On the best conditioned bases of quadratic polynomials

Simon Foucart

University of Cambridge, Department of Applied Mathematics and Theoretical Physics, Wilberforce Road, Cambridge, CB3 0WA, UK

Abstract

It is known that, for $\mathcal{P}_n$, the subspace of $\mathcal{C}([-1,1])$ of all polynomials of degree at most n, the least basis condition number $\kappa_\infty(\mathcal{P}_n)$ (also called the Banach-Mazur distance between $\mathcal{P}_n$ and ℓ_∞^{n+1}), is bounded from below by the projection constant of $\mathcal{P}_n$ in $\mathcal{C}([-1,1])$, while the condition number of the Lagrange basis coincides with the norm of the corresponding interpolating projection. Consequently, the Lagrange basis condition number (say, at the Chebyshev extrema), growing like $\mathcal{O}(\ln n)$, is of optimal order, and for $n=2$,

$$1.2201\ldots \leq \kappa_\infty(\mathcal{P}_2) \leq 1.25.$$

In this paper we prove that there is a basis $\underline{u}$ of $\mathcal{P}_2$ such that

$$\kappa_\infty(\underline{u}) \approx 1.24839.$$

This result means that no Lagrange basis of $\mathcal{P}_2$ is best conditioned. It also seems likely that the previous value is actually the least basis condition number of $\mathcal{P}_2$, which therefore would not equal the projection constant of $\mathcal{P}_2$ in $\mathcal{C}([-1,1])$.

As for trigonometric polynomials of degree at most 1, we present numerical evidence that the Lagrange bases at equidistant points are best conditioned.

Key words: Banach-Mazur distance, condition number, projection constant

Email address: sf259@cam.ac.uk (Simon Foucart).

1 Preliminaries

Let $\underline{u} := (u_1, \dots, u_n)$ be a basis of a finite-dimensional subspace U of a Banach space $(X, \|\bullet\|)$. The ℓ_∞-condition number of $\underline{u}$ is by definition:

$$\kappa_\infty(\underline{u}) := \sup_{a \in \ell_\infty^n \setminus \{0\}} \frac{\|\sum_{i=1}^n a_i u_i\|}{\|a\|_\infty} \times \sup_{a \in \ell_\infty^n \setminus \{0\}} \frac{\|a\|_\infty}{\|\sum_{i=1}^n a_i u_i\|} =: s_1(\underline{u}) \times s_2(\underline{u})\,. \quad (1)$$

In approximation theory, some efforts were put into evaluating the condition number of certain bases. In particular, Gautschi considered the power basis [6] and orthogonal bases [5] of $\mathcal{P}_n$, the space of all algebraic polynomials of degree at most n, and de Boor initiated the studies of the B-spline basis condition number, which attracted a great deal of attention (see [8,9] and references therein).

The condition number of a basis $\underline{u}$ of U is pertinent not only because it characterizes, to a certain extent, the stability of numerical computations with $\underline{u}$, but also because one can associate to $\underline{u}$ a projection P from X onto U satisfying

$$\|P\| \le \kappa_\infty(\underline{u}). \quad (2)$$

Indeed, giving the basis $\underline{u}$ is giving an isomorphism $T : U \to \ell_\infty^n$, and by the Hahn-Banach theorem applied componentwise, there exists a norm-preserving extension of T, say $\widetilde{T} : X \to \ell_\infty^n$. Then $P := T^{-1}\widetilde{T}$ is a projection from X onto U which satisfies $\|P\| \le \|T^{-1}\|\|T\| = \kappa_\infty(\underline{u})$.

Therefore, when searching for a projection with minimal norm, it is reasonable to examine the least of these condition numbers, that is the value

$$\kappa_\infty(U) := \inf \kappa_\infty(\underline{u}) = \inf \{\|T\|\|T^{-1}\|,\ T : U \to \ell_\infty^n \text{ isomorphism}\},$$

which is known to the Banach space geometer as the Banach-Mazur distance between U and ℓ_∞^n. Thus, for the (relative) projection constant of U in X, defined by

$$p(U, X) := \inf \{\|P\|,\ P : X \twoheadrightarrow U \text{ projection}\},$$

equation (2) implies the upper estimate

$$p(U, X) \le \kappa_\infty(U)\,.$$

We are interested in the possibility of a deeper relation between those two constants, in particular we wonder if equality can occur. With no further assumptions on the spaces, the answer is clearly no. Indeed, even an inequality of the type $\kappa_\infty(U) \le C\, p(U, X)$ is wrong in general, since for example $p(\ell_2^n, \ell_2) = 1$ while $\kappa_\infty(\ell_2^n) = \sqrt{n}$ [11, chapter 9]. Furthermore, Szarek [10] showed that a much weaker inequality $\kappa_\infty(U) \le C\, p(U)$ also fails to be true. Here $p(U)$, the

(absolute) projection constant of U is, in a sense, the supremum of the relative projection constants of U, more precisely

$$p(U) := \sup\{p(i(U), X),\ X \text{ Banach},\ i : U \to X \text{ isometric embedding}\}.$$

However, if we impose $X = \mathcal{C}$ ($\mathcal{C}$ representing $\mathcal{C}([-1,1])$ or $\mathcal{C}(\mathbb{T})$), there is the estimate, noticed by de Boor in [1, p. 19]:

$$\kappa_\infty(U) \le p_{\text{int}}(U, \mathcal{C}) \tag{3}$$

where

$$p_{\text{int}}(U, \mathcal{C}) := \inf\{\|P\|,\ P : \mathcal{C} \twoheadrightarrow U \text{ interpolating projection}\}$$

is the interpolating projection constant of U in $\mathcal{C}$. Let us outline the arguments. If $P = \sum_{i=1}^n \bullet(x_i)\ell_i$ is an interpolating projection ($\underline{\ell}$ is the Lagrange basis of U at the points $x_1, \dots, x_n \in [-1,1]$, i.e. it satisfies $\ell_i(x_j) = \delta_{i,j}$), then $\|P\| = \|\sum_{i=1}^n |l_i|\|_{[-1,1]}$. On the other hand, with s_1 and s_2 being defined in (1), we easily find $s_1(\underline{l}) = \|\sum_{i=1}^n |l_i|\|_{[-1,1]}$ and $s_2(\underline{l}) \le 1$. Hence, $\kappa_\infty(\underline{\ell}) \le \|P\|$, with equality if $1 \in U$, and we get (3) by taking the infimum over P.

So, for a finite-dimensional subspace U of $\mathcal{C}$,

$$p(U, \mathcal{C}) \le \kappa_\infty(U) \le p_{\text{int}}(U, \mathcal{C}),$$

and an inequality of the type $\kappa_\infty(U) \le C\, p(U, \mathcal{C})$ becomes true for subspaces U of $\mathcal{C}$ whose projection constants are both of the same order. The first to come to mind are the spaces $\mathcal{P}_n$, since in this case both constants are of order $\mathcal{O}(\ln n)$. More precisely (see Cheney and Light [4, chapter 3]):

$$\frac{2}{\pi^2}\ln n - \frac{1}{2} \le p(\mathcal{P}_n, \mathcal{C}) \quad \text{and} \quad p_{\text{int}}(\mathcal{P}_n, \mathcal{C}) \le \frac{2}{\pi}\ln(n+1) + 1,$$

hence the condition number of the Lagrange basis at the Chebyshev extrema is of optimal order, since it grows like $\frac{2}{\pi}\ln n$ [2].

2 Objectives

While the numerical value of the interpolating projection constant $p_{\text{int}}(\mathcal{P}_n, \mathcal{C})$ is known [2] at least up to $n = 200$, the projection constant $p(\mathcal{P}_n, \mathcal{C})$ has only been calculated [3] in the case $n = 2$. So the only available pair of exact constants is

$$1.2201 \approx p(\mathcal{P}_2, \mathcal{C}) \le \kappa_\infty(\mathcal{P}_2) \le p_{\text{int}}(\mathcal{P}_2, \mathcal{C}) = \frac{5}{4}, \tag{4}$$

where $\frac{5}{4}$ is the value of the norm of the interpolating projection at the Chebyshev extrema -1, 0 and 1, and at some of their dilatations and shifts.

We are going to prove that there is a basis $\underline{u}$ of $\mathcal{P}_2$ such that

$$\kappa_\infty(\underline{u}) \approx 1.248394563,$$

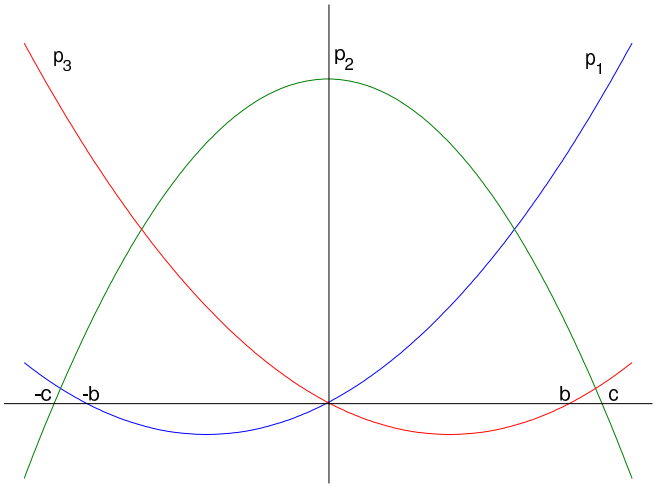
hence the second inequality of (4) is strict, i.e. no Lagrange basis of $\mathcal{P}_2$ is best conditioned. We also believe that this basis $\underline{u}$ is in fact best conditioned, so that the first inequality of (4) is strict as well.

To this end, let us denote by $\underline{p}(b,c,d)$ the following symmetric basis of $\mathcal{P}_2$:

$$p_1(x) := \frac{x(x+b)}{2d}, \quad p_2(x) := c^2 - x^2, \quad p_3(x) := \frac{x(x-b)}{2d}, \qquad x \in [-1,1],$$

where

$$b, c, d \in (0, +\infty).$$



For example, $\frac{1}{b^2}\underline{p}(b,b,1)$ is the interpolating basis at the points $-b$, 0, b.

The main part of this paper is devoted to the proof of the following statement:

Theorem 1 $\min\limits_{b,c,d>0} \kappa_\infty(\underline{p}(b,c,d)) \approx 1.248394563 < \frac{5}{4}$.

3 Determining the first supremum

It is readily seen that the first supremum in (1) is

$$s_1 := s_1(\underline{p}(b,c,d)) = \max_{\pm,\mp}(\|p_{\pm,\mp}\|_{[0,1]}), \qquad \text{where} \qquad p_{\pm,\mp} := p_1 \pm p_2 \mp p_3.$$

More precisely, we have, for $x \in [0,1]$,

$$\begin{aligned}
p_{+,+}(x) &= \left(\frac{1}{d} - 1\right) x^2 + C \ \text{ either increases or decreases with } x,\\
p_{-,+}(x) &= \left(\frac{1}{d} + 1\right) x^2 - C \ \text{ increases with } x,\\
p_{-,-}(x) &= \ x^2 + \frac{b}{d}x - C \quad \text{increases with } x,\\
p_{+,-}(x) &= -x^2 + \frac{b}{d}x + C\,.
\end{aligned}$$

We get $\|p_{+,+}\|_{[0,1]} = \max(|p_{+,+}(0)|, |p_{+,+}(1)|) = \max\left(C, \left|\frac{1}{d} - 1 + C\right|\right)$ and $\|p_{-,+}\|_{[0,1]} = \max(|p_{-,+}(0)|, |p_{-,+}(1)|) = \max\left(C, \left|\frac{1}{d} + 1 - C\right|\right)$, hence

$$\max(\|p_{+,+}\|_{[0,1]}, \|p_{-,+}\|_{[0,1]}) = \max\left(C, \frac{1}{d} + |C-1|\right). \tag{5}$$

We also have $\|p_{-,-}\|_{[0,1]} = \max(|p_{-,-}(0)|, |p_{-,-}(1)|) = \max\left(C, \left|\frac{b}{d} + 1 - C\right|\right)$. Now, if $x^* := \frac{b}{2d}$, the critical point of $p_{+,-}$, is in the interval $[0,1]$, we get $\|p_{+,-}\|_{[0,1]} = \max(|p_{+,-}(x^*)|, |p_{+,-}(1)|) = \max\left(\frac{b^2}{4d^2} + C, \left|\frac{b}{d} - 1 + C\right|\right)$, and then

$$\max(\|p_{-,-}\|_{[0,1]}, \|p_{+,-}\|_{[0,1]}) = \max\left(\frac{b}{d} + |C-1|, \frac{b^2}{4d^2} + C\right). \tag{6}$$

If otherwise $x^* = \frac{b}{2d} \geq 1$, we have

$$\max(\|p_{-,-}\|_{[0,1]}, \|p_{+,-}\|_{[0,1]}) = \max\left(C, \frac{b}{d} + |C-1|\right). \tag{7}$$

In view of (5), (6) and (7), the following proposition holds:

Proposition 2 *If $b \leq 2d$, we have*

$$s_1 = \max\left(\frac{\max(1,b)}{d} + |C-1|, \frac{b^2}{4d^2} + C\right).$$

If otherwise $b \geq 2d$, we have

$$s_1 = \max\left(\frac{\max(1,b)}{d} + |C-1|, C\right).$$

4 Determining the second supremum

For any linear map F from one finite-dimensional normed vector space U to another, there exists an extreme point v of the unit ball of U such that $\|F(v)\| = \|F\|$. By the result of Konheim and Rivlin [7] adapted to the case $n = 2$, the extreme points of the unit ball of $\mathcal{P}_2$ are the family $\{\pm 1\} \cup \{\pm q_t,\, t \in [-1,1]\}$, where

$$q_t(x) := \min\left(\frac{2}{(1+t)^2}, \frac{2}{(1-t)^2}\right)(x-t)^2 - 1, \qquad x \in [-1,1],$$

is the polynomial of norm 1 that interpolates -1 twice at $x = t$, and 1 at one of the endpoints $x = \pm 1$. Therefore, if we set

$$1 =: \sum_{i=1}^{3} h_i p_i \qquad \text{and} \qquad q_t =: \sum_{i=1}^{3} f_i(t) p_i\,,$$

the second supremum in (1) becomes

$$s_2 := s_2(\underline{p}(b,c,d)) = \max\left(\|f_1\|_{[-1,1]}, \|f_2\|_{[-1,1]}, \|f_3\|_{[-1,1]}, \|h\|_\infty\right),$$

and by symmetry

$$s_2 = \max\left(\|f_1\|_{[0,1]}, \|f_2\|_{[0,1]}, \|f_3\|_{[0,1]}, \|h\|_\infty\right). \tag{8}$$

From the expansions of 1, x and x^2 in the basis $\underline{p}(b,c,d)$, we derive that, for $t \in [0,1]$ and with $C := c^2$,

$$\begin{aligned}
h &= \frac{1}{C}(d,1,d),\\
f_1(t) &= \frac{d}{C}\left(-1 + 2\,\frac{t^2 - 2\frac{C}{b}t + C}{(1+t)^2}\right),\\
f_2(t) &= \frac{1}{C}\left(-1 + 2\,\frac{t^2}{(1+t)^2}\right),\\
f_3(t) &= \frac{d}{C}\left(-1 + 2\,\frac{t^2 + 2\frac{C}{b}t + C}{(1+t)^2}\right).
\end{aligned} \tag{9}$$

The following lemma will provide us with the exact value of s_2 in the case $b \leq c \leq 1$; in all the other cases, using only straightforward lower bounds for s_2, we will obtain $\frac{5}{4}$ as a lower bound for the condition number.

Lemma 3 *Let, for* $t \in (-1, +\infty)$,

$$u_{\pm\gamma}(t) := -1 + 2\frac{t^2 \pm 2\gamma t + \delta}{(1+t)^2}, \quad \textit{where} \quad 0 \leq \delta \leq 1, \quad \gamma \geq 0, \quad \textit{and} \quad \gamma^2 \geq \delta.$$

Then

$$\max\left(1, \|u_\gamma\|_{[0,1]}, \|u_{-\gamma}\|_{[0,1]}\right) = \frac{(1+\gamma)^2 + (\gamma^2 - \delta)}{(1+\gamma)^2 - (\gamma^2 - \delta)}.$$

Proof. Let us notice that

$$u_{\pm\gamma}(0) = -1 + 2\delta \in [-1, 1] \quad \text{and} \quad \lim_{t \to +\infty} u_{\pm\gamma}(t) = 1.$$

If $\gamma = 1$, then $u_\gamma(t) = 1 - \frac{2(1-\delta)}{(1+t)^2}$ increases with t, so that $\|u_\gamma\|_{[0,1]} \leq 1$.

If otherwise $\gamma \neq 1$, we calculate

$$u'_\gamma(t) = \frac{4(1-\gamma)}{(1+t)^3}(t - t^*_+), \quad \text{where} \quad t^*_+ := \frac{\delta - \gamma}{1 - \gamma},$$
$$u'_{-\gamma}(t) = \frac{4(1+\gamma)}{(1+t)^3}(t - t^*_-), \quad \text{where} \quad t^*_- := \frac{\delta + \gamma}{1 + \gamma}.$$

Then u_γ is increasing on $[0, 1]$ because in the case $1 - \gamma > 0$, we have $t^*_+ \leq 0$, and in the case $1 - \gamma < 0$, we have $t^*_+ \geq 1$. Thus

$$\max\left(1, \|u_\gamma\|_{[0,1]}\right) = \max\left(1, u_\gamma(1)\right) = \max\left(1, \frac{-1 + 2\gamma + \delta}{2}\right).$$

Besides, since $\gamma \geq 0$, we have $t^*_- \in [0, 1]$, and

$$\max\left(1, \|u_{-\gamma}\|_{[0,1]}\right) = \max\left(1, -u_{-\gamma}(t^*_-)\right) = \frac{(1+\gamma)^2 + (\gamma^2 - \delta)}{(1+\gamma)^2 - (\gamma^2 - \delta)}.$$

(It is easy to determine $u_{-\gamma}(t^*_-)$ if we note that $t^{*2}_- - 2\gamma t^*_- + \delta = (1 + t^*_-)(t^*_- - \gamma)$.) Then, one can check that

$$\frac{-1 + 2\gamma + \delta}{2} \leq \frac{(1+\gamma)^2 + (\gamma^2 - \delta)}{(1+\gamma)^2 - (\gamma^2 - \delta)},$$

and this completes the proof.

In view of (8) and (9), this lemma implies the following proposition:

Proposition 4 *If $b \le c \le 1$, we have*

$$s_2 = \frac{1}{C}\max(1, d\lambda)$$

where

$$\lambda := \lambda(b, C) := \frac{(b+C)^2 + C(C-b^2)}{(b+C)^2 - C(C-b^2)}.$$

5 Minimizing the condition number

Let us first notice that, in order to get $\kappa := \kappa_\infty(p(b,c,d)) \le \frac{5}{4}$, we can always assume that $C \le 4$ and that $b \le 2d$. Indeed, if $C \ge 4$, we have $s_2 \ge f_1(0) = d\left(2 - \frac{1}{C}\right) \ge \frac{7}{4}d$, and since $s_1 \ge \frac{1}{d} + |C-1| \ge \frac{1}{d}$, we obtain $\kappa = s_1 \times s_2 \ge \frac{7}{4}$. Now if $C \le 4$ and $b \ge 2d$, from the inequalities $s_1 \ge \frac{b}{d} + C - 1 \ge 1 + C$ and $s_2 \ge \frac{1}{C}$, we obtain $\kappa \ge 1 + \frac{1}{C} \ge \frac{5}{4}$.

Let us now mention the following fact, which we will use several times: a function of the type $\frac{\alpha}{x} + \beta x$, with $\alpha, \beta > 0$, defined for $x > 0$, is decreasing until $\sqrt{\frac{\alpha}{\beta}}$, where it achieves the value $2\sqrt{\alpha\beta}$, and increasing thereafter.

1) **The case $c \le b$.**

Since we may assume that $b \le 2d$ in order to get $\kappa \le \frac{5}{4}$, we have $s_1 \ge \frac{b^2}{4d^2} + C$. Then, since $C = c^2 \le b^2$, we get

$$s_1 \ge C\left(\frac{1}{4d^2} + 1\right) \qquad \text{and} \qquad s_2 \ge \frac{1}{C}\max(1, d).$$

1a) If $d \le 1$, then

$$\kappa = s_1 \times s_2 \ge \frac{1}{4d^2} + 1 \ge \frac{1}{4} + 1 = \frac{5}{4}.$$

1b) If $d \ge 1$, then

$$\kappa = s_1 \times s_2 \ge \frac{1}{4d} + d \ge \frac{1}{4} + 1 = \frac{5}{4},$$

because $\frac{1}{4d} + d$ is an increasing function of d on $\left[\frac{1}{2}, +\infty\right)$.
Thus, in search of a condition number smaller than $\frac{5}{4}$, we must assume that $b \le c$.

2) **The case $b \le c$ and $c \ge 1$.**

In view of $f_3(1) = \frac{d}{C}\frac{-1+2\frac{C}{b}+C}{2} \geq \frac{d}{b}$, and since we may assume that $b \leq 2d$ in order to get $\kappa \leq \frac{5}{4}$, we have

$$s_1 \geq \frac{b^2}{4d^2} + C \qquad \text{and} \qquad s_2 \geq \max\left(\frac{1}{C}, \frac{d}{b}\right).$$

2a) If $d \leq \frac{b}{C}$, then

$$\kappa \geq \frac{1}{C}\frac{b^2}{4d^2} + 1 \geq \frac{C}{4} + 1 \geq \frac{5}{4}.$$

2b) If $d \geq \frac{b}{C}$, then

$$\kappa \geq \frac{b}{4}\frac{1}{d} + \frac{C}{b}d \geq \frac{C}{4} + 1 \geq \frac{5}{4},$$

because $\frac{b}{4}\frac{1}{d} + \frac{C}{b}d$ is an increasing function of d on $\left[\frac{b}{2c}, +\infty\right)$, and $\frac{b}{2c} \leq \frac{b}{C}$, since $C \leq 4$.
Therefore, only the case $b \leq c \leq 1$ can lead to a condition number smaller than $\frac{5}{4}$.

3) **The case $b \leq c \leq 1$.**

First of all, let us notice that $b \leq \frac{2c}{\lambda}$. Indeed, this is equivalent to

$$b \leq \frac{2c(b^2 + 2bC + b^2C)}{b^2 + 2bC + 2C^2 - b^2C}, \qquad \text{or} \qquad b^2 + 2bc^2 + 2c^4 - b^2c^2 \leq 2bc + 4c^3 + 2bc^3,$$

which is true because $b^2 \leq 2bc$ and $2bc^2 + 2c^4 \leq 4c^3$.
In particular, $\frac{b}{2} \leq \frac{1}{\lambda}$, and from propositions 4 and 2, we need to consider three cases for the expression of κ.
3a) For $d \in \left(0, \frac{b}{2}\right]$, since $b \geq 2d$ and $d\lambda \leq 1$,

$$\kappa = \frac{1}{C}\max\left(\frac{1}{d} + 1 - C, C\right)$$

is a decreasing function of d.
3b) For $d \in \left[\frac{b}{2}, \frac{1}{\lambda}\right]$, since $b \leq 2d$ and $d\lambda \leq 1$,

$$\kappa = \frac{1}{C}\max\left(\frac{1}{d} + 1 - C, \frac{b^2}{4d^2} + C\right)$$

is a decreasing function of d.
3c) For $d \in \left[\frac{1}{\lambda}, +\infty\right)$, since $b \leq 2d$ and $d\lambda \geq 1$,

$$\kappa = \frac{1}{C}\max\left(\lambda + (1-C)d\lambda, \frac{b^2\lambda}{4d} + C\lambda d\right)$$

is an increasing function of d. It is indeed obvious that the first function under the maximum is increasing, and the fact that $\frac{1}{\lambda} \geq \frac{b}{2c}$ implies that the second one is increasing too.

Hence, for fixed b and c, the optimal choice is $d = \frac{1}{\lambda}$, and we now need to minimize, under the conditions $0 < b \leq c \leq 1$, $C = c^2$,

$$\kappa_\infty \left(\underline{p} \left(b, c, \frac{1}{\lambda} \right) \right) = \max(F(b,C), G(b,C))$$

where

$$F(b,C) := \frac{\lambda + 1}{C} - 1 \quad \text{and} \quad G(b,C) := \frac{b^2\lambda^2}{4C} + 1.$$

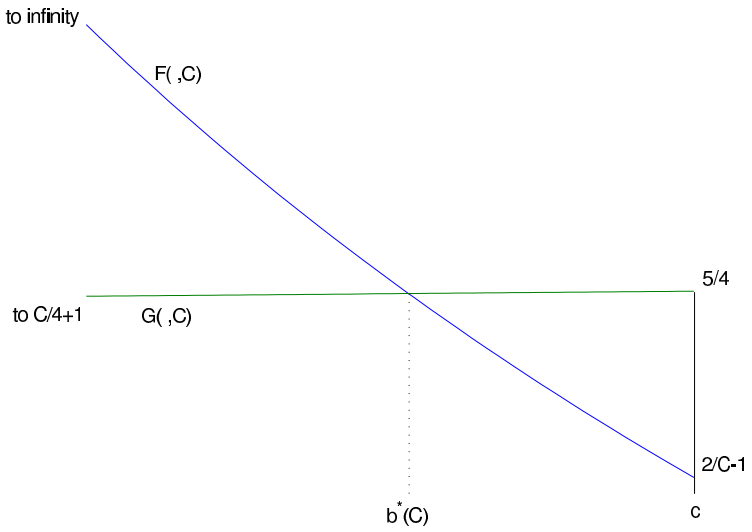
One calculates

$$(b^2 + 2bC + b^2C)^2 \frac{\partial \lambda}{\partial b} = -4C^2(b + C)(b + 1) \leq 0,$$

so that λ, and therefore F, decreases with b. One also gets

$$(b + 2C + bC)^2 \frac{\partial (b\lambda)}{\partial b} = (1 - C)(b(b + bC + 4C) + 2C^2) \geq 0,$$

so that $b\lambda$, and therefore G, increases with b.



Let us remark that $\lambda(c,C) = 1$, thus $F(b,C) \geq F(c,C) = \frac{2}{C} - 1$, so that, in order to get $\kappa \leq \frac{5}{4}$, we may assume $C \geq \frac{8}{9}$. Then $F(c,C) \leq \frac{5}{4} = G(c,C)$, and, on the other hand, since $b\lambda(b,C) \underset{b \downarrow 0}{\longrightarrow} C$, we have $G(0,C) = \frac{C}{4} + 1 < \lim_{b \downarrow 0} F(b,C) = +\infty$. Therefore, for each $C \in \left[\frac{8}{9}, 1\right]$, there exists a unique $b^*(C) \in [0,c]$ such that $F(b^*(C),C) = G(b^*(C),C) =: H(C)$, and we have

$$\kappa \geq \kappa_\infty \left(\underline{p} \left(b^*(C), c, \frac{1}{\lambda(b^*(C),C)} \right) \right) = H(C).$$

Finally, for any $C \in \left(\frac{8}{9}, 1\right)$, we have $H(C) = G(b^*(C), C) < G(c, C) = \frac{5}{4}$, which already proves that

$$\min_{b,c,d>0} \kappa_\infty(\underline{p}(b,c,d)) < \frac{5}{4},$$

so that

$$\kappa_\infty(\mathcal{P}_2) < p_{\text{int}}(\mathcal{P}_2, \mathcal{C}).$$

6 Numerical value

The minimal value of $\kappa_\infty(\underline{p}(b,c,d))$, i.e. the minimal value of H on $\left[\frac{8}{9}, 1\right]$, can be found numerically. Let $C^* \in \left[\frac{8}{9}, 1\right]$ be such that $H(C^*) = \min_{\frac{8}{9} \leq C \leq 1} H(C)$, it must satisfies $H'(C^*) = 0$, because C^* is neither $\frac{8}{9}$ nor 1. Now, differentiating the relation $F(b^*(C), C) = G(b^*(C), C)$ with respect to C, we get, for all $C \in \left[\frac{8}{9}, 1\right]$,

$$b^{*\prime}(C) \times \left[\frac{\partial F}{\partial b} - \frac{\partial G}{\partial b}\right]\Big(b^*(C), C\Big) + \left[\frac{\partial F}{\partial C} - \frac{\partial G}{\partial C}\right]\Big(b^*(C), C\Big) = 0.$$

On the other hand, we have, for all $C \in \left[\frac{8}{9}, 1\right]$,

$$H'(C) = b^{*\prime}(C) \times \frac{\partial F}{\partial b}\Big(b^*(C), C\Big) + \frac{\partial F}{\partial C}\Big(b^*(C), C\Big).$$

Hence, annihilating the determinant of the previous system, we conclude that $H'(C) = 0$ if and only if $\left[\frac{\partial F}{\partial b}\frac{\partial G}{\partial C} - \frac{\partial F}{\partial C}\frac{\partial G}{\partial b}\right](b^*(C), C) = 0$. As a result, $C \in \left[\frac{8}{9}, 1\right]$ satisfies $H'(C) = 0$ if and only if $(b^*(C), C)$ is solution of the following (polynomial, after simplification) system:

$$\begin{cases} F(b,C) - G(b,C) &= 0, \\ \left[\frac{\partial F}{\partial b}\frac{\partial G}{\partial C} - \frac{\partial F}{\partial C}\frac{\partial G}{\partial b}\right](b,C) &= 0. \end{cases}$$

Using the Groebner package from Maple, one finds that this system is equivalent to

$$144\,C^8 + 6498\,C^7 + 25839\,C^6 - 25108\,C^5 + 9827\,C^4 \\ - 17192\,C^3 + 2336\,C^2 + 1088\,C - 192 = 0,$$

and

$$60\,b^8 - 906\,b^7 - 1452\,b^6 + 2261\,b^5 + 6451\,b^4 \\ + 568\,b^3 - 3704\,b^2 - 1408\,b - 192 = 0,$$

which, in the prescribed domain, have the unique solution $C \approx 0.9402938300$, $b \approx 0.8675381234$. Computing F for these b and C gives us the following value

$$\min_{b,c,d>0} \kappa_\infty(\underline{p}(b,c,d)) \approx 1.248394563.$$

7 Concluding remarks

1) We tried to cover the case of symmetric bases completely by introducing an additional parameter a, $|a| \leq b$, and minimizing the condition number over the bases:

$$p_1(x) := \frac{(x+a)(x+b)}{2d},\ p_2(x) := c^2 - x^2,\ p_3(x) := \frac{(x-a)(x-b)}{2d},\ x \in [-1,1].$$

The technique is the same, though the calculations become rather more intricated: whereas, when $b \leq 1$ and $c \leq 1$, we could show theoretically that the case $a \leq 0$ does not lead to any improvement, the same conclusion for $a \geq 0$ was obtained numerically, using the Matlab function fminimax. Hence if, as we believe, best conditioned bases are symmetric, it is very likely that our optimal basis is actually the best conditioned basis of $\mathcal{P}_2$, and then that the first inequality of (4) is also strict.

2) The same technique can be applied to determine the condition number of $\mathcal{T}_1$, the space of trigonometric polynomials of degree at most 1. Here, the extreme points of the unit ball of $\mathcal{T}_1$ are the family $\{\pm 1\} \cup \{\sin(\bullet - t), t \in \mathbb{T}\}$. It leads to an expression of s_1 as a maximum of 4 functions, and an expression of s_2 as a maximum of 6 functions, hence an expression of κ as a maximum of 24 functions. If we trust once again the Matlab function fminimax, we conclude, quite surpisingly, that the Lagrange bases at equidistant points are best conditioned in $\mathcal{T}_1$, with

$$\kappa_\infty(\mathcal{T}_1) = p_{\text{int}}(\mathcal{T}_1, \mathcal{C}) = \frac{5}{3}.$$

Let us note that this is not the projection constant of $\mathcal{T}_1$ in $\mathcal{C}$. Indeed, it is known that the Fourier projection from $\mathcal{C}$ onto $\mathcal{T}_n$ is minimal, and we easily derive

$$p(\mathcal{T}_1, \mathcal{C}) = \frac{1}{3} + \frac{2\sqrt{3}}{\pi} \approx 1.435991124.$$

3) Finally, the equalities $F = G$ and $d\lambda = 1$, which occur in particular in the optimal case, mean that

$$\|p_{-,+}\|_{[-1,1]} = \|p_{+,-}\|_{[-1,1]} = \|p_{-,-}\|_{[-1,1]}$$

and

$$\|h\|_\infty = \|f_2\|_{[-1,1]} = |f_2(0)| = \|f_1\|_{[-1,1]} = |f_1(t_-^*)| = \|f_3\|_{[-1,1]} = |f_3(-t_-^*)|.$$

The latter implies that the polynomials 1, $q_0 = T_2$, $q_{t_-^*}$ and $q_{-t_-^*}$ (with $t_-^* \approx 0.9713483443$) are extremal for s_2, while the former implies that $(1,-1,1)$, $(1,1,-1)$ and $(1,-1,-1)$ are extremal for s_1. Thus, for our optimal basis, there are 3 independent extreme points of the unit ball of $\mathcal{P}_2$ that are extremal for s_2 and there are 3 independent extreme points of the unit ball of ℓ_∞^3 that are extremal for s_1. The same holds for Lagrange bases of $\mathcal{T}_1$ at equidistant points, which are, according to our computations, the best ℓ_∞-conditioned, and it also holds for the canonical basis of ℓ_p^n, $2 \leq p < +\infty$, which is best ℓ_∞-conditioned. It would be interesting to know if the best conditioned bases of all finite-dimensional vector spaces always satisfy this criterion.

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