) = x$) the *moments* of the functional I_x (that is I_g with $g(x) = x$) are asymptotically

$$\begin{aligned}\mu_m(\omega) = I_x[x^m] &\sim \begin{cases} (\mathrm{i}\omega)^{-1}(\mathrm{e}^{\mathrm{i}\omega} - 1) + \mathcal{O}\big(\omega^{-2}\big)\,, & m = 0, \\ (\mathrm{i}\omega)^{-1}\mathrm{e}^{\mathrm{i}\omega} + \mathcal{O}\big(\omega^{-2}\big)\,, & m \geq 1 \end{cases} \\ &= \frac{1^m \mathrm{e}^{\mathrm{i}\omega} - 0^m}{\mathrm{i}\omega} + \mathcal{O}\big(\omega^{-2}\big)\,, \qquad \omega \to \infty. \end{aligned} \tag{1.4}$$

In other words, the dependence of leading term in the asymptotic expansion upon m is through the powers 0^m and 1^m. Intuitively speaking, this is precisely the reason why choosing $c_1 = 0$ and $c_\nu = 1$ as quadrature nodes eliminates the leading expansion term from the error $Q^{\mathrm{F}}[f] - I_x[f]$.

Why are the moments of I_x of the asymptotic form (1.4)? Is this behaviour shared by other oscillators I_g? And, if not, what is an appropriate generalisation of (1.4), valid for a wider range of functions g yet amenable to analysis and to exploitation in enhancing the behaviour of the Filon quadrature (1.3)? These are the issues at the centre of the present paper.

In Section 2 we replace (1.4) with a more general *geometric model*

$$\mu_m(\omega) \sim \sum_{j=1}^{r} v_j(\omega) d_j^m \omega^{-\alpha_j} + \mathcal{O}\big(\omega^{-\beta}\big)\,, \qquad \omega \to \infty, \tag{1.5}$$

where $v_1, v_2, \ldots, v_r \in \mathrm{L}_\infty[0, \infty)$ are typically periodic functions and $0 < \alpha_1, \alpha_2, \ldots, \alpha_r < \beta$. We prove that, given an appropriate choice of quadrature points, leading terms are annihilated in the asymptotic expansion of the error $Q[f] - I_g[f]$. Section 3 explores the case when $g' \neq 0$ in $[0, 1]$. Using elementary means, we prove that I_g is consistent with the geometric model (1.5) with $r = 2$, $d_1 = 0$, $d_2 = 1$. This extends (1.4) in a straightforward manner.

Exploring the case when g has stationary points in $(0, 1)$ requires the combination of the analysis of Section 2, which caters for the endpoints, with the method of stationary phase (Hinch 1991, Olver 1974, Stein 1993). This is the theme of Section 4, where we demonstrate that, as long as $g'(0), g'(1) \neq 0$, the functional I_g is always consistent with the geometric model (1.5).

Section 5 is devoted to a raft of additional issues arising in this paper. Firstly, we consider functions g that fall outside the scope of our analysis. Often our results can be recovered, even when g is not smooth. However, if g' vanishes at an endpoint then our analysis breaks down: in that case (1.5) is no longer true and the performance of the Filon method (1.3) cannot be enhanced by a judicious choice of quadrature points. Secondly, returning to the framework of Sections 2–4, we debate how to exploit the freedom left in the choice of nodes, once some of them have been used to improve the error asymptotics. Thirdly, we present a method whose weights are computed directly from our knowledge of the asymptotic behaviour of the moments and discuss its relative advantages *vis-á-vis* the Filon quadrature. Finally, we explore a generalisation of (1.2) to integrals with large parameters, which need not be highly oscillatory.

This is a second in a sequence of papers that address the Filon method and its generalisations. Forthcoming papers will be devoted to methods that employ derivatives, as well as to error estimation, multivariate quadrature and computation of singular integrands.

2 The geometric model

Let g be a real-valued, nonzero function and f a complex-valued function, both smooth. We consider the linear functional I_g given by (1.2) and denote its moments by

$$\mu_m(\omega) = I_g[x^m], \qquad m \geq 0.$$

In particular, we are interested in functions g consistent with the *geometric model* (1.5), which we restate for continuity of exposition,

$$\mu_m(\omega) \sim \sum_{j=1}^{r} v_j(\omega) d_j^m \omega^{-\alpha_j} + \mathcal{O}\big(\omega^{-\beta}\big), \qquad m \geq 0, \quad \omega \to \infty. \tag{2.1}$$

Here $v_1, v_2, \ldots, v_r \in \mathrm{L}_\infty[0, \infty)$, the points $d_1, d_2, \ldots, d_m \in [0, 1]$ are distinct and

$$0 < \alpha_1, \alpha_2, \ldots, \alpha_r < \beta,$$

therefore $\mathcal{O}\big(\omega^{-\beta}\big)$ is the tail of the asymptotic expansion. Note that the $\mathcal{O}\big(\omega^{-\beta}\big)$ term depends on m, therefore the expansion (2.1) is not uniform for $m \geq 0$.

Theorem 1 *Let the function g be consistent with (2.1), $\nu \geq r$ and*

$$d_1, d_2, \ldots, d_r \in \{c_1, c_2, \ldots, c_\nu\}.$$

Then the error of the Filon method (1.3) is asymptotically

$$E[f] = Q^{\mathrm{F}}[f] - I_g[f] = \mathcal{O}\big(\omega^{-\beta}\big), \qquad \omega \to \infty. \tag{2.2}$$

Proof We assume that the function f is analytic: the proof can be easily extended to a non-analytic $f \in \mathrm{C}^\infty[0, 1]$ by a standard density argument. Set

$$\rho_m(\omega) = \sum_{l=1}^{\nu} b_l(\omega) c_l^m - \mu_m(\omega), \qquad m \geq 0$$

and note that the construction of the Filon method implies that

$$\rho_0, \rho_1, \ldots, \rho_{\nu-1} \equiv 0.$$

Moreover, recalling that f is analytic and expanding f into Taylor series,

$$f(x) = \sum_{m=0}^{\infty} \frac{f_m}{m!} x^m, \qquad x \in [0, 1],$$

we observe that

$$E[f] = \sum_{m=\nu}^{\infty} \frac{f_m}{m!} \rho_m(\omega). \tag{2.3}$$

Letting

$$\gamma(t) = \prod_{l=1}^{\nu}(t - c_l) = \sum_{k=0}^{\nu} \gamma_k t^k,$$

we multiply (2.1) (with m replaced by $m+k$) by γ_k and sum up for $k = 0, 1, \ldots, \nu$. The outcome is

$$\sum_{k=0}^{\nu} \gamma_k \mu_{m+k}(\omega) \sim \sum_{j=1}^{r} \gamma(d_j) v_j(\omega) \omega^{-\alpha_j} d_j^m + \mathcal{O}\big(\omega^{-\beta}\big), \qquad m \geq 0. \tag{2.4}$$

Likewise, multiplying ρ_{m+k} by γ_k and summing up, we have

$$\sum_{k=0}^{\nu} \gamma_k \mu_{m+k}(\omega) = \sum_{l=1}^{\nu} b_l(\omega)\gamma(c_l) - \sum_{k=0}^{\nu} \gamma_k \rho_{m+k}(\omega)$$

and, since $\gamma(c_l) = 0$ and $d_1, \ldots, d_r \in \{c_1, \ldots, c_\nu\}$, substitution in (2.4) yields

$$\sum_{k=0}^{\nu} \gamma_k \rho_{m+k}(\omega) \sim \mathcal{O}\big(\omega^{-\beta}\big), \qquad \omega \to \infty. \tag{2.5}$$

Considering (2.5) as a recurrence relation with the initial conditions $\rho_0, \ldots, \rho_{\nu-1} \equiv 0$ readily affirms that $\rho_m(\omega) \sim \mathcal{O}\big(m\omega^{-\beta}\big)$, $m \geq 0$, $\omega \to \infty$. Substitution in (2.3) and the analyticity of f, hence of f', thus yield $E[f] \sim \mathcal{O}\big(\omega^{-\beta}\big)$ and complete the proof. $\square$

The condition that *each* d_l is a quadrature node is essential to the theorem. For suppose that

$$\mathcal{J} = \{j \,:\, d_j \notin \{c_1, c_2, \ldots, c_\nu\}\} \neq \emptyset.$$

Then, in place of (2.5), we have

$$\sum_{k=0}^{\nu} \gamma_k \rho_{m+k}(\omega) \sim -\sum_{j\in\mathcal{J}} \gamma(d_j) v_j(\omega) \omega^{-\alpha_j} d_j^m + \mathcal{O}\big(\omega^{-\beta}\big), \qquad \omega \to \infty.$$

Note, however, that for any d such that $\gamma(d) \neq 0$, a solution of the linear recurrence

$$\sum_{k=0}^{\nu} \gamma_k y_{m+k} = v d^m, \qquad m \geq 0,$$

is $y_m = v d^m / \gamma(d)$, $m \geq 0$. Therefore the general solution of the asymptotic recurrence is

$$\rho_m(\omega) \sim \sum_{l=1}^{\nu} \sigma_l(\omega) c_l^m - \sum_{j\in\mathcal{J}} v_j(\omega)\omega^{-\alpha_j} d_j^m + \mathcal{O}\big(\omega^{-\beta}\big), \qquad m \geq 0, \quad \omega \to \infty,$$

where $\sigma_1, \sigma_2, \ldots, \sigma_\nu$ are the solution of a nonsingular Vandermonde system forcing compliance with the initial conditions $\rho_0, \ldots, \rho_{\nu-1} \equiv 0$. Therefore

$$\rho_m(\omega) \sim \mathcal{O}\big(\omega^{-\hat{\alpha}}\big), \qquad \omega \to \infty,$$

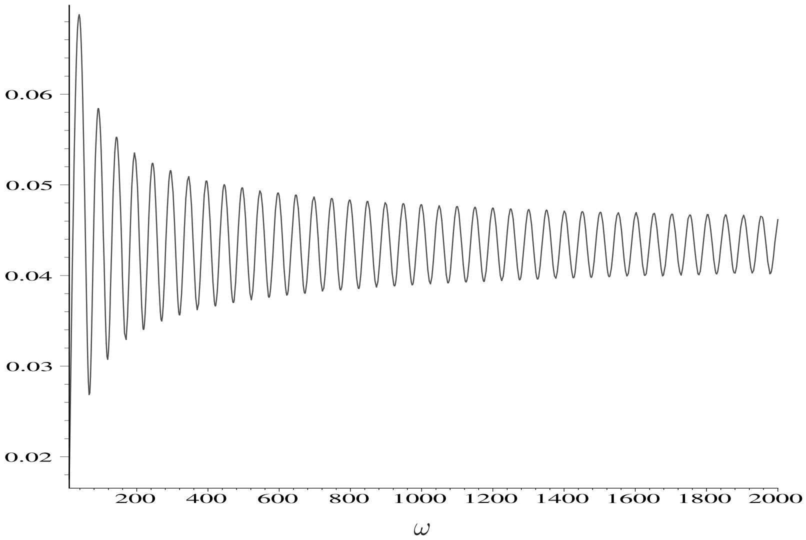

Figure 1: The absolute value of the error in Filon quadrature of $\int_0^1 \mathrm{e}^x \mathrm{e}^{\frac{1}{2}\mathrm{i}\omega(x-\frac{1}{2})^2}\mathrm{d}x$, multiplied by $\omega^{3/2}$, for $\nu = 3$, $c_1 = 0$, $c_2 = \frac{1}{2}$, $c_3 = 1$ and different values of ω.

where $\hat{\alpha} = \min_{j \in \mathcal{J}} \alpha_j < \beta$.

The regular oscillator $g(x) = x$ is an obvious example of compliance with the geometric model (2.1), with

$$r = 2, \qquad d_1 = 0, \quad d_2 = 1, \qquad v_1(\omega) \equiv \mathrm{i}, \quad v_2(\omega) = -\mathrm{i}\mathrm{e}^{\mathrm{i}\omega}.$$

A far less trivial example is the quadratic function $g(x) = \frac{1}{2}(x - \frac{1}{2})^2$. The first few moments can be computed by a symbolic package and the general asymptotic form of μ_m follows from the recurrence relation

$$\mu_{m+1}(\omega) - \tfrac{1}{2}\mu_m(\omega) + \frac{m}{\mathrm{i}\omega}\mu_{m-1}(\omega) = \frac{\mathrm{e}^{\frac{1}{8}\mathrm{i}\omega}}{\mathrm{i}\omega}, \qquad m \geq 1,$$

which can be easily obtained by integration by parts from

$$\mu_{m+1}(\omega) - \tfrac{1}{2}\mu_m(\omega) = \frac{1}{\mathrm{i}\omega}\int_0^1 x^m \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\frac{1}{2}\mathrm{i}\omega(x-\frac{1}{2})^2}\mathrm{d}x.$$

Thus, the moments are

$$\mu_0(\omega) = \mathrm{i}\,\mathrm{erf}(\tfrac{1}{4}\sqrt{-2\mathrm{i}\omega})\sqrt{-\frac{2\pi\mathrm{i}}{\omega}},$$

$$\mu_m(\omega) \sim \tfrac{1}{2^m}\mathrm{i}\,\mathrm{erf}(\tfrac{1}{4}\sqrt{-2\mathrm{i}\omega})\sqrt{-\frac{2\pi\mathrm{i}}{\omega}} + \tfrac{2^{m-1}-1}{2^{m-2}}\frac{\mathrm{e}^{\frac{1}{8}\mathrm{i}\omega}}{\mathrm{i}\omega} + \mathcal{O}\left(\omega^{-3/2}\right), \qquad m \geq 1, \quad \omega \to \infty.$$

On the face of it, the geometric model is satisfied with $r = 1$ and $d_1 = \frac{1}{2}$, therefore Filon's quadrature (with a node at $\frac{1}{2}$) produces a $\mathcal{O}\big(\omega^{-1}\big)$ error. This, however, is much too pessimistic, as transpires with some extra effort. Since

$$\operatorname{erf} z \sim 1 - \frac{\mathrm{e}^{-z^2}}{\sqrt{\pi} z}, \qquad |z| \to \infty, \quad |\arg z| < \frac{3\pi}{4}$$

(Abramowitz & Stegun 1964, p. 298), we can derive the next term in the asymptotic expansion and observe that it is also consistent with the geometric model (2.1),

$$\mu_m(\omega) \sim \tfrac{\mathrm{i}}{2^m}\sqrt{-\frac{2\mathrm{i}\pi}{\omega}} - \frac{2\mathrm{i}\mathrm{e}^{\frac{1}{8}\mathrm{i}\omega}}{\omega}(0^m + 1^m) + \mathcal{O}\Big(\omega^{-3/2}\Big), \qquad m \ge 0, \quad \omega \to \infty.$$

Therefore $r = 3$,

$$d_1 = 0,\ d_2 = \tfrac{1}{2},\ d_3 = 1, \qquad v_1(\omega) = -\frac{2\mathrm{i}\mathrm{e}^{\frac{1}{8}\mathrm{i}\omega}}{\omega},\ v_2(\omega) = \mathrm{i}\sqrt{-\frac{2\mathrm{i}\pi}{\omega}},\ v_3(\omega) = -\frac{2\mathrm{i}\mathrm{e}^{\frac{1}{8}\mathrm{i}\omega}}{\omega}$$

and the Filon quadrature with $\nu \ge 3$, with nodes including $0, \frac{1}{2}, 1$, has an asymptotic error of $\mathcal{O}\big(\omega^{-3/2}\big)$. This is confirmed in Figure 1, where we have integrated

$$\begin{aligned} I_{\frac{1}{2}(x-\frac{1}{2})^2}[\mathrm{e}^x] &= -\sqrt{-\frac{\pi}{2\mathrm{i}\omega}}\mathrm{e}^{(\omega+\mathrm{i})/(2\omega)}\left[\operatorname{erf}\left(\frac{\sqrt{2}(2+\mathrm{i}\omega)}{4\sqrt{-\mathrm{i}\omega}}\right) + \operatorname{erf}\left(\frac{\sqrt{2}(-2+\mathrm{i}\omega)}{4\sqrt{-\mathrm{i}\omega}}\right)\right] \\ &\sim \sqrt{-\frac{2\pi\mathrm{e}}{\mathrm{i}\omega}} + \frac{2\mathrm{i}}{\omega}(\mathrm{e}-1)\mathrm{e}^{\frac{1}{8}\mathrm{i}\omega} + \mathcal{O}\Big(\omega^{-3/2}\Big), \qquad \omega \to \infty, \end{aligned}$$

using Filon quadrature with $\nu = 3$, $d_1 = 0$, $d_2 = \frac{1}{2}$ and $d_3 = 1$. Note as an interesting aside that, although the integrand oscillates rapidly, the leading asymptotic term of the exact integral is nonoscillatory. This, of course, does not mean that classical quadrature methods, e.g. interpolatory quadrature, are likely to be of any use, since their error is expressible as a scaled derivative of the integrand, which is large for $\omega \gg 1$.

On the face of it, this is a highly nontrivial example that requires careful 'massaging' and ultimately exhibits a degree of serendipity. As a matter of fact, once the general theory is worked out, as it will be in the next two sections, the correct asymptotic behaviour of the moments follows in a fully transparent manner.

Before we embark on our analysis, it is instructive to indicate what can we expect. A crucial insight into the asymptotic behaviour of $I_g[f]$ is provided by a classical result from harmonic analysis which should have been perhaps more familiar to numerical analysts and whose proof can be found, for example, in (Stein 1993, p. 332).

Lemma 2 (van der Corput) *Suppose that g is a real-valued smooth function in $[0, 1]$ and that $|g^{(s)}(x)| \ge 1$ for $x \in (0, 1)$ and some $s \ge 1$. Then*

$$\left|\int_0^1 \mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x\right| \le \xi_s \omega^{-1/s}, \qquad \omega > 0, \tag{2.6}$$

holds for

1. $s = 1$ *and monotone* g'*; or*

2. $s \geq 2$.

Moreover, the optimal bound ξ_s *is independent of* g *and* ω.

An immediate generalisation of the lemma replaces the condition $|g^{(s)}(x)| \geq 1$, $x \in (0,1)$, by $g^{(s)}(x) \neq 0$, $x \in [0,1]$: writing

$$g(x) = \sigma \tilde{g}(x), \qquad \text{where} \qquad \sigma = \min_{0 \leq x \leq 1} |g^{(s)}(x)|$$

and rescaling ω, we obtain, in place of (2.6), the bound

$$\left| \int_0^1 \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x \right| \leq \xi_s (\sigma\omega)^{-1/s}, \qquad \omega > 0. \tag{2.7}$$

The inequality (2.6) can be employed to bound $|I_g[f]|$ for any smooth function f (Stein 1993, p. 334) and this can be recast using (2.7) in place of (2.6).

Corollary 1 *Let both* f *and* g *be smooth, real-valued functions in* $[0,1]$ *and assume that* $g^{(s)}(x) \neq 0$*,* $x \in [0,1]$*, for some* $s \geq 1$*. If* $s = 1$ *assume in addition that* g' *is monotone. Then*

$$|I_g[f]| = \left| \int_0^1 f(x) \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x \right| \leq \xi_s (\sigma\omega)^{-1/s} \left[f(1) + \int_0^1 |f'(x)| \mathrm{d}x \right]. \tag{2.8}$$

The inequality (2.8) indicates that the rate of decay of $I_g[f]$ is governed by $\mathcal{O}\big(\omega^{-1/s}\big)$. Although not cast in a language of the geometric model, (2.8) can be used to derive many of the results in the sequel. We do not follow this route, since an alternative approach leads to substantially stronger results. Yet, the importance of the van der Corput lemma and its corollary is in indicating that, unless g is strictly monotone in $[0,1]$, the leading term in the asymptotic expansion is governed by the nature of its stationary points. First, however, we discuss the 'plain vanilla' case of strictly monotone g.

3 The case $g'(x) \neq 0$, $x \in [0,1]$

Our point of departure is a result which, while trivial, plays fundamental role in our analysis.

Lemma 3 *Suppose that* g *is a real-valued smooth function in* $[0,1]$ *and that* $g'(x) \neq 0$*,* $x \in [0,1]$*. Then*

$$I_g[f] \sim \frac{1}{\mathrm{i}\omega} \left[\frac{\mathrm{e}^{\mathrm{i}\omega g(1)} f(1)}{g'(1)} - \frac{\mathrm{e}^{\mathrm{i}\omega g(0)} f(0)}{g'(0)} \right] + \mathcal{O}\big(\omega^{-2}\big), \qquad \omega \to \infty. \tag{3.1}$$

Proof Since $g' \neq 0$, we write I_g in the form

$$I_g[f] = \frac{1}{\mathrm{i}\omega} \int_0^1 \frac{f(x)}{g'(x)} \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x$$

and integrate by parts. Therefore

$$I_g[f] = \frac{1}{\mathrm{i}\omega}\left[\frac{\mathrm{e}^{\mathrm{i}\omega g(1)} f(1)}{g'(1)} - \frac{\mathrm{e}^{\mathrm{i}\omega g(0)} f(0)}{g'(0)}\right] - \frac{1}{\mathrm{i}\omega} I_g\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right)\right].$$

Since it follows from our analysis that $I_g[f] \sim \mathcal{O}(\omega^{-1})$ and $(f/g')'$ is, like f, smooth in $[0,1]$, we deduce that, likewise, $I[(f/g')'] \sim \mathcal{O}(\omega^{-1})$. Substituting this estimate in the last displayed formula confirms that (3.1) is true. □

Recall our aim: to explore the satisfaction of the geometric model (2.1) by the moments of I_g. Straightforward application of (3.1) demonstrates that this is the case, provided that g is strictly monotone in $[0,1]$.

Theorem 4 *Let us suppose that g' is nonzero in $[0,1]$. Then*

$$\mu_m(\omega) \sim \frac{1}{\mathrm{i}\omega}\left[1^m \frac{\mathrm{e}^{\mathrm{i}\omega g(1)}}{g'(1)} - 0^m \frac{\mathrm{e}^{\mathrm{i}\omega g(0)}}{g'(0)}\right] + \mathcal{O}(\omega^{-2}), \qquad m \ge 0, \quad \omega > 0. \tag{3.2}$$

(As before, the $\mathcal{O}(\omega^{-2})$ term depends upon m.)

We deduce that, as long as $g' \neq 0$, the moments are consistent with the geometric model (2.1) with $r = 2$ and

$$d_1 = 0, \quad d_2 = 1, \qquad v_1(\omega) = \frac{\mathrm{e}^{\mathrm{i}\omega g(0)}}{\mathrm{i}g'(0)}, \quad v_2(\omega) = \frac{\mathrm{e}^{\mathrm{i}\omega g(1)}}{\mathrm{i}g'(1)}.$$

An example of (3.2), already familiar from (1.4), is the regular oscillator $g(x) = x$, the theme of (Iserles 2004). Less trivial is $g(x) = \log(1+x)$, whereby

$$\begin{aligned}
\mu_0(\omega) &= \frac{2^{1+\mathrm{i}\omega} - 1}{1 + \mathrm{i}\omega} \sim \frac{2^{1+\mathrm{i}\omega} - 1}{\mathrm{i}\omega} + \mathcal{O}(\omega^{-2}), \\
\mu_1(\omega) &= \frac{\mathrm{i}\omega 2^{1+\mathrm{i}\omega} + 1}{(1+\mathrm{i}\omega)(2+\mathrm{i}\omega)} \sim \frac{2^{1+\mathrm{i}\omega}}{\mathrm{i}\omega} + \mathcal{O}(\omega^{-2}), \\
\mu_2(\omega) &= \frac{(2 + \mathrm{i} - \omega^2)2^{1+\mathrm{i}\omega} - 2}{(1+\mathrm{i}\omega)(2+\mathrm{i}\omega)(3+\mathrm{i}\omega)} \sim \frac{2^{1+\mathrm{i}\omega}}{\mathrm{i}\omega} + \mathcal{O}(\omega^{-2})
\end{aligned}$$

is consistent with (3.2).

Although many results of this section can be derived by the alternative route of the van der Corput Lemma 2, our approach is more general. Thus, using Corollary 1, we could have proved (3.2) (with $\mathcal{O}(\omega^{-2})$ replaced by the weaker $\mathcal{O}(\omega^{-1/1/s})$) for strictly monotone g such that $g^{(s)} \neq 0$ for some $s \ge 2$. In our setting the latter condition is not required.

4 Stationary points in $(0,1)$

In the last section we have seen how integration by parts can be used to bring strictly monotone g within the realm of the geometric model (2.1). We presently turn our attention to a function g that possesses stationary points in the open interval $(0,1)$. The main additional tool that we bring to bear on the problem is *the method of stationary phase.* This method, whose progeny can be traced to Lord Kelvin (Olver 1974), is also sometimes known as *the method of critical points.*

Lemma 5 (The method of stationary phase) *Suppose that $\delta \geq 2$ and*

$$g(d) = g'(d) = \cdots = g^{(\delta-1)}(d) = 0, \qquad g^{(\delta)}(d) \neq 0 \tag{4.1}$$

for some $d \in (0,1)$. In addition we stipulate that $g'(x) \neq 0$ for all $x \in (0,1) \setminus \{d\}$. Then

$$I_g[f] = \int_0^1 f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x \sim \omega^{-1/\delta} \sum_{n=0}^{\infty} a_n[f]\omega^{-n/\delta}, \qquad \omega \to \infty, \tag{4.2}$$

for every f with sufficiently small compact support in the neighbourhood of d.

Rigourous proof of Lemma 5 is given in (Stein 1993, p. 334). The following points highlight a number of relevant aspects of the lemma.

1. Since
$$\mathrm{e}^{\mathrm{i}\omega g(x)} = \mathrm{e}^{\mathrm{i}\omega g(d)}\mathrm{e}^{\mathrm{i}\omega[g(x)-g(d)]},$$
it trivially follows that we can drop the requirement $g(d) = 0$ in (4.1), while multiplying the coefficients $a_n[f]$ in (4.2) by $\mathrm{e}^{\mathrm{i}\omega g(d)}$.

2. The requirement that the support of f is localised in a neighbourhood of the stationary point is typically omitted in most expositions of the method of stationary phase. Sometimes this is justified, e.g. when the integration is carried out in $(-\infty, \infty)$, rather than $[0,1]$ (Hinch 1991, Olver 1974): the localisation is the price we need to pay for imposing finite endpoints! But often it is an unwelcome consequence of physical 'intuition' laced with careless hand-waiving. Be is at it may, compact support and the setting in (Stein 1993) provide the right framework for our analysis and for the eventual treatment of the case of several stationary points in $(0,1)$.

3. The linear operators $a_n[f]$ can be often derived explicitly for some values of n. In particular, in the important special case $\delta = 2$ we have
$$a_0[f] = \mathrm{e}^{\mathrm{i}\omega g(d)} \sqrt{-\frac{2\pi}{\mathrm{i}g''(d)}}\, f(d), \tag{4.3}$$
while the method of proof in (Stein 1993) demonstrates that for *even* $\delta \geq 2$ it is true that $a_n[f] \equiv 0$ for all odd $n \geq 1$. In that case (4.2) reads
$$I_g[f] \sim \omega^{-1/\delta} \sum_{n=0}^{\infty} a_{2n}[f]\omega^{-2n/\delta}, \qquad \omega \to \infty.$$

4. Another consequence of the method of proof in (Stein 1993) is that, regardless of the value of δ, $a_0[f] = \tilde{a}_0(\omega)f(d)$, where $\tilde{a}_0$ is independent of f: for example, for $\delta = 2$ (4.3) shows that $\tilde{a}_0(\omega) = \mathrm{e}^{\mathrm{i}\omega g(d)}\sqrt{-2\pi/[\mathrm{i}g''(d)]}$. This fact, which can be 'proved' in a hand-waiving fashion by observing that $a_0[f]$ is a linear operator, is fundamental in the proof of Theorem 6 in the sequel.

While Theorem 4 extracts the contribution to the moments accruing from the endpoints, the method of stationary phase captures the contribution of stationary points in $(0, 1)$. In greater generality, let us extend the framework of Lemma 5 and assume the existence of $k \geq 1$ distinct points $\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_k \in (0, 1)$ such that

$$g'(\tilde{d}_j) = g''(\tilde{d}_j) = \cdots = g^{(\delta_j - 1)}(\tilde{d}_j) = 0, \quad g^{(\delta_j)}(\tilde{d}_j) \neq 0, \qquad j = 1, 2, \ldots, k. \tag{4.4}$$

Furthermore, $g'(x) \neq 0$ elsewhere in $[0, 1]$.

Given $f \in \mathrm{C}^\infty[0, 1]$, which need not have compact support, we partition it into

$$f(x) = \sum_{j=0}^{k} f_j(x).$$

Here $f_j(x) = f(x)\theta_{\varepsilon,\tilde{d}_j}(x)$, $j = 1, 2, \ldots, k$ and $f_0(x) = f(x) - \sum_{j=1}^{k} f_j(x)$, where $\theta_{\varepsilon,d} \in \mathrm{C}^\infty[0, 1]$ is a *bump function:* $\theta_{\varepsilon,d}(x) \equiv 1$ for $|x - d| < \frac{1}{2}\varepsilon$ and $\theta_{\varepsilon,d}(x) \equiv 0$, $|x - d| > \varepsilon$ (Hirsch 1976, p. 41). Note thus that each f_j for $j = 1, 2, \ldots, k$ is supported by an ε-neighbourhood of $\tilde{d}_j$. We choose $\varepsilon > 0$ so that Lemma 5 and the asymptotic expansion (4.2) are valid at all $\tilde{d}_j$s,

$$I_g[f_j] \sim \omega^{-1/\delta_j} \sum_{n=0}^{\infty} a_n[f_j]\omega^{-n/\delta_j}, \qquad j = 1, 2, \ldots, k, \quad \omega \to \infty.$$

Moreover, $I_g[f_0]$ can be estimated using Lemma 5. The intuitive reason is clear, since f_0 is identically zero in a $\frac{1}{2}\varepsilon$-neighbourhood of each stationary point. A more rigourous argument is as follows. The support of f_0 can be partitioned into $k + 1$ disjoint intervals where g is strictly monotone, thus within the conditions of Lemma 3. Therefore the expansion

$$\begin{aligned} I_g[f_0] &\sim \frac{1}{\mathrm{i}\omega} \sum_{i=1}^{k+1} \left[\frac{\mathrm{e}^{\mathrm{i}\omega g(\beta_i)}}{g'(\beta_i)} f_0(\beta_i) - \frac{\mathrm{e}^{\mathrm{i}\omega g(\alpha_i)}}{g'(\alpha_i)} f_0(\alpha_i) \right] + \mathcal{O}\big(\omega^{-2}\big) \\ &= \frac{1}{\mathrm{i}\omega} \left[\frac{\mathrm{e}^{\mathrm{i}\omega g(1)}}{g'(1)} f(1) - \frac{\mathrm{e}^{\mathrm{i}\omega g(0)}}{g'(0)} f(0) \right] + \mathcal{O}\big(\omega^{-2}\big), \qquad \omega \to \infty \end{aligned}$$

follows from Lemma 3, the support of f_0 being $\bigcup_{i=1}^{k+1}(\alpha_i, \beta_i)$. Since $I_g[f] = \sum_{j=0}^{k} I_g[f_j]$, letting $f(x) = x^m$ for $m \geq 0$ confirms at once consistency with the geometric model (2.1). We have thus proved the following result on the asymptotic behaviour of the moments of g.

Theorem 6 *Suppose that (4.4) holds and that $g' \neq 0$ elsewhere in $[0, 1]$. Then the geometric model (2.1) is satisfied for $r = k + 2$ and*

$$d_1 = 0, \qquad d_j = \tilde{d}_{j-1}, \quad j = 2, 3, \ldots, r - 1, \qquad d_r = 1.$$

Specifically, $\alpha_1 = \alpha_r = 1$, $\alpha_j = 1/\delta_{j-1}$, $j = 2, 3, \ldots, r - 1$ and

$$\beta = \min \left\{ 2, 2 \min_{\delta_j \text{ odd}} \frac{1}{\delta_j}, 3 \min_{\delta_j \text{ even}} \frac{1}{\delta_j} \right\}. \tag{4.5}$$

The proof of the theorem follows at once from our discussion. In particular,

$$v_1(\omega) = -\frac{\mathrm{e}^{\mathrm{i}\omega g(0)}}{\mathrm{i}g'(0)}, \qquad v_r(\omega) = \frac{\mathrm{e}^{\mathrm{i}\omega g(1)}}{\mathrm{i}g'(1)}$$

and note different treatment of odd and even δ_j, motivated by

$$I_g[f] \sim a_0[f]\omega^{-1/\delta} + \mathcal{O}\left(\omega^{-2/\delta}\right), \qquad \omega \to \infty$$

or

$$I_g[f] \sim a_0[f]\omega^{-1/\delta} + \mathcal{O}\left(\omega^{-3/\delta}\right), \qquad \omega \to \infty$$

in (4.2), depending on whether δ is odd or even, respectively. Note further that, for $\delta_{j-1} = 2$, (4.3) implies that

$$v_j(\omega) = \mathrm{e}^{\mathrm{i}\omega g(d)} \sqrt{-\frac{2\pi}{\mathrm{i}g''(d_j)}}.$$

Corollary 2 *Suppose that g is within the conditions of Theorem 6 and that I_g is approximated by Filon's quadrature with $\nu \geq k+2$ and*

$$0, \tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_k, 1 \in \{c_1, c_2, \ldots, c_\nu\}.$$

Then the quadrature error is asymptotically $\mathcal{O}\left(\omega^{-\beta}\right)$, where β is given by (4.5).

We have already seen in Section 2 one example of g with a stationary point in $(0,1)$, namely $g(x) = \frac{1}{2}(x-\frac{1}{2})^2$. Another example is $g(x) = x(1-x)$, again with a single stationary point at $\tilde{d}_1 = \frac{1}{2}$. We now have

$$\mu_m(\omega) \sim -\tfrac{1}{2^m}\frac{\mathrm{i}\sqrt{\pi\mathrm{i}}\mathrm{e}^{\frac{1}{4}\mathrm{i}\omega}\mathrm{erf}\,\frac{1}{2}\sqrt{\mathrm{i}\omega}}{\omega^{1/2}} - \frac{1^m - 2(\frac{1}{2})^m + 0^m}{\mathrm{i}\omega} + \mathcal{O}\left(\omega^{-3/2}\right), \quad m \geq 0,\ \omega \to \infty.$$

On the face of things, this is not immediately reconciled with (2.1). Further 'massage', replacing the error function by its asymptotic approximation from (Abramowitz & Stegun 1964, p. 298), yields

$$\mu_m(\omega) \sim -\tfrac{1}{2^m}\frac{\mathrm{i}\sqrt{\mathrm{i}\pi}\mathrm{e}^{\frac{1}{4}\mathrm{i}\omega}}{\omega^{1/2}} - \frac{1^m + 0^m}{\mathrm{i}\omega} + \mathcal{O}\left(\omega^{-3/2}\right), \qquad m \geq 0, \quad \omega \to \infty,$$

in conformity with (2.1).

A more challenging example is $g(x) = (x-\frac{1}{2})^4$, whereby $\delta = 4$. The first few moments are

$$\begin{aligned}
\mu_0(\omega) &= \frac{\pi\sqrt{2}}{2\Gamma(\frac{3}{4})(-\mathrm{i}\omega)^{1/4}} - \frac{\Gamma(\frac{1}{4}, -\frac{1}{16}\mathrm{i}\omega)}{2(-\mathrm{i}\omega)^{1/4}},\\
\mu_1(\omega) &= \frac{\pi\sqrt{2}}{4\Gamma(\frac{3}{4})(-\mathrm{i}\omega)^{1/4}} - \frac{\Gamma(\frac{1}{4}, -\frac{1}{16}\mathrm{i}\omega)}{4(-\mathrm{i}\omega)^{1/4}},\\
\mu_2(\omega) &= \frac{\pi\sqrt{2}}{8\Gamma(\frac{3}{4})(-\mathrm{i}\omega)^{1/4}} - \frac{\Gamma(\frac{1}{4}, -\frac{1}{16}\mathrm{i}\omega)}{8(-\mathrm{i}\omega)^{1/4}} + \frac{\Gamma(\frac{3}{4})}{2(-\mathrm{i}\omega)^{3/4}} - \frac{\Gamma(\frac{3}{4}, -\frac{1}{16}\mathrm{i}\omega)}{2(-\mathrm{i}\omega)^{3/4}},\\
\mu_3(\omega) &= \frac{\pi\sqrt{2}}{16\Gamma(\frac{3}{4})(-\mathrm{i}\omega)^{1/4}} - \frac{\Gamma(\frac{1}{4}, -\frac{1}{16}\mathrm{i}\omega)}{16(-\mathrm{i}\omega)^{1/4}} + \frac{3\Gamma(\frac{3}{4})}{4(-\mathrm{i}\omega)^{3/4}} - \frac{3\Gamma(\frac{3}{4}, -\frac{1}{16}\mathrm{i}\omega)}{4(-\mathrm{i}\omega)^{3/4}}
\end{aligned}$$

and so on. This can be streamlined by replacing the incomplete Gamma function by its asymptotic expansion,

$$\Gamma(a,z) \sim z^{a-1}\mathrm{e}^{-z}[1+\mathcal{O}(z^{-1})], \qquad |z|\to\infty, \quad |\arg z| \le \frac{3\pi}{2}$$

(Abramowitz & Stegun 1964, p. 263). The outcome is

$$\begin{aligned}
\mu_0(\omega) &\sim \frac{\pi\sqrt{2}}{2\Gamma(\frac{3}{4})(-\mathrm{i}\omega)^{1/4}} - \frac{4\mathrm{i}\mathrm{e}^{\frac{1}{16}\mathrm{i}\omega}}{\omega} + \mathcal{O}\left(\omega^{-5/4}\right),\\
\mu_1(\omega) &\sim \tfrac{1}{2}\frac{\pi\sqrt{2}}{2\Gamma(\frac{3}{4})(-\mathrm{i}\omega)^{1/4}} - \frac{2\mathrm{i}\mathrm{e}^{\frac{1}{16}\mathrm{i}\omega}}{\omega} + \mathcal{O}\left(\omega^{-5/4}\right),\\
\mu_2(\omega) &\sim \tfrac{1}{2^2}\frac{\pi\sqrt{2}}{2\Gamma(\frac{3}{4})(-\mathrm{i}\omega)^{1/4}} + \tfrac{1}{2}\frac{\Gamma(\frac{3}{4})}{(-\mathrm{i}\omega)^{3/4}} - \frac{\mathrm{i}\mathrm{e}^{\frac{1}{16}\mathrm{i}\omega}}{\omega} + \mathcal{O}\left(\omega^{-5/4}\right),\\
\mu_3(\omega) &\sim \tfrac{1}{2^3}\frac{\pi\sqrt{2}}{2\Gamma(\frac{3}{4})(-\mathrm{i}\omega)^{1/4}} + \tfrac{3}{4}\frac{\Gamma(\frac{3}{4})}{(-\mathrm{i}\omega)^{3/4}} - \frac{\mathrm{i}\mathrm{e}^{\frac{1}{16}\mathrm{i}\omega}}{2\omega} + \mathcal{O}\left(\omega^{-5/4}\right).
\end{aligned}$$

Two observations are in order. Firstly, since δ is even, the $\mathcal{O}(\omega^{-1/2})$ term is nil, but this is not the case with the $\mathcal{O}(\omega^{-1})$ term, which originates in the endpoints. Secondly, although formally (2.1) is satisfied with $r=3$, it makes little sense to use endpoints as Filon quadrature nodes because the $\mathcal{O}(\omega^{-3/4})$ error term cannot be eliminated by our approach, regardless of the choice of quadrature nodes, and it dominates any savings that might accrue by incorporating the endpoints.

Our last example demonstrates that presence of stationary points *outside* $[0,1]$ does not interfere with the asymptotic expansion of the moments, hence such points can be disregarded. Thus, let $g(x)=x+x^2$, with $g'(-\frac{1}{2})=0$. In that case

$$\begin{aligned}
\mu_0(\omega) &= -\tfrac{1}{2}\frac{\sqrt{\pi}\mathrm{e}^{-\frac{1}{4}\mathrm{i}\omega}}{\sqrt{-\mathrm{i}\omega}}[\operatorname{erf}(-\tfrac{3}{2}\sqrt{-\mathrm{i}\omega}) - \operatorname{erf}(-\tfrac{1}{2}\sqrt{-\mathrm{i}\omega})],\\
\mu_1(\omega) &= \tfrac{1}{4}\frac{\sqrt{\pi}\mathrm{e}^{-\frac{1}{4}\mathrm{i}\omega}}{\sqrt{-\mathrm{i}\omega}}[\operatorname{erf}(-\tfrac{3}{2}\sqrt{-\mathrm{i}\omega}) - \operatorname{erf}(-\tfrac{1}{2}\sqrt{-\mathrm{i}\omega})] + \tfrac{1}{2}\frac{\mathrm{e}^{2\mathrm{i}\omega}-1}{\mathrm{i}\omega},\\
\mu_2(\omega) &= -\tfrac{1}{8}\frac{\sqrt{\pi}\mathrm{e}^{-\frac{1}{4}\mathrm{i}\omega}}{\sqrt{-\mathrm{i}\omega}}[\operatorname{erf}(-\tfrac{3}{2}\sqrt{-\mathrm{i}\omega}) - \operatorname{erf}(-\tfrac{1}{2}\sqrt{-\mathrm{i}\omega})] + \tfrac{1}{4}\frac{\mathrm{e}^{2\mathrm{i}\omega}+1}{\mathrm{i}\omega}\\
&\quad + \tfrac{1}{4}\frac{\sqrt{\pi}\mathrm{e}^{-\frac{1}{4}\mathrm{i}\omega}}{\mathrm{i}\omega\sqrt{-\mathrm{i}\omega}}[\operatorname{erf}(-\tfrac{3}{2}\sqrt{-\mathrm{i}\omega}) - \operatorname{erf}(-\tfrac{1}{2}\sqrt{-\mathrm{i}\omega})]
\end{aligned}$$

and so on. Seemingly, the leading term decays as $(-\frac{1}{2})^m$ and, somehow, a stationary point outside the interval leaves an enduring imprint on the moments. This is illusory since, once the error function is replaced by an asymptotic expansion, the 'rogue' terms disappear and it follows that

$$\mu_m(\omega) \sim \frac{\frac{1}{3}\mathrm{e}^{2\mathrm{i}\omega}1^m - 0^m}{\mathrm{i}\omega} + \mathcal{O}\left(\omega^{-2}\right), \qquad m\ge 0,$$

consistently with (3.2).

5 Further comments

Concluding this paper, we address in this section four issues that arise naturally from our work.

5.1 More general functions g

What is the scope of Theorem 6? An assumption open to an easy challenge is the smoothness of g. Clearly, we need to exclude 'flat' functions g, with $g' = 0$ on a set of positive measure, since then $\mu_m(\omega) = \mathcal{O}(1)$. Yet, some non-smooth functions are apparently consistent with the geometric model (2.1) and even with Theorem 6. For example, the *chapeau function* $g(x) = \frac{1}{2}[1 - |2x - 1|]$ yields

$$\mu_m(\omega) \sim \frac{1}{\mathrm{i}\omega}(-0^m + \tfrac{1}{2^{m-1}}\mathrm{e}^{\frac{1}{2}\mathrm{i}\omega} - 1^m) + \mathcal{O}\big(\omega^{-2}\big), \qquad m \geq 0, \quad \omega \to \infty,$$

while

$$g(x) = \begin{cases} -x, & x \in [0, \frac{1}{2}), \\ 1 - x, & x \in [\frac{1}{2}, 1], \end{cases}$$

which is not even continuous, results in

$$\mu_m(\omega) \sim \frac{1}{\mathrm{i}\omega}(0^m + \tfrac{1}{2^{m-1}}\mathrm{i}\sin\tfrac{\omega}{2} - 1^m) + \mathcal{O}\big(\omega^{-2}\big), \qquad m \geq 0, \quad \omega \to \infty.$$

There is at present no general theory to cater for nonsmooth functions g but the following argument goes a long way towards a resolution of this issue. Any piecewise-smooth function can be approximated arbitrarily close (in an L_∞ sense) by a *comonotone* algebraic polynomial that shares its stationary points (Beatson & Leviatan 1983). In other words, there exists a sequence $\{g_j\}_{j\geq 0}$ of polynomials with the same stationary points as g and such that $g_j \to g$ uniformly in $[0, 1]$. In that case Theorem 6 can be extended by a standard limiting argument.

A more substantive restriction is that no stationary point may occur at an endpoint. This is an essential requirement since, once g' vanishes at an endpoint, the geometric model (2.1) need not be valid. The simplest example is $g(x) = \frac{1}{2}x^2$, whence

$$\mu_0(\omega) = \frac{(1+\mathrm{i})\sqrt{\pi}}{2}\frac{\mathrm{erf}(\frac{1}{2}(1-\mathrm{i})\omega^{1/2})}{\omega^{1/2}} \sim -\frac{(1+\mathrm{i})\sqrt{\pi}}{2\omega^{1/2}} + \frac{\mathrm{e}^{\frac{1}{2}\mathrm{i}\omega}}{\mathrm{i}\omega} + \mathcal{O}\Big(\omega^{-3/2}\Big), \quad \omega \to \infty,$$

$$\mu_1(\omega) = \frac{\mathrm{e}^{\frac{1}{2}\mathrm{i}\omega} - 1}{\mathrm{i}\omega},$$

$$\mu_m(\omega) \sim \frac{\mathrm{e}^{\frac{1}{2}\mathrm{i}\omega}}{\mathrm{i}\omega} + \mathcal{O}\big(\omega^{-2}\big), \qquad m \geq 2, \quad \omega \to \infty.$$

Clearly, geometric progression is valid only subject to the minimalist interpretation of $r = 1$, $d_1 = 0$ and $\mu_m(\omega) \sim -(1+\mathrm{i})\sqrt{\pi}0^m/(2\omega^{1/2}) + \mathcal{O}\big(\omega^{-1}\big)$. No choice of nodes in Filon's quadrature (1.3) annihilates the $\mathcal{O}\big(\omega^{-1}\big)$ component in the error term.

5.2 The behaviour for moderate $\omega > 0$

Although our concern in this paper is with the computation of integrals (1.2) with large values of ω, it is legitimate to investigate the behaviour in the entire range of frequencies ω. (Without loss of generality we assume that $\omega > 0$.) A natural dichotomy is "Filon's quadrature for high oscillation, Gauss–Legendre quadrature otherwise", but it has been already demonstrated in (Iserles 2004) that this course of action is naive. Indeed, if the nodes $c_1, c_2, \ldots, c_\nu$ lead to an order-p Gauss–Legendre quadrature (a phenomenon which is closely related to orthogonality properties of the collocation polynomial (Davis & Rabinowitz 1984)), it has been proved in (Iserles 2004) that the order of Filon's quadrature for (1.1) (that is, for $g(x) = x$) is also p. Order has to do with exact reproduction of polynomial functions f or, alternatively, with the introduction of a small parameter h and, in place of (1.1), the integration of

$$\frac{1}{h}\int_0^h f(x)\mathrm{e}^{\mathrm{i}\omega x}\mathrm{d}x = \int_0^1 f(hx)\mathrm{e}^{\mathrm{i}h\omega x}\mathrm{d}x.$$

In this case, order p of interpolatory and Filon quadratures means that the error of *both* methods for fixed ω and $h \to 0$ is $\mathcal{O}(h^p)$.

It thus makes sense to investigate in this context the case of a more general function g. One motivation for this course of action is that, once $\nu > r$, we need a plausible criterion to select the remaining $\nu - r$ quadrature points that are neither endpoints nor stationary points of g. A strategy that maximises the order for small $\omega > 0$ makes a great deal of sense in this context. Yet, the results of (Iserles 2004) do not translate intact to the realm of irregular oscillators. Our current understanding of this matter is at best incomplete. Rather than addressing the issue in its totality, we just present an example that, at the very least, implies that the general picture is interesting. A judicious choice of nodes may increase 'classical' order in a non-oscillatory regime, while possibly falling short of the order of the interpolatory quadrature with the same nodes.

We revisit the case $g(x) = x(1-x)$, which we have already discussed in a different context in Section 4. Since $r = 3$, let us consider first $\nu = 3$ with quadrature nodes $c_1 = 0$, $c_2 = \frac{1}{2}$, $c_3 = 1$ determined from the asymptotic considerations of Theorem 6. It is trivial to verify that the order of the interpolatory quadrature with the above nodes is $p = 4$. In other words, such a method integrates exactly all cubic polynomials when $\omega = 0$. For the record, the weights of the Filon quadrature are

$$\begin{aligned}
b_1(\omega) &= -\frac{1}{\mathrm{i}\omega} + \frac{\sqrt{\pi}\mathrm{e}^{\frac{1}{4}\mathrm{i}\omega}\mathrm{erf}\frac{1}{2\sqrt{\mathrm{i}\omega}}}{(\mathrm{i}\omega)^{3/2}},\\
b_2(\omega) &= \frac{\sqrt{\pi}\mathrm{e}^{\frac{1}{4}\mathrm{i}\omega}\mathrm{erf}\frac{1}{2\sqrt{\mathrm{i}\omega}}}{\sqrt{\mathrm{i}\omega}} + \frac{2}{\mathrm{i}\omega} - \frac{2\sqrt{\pi}\mathrm{e}^{\frac{1}{4}\mathrm{i}\omega}\mathrm{erf}\frac{1}{2\sqrt{\mathrm{i}\omega}}}{(\mathrm{i}\omega)^{3/2}},\\
b_3(\omega) &= -\frac{1}{\mathrm{i}\omega} + \frac{\sqrt{\pi}\mathrm{e}^{\frac{1}{4}\mathrm{i}\omega}\mathrm{erf}\frac{1}{2\sqrt{\mathrm{i}\omega}}}{(\mathrm{i}\omega)^{3/2}}.
\end{aligned}$$

By direct calculation, $Q^{\mathrm{F}}[x^m] = I_{x(1-x)}[x^m]$, $m = 0, 1, 2, 3$, and *also the Filon quadrature is of order 4.*

Emboldened by this, we consider next the case $\nu = 5$, with two extra nodes, $\frac{1}{2} \pm \frac{\sqrt{21}}{14}$, chosen so that the underlying interpolatory scheme is of order 8. Filon weights are quite complicated, yet this prevents neither their calculation nor manipulation with symbolic software: the outcome is that the order of the Filon quadrature is just 6: one more than ν but two less than the order of the interpolatory quadrature. The general answer is, thus, more complicated than for $g(x) = x$ and by this stage we refrain from even conjecturing what it might be.

The situation is somewhat different when $\omega > 0$ is neither very small nor very large. In that case we have $\omega = \mathcal{O}(1)$ and it has been shown in (Iserles 2004) that for the Fourier oscillator $g(x) = x$ interpolatory quadrature is of absolutely no use: matters are unlikely to be better for more complicated oscillators. (An alternative argument is that, like in approximation of the Fourier transform by discrete Fourier transform, interpolatory quadrature requires $\mathcal{O}(\omega^{-1})$ points to prevent aliasing of different frequencies.) Filon's quadrature, on the other hand, is substantially better but not as good as for large ω: again, this has been investigated in depth only for $g(x) = x$, but numerical experiments confirm that this phenomenon is of wider validity. It might come as a surprise that moderate oscillation is less tractable, at least by existing methods, than rapid oscillation but it makes mathematical sense. For small $\omega > 0$ the behaviour is modelled well by a Taylor expansion, hence the efficacy of Gaussian quadrature. For $\omega \gg 1$ the Taylor theorem is of little use but we can employ an asymptotics expansion instead. It is the intermediate regime, when neither a Taylor nor an asymptotic expansion are adequate, that represents an enduring challenge.

5.3 An asymptotic quadrature scheme

The results of this paper suggest the following approximation to $I_g[f]$, an alternative to Filon's method, which we call *the asymptotic method.* Suppose that the function g obeys the geometric model (2.1). Then, given any analytic function $f(x) = \sum_{m=0}^{\infty} f_m x^m$, simple calculation affirms that

$$I_g[f] = \sum_{l=1}^{r} v_l(\omega) f(d_l) \omega^{-\alpha_l} + \mathcal{O}(\omega^{-\beta}), \qquad \omega \to \infty. \tag{5.1}$$

Assuming that the α_ls and v_ls are known, we can truncate the right-hand side of (5.1) to produce the quadrature

$$Q^{\mathrm{A}}[f] = \sum_{l=1}^{r} v_l(\omega) f(d_l) \omega^{-\alpha_l}. \tag{5.2}$$

Note that, by design,

$$Q^{\mathrm{A}}[f] - I_g[f] = \mathcal{O}(\omega^{-\beta}), \qquad \omega \to \infty,$$

therefore the new method shares the advantageous asymptotic behaviour of the Filon quadrature (1.3). It is important to bear in mind that (5.2) is different from Filon's method. As the simplest example we consider $g(x) = x$, $\nu = 2$, whereby $r = 2$, $d_1 = 0$, $d_2 = 1$ and

$$\begin{aligned} Q^{\mathrm{A}}[f] &= -\frac{1}{\mathrm{i}\omega} f(0) + \frac{\mathrm{e}^{\mathrm{i}\omega}}{\mathrm{i}\omega} f(1), \\ Q^{\mathrm{F}}[f] &= \left(-\frac{1}{\mathrm{i}\omega} - \frac{\mathrm{e}^{\mathrm{i}\omega} - 1}{\omega^2} \right) f(0) + \left(\frac{\mathrm{e}^{\mathrm{i}\omega}}{\mathrm{i}\omega} + \frac{\mathrm{e}^{\mathrm{i}\omega} - 1}{\omega^2} \right) f(1). \end{aligned}$$

Other things being equal, Filon's quadrature has the edge over the asymptotic method for a number of reasons. Firstly, it produces high-quality approximation also for low frequencies $\omega > 0$, a phenomenon which is completely understood for $g(x) = x$ and has been briefly addressed in the previous subsection: note that, even if the exact order is unknown for general g, it is clear from construction that it always is at least ν. The asymptotic scheme, however,

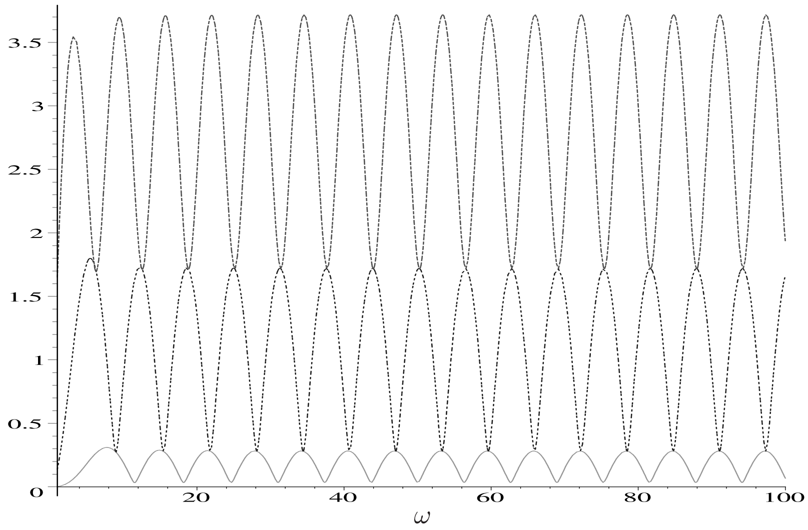

Figure 2: The absolute value of the error in Filon quadrature of $\int_0^1 \mathrm{e}^x \mathrm{e}^{\frac{1}{2}\mathrm{i}\omega x}\mathrm{d}x$, multiplied by ω, for three schemes: Q^{A} (the upper, dash line), Q^{F} with $\nu = 2$ (the middle, dash-dot line) and Q^{F} with $\nu = 3$ and $\boldsymbol{c} = [0, \frac{1}{2}, 1]$ (the bottom, solid line) and different values of ω.

having put all its money on $\omega \gg 1$, is completely useless for low frequencies and its coefficients blow up. Secondly, as indicated by Figure 2, Filon's method is likely to have smaller error constant. A third advantage is also apparent from Figure 2: while the asymptotic method (5.2) is restricted to just r points, the Filon quadrature can be implemented with any $\nu \geq r$, while sharing optimal asymptotic rate of decay. Adding further points is likely to decrease the error, as well as enhancing performance for low frequencies. Finally, the design of Filon methods is easier: we need just to evaluate the moments and use Theorem 6 to identify the d_ls. Insofar as the asymptotic quadrature is concerned, we also require the $v_l(\omega)$s and the α_ls. In principle, this information can be derived from the moments $\mu_0, \mu_1, \ldots, \mu_{r-1}$ but in practice, as we have seen in a number of detailed examples above, this procedure requires further asymptotic analysis.

So, what is the point in the method (5.2) and why mention it at all? It has one clear advantage, simplicity. More intriguing is that it can be often used when the moments are not available explicitly, since the functions v_l are often known from the discussion following Theorem 6. Consider for example the function $g(x) = x \log x$. It is possible, representing the exponential as a Taylor series in x and employing repeated integration by parts, to expand its moments in a Taylor series about the origin,

$$\mu_m(\omega) = \sum_{k=0}^{\infty} \frac{(-\mathrm{i}\omega)^k}{(m+k+1)^{k+1}}, \qquad m \geq 0,$$

but this does not help (at least insofar as the present author is concerned) in elucidating their asymptotic expansion as $\omega \to \infty$. The plot of the real and imaginary parts of μ_0 is displayed

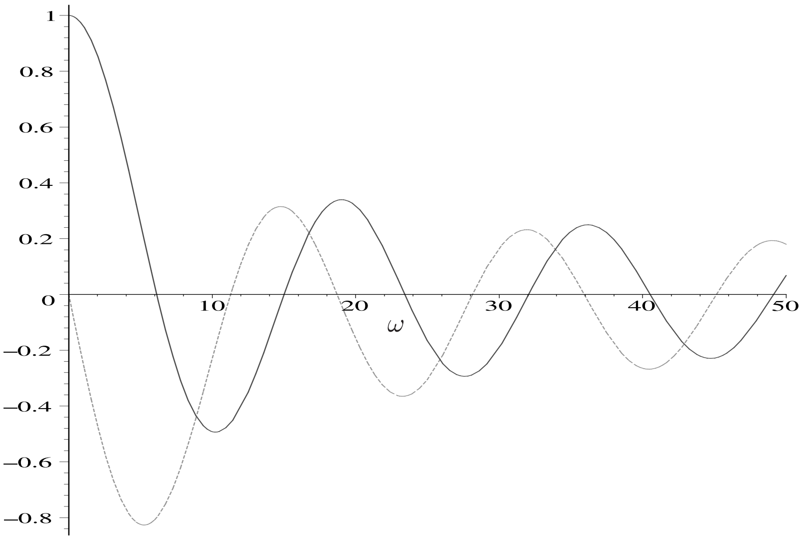

Figure 3: The zeroth moment corresponding to $g(x) = x \log x$: the real part as a solid line and the imaginary part dashed.

in Figure 3: evidently, this is a 'nice' function, possibly related to one of the standard special functions of mathematical analysis.

We might be ignorant of an asymptotic *expansion,* but we can deduce asymptotic *behaviour* from the theory of this paper and this suffices to construct the quadrature Q^{A}. Thus, $g'(\mathrm{e}^{-1}) = 0$, $g''(\mathrm{e}^{-1}) \neq 0$ and $g' \neq 0$ elsewhere in $[0, 1]$. Moreover, since $g'(0)$ is unbounded, a limiting argument shows that $v_1 \equiv 0$. Therefore the quadrature

$$Q^{\mathrm{A}}[f] = \mathrm{e}^{-\mathrm{i}\omega \mathrm{e}^{-1}} \sqrt{-\frac{2\pi}{\mathrm{i}\mathrm{e}\omega}} f(\mathrm{e}^{-1}) + \frac{1}{\mathrm{i}\omega} f(1)$$

carries an asymptotic error of $\mathcal{O}(\omega^{-3/2})$. This is confirmed by Figure 4. Note that in that figure (and also in Figure 3) we have approximated the moments using Taylor series with 200 terms and 40 significant digits. Using just the two terms in Q^{A} and standard IEEE floating-point precision is somewhat easier. . . .

Another interesting example of this kind is $g(x) = \sin 2\pi x$. Although it is possible to use standard theory to prove that $\mu_0(\omega) = \mathrm{J}_0(\omega)$, a Bessel function, the explicit form of μ_m for $m \geq 1$ is apparently unknown. Nonetheless, we can construct Q^{A} by identifying the two stationary points at $\frac{1}{4}$ and $\frac{3}{4}$, both with $\delta = 2$, and using the theory of Section 4:

$$Q^{\mathrm{A}}[f] = \left[\frac{\mathrm{e}^{\mathrm{i}\omega}}{\sqrt{2\pi\mathrm{i}}} f(\tfrac{1}{4}) + \frac{\mathrm{e}^{-\mathrm{i}\omega}}{\sqrt{-2\pi\mathrm{i}}} f(\tfrac{3}{4}) \right] \omega^{-\frac{1}{2}} + \frac{1}{2\pi\mathrm{i}\omega} [f(1) - f(0)],$$

with an error of $\mathcal{O}(\omega^{-3/2})$.

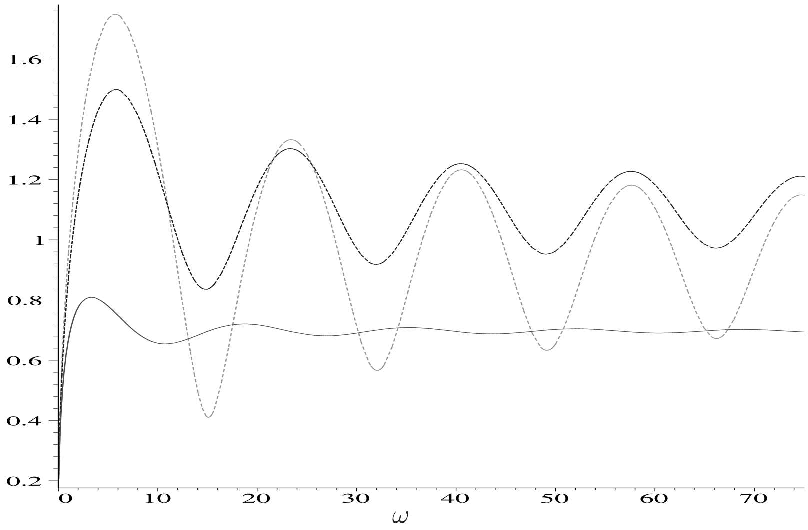

Figure 4: The quantities $\omega^{-3/2}|Q^{\mathrm{A}}[x^m] - \mu_m(\omega)|$ for $m = 1, 2, 3$ (solid, dashed and dotted lines, respectively), for $g(x) = x \log x$.

5.4 Calculation of the moments

As long as ν is small and the function g sufficiently simple, the moments can be typically expressed explicitly in terms of familiar transcendental functions and this can be done conveniently using a symbolic algebra package. This is the case, for example, when g is a cubic polynomial. However, the implementation of Filon methods for general g and ν requires numerical computation of moments. For large ν, however, this computation is typically ill conditioned and it is imperative to use modified moments instead (Gautschi 1996). Moreover, the issue of using *approximate* moments and its impact on the performance of Filon's quadrature has not been investigated yet, the example of last subsection notwithstanding, hence caution is the byword.

5.5 Generalised Fourier oscillators

Letting σ be a complex-valued function such that $\sigma(t), t^{-1}\sigma(t) \in \mathrm{L}_1(-\infty, \infty)$, we set

$$S(y) = \mathcal{F}[\sigma](y) = \int_{-\infty}^{\infty} \sigma(t) \mathrm{e}^{\mathrm{i}yt} \mathrm{d}t,$$

the *Fourier transform* of σ. Given a real function g in $[0, 1]$, we are interested in the quadrature of the integral

$$\mathcal{I}_{\sigma,g}[f] = \int_0^1 f(x) S(\omega g(x)) \mathrm{d}x \tag{5.3}$$

where, usually, $\omega \gg 1$. Since we may exchange the order of integration,

$$\mathcal{I}_{\sigma,g}[f] = \int_0^1 f(x) \int_{-\infty}^{\infty} \sigma(t) \mathrm{e}^{\mathrm{i}\omega t g(x)} \mathrm{d}t \mathrm{d}x = \int_{-\infty}^{\infty} \sigma(t) I_g[f](t\omega) \mathrm{d}t.$$

Suppose, though, that the moments of I_g are consistent with the geometric model (2.1) and that $0 < \alpha_1, \alpha_2, \ldots, \alpha_r \le 1$: the latter condition is implicit within the framework of Theorem 6. Recalling from the definition of the geometric model that $v_j \in \mathrm{L}_\infty[0,\infty)$, $j = 1, 2, \ldots, r$, in the present context we require $v_j \in \mathrm{L}_\infty(-\infty,\infty)$, a very minor restriction of generality. Then

$$\begin{aligned} & u_j(\omega) = \int_{-\infty}^{\infty} \sigma(t) v_j(\omega t) t^{-\alpha_j} \mathrm{d}t \\ \Rightarrow \qquad & \|u_j\|_{\mathrm{L}_\infty[0,\infty)} \le \|v_j\|_{\mathrm{L}_\infty(-\infty,\infty)} \int_{-\infty}^{\infty} \frac{|\sigma(t)|}{|t|} \mathrm{d}t < \infty, \end{aligned}$$

consequently

$$\mu_m(\omega; \mathcal{I}_{\sigma,g}) \sim \sum_{j=1}^{r} u_j(\omega) d_j^m \omega^{-\alpha_j} + \mathcal{O}\big(\omega^{-\beta}\big), \qquad m \ge 0, \quad \omega \to \infty,$$

where $\mu_m(\omega; \mathcal{I}_{\sigma,g})$ is the mth moment of the linear functional (5.3). We thus recover the geometric model for $\mathcal{I}_{\sigma,g}$, as long as it is valid for I_g. In other words, provided that we approximate (5.3) with the Filon method (1.3), where the weights are chosen so that $Q^{\mathrm{F}}[x^m] = \mu_m(\omega; \mathcal{I}_{\sigma,g})$, $m = 0, 1, \ldots, \nu - 1$, and assuming further that $\nu \ge r$ and $d_1, d_2, \ldots, d_r$ are all quadrature nodes, it is true that

$$Q^{\mathrm{F}}[f] = \mathcal{I}_{\sigma,g}[f] + \mathcal{O}\big(\omega^{-\beta}\big), \qquad \omega \to \infty$$

for every smooth function f.

Interesting examples of functions σ which lend themselves to this approach are

$$\begin{aligned}
\sigma(t) &= \frac{2t}{\mathrm{i}\sqrt{\pi}} \mathrm{e}^{-t^2} && \Rightarrow && S(y) = y \mathrm{e}^{-\frac{1}{4}y^2}, \\
\sigma(t) &= \frac{4t}{\sqrt{\pi(1+\mathrm{i})}} \mathrm{e}^{-(1+\mathrm{i})t^2} && \Rightarrow && S(y) = y \mathrm{e}^{-\frac{1}{8}(1+\mathrm{i})y^2}, \\
\sigma(t) &= \frac{4t}{\sqrt{\pi}} \sin t \mathrm{e}^{-t^2} && \Rightarrow && S(y) = (1-y)\mathrm{e}^{-\frac{1}{2}(y-1)^2} + (1+y)\mathrm{e}^{-\frac{1}{2}(y+1)^2}, \\
\sigma(t) &= \frac{4t}{\mathrm{i}\sqrt{\pi}} \cos t \mathrm{e}^{-t^2} && \Rightarrow && S(y) = (y-1)\mathrm{e}^{-\frac{1}{2}(y-1)^2} + (y+1)\mathrm{e}^{-\frac{1}{2}(y+1)^2}, \\
\sigma(t) &= \frac{1}{4\mathrm{i}} t \mathrm{e}^{-|t|} && \Rightarrow && S(y) = \frac{y}{|y^2 - 1 + 2\mathrm{i}y|^2}, \\
\sigma(t) &= \frac{\mathrm{i}}{48} t(t^2 - 12)\mathrm{e}^{-|t|} && \Rightarrow && S(y) = \frac{y^3(3+y^2)}{|(1 - 6y^2 + y^4) - 4\mathrm{i}y(1-y)|^2}, \\
\sigma(t) &= \frac{1}{4\mathrm{i}} \mathrm{e}^{-|t|} \sin t && \Rightarrow && S(y) = \frac{y}{|y^2 - 2 + 2\mathrm{i}y|^2}.
\end{aligned}$$

It is evident from the above that, unlike the kernel $\mathrm{e}^{\mathrm{i}\omega g(x)}$, $S(\omega g(x))$ need not oscillate. The common structural denominator to the kernels considered in this paper is, indeed, the presence of a large parameter, rather than high oscillation.

The benefits of the generalisation from I_g to $\mathcal{I}_{\sigma,g}$ are presently unclear and the author cannot point out to any existing applications. Yet, since it can be accomplished with so little extra effort, it makes sense to include it in this paper.

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