EXPONENTIAL BROWNIAN MOTION AND DIVIDED DIFFERENCES

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ABSTRACT. We calculate an analytic value for the correlation coefficient beween a geometric, or exponential, Brownian motion and its time-average, a novelty being our use of divided differences to elucidate formulae. This provides a simple approximation for the value of certain Asian options regarding them as exchange options. We also illustrate that the higher moments of the time-average can be expressed neatly as divided differences of the exponential function via the Hermite–Genocchi integral relation, as well as demonstrating that these expressions agree with those obtained by Oshanin and Yor when the drift term vanishes.

1. INTRODUCTION

We begin with geometric, or exponential, Brownian motion, defined by

$$S(t) = e^{(r-\frac{\sigma^2}{2})t+\sigma B_t}, \qquad t \geq 0, \tag{1.1}$$

where $B : [0, \infty) \to \mathbb{R}$ is Brownian motion. In other words, it is a stochastic process, or *random function*, for which $B_0 = 1$, its increments are independent, and, for $0 \leq s < t$, the increment $B_t - B_s$ is normally distributed with mean zero and variance $t - s$. The basic properties of Brownian motion are explained in Section 37 of [1], while Karatzas and Shreve have provided a comprehensive treatise [7]. We have decided to use the probabilists' notation B_t rather than the analyst's $B(t)$, and hope that this does not provoke confusion in half our audience.

We shall study the *time average*

$$A(T) := \frac{1}{T}\int_0^T S(t)\,dt \tag{1.2}$$

using the calculus of divided differences, a standard tool in approximation theory. We have provided fairly full explanations of even elementary points in the hope that this will enhance the paper's use to both the mathematical finance and approximation theoretic communities; in particular, almost of the required divided difference theory is derived in Section 4.

We first observe the familiar result

$$\mathbb{E}S(T) = e^{(r-\sigma^2/2)T}\mathbb{E}e^{\sigma T^{1/2}Z} = e^{(r-\sigma^2/2)T}e^{\sigma^2 T/2} = e^{rT}. \tag{1.3}$$

Here Z denotes a generic $N(0, 1)$ Gaussian random variable and we have used the standard fact that

$$\mathbb{E}e^{\lambda Z} = (2\pi)^{-1/2}\int_{\mathbb{R}} e^{\lambda\tau}e^{-\tau^2/2}\,d\tau = (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-\frac{1}{2}\{(\tau-\lambda)^2-\lambda^2\}}\,d\tau = e^{\lambda^2/2}. \tag{1.4}$$

Similarly,

$$\begin{aligned}\mathbb{E}A(T) &= T^{-1}\int_0^T \mathbb{E}S(t)\,dt \\ &= \frac{e^{rT}-1}{rT}.\end{aligned} \tag{1.5}$$

In *divided difference notation*, we have shown that

$$\mathbb{E}A(T) = \exp[0, rT]. \tag{1.6}$$

Divided differences are important tools in polynomial interpolation theory and spline theory and are treated in many books. In particular, most of the material used here can be found in Davis [2] and Powell [9]. We remind the reader that $f[a_0, a_1, \ldots, a_n]$ is the highest coefficient of the unique polynomial of degree n interpolating f at distinct points $a_0, \ldots, a_n \in \mathbb{R}$. Hence $f[a_0] = f(a_0)$ and

$$f[a_0, a_1] = \frac{f(a_1) - f(a_0)}{a_1 - a_0}.$$

Further, it is evident that a divided difference does not depend on the order in which the points $a_0, a_1, \ldots, a_n$ are chosen. Divided differences also satisfy a certain integral relation, namely the Hermite–Genocchi formula, which we shall use to express the iterated integrals arising when we compute moments of the time-average $A(T)$. Section 4 contains further divided difference theory required by this paper.

2. The Correlation Coefficient between the time-average and the asset

We shall compute the correlation coefficient between $S(T)$ and $A(T)$. Specifically, we calculate

$$R := \frac{\mathbb{E}\left(S(T)A(T)\right) - \mathbb{E}\left(S(T)\right)\mathbb{E}\left(A(T)\right)}{\sqrt{\operatorname{var} S(T) \operatorname{var} A(T)}}. \tag{2.1}$$

We find a succinct divided difference expression for R.

Theorem 2.1. *The correlation coefficient* (2.1) *is given by*

$$R \equiv R(rT, \sigma^2 T) = \frac{\exp[rT, 2rT, (2r+\sigma^2)T]}{\sqrt{2\exp[2rT, (2r+\sigma^2)T]\exp[0, rT, 2rT, (2r+\sigma^2)T]}}. \tag{2.2}$$

Let us begin our derivation.

Lemma 2.2. *If $0 \le a \le b$, then*

$$\mathbb{E}S(a)S(b) = \exp\Big(a(r+\sigma^2) + br\Big). \tag{2.3}$$

Proof. We have

$$\begin{aligned}
\mathbb{E}S(a)S(b) &= \mathbb{E}S(a)^2 e^{(b-a)(r-\sigma^2/2)+\sigma\sqrt{b-a}Z_{b-a}} \\
&= \mathbb{E}S(a)^2 \mathbb{E}e^{(b-a)(r-\sigma^2/2)+\sigma\sqrt{b-a}Z_{b-a}} \\
&= e^{(2r+\sigma^2)a} e^{(b-a)r} \\
&= e^{a(r+\sigma^2))} e^{br}. \qquad (2.4)
\end{aligned}$$

□

Proposition 2.3. *We have*

$$\mathbb{E}S(T)A(T) = \exp[rT, (2r+\sigma^2)T]. \tag{2.5}$$

Proof. Applying Lemma 2.2, we obtain

$$\begin{aligned}
\mathbb{E}S(T)A(T) &= T^{-1}\int_0^T \mathbb{E}S(t)S(T)\,dt \\
&= T^{-1}\int_0^T e^{(r+\sigma^2)t}e^{rT}\,dt \\
&= \exp[rT,(2r+\sigma^2)T].
\end{aligned}$$

□

Proposition 2.4.

$$\mathbb{E}(A(T)^2) = 2\exp[0,rT,(2r+\sigma^2)T]. \tag{2.6}$$

Proof. We find

$$\begin{aligned}
\mathbb{E}(A(T)^2) &= T^{-2}\int_0^T \Big(\int_0^T \mathbb{E}S(t_1)S(t_2)\,dt_2\Big)\,dt_1 \\
&= 2T^{-2}\int_0^T \Big(\int_0^{t_1} \mathbb{E}S(t_1)S(t_2)\,dt_2\Big)\,dt_1. \qquad (2.7)
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{E}(A(T)^2) &= 2T^{-2}\int_0^T \Big(\int_0^{t_1} e^{r(t_1+t_2)}e^{\sigma^2 t_2}\,dt_2\Big)\,dt_1 \\
&= 2T^{-2}\int_0^T e^{rt_1}\Big(\frac{e^{(r+\sigma^2)t_1}-1}{r+\sigma^2}\Big)\,dt_1 \\
&= \frac{2}{(r+\sigma^2)T}\big[\exp[0,(2r+\sigma^2)T]-\exp[0,rT]\big] \\
&= 2\exp[0,rT,(2r+\sigma^2)T], \qquad (2.8)
\end{aligned}$$

using the divided difference recurrence (4.1) to obtain the final line. □

Any reader still doubtful of the simplification provided by divided difference notation might consider the alternative expression

$$\mathbb{E}\left(A(T)^2\right) = \frac{2e^{(2r+\sigma^2)T}}{(r+\sigma^2)(2r+\sigma^2)T^2} + \frac{2}{rT^2}\left(\frac{1}{2r+\sigma^2} - \frac{e^{rT}}{r+\sigma^2}\right).$$

There is a similar divided difference relation for $\mathbb{E}\left(A(T)^m\right)$, described in the next section, but we now complete our derivation of Theorem 2.1.

Proof of Theorem 2.1. Applying (1.4, 1.5, 2.5) and (4.1), we obtain

$$\begin{aligned}
\mathbb{E}S(T)A(T) - \mathbb{E}S(T)\mathbb{E}A(T) &= \exp[rT,(2r+\sigma^2)T] - e^{rT}(e^{rT}-1)/(rT) \\
&= \exp[rT,(2r+\sigma^2)T] - \exp[rT,2rT] \\
&= \sigma^2T\exp[rT,2rT,(2r+\sigma^2)T]. \qquad (2.9)
\end{aligned}$$

Further,

$$\operatorname{var} S(T) = \mathbb{E}(S(T)^2) - (\mathbb{E}S(T))^2 = e^{(2r+\sigma^2)T} - e^{2rT} = \sigma^2T\exp[2rT,(2r+\sigma^2)T], \tag{2.10}$$

and, by (1.5, 2.6),

$$\begin{aligned}
\operatorname{var} A(T) &= 2\exp[0,rT,(2r+\sigma^2)T] - \left(\frac{e^{rT}-1}{rT}\right)^2 \\
&= 2\exp[0,rT,(2r+\sigma^2)T] - 2\exp[0,rT,2rT] \\
&= 2\sigma^2T\exp[0,rT,2rT,(2r+\sigma^2)T], \qquad (2.11)
\end{aligned}$$

using the divided difference recurrence (4.1) once more. Hence

$$R = \frac{\exp[rT, 2rT, (2r+\sigma^2)T]}{\sqrt{2\exp[2rT, (2r+\sigma^2)T]\exp[0, rT, 2rT, (2r+\sigma^2)T]}}. \tag{2.12}$$

□

It is remarkable that the divided differences appearing in (2.12) are coefficients of the cubic polynomial interpolating the exponential function at $0, rT, 2rT, (2r+\sigma^2)T$. We make two further observations:

(1) Armed with an analytic expression for the correlation coefficient, we can apply Margrabe's exchange option valuation [8] to derive the values of Asian options, if we are willing to accept that the time–average is suitably approximated by geometric Brownian motion. We are investigating the numerics of this rather simple approximation at present and preliminary results are surprisingly promising.
(2) The correlation coefficient $R(rT, \sigma^2 T)$ seems to be rather close to unity. Typical values of r, σ and T produce values of R in the $0.8 - 0.9$ range. We cannot, at present, fully explain this high correlation.

3. Computing higher moments of $A(T)$

We now demonstrate that the neat divided difference formulae obtained for the first and second moments of $A(T)$ are *not* coincidences, but part of a greater pattern from which arise new formulae generalizing the moment calculations of Oshanin [4] and Yor [11].

We begin with the iterated integral

$$\mathbb{E}A(T)^m = T^{-m}\int_0^T d\tau_m \int_0^T d\tau_{m-1}\cdots\int_0^T d\tau_1\ \mathbb{E}S(\tau_1)\cdots S(\tau_m). \tag{3.1}$$

Now, given any point $(\tau_1, \ldots, \tau_m) \in [0,T]^m$, let us sort its components into increasing order, obtaining $(t_1, \ldots, t_n)$ (say). Then

$$\mathbb{E}S(\tau_1)\cdots S(\tau_m) = \mathbb{E}S(t_1)\cdots S(t_m)$$

and

$$\mathbb{E}A(T)^m = m!T^{-m}\int_0^T dt_m \int_0^{t_m} dt_{m-1}\cdots\int_0^{t_2} dt_1\mathbb{E}S(t_1)\cdots S(t_m). \tag{3.2}$$

Our first task is to calculate the integrand, which we complete after a simple lemma.

Lemma 3.1. *For any positive integer k, we have*

$$\mathbb{E}\left[S(t)^k\right] = \exp\Big(krt + \frac{\sigma^2 t}{2}k(k-1)\Big). \tag{3.3}$$

Proof. This is almost immediate from (1.4):

$$\mathbb{E}S(t)^k = \mathbb{E}e^{k(r-\sigma^2)t+\sigma k\sqrt{t}Z} = e^{k(r-\sigma^2/2)t+\sigma^2k^2t/2} = e^{krt+\sigma^2tk(k-1)/2},$$

where $Z \sim N(0,1)$. □

Proposition 3.2. *If $0 \le t_1 \le t_2 \le \cdots \le t_m$, then*

$$\mathbb{E}S(t_1)S(t_2)\cdots S(t_m) = \exp\Big(\sum_{k=1}^m \left(r + (m-k)\sigma^2\right) t_k\Big). \tag{3.4}$$

Proof. Lemma 2.2 comprises the case $m = 2$. We complete the proof by induction on the number of terms m, first observing that, by a standard property of geometric Brownian motion,

$$\mathbb{E}S(t_1)S(t_2)\cdots S(t_m) = \mathbb{E}S(t_1)^m \mathbb{E}S(t_2 - t_1)\cdots S(t_m - t_1). \tag{3.5}$$

Applying Lemma 3.1 and our induction hypothesis, we obtain

$$\mathbb{E}S(t_1)S(t_2)\cdots S(t_m) = \exp\Big(mrt_1 + \sigma^2 t_1 m(m-1)/2 + \sum_{\ell=2}^{m} \left(r + (m-\ell)\sigma^2\right)(t_\ell - t_1)\Big). \tag{3.6}$$

The t_1 coefficient in the exponent is given by

$$mr - (m-1)r + \sigma^2 t_1 \left(\frac{1}{2}m(m-1) - \sum_{\ell=1}^{m-2} \ell\right) = r + \sigma^2 t_1 (m-1),$$

using the elementary fact that $m(m-1)/2 = 1 + 2 + \cdots + m - 1$. The coefficients of $t_2, \ldots, t_m$ are as already stated in (3.4). $\square$

Thus the desired integral (3.2) becomes

$$\begin{aligned} \mathbb{E}A(T)^m &= m!T^{-m} \int_0^T dt_m \int_0^{t_m} dt_{m-1} \cdots \int_0^{t_2} dt_1 \mathbb{E}S(t_1)\cdots S(t_m) \\ &= m! \int_0^1 dt_m \int_0^{t_{m-1}} \cdots \int_0^{t_2} dt_1 \exp(\alpha_1 t_1 + \cdots \alpha_m t_m), \end{aligned} \tag{3.7}$$

where

$$\alpha_k = \left(r + (m-k)\sigma^2\right) T, \qquad k = 1, \ldots, m. \tag{3.8}$$

The integral displayed in (3.7) can now be identified as a divided difference using a variant form of the Hermite–Genocchi integral relation.

Theorem 3.3. *Let*

$$b_k := kr + \sigma^2 k(k-1)/2, \qquad k = 0, 1, \ldots. \tag{3.9}$$

Then

$$\mathbb{E}(A(T))^m = m! \exp[b_0 T, b_1 T, \ldots, b_m T], \qquad m \geq 0. \tag{3.10}$$

Proof. Apply Corollary 4.5 to (3.7) and (3.8), using the elementary relation $\sum_{k=1}^{j} k = j(j+1)/2$. $\square$

The statement of Theorem 3.3 simplifies when $r = \sigma^2$, for then the drift term in (1.1) vanishes, that is, we consider $S(t) = \exp(\sigma\sqrt{t}Z_t)$ alone; this is the special case studied by Oshanin [4] and Yor [11].

Theorem 3.4. *If we set $r = \sigma^2/2$ in Theorem 3.3, then we obtain*

$$\begin{aligned} \mathbb{E}\left[A(T)^m\right] &= m! \exp[0, rT, 2^2 rT, 3^2 rT, \ldots, m^2 rT] \\ &= m! H_{\sqrt{rT}}[-m, \ldots, -1, 0, 1, \ldots, m], \end{aligned} \tag{3.11}$$

where $H_c(x) := \exp(c^2 x^2)$, $x \in \mathbb{R}$, for any positive c.

Proof. We simply set $r = \sigma^2$ in Theorem 3.3 and apply (4.13). $\square$

We can now apply Corollary 4.10 to derive the formula given in equation (14) of [4].

Theorem 3.5. *If we set $r = \sigma^2/2$, then*

$$\mathbb{E}\left[A(T)^m\right] = \left(\frac{\Gamma(m)}{\Gamma(2m)}\right) r^{-m} \left(-\frac{1}{2}(-1)^m \binom{2m}{m} + \sum_{\ell=0}^{m} \binom{2m}{\ell} (-1)^\ell e^{rT(m-\ell)^2}\right). \tag{3.12}$$

Proof. Applying Corollary 4.10 to Theorem 3.4, we obtain

$$\mathbb{E}\left[A(T)^m\right] = \left(\frac{m!}{(2m)!}(rT)^{-m} \sum_{k=0}^{2m} \binom{2m}{k} (-1)^k e^{rT(k-m)^2}\right)$$

$$= \left(\frac{\Gamma(m)}{\Gamma(2m)}\right) (rT)^{-m} \left(-\frac{1}{2}(-1)^m \binom{2m}{m} + \sum_{\ell=0}^{m} \binom{2m}{\ell} (-1)^\ell e^{rT(m-\ell)^2}\right), \tag{3.13}$$

after some straightforward algebraic manipulation. □

If we now replace rT by α and m by j in (3.12), then we obtain equation (14) of Oshanin [4].

4. Divided Difference Theory

Most of the properties of divided differences required here can be found in [2] and Chapter 5 of [9]. However, proofs of the Hermite–Genocchi integral relation are less easily available in the Anglophone mathematical literature, as is our particular variant of it, although the specialist can find much useful material in the treatise of DeVore and Lorentz [3]. We have therefore provided a derivation for the convenience of the reader. The Hermite–Genocchi formula and its consequences are still very much topics of current research; see, for example, [10]. Furthermore, the result is better served in other European languages; see, for instance, [5] for a French translation of a Russian classic, or indeed the original [6].

We recall the *divided difference recurrence relation*

Theorem 4.1.

$$f[a_0, a_1, \ldots, a_n] = \frac{f[a_1, \ldots, a_n] - f[a_0, \ldots, a_{n-1}]}{a_n - a_0}, \tag{4.1}$$

for any distinct complex numbers $a_0, \ldots, a_n$.

Proof. See, for instance, [9], Theorem 5.3. □

If f is sufficiently differentiable, then we can define divided differences for coincident points. We see that

$$\lim_{a_1 \to a_0} f[a_0, a_1] = f'(a_0).$$

Further, the elementary relation

$$f[a_0, a_1] = \frac{f(a_1) - f(a_0)}{a_1 - a_0} = \int_0^1 f'((1-t)a_0 + ta_1)\, dt, \qquad \text{when } a_0, a_1 \in \mathbb{R}, \tag{4.2}$$

can be generalized to obtain the *Hermite–Genocchi formula.*

Theorem 4.2 (Hermite–Genocchi)**.** *Let $f \in C^{(n)}(\mathbb{R})$ and let $a_0, a_1, \ldots, a_n$ be (not necessarily distinct) real numbers Then, for $n \geq 1$,*

$$f[a_0, a_1, \ldots, a_n]$$
$$= \int_{S_n} f^{(n)}(t_0 a_0 + t_1 a_1 + \cdots + t_n a_n)\, dt_1 \cdots dt_n,$$
$$= \int_0^1 dt_1 \int_0^{1-t_1} dt_2 \cdots \int_0^{1-\sum_{k=1}^{n-1} t_k} dt_n f^{(n)}(t_0 a_0 + t_1 a_1 + \cdots + t_n a_n) \tag{4.3}$$

where the domain of integration is the simplex

$$S_n = \{t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}_+^n \colon \sum_{k=1}^n t_k \le 1\} \tag{4.4}$$

and

$$t_0 = 1 - \sum_{k=1}^n t_k.$$

Proof. We shall prove (4.3) by induction on n, observing that

$$\int_{S_1} f'(t_0 a_0 + t_1 a_1)\, dt_1 = \int_0^1 f'(a_0 + t_1(a_1 - a_0))\, dt_1 = \frac{f(a_1) - f(a_0)}{a_1 - a_0} = f[a_0, a_1].$$

To extend the formula to higher order divided differences, we note that

$$f[a_0, a_1, \ldots, a_n, a_{n+1}] = \frac{f[a_1, a_2, \ldots, a_{n+1}] - f[a_0, a_1, \ldots, a_n]}{a_{n+1} - a_0} = g[a_0, a_{n+1}], \tag{4.5}$$

where

$$g(x) = f[a_1, \ldots, a_n, x], \qquad x \in \mathbb{R}. \tag{4.6}$$

Now

$$g(x) = \int_{S_n} f^{(n)}(x t_0 + a_1 t_1 + \cdots + a_n t_n)\, dt_1 \cdots dt_n$$

so that

$$g'(x) = \int_{S_n} t_0 f^{(n+1)}(x t_0 + a_1 t_1 + \cdots + a_n t_n)\, dt_1 \cdots dt_n.$$

Therefore

$$\begin{aligned}
&f[a_0, a_1, \ldots, a_n, a_{n+1}] \\
&\quad= \int_0^1 d\tau\ g'((1-\tau)a_0 + \tau a_{n+1}) \\
&\quad= \int_0^1 d\tau \int_{S_n} dt_1 \cdots dt_n\ t_0 f^{(n+1)}([(1-\tau)a_0 + \tau a_{n+1}]t_0 + a_1 t_1 + \cdots a_n t_n) \\
&\quad= \int_{S_n} dt_1 \cdots dt_n \int_0^1 d\tau\ t_0 f^{(n+1)}([(1-\tau)t_0 a_0 + \sum_{\ell=1}^n a_\ell t_\ell + \tau t_0 a_{n+1}) \\
&\quad= \int_0^1 dt_1 \int_0^{1-t_1} dt_2 \cdots \int_0^{1-\sum_{k=1}^{n+1} t_k} dt_{n+1} f^{(n+1)} \left(T_0 a_0 + \sum_{k=1}^{n+1} t_k a_k \right) \\
&\quad= \int_{S_{n+1}} f^{(n+1)} \left(T_0 a_0 + t_1 a_1 + \cdots + t_{n+1} a_{n+1} \right) dt_1 \cdots dt_{n+1},
\end{aligned}$$

where we have used the substitution $t_{n+1} = t_0 \tau$ and the notation $T_0 = 1 - \sum_{k=1}^{n+1} t_k$. $\square$

We shall need a variant form of the Hermite–Genocchi integral relation for which the following notation is useful. Given any real $n \times n$ nonsingular matrix V, with columns $v_1, \ldots, v_n$, we let $K(V)$ denote the closed convex hull of $0, v_1, \ldots, v_n$, i.e.

$$K(V) := \mathrm{conv}\{0, v_1, \ldots, v_n\}.$$

In this notation, the Hermite–Gnocchi integral relation states that

$$f[a_0, a_1, \ldots, a_n] = \int_{K(I_n)} f^{(n)} \left(a_0 + (a - a_0 e)^T y \right) dy, \tag{4.7}$$

where

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \qquad e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

and I_n denotes the $n \times n$ identity matrix. Integrating the nth derivative over the simplex $K(V)$ yields a useful variant form of Hermite–Genocchi.

Theorem 4.3. *Let $V \in \mathbb{R}^{n\times n}$ be any nonsingular matrix. Then*

$$\frac{1}{|\det V|} \int_{K(V)} f^{(n)}\left(a^T y\right) \, dy = f[0, (V^T a)_1, \ldots, (V^T a)_n], \tag{4.8}$$

where $(V^T a)_k$ denotes the kth component of the vector $V^T a$.

Proof. Substituting $y = Vz$, Hermite–Genocchi implies the relation

$$\int_{K(V)} f^{(n)}\left((V^T a)^T z\right) \, dz = f[0, (V^T a)_1, \ldots, (V^T a)_n].$$

□

Corollary 4.4. *For any function $f \in C^{(n)}(\mathbb{R})$, we have*

$$\begin{aligned} &\int_0^1 dx_n \int_0^{x_n} dx_{n-1} \cdots \int_0^{x_2} dx_1 f^{(n)}\left(\sum_{k=1}^n a_k x_k\right) \\ &= f[0, a_n, a_n + a_{n-1}, \ldots, a_n + a_{n-1} + \cdots + a_1]. \end{aligned} \tag{4.9}$$

Proof. Set

$$V = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

in Theorem 4.3. □

The exponential function is a particularly important case for us, in which case the Hermite–Genocchi formula becomes

$$\exp[a_0, a_1, \ldots, a_n] = \int_{S_n} e^{t_0 a_0 + t_1 a_1 + \cdots + t_n a_n} \, dt_1 \cdots dt_n \tag{4.10}$$

and Corollary 4.4 takes the following form.

Corollary 4.5. *We have*

$$\begin{aligned} &\int_0^1 dx_n \int_0^{x_n} dx_{n-1} \cdots \int_0^{x_2} dx_1 \exp\left(\sum_{k=1}^n a_k x_k\right) \\ &= \exp[0, a_n, a_n + a_{n-1}, \ldots, a_n + a_{n-1} + \cdots + a_1]. \end{aligned} \tag{4.11}$$

Proof. Let f be the exponential function in Corollary 4.4. □

We shall also need two simple preliminary results. Let us use $\mathbb{P}_n$ to denote the vector space of polynomials of degree n.

Lemma 4.6. *We have*

$$\exp(\mu) \exp[\lambda_0, \ldots, \lambda_m] = \exp[\lambda_0 + \mu, \ldots, \lambda_m + \mu], \tag{4.12}$$

where $\lambda_0, \ldots, \lambda_m$ and μ can be any complex numbers.

Proof. Immediate. □

Lemma 4.7. *Let $f\colon \mathbb{C} \to \mathbb{C}$ and let $a_1, \ldots, a_n$ be distinct nonzero complex numbers. Then*

$$f[0, a_1^2, \ldots, a_n^2] = g[-a_n, \ldots, -a_1, 0, a_1, \ldots, a_n], \tag{4.13}$$

where $g(z) = f(z^2)$, for $z \in \mathbb{C}$.

Proof. Let $p \in \mathbb{P}_n$ interpolate f at $0, a_1^2, \ldots, a_n^2$. Then $q(z) := p(z^2)$ is a polynomial of degree $2n$ satisfying $q(\pm a_j) = p(a_j^2) = f(a_j^2) = g(\pm a_j)$, for $j = 0, \ldots, n$, setting $a_0 = 0$, for convenience. The result then follows from uniqueness of the interpolating polynomial.

□

It is well-known that a divided difference at equally spaced points can be expressed in a particularly simple form using the forward difference operator

$$\Delta_h f(x) := f(x+h) - f(x),$$

which we shall need when demonstrating the equivalence between our moment calculations and those of Oshanin [4] and Yor [11]. The next proposition is well-known and can be found in [2], but we again include its short proof for the reader's convenience.

Proposition 4.8. *Let $f\colon \mathbb{R} \to \mathbb{R}$, let h be any positive constant and let n be a non-negative integer. Then*

$$f[x, x+h, x+2h, \ldots, x+nh] = \frac{\Delta_h^n f(x)}{n! h^n}. \tag{4.14}$$

Proof. It is easily checked that $f[x, x+h] = \Delta_h f(x)/h$. Further, if we assume (4.14) for $n-1$, then the divided difference recurrence relation implies that

$$\begin{aligned}
&f[x, x+h, \ldots, x+nh] \\
&= \frac{f[x+h, \ldots, x+nh] - f[x, x+h, \ldots, x+(n-1)h]}{nh} \\
&= \frac{\Delta_h f[x, \ldots, x+(n-1)h]}{nh} \\
&= \frac{1}{nh} \Delta_h \left(\frac{\Delta_h^{n-1} f(x)}{(n-1)! h^{n-1}} \right) \\
&= \frac{\Delta_h^n f(x)}{n! h^n}.
\end{aligned}$$

Thus the result follows by induction. □

Corollary 4.9. *Let $f\colon \mathbb{R} \to \mathbb{R}$ and let h be any positive constant. Then*

$$f[x, x+h, x+2h, \ldots, x+nh] = \frac{1}{n! h^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+kh). \tag{4.15}$$

Proof. We define the *forward shift* operator

$$E_h f(x) := f(x+h), \qquad x \in \mathbb{R},$$

and observe that, by the binomial theorem,

$$\Delta_h^n f(x) = (E_h - 1)^n f(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} E_h^k f(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+kh).$$

□

Corollary 4.10. *Let $f\colon \mathbb{R} \to \mathbb{R}$ and let h be any positive number. Then*
(4.16)
$$f[-nh, -(n-1)h, \ldots, -h, 0, h, \ldots, nh] = \frac{1}{(2n)!h^{2n}} \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k f((k-n)h).$$

Proof. This is an immediate consequence of Corollary 4.9. □

We shall also need the *Leibnitz* relation for divided differences of a product when deriving the recurrence differential equation for moments.

Theorem 4.11 (Leibnitz)**.** *Let D be any subset of $\mathbb{C}$ containing the distinct points $z_0, z_1, \ldots, z_n$ and let v and w be complex-valued functions on D. If $u = v \cdot w$, then*

$$u[z_0, \ldots, z_n] = \sum_{k=0}^{n} v[z_0, \ldots, z_k] w[z_k, \ldots, z_n]. \tag{4.17}$$

Proof. Let $p \in \mathbb{P}_n$ be the unique polynomial interpolant for u written in standard Newton form, that is

$$p(z) = v[z_0] + v[z_0, z_1](z - z_0) + \cdots v[z_0, z_1, \ldots, z_n](z - z_0) \cdots (z - z_{n-1}). \tag{4.18}$$

We shall let $q \in \mathbb{P}_n$ be the unique polynomial interpolating w, but with the points chosen in the order $z_n, z_{n-1}, \ldots, z_0$, that is,

$$q(z) = w[z_n] + w[z_n, z_{n-1}](z - z_n) + \cdots + w[z_n, \ldots, z_0](z - z_n) \cdots (z - z_1). \tag{4.19}$$

Now their product $p \cdot q$ is a polynomial of degree $2n$. Dividing this polynomial by $(z - z_0) \cdots (z - z_n)$, we obtain

$$p(z)q(z) = r(z) + s(z)(z - z_0) \cdots (z - z_n),$$

where $r \in \mathbb{P}_n$. We see that $u(z_j) = v(z_j)w(z_j) = p(z_j)q(z_j) = r(z_j)$, for $0 \le j \le n$. Hence, by uniqueness of the polynomial interpolant for u in $\mathbb{P}_n$, we obtain

$$r(z) = u[z_0] + \cdots + u[z_0, \ldots, z_n](z - z_0) \cdots (z - z_n). \tag{4.20}$$

We obtain (4.17) by equating the coefficients of z^n in (4.20) and the product of the expressions in (4.18) and (4.19), modulo $(z - z_0) \cdots (z - z_n)$. □

5. A Recurrence Relation

The Feynman–Kac formula suggests that $E_n(t) := \mathbb{E}\left(A(t)^n\right)$ should satisfy a certain differential equation, which we shall obtain via Hermite–Genocchi.

Theorem 5.1. *Let $\{c_n\}_{n=1}^{\infty}$ be any strictly increasing sequence of positive numbers and define $e_n : (0, \infty) \to \mathbb{R}$ by the divided difference*

$$e_n(t) = \exp[0, c_1 t, \ldots, c_n t], \qquad t > 0, \quad n \ge 0. \tag{5.1}$$

Then

$$t e_n'(t) = e_n(t)\left(c_n t - n\right) + e_{n-1}(t), \qquad \textit{for } n \ge 1. \tag{5.2}$$

Proof. Applying the Hermite–Genocchi formula, we obtain

$$e_n(t) = \int_{K(I_n)} \exp\left(t c^T y\right)\, dy, \tag{5.3}$$

where $b = \left(c_1, \ldots, c_n\right)^T$, and differentiating (5.3) yields

$$e_n'(t) = \int_{K(I_n)} \exp(t c^T y)(c^T y)\, dy. \tag{5.4}$$

Now writing $g(s) = s$ and applying Leibnitz's formula for divided differences, we find

$$\begin{aligned}(g \cdot \exp)[0, c_1 t, \dots, c_n t] &= g[0] \exp[0, c_1 t, \dots, c_n t] + g[0, c_1 t] \exp[c_1 t, \dots, c_n t] \\ &= \exp[c_1 t, \dots, c_n t]. \end{aligned} \tag{5.5}$$

Further, the relation $(g \cdot \exp)^{(n)} = g \cdot \exp + n \exp$ and (5.5) imply

$$\begin{aligned} t e_n'(t) &= \int_{K(I_n)} (g \cdot \exp)^{(n)} (t c^T y) \, dy - n \int_{K(I_n)} \exp(t c^T y) \, dy \\ &= (g \cdot \exp)[0, c_1 t, \dots, c_n t] - n \exp[0, c_1 t, \dots, c_n t] \\ &= \exp[c_1 t, \dots, c_n t] - n e_n(t). \end{aligned} \tag{5.6}$$

However,

$$e_n(t) = \frac{\exp[c_1 t, \dots, c_n t] - e_{n-1}(t)}{c_n t}, \tag{5.7}$$

by the divided difference recurrence relation, so that

$$\exp[c_1 t, \dots, c_n t] = c_n t e_n(t) + e_{n-1}(t). \tag{5.8}$$

Substituting (5.8) in (5.6) provides (5.2). □

The corresponding differential equation for E_n is now immediate.

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