Moment-free numerical integration of highly oscillatory functions

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Abstract

The aim of this paper is to derive new methods for numerically approximating the integral of a highly oscillatory function. We begin with a review of the asymptotic and Filon-type methods developed by Iserles and Nørsett. Using a method developed by Levin as a point of departure, we construct a new method that utilizes the same information as the Filon-type method, and obtains the same asymptotic order, while not requiring the computation of moments. We also show that a special case of this method has the property that the asymptotic order increases with the addition of sample points within the interval of integration, unlike all the preceding methods whose orders depend only on the endpoints.

1   Introduction

A highly oscillatory integral is defined as

$$I[f] = \int_a^b f(x)\, \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x$$

for $f, g \in C^\infty$ and frequency $\omega \gg 1$. In this paper we only consider the case of $g'(x) \neq 0$ for $a \leq x \leq b$, in other words, when g has no stationary points. The most immediate candidate for numerically approximating this integral might be Gaussian quadrature. Unfortunately, if we subdivide $[a, b]$ into panels of length $h > 0$ we can choose ω large enough such that the approximation is completely meaningless, as the node points are essentially random samples in the range of oscillation. The error of this approximation is $\mathcal{O}(1)$ as $\omega \to \infty$, which compares to an error of $\mathcal{O}(\omega^{-1})$ when approximating $I[f]$ by zero (Iserles and Nørsett, 2004). It is safe to say that any approximation that is less accurate than equating the integral to zero is fairly useless. Letting h depend on ω, on the other hand, results in an enormous amount of computation for large ω. Fortunately, there is another way.

We begin with a review of two methods described in (Iserles and Nørsett, 2004), the asymptotic method and the Filon-type method, which both have an error of order $\mathcal{O}(\omega^{-s-1})$ for any fixed positive integer s and increasing frequency ω. The implication is that, in stark contrast to Gaussian quadrature, the larger the frequency the more accurate the approximation. Using these two methods as an inspiration and extending the work of David Levin, we derive another method that also has an error of order $\mathcal{O}(\omega^{-s-1})$. Like the Filon-type method, this new method uses interior points as well as the endpoints for deriving the approximation, while unlike the Filon-type method it does not require moments. We also show that a special case of this method has the property that using interior points, in addition to endpoints, further increases the order of error.

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2 The asymptotic method

The foundation of the proofs in this paper lies in the observation, as described in (Iserles and Nørsett, 2004), that

$$I[f] = \int_a^b f(x)\, \mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x = \frac{1}{\mathrm{i}\omega} \int_a^b \frac{f(x)}{g'(x)} \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x$$

$$= \frac{1}{\mathrm{i}\omega} \left[\frac{f(x)}{g'(x)} \mathrm{e}^{\mathrm{i}\omega g(x)} \right]_a^b - \frac{1}{\mathrm{i}\omega} \int_a^b \frac{\mathrm{d}}{\mathrm{d}x} \left[\frac{f(x)}{g'(x)} \right] \mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x = Q^A[f] - \frac{1}{\mathrm{i}\omega} I \left[\frac{\mathrm{d}}{\mathrm{d}x} \frac{f}{g'} \right],$$

where $Q^A[f]$ is defined as $\frac{1}{\mathrm{i}\omega} \left[\frac{f(x)}{g'(x)} \mathrm{e}^{\mathrm{i}\omega g(x)} \right]_a^b$. Since we assume that $g'(x) \neq 0$ for all $x \in [a,b]$ there is no issue with dividing by g'. This equation states that if we approximate $I[f]$ by $Q^A[f]$ we have an order of error of $\mathcal{O}(\omega^{-2})$, again using the fact that $I[f] = \mathcal{O}(\omega^{-1})$ for bounded f (Iserles and Nørsett, 2004). But the error term of this approximation can likewise be approximated, now by $\frac{1}{\mathrm{i}\omega} Q^A \left[\frac{\mathrm{d}}{\mathrm{d}x} \frac{f}{g'} \right]$. Hence we have derived a new approximation of $I[f]$, namely $Q^A[f] - \frac{1}{\mathrm{i}\omega} Q^A \left[\frac{\mathrm{d}}{\mathrm{d}x} \frac{f}{g'} \right]$, which has an order of error of $\mathcal{O}(\omega^{-3})$. Clearly, if we continue this process of approximating the error terms using Q^A, after s steps we obtain error $\mathcal{O}(\omega^{-s-1})$. Thus we have derived an asymptotic expansion:

Theorem 1 *Let $f \in C^\infty$ and $g'(x) \neq 0$ for $a \leq x \leq b$. Define σ_k as*

$$\sigma_1[f](x) = \frac{f(x)}{g'(x)}, \qquad \sigma_{k+1}[f](x) = \frac{\sigma_k[f]'(x)}{g'(x)}, \qquad k \geq 1.$$

Then, for $\omega \to \infty$,

$$I[f] \sim -\sum_{k=1}^{\infty} \frac{1}{(-\mathrm{i}\omega)^k} \left\{ \sigma_k[f](b)\, \mathrm{e}^{\mathrm{i}\omega g(b)} - \sigma_k[f](a)\, \mathrm{e}^{\mathrm{i}\omega g(a)} \right\}.$$

A formal proof of this theorem can be found in (Iserles and Nørsett, 2004). We define

$$Q_s^A[f] = -\sum_{k=1}^{s} \frac{1}{(-\mathrm{i}\omega)^k} \left\{ \sigma_k[f](b)\, \mathrm{e}^{\mathrm{i}\omega g(b)} - \sigma_k[f](a)\, \mathrm{e}^{\mathrm{i}\omega g(a)} \right\},$$

i.e. the s-step partial sum of the asymptotic expansion. From the omitted proof of the theorem we know that the error $I[f] - Q_s^A[f]$ is equal to $\frac{1}{(-\mathrm{i}\omega)^s} \int_a^b g'(x)\, \sigma_{s+1}[f](x)\, \mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x$ and by the definition of asymptotic expansions we know that $I[f] - Q_s^A[f] \sim \mathcal{O}(\omega^{-s-1})$. Hence the error of the approximation tends to zero as ω^{-s-1}. In other words, the more oscillatory the integrand, the more accurately we can approximate the integral! This flies in the face of the common intuition, based on the problems associated with Gaussian quadrature, that oscillations make numerical integration difficult.

From this theorem we now derive a corollary that will be used to find the order of error for the Filon-type and Levin-type methods.

Corollary 1 *Suppose $0 = f(a) = f(b) = f'(a) = f'(b) = \cdots = f^{(s-1)}(a) = f^{(s-1)}(b)$ for some positive integer s. Furthermore, allow f to depend on ω, and suppose that every function in the set $\{f, \cdots, f^{(s+1)}\}$ is of asymptotic order $\mathcal{O}(\omega^{-n})$, $\omega \to \infty$, for some fixed n. Then, as $\omega \to \infty$,*

$$I[f] \sim \mathcal{O}(\omega^{-n-s-1}).$$

2

Proof Fix $s \geq 1$. By simple inductive reasoning, we see that each $\sigma_k[f]$ is a linear combination of terms independent of ω multiplied by functions in the set $\{f, \cdots, f^{(k-1)}\}$. As a result $0 = \sigma_k[f](a) = \sigma_k[f](b)$ for all $k \leq s$, and it follows that $Q_s^A[f]$ is 0. Hence

$$I[f] = -\frac{1}{(-i\omega)^{s+1}} \left\{ \sigma_{s+1}[f](b) \, e^{i\omega g(b)} - \sigma_{s+1}[f](a) \, e^{i\omega g(a)} \right\} + \frac{1}{(-i\omega)^{s+1}} \int_a^b g'(x) \, \sigma_{s+2}[f](x) \, e^{i\omega g(x)} dx.$$

The first two terms are $\mathcal{O}(\omega^{-n-s-1})$. We know that $|\sigma_{s+2}[f]| \leq \frac{K}{\omega^n}$ for some constant K, since $\sigma_{s+2}[f]$ is a combination of f and its first $s+1$ derivatives. Thus the integral term is also $\mathcal{O}(\omega^{-n-s-1})$, since

$$\left| \frac{1}{(-i\omega)^{s+1}} \int_a^b g'(x) \, \sigma_{s+2}[f](x) \, e^{i\omega g(x)} dx \right| \leq \frac{K}{\omega^{n+s+1}} (b-a) = \mathcal{O}(\omega^{-n-s-1}).$$

$\square$

Note that this corollary applies equally well when f and its derivatives are independent of ω, in which case we take $n = 0$. Unless otherwise stated, we assume $n = 0$ when this corollary is used.

Remark In Corollary 1 it is necessary to impose the order requirement on both f and its first s derivatives: $f = \mathcal{O}(\omega^{-n})$ with no restriction on its derivatives is not sufficient. For example consider $f(x) = \omega^{-n} e^{i\omega^n x}$. Then $f^{(k)}(x) = i^{k-1} \omega^{(k-1)n} e^{i\omega^n x} = \mathcal{O}(\omega^{(k-1)n})$ and the corollary does not hold.

3 The Filon-type method

The shortcoming with using an asymptotic expansion as an approximation is that in general $Q_s^A[f]$ diverges for fixed ω as $s \to \infty$. In other words, for fixed ω the accuracy of approximating an integral by the partial sums Q_s^A is limited. To work around this weakness we derive a Filon-type method, which extends the work of Filon as described in (Iserles and Nørsett, 2004).

Theorem 2 *Let s be some positive integer, let $\{x_k\}_0^\nu$ be a set of node points such that $a = x_0 < x_1 < \cdots < x_\nu = b$, and let $\{m_k\}_0^\nu$ be a set of multiplicities associated with those node points such that $m_0, m_\nu \geq s$. Suppose that $v(x) = \sum_{k=0}^n c_k x^k$, where $n = \sum_{k=0}^\nu m_k - 1$, is the solution to the system of equations*

$$v(x_k) = f(x_k)$$
$$v'(x_k) = f'(x_k)$$
$$\vdots$$
$$v^{(m_k-1)}(x_k) = f^{(m_k-1)}(x_k)$$

for every integer $0 \leq k \leq \nu$. Then

$$I[f] - Q^F[f] \sim \mathcal{O}(\omega^{-s-1}),$$
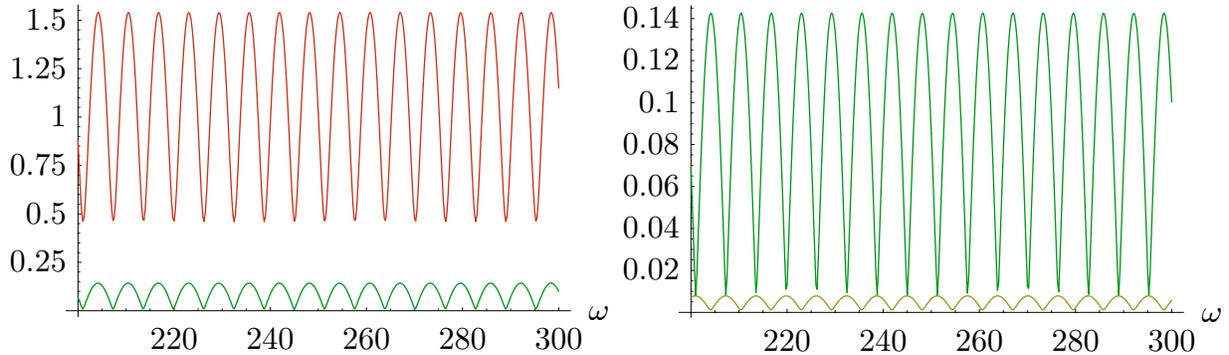
where

$$Q^F[f] \equiv I[v] = \sum_{k=0}^n c_k I\left[x^k\right].$$

Figure 1: The error scaled by ω^3 of $Q_2^A[f]$ (left figure, top), $Q^F[f]$ with only endpoints and multiplicities all 2 (left figure, bottom)/(right figure, top), and $Q^F[f]$ with nodes $\left\{0, \frac{1}{2}, 1\right\}$ and multiplicities $\{2, 1, 2\}$ (right figure, bottom) for $I[f] = \int_0^1 \cos(x)\, e^{i\omega x} dx$.

Proof Note that $I[f] - Q^F[f] = I[f] - I[v] = I[f - v]$. By the definition of v, the hypotheses of Corollary 1 hold for the function $f - v$, hence $I[f - v] \sim \mathcal{O}(\omega^{-s-1})$. $\qquad\qquad\square$

In other words, we interpolate f by a polynomial v using Hermite interpolation. Since we are assuming that moments are available and $Q^F[f]$ is a linear combination of moments, we know that $Q^F[f]$ can be computed. The obvious question then is if it has the same order of error as the asymptotic method, as well as requiring the same number of derivatives, why bother? The answer is that in many situations the accuracy of the Filon-type method is significantly higher than that of the asymptotic method, even though it is of the same order. We also have the ability to add interior node points to further increase the accuracy, and it is clear that $Q^F[f]$ converges to $I[f]$ whenever the interpolating polynomial v converges uniformly to f.

We now compare asymptotic and Filon-type methods numerically. For example, consider the case of the Fourier oscillator $g(x) = x$, and let $f(x) = \cos x$ over the interval $[0, 1]$. In Figure 1 we compare several methods of order 3: $Q_2^A[f]$, $Q^F[f]$ with nodes $\{0, 1\}$ and multiplicities $\{2, 2\}$, and $Q^F[f]$ with nodes $\left\{0, \frac{1}{2}, 1\right\}$ and multiplicities $\{2, 1, 2\}$. Even when sampling f only at the endpoints of the interval, the Filon-type method represents a significant improvement over the asymptotic method, having approximately one twelfth the error, while using exactly the same information. Adding a single interpolation point resulted in an error almost indistinguishable from zero when compared to the asymptotic method. Adding additional node points continues to have a similar effect.

Unfortunately, it is not always true that the Filon-type method is more accurate than the asymptotic method. Take the case of the Fourier oscillator and $f(x) = \frac{1}{1+10x^2}$, now over the interval $[-1, 1]$. This suffers from Runge's phenomenon, as described in (Powell, 1981), where certain non-oscillatory functions have oscillating interpolation polynomials. Since the Filon-type method is based on interpolation, it is logical that the accuracy of $Q^F[f]$ is directly related to how accurate the interpolation is. In Figure 2 we see that adding additional nodes actually reduces the accuracy of $Q^F[f]$. It should be noted that in this example $Q^F[f]$ with only endpoints and $Q_1^A[f]$ are equivalent, which can be trivially proved by finding the explicit formula for $Q^F[f]$. Thus $Q_1^A[f]$ is the best method of the three tried.

We know that using Chebyshev interpolation points, also described in (Powell, 1981), helps reduce the magnitude of Runge's phenomenon. Using this choice for nodes, along with the required
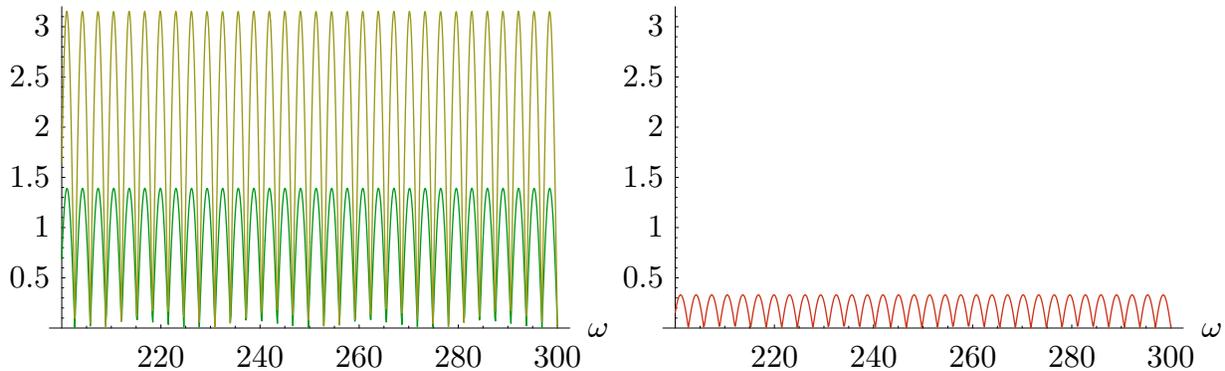
Figure 2: The error scaled by ω^2 of $Q^F[f]$ with only endpoints (right figure), endpoints and two additional evenly spaced points (left figure, bottom), and endpoints and four additional evenly spaced points (left figure, top), where all multiplicities are 1 for $I[f] = \int_0^1 \frac{1}{1+10x^2} e^{i\omega x} dx$.
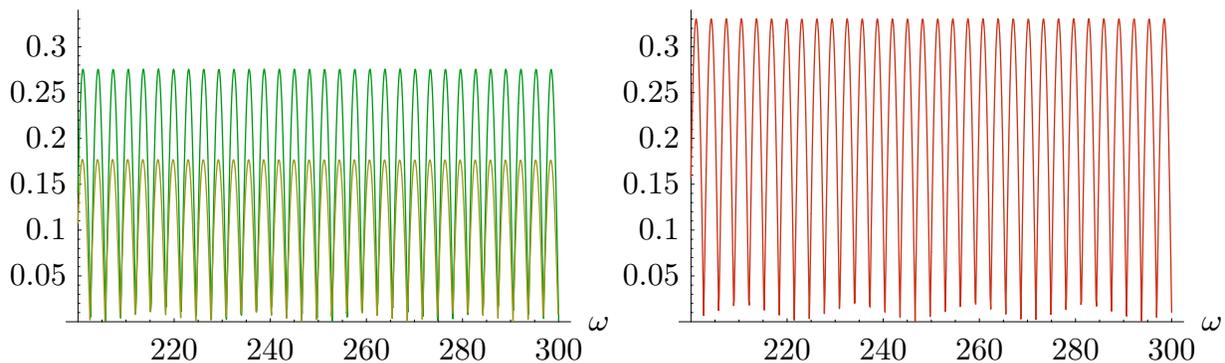

Figure 3: The error scaled by ω^2 of $Q^F[f]$ with only endpoints (right figure), endpoints and 2 additional Chebyshev interpolation points (left figure, top), and endpoints and 4 additional Chebyshev interpolation points (left figure, bottom), where all multiplicities are 1 for $I[f] = \int_0^1 \frac{1}{1+10x^2} e^{i\omega x} dx$.

endpoint nodes, results in the errors seen in Figure 3. Now adding additional node points results in a more accurate approximation. This certainly is a huge improvement over Figure 2, but the Filon-type methods definitely do not have the same magnitude of improvement over the asymptotic method that they did in Figure 1.

Another option, with regards to Runge's phenomenon, is to use cubic splines in place of interpolation. Unfortunately this suffers from the fact that a cubic spline can only match up to the first derivative at the endpoints, hence the order is at most $\mathcal{O}(\omega^{-3})$ in the present framework. Since we are only considering methods with arbitrarily high order of convergence for increasing ω, we will not explore the use of cubic or higher-degree splines.

4 The Levin-type method

The Filon-type method requires that moments are easily computable, which is not necessarily the case. Fortunately, we can work around this problem by expanding on the method developed by

David Levin in (Levin, 1997). What follows is a brief, and simplified, synopsis of the method described in that paper. Suppose we have a function $F(x)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[F(x)\,\mathrm{e}^{\mathrm{i}\omega g(x)}\right] = f(x)\,\mathrm{e}^{\mathrm{i}\omega g(x)}. \tag{4.1}$$

It follows immediately that $I[f] = \left[F(x)\,\mathrm{e}^{\mathrm{i}\omega g(x)}\right]_a^b$. If we approximate F by some function v, then we can approximate the integral by $Q^L[f] = \left[v(x)\,\mathrm{e}^{\mathrm{i}\omega g(x)}\right]_a^b$. By expanding out the derivative on the left hand side of (4.1) and canceling the $\mathrm{e}^{\mathrm{i}\omega g(x)}$ terms we obtain the equation $L[F](x) = f(x)$, where L is the operator defined by $L[F] = F' + \mathrm{i}\omega g'F$. Now let $v(x) = \sum_{k=0}^n c_k x^k$ be the collocation polynomial which is the solution to the system of equations $L[v](x_k) = f(x_k)$ at points $a = x_0 < x_1 < \cdots < x_\nu = b$. Then $Q^L[f]$ approximates $I[f]$ with error $\mathcal{O}(\omega^{-2})$.

The natural extension to the Levin method is to emulate the Filon-type method of the preceding section and match not only the value of f and $L[v]$ at the node points, but also the values of the derivatives of f and $L[v]$, up to given multiplicity. We prove in this section that, if we again match the function values f and $L[v]$ and the first $s - 1$ derivatives at the endpoints, then we obtain an order of error of $\mathcal{O}(\omega^{-s-1})$. Since the proof of the following theorem does not rely on v being a polynomial, we allow v to be a linear combination of a set of basis functions matching certain criteria — a generalization that will be exploited in Section 5.

Theorem 3 *Suppose that $g'(x) \neq 0$ for $x \in [a,b]$. Let $\{\psi_k\}_0^n$ be a basis of functions independent of ω and let s be some positive integer. Furthermore let $\{x_k\}_0^\nu$ be a set of node points such that $a = x_0 < x_1 < \cdots < x_\nu = b$ and $\{m_k\}_0^\nu$ a set of multiplicities associated with those node points such that $m_0, m_\nu \geq s$. Suppose that $v = \sum_{k=0}^n c_k \psi_k$, where $n = \sum_{k=0}^\nu m_k - 1$, is the solution to the system of collocation equations*

$$L[v](x_k) = f(x_k)$$
$$\frac{\mathrm{d}L[v]}{\mathrm{d}x}(x_k) = f'(x_k)$$
$$\vdots$$
$$\frac{\mathrm{d}^{m_k-1}L[v]}{\mathrm{d}x^{m_k-1}}(x_k) = f^{(m_k-1)}(x_k)$$

for every integer $0 \leq k \leq n$ and $L[v] = v' + \mathrm{i}\omega g'v$. Define

$$\mathbf{g}_k = \left[\left(g'\psi_k\right)(x_0), \cdots, \left(g'\psi_k\right)^{(m_0-1)}(x_0), \cdots, \left(g'\psi_k\right)(x_\nu), \cdots, \left(g'\psi_k\right)^{(m_\nu-1)}(x_\nu)\right]^\top.$$

If the vectors $\{\mathbf{g}_0, \cdots, \mathbf{g}_n\}$ are linearly independent, then for sufficiently large ω the system has a unique solution and

$$I[f] - Q^L[f] \sim \mathcal{O}(\omega^{-s-1}),$$

where

$$Q^L[f] \equiv \left[v(x)\,\mathrm{e}^{\mathrm{i}\omega g(x)}\right]_a^b = v(b)\,\mathrm{e}^{\mathrm{i}\omega g(b)} - v(a)\,\mathrm{e}^{\mathrm{i}\omega g(a)}. \tag{4.2}$$

Proof We know that $I[f] - Q^L[f] = I[f] - I[L[v]] = I[f - L[v]]$. Hence we use Corollary 1, in a manner similar to the proof of Theorem 2. Unfortunately, $L[v]$ depends on ω so we need to show that all the functions in the set $\left\{f - L[v], \cdots, f^{(s+1)} - L[v]^{(s+1)}\right\}$ are bounded for increasing ω. Since $\left\{f, \cdots, f^{(s+1)}\right\}$ are by definition independent of ω, we need only show that $\left\{L[v], \cdots, L[v]^{(s+1)}\right\}$ are $\mathcal{O}(1)$.

6

The vector of coefficients $\mathbf{c} = [c_0, \cdots, c_n]^\top$ solves the system of equation $A\mathbf{c} = \mathbf{f}$, where

$$
A = \begin{pmatrix}
L[\psi_0](x_0) & \cdots & L[\psi_n](x_0) \\
\vdots & \ddots & \vdots \\
L[\psi_0]^{(m_0-1)}(x_0) & \cdots & L[\psi_n]^{(m_0-1)}(x_0) \\
\vdots & \ddots & \vdots \\
L[\psi_0](x_\nu) & \cdots & L[\psi_n](x_\nu) \\
\vdots & \ddots & \vdots \\
L[\psi_0]^{(m_\nu-1)}(x_\nu) & \cdots & L[\psi_n]^{(m_\nu-1)}(x_\nu)
\end{pmatrix}, \qquad
\mathbf{f} = \begin{pmatrix}
f(x_0) \\
\vdots \\
f^{(m_0-1)}(x_0) \\
\cdots \\
f(x_\nu) \\
\vdots \\
f^{(m_\nu-1)}(x_\nu)
\end{pmatrix}.
$$

For notational brevity we regard matrices as row vectors whose entries are column vectors. If we let

$$
\mathbf{p}_k = \left[\psi_k'(x_0), \cdots, \psi_k^{(m_0)}(x_0), \cdots, \psi_k'(x_\nu), \cdots, \psi_k^{(m_\nu)}(x_\nu) \right]^\top,
$$

then $A = [\mathbf{p}_0 + \mathrm{i}\omega\mathbf{g}_0, \cdots, \mathbf{p}_n + \mathrm{i}\omega\mathbf{g}_n]$. In later proofs we will also use $\mathbf{a}_k = \mathbf{p}_k + \mathrm{i}\omega\mathbf{g}_k$ to denote the $(k+1)$th column of A.

Using Cramer's rule we find that $c_k = \frac{\det D_k}{\det A}$ for $0 \le k \le n$, where D_k is the matrix A with the $(k+1)$th column replaced by $\mathbf{f}$. Note that all the entries of the matrix D_k, except for a single column, are of order $\mathcal{O}(\omega)$. Hence it is obvious from the definition of the determinant that $\det D_k = \mathcal{O}(\omega^n)$.

We now show that $\frac{1}{\det A} = \mathcal{O}(\omega^{-n-1})$. We know that

$$
\det A = \det [\mathrm{i}\omega\mathbf{g}_0, \cdots, \mathrm{i}\omega\mathbf{g}_n] + \mathcal{O}(\omega^n) = (\mathrm{i}\omega)^{n+1} \det [\mathbf{g}_0, \cdots, \mathbf{g}_n] + \mathcal{O}(\omega^n)
$$

But, by the hypothesis, the columns of this determinant are linearly independent, hence this determinant is not zero and $\det A$ is an $(n+1)$th degree polynomial in ω. If ω is sufficiently large, then the ω^{n+1} term overwhelms the $\mathcal{O}(\omega^n)$ term and $\det A \ne 0$, which proves that the system has a unique solution. Furthermore $\frac{1}{\det A} = \mathcal{O}(\omega^{-n-1})$, and we have shown that $c_k = \mathcal{O}(\omega^{-1})$. Since each ψ_k is independent of ω, it follows that v and its derivatives are also $\mathcal{O}(\omega^{-1})$. Thus,

$$
L[v]^{(j)} = v^{(j+1)} + \mathrm{i}\omega \sum_{k=0}^{j} g^{(k+1)} v^{(j-k)} = \mathcal{O}(\omega^{-1}) + \mathcal{O}(1) = \mathcal{O}(1)
$$

for all $0 \le j \le s+1$. We have satisfied the hypotheses of Corollary 1 and the theorem follows. □

Theorem 4 provides somewhat simpler conditions on the basis $\{\psi_k\}_0^n$ in the preceding theorem. It is especially helpful as it ensures that the standard polynomial basis can be used with the Levin-type method and any choice of nodes and multiplicities. Recall from (Powell, 1981) that stating that a basis is a Chebyshev set is equivalent to stating that it spans a set M that satisfies the Haar condition, or in other words that every function $u \in M$ has less than $n+1$ roots to the equations $u(x) = 0$ in the interval $[a, b]$.

Theorem 4 *Suppose that the basis $\{\psi_k\}_0^n$ is a Chebyshev set. Then the conditions on $\{\mathbf{g}_k\}_0^n$ of the preceding theorem are satisfied for all choices of $\{x_k\}_0^\nu$ and $\{m_k\}_0^\nu$.*

Proof Let M be equal to the span of $\{\psi_k\}_0^n$. We begin by showing that $\{g'\psi_k\}_0^n$ is a Chebyshev set. Note that $\{g'\psi_k\}_0^n$ is a family of linearly independent functions, since $\sum c_k g'\psi_k = g' \sum c_k \psi_k$
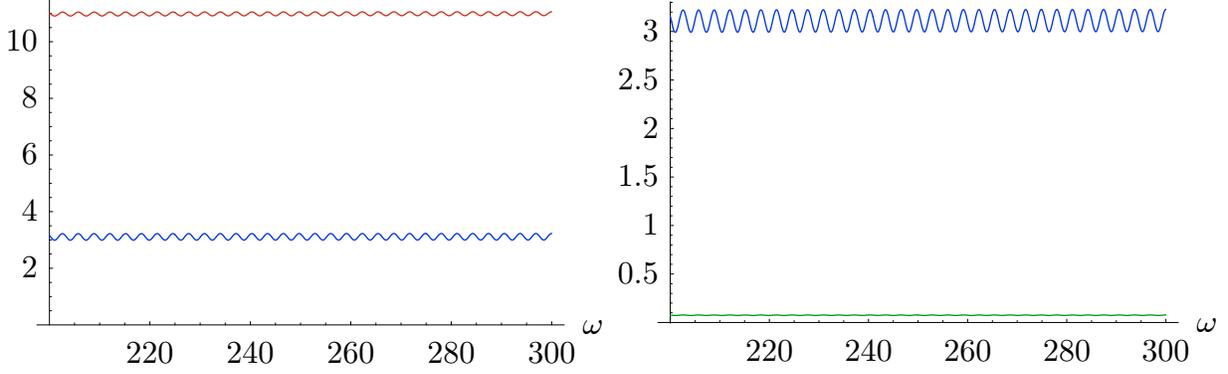
Figure 4: The error scaled by ω^3 of $Q_2^A[f]$ (left figure, top), $Q^L[f]$ (left figure, bottom)/(right figure, top) and $Q^F[f]$ (right figure, bottom) both with only endpoints and multiplicities two for $I[f] = \int_0^1 \cos(x)\, e^{i\omega(x^2+x)} dx$.

and $g' \neq 0$. Let $\tilde{M} = \operatorname{span} \{g'\psi_k\}_0^n$ and $\tilde{u} \in \tilde{M}$, where $\tilde{u}$ is not identically zero. We know that $\tilde{u} = g'u$ for some $u \in M$, and u is equal to zero less than $n+1$ times. But if $u(x) \neq 0$ then $\tilde{u}(x) \neq 0$. Thus $\tilde{M}$ satisfies the Haar condition. It follows that the vectors $[g'(y_0)\,\psi_k(y_0), \cdots, g'(y_n)\,\psi_k(y_n)]^\top$ for $0 \leq k \leq n$ are linearly independent for any choice of nodes $\{y_k\}_0^n$ (Powell, 1981). Thus, by a trivial limiting argument, we know that $\{\mathbf{g}_0, \cdots, \mathbf{g}_n\}$ are linearly independent. $\qquad\square$

The simplest and most obvious choice for $\{\psi_k\}$ is the standard basis of polynomials, which we know is a Chebyshev set. In fact this choice is equivalent to the Filon-type method for the Fourier oscillator case. This was proved in (Xiang, 2005) for the original Levin method (i.e. multiplicities all one), and the proof is trivial to generalize for the preceding Levin-type method. For the remainder of this section we assume that $\{\psi_k\}$ is the standard basis of polynomials.

How does the Levin-type method compare numerically to the asymptotic and Filon-type methods? Consider the case with $g(x) = x^2 + x$ and $f(x) = \cos x$. We fix s equal to two, hence the endpoints for the Filon-type and Levin-type methods must have multiplicity at least two, and we obtain Figure 4. This figure suggests that, in reasonable situations, the Levin-type method is a clear improvement over the asymptotic method, though not quite as accurate as the Filon-type method.

Figure 5 compares the Levin-type method and the Filon-type method with the addition of two sample points. This graph helps emphasize the effectiveness of adding node points within the interval of integration. With just two node points, only one of which has multiplicity greater than one, the error of $Q^L[f]$ is less than a sixth of what it was. In fact it is fairly close to the former $Q^F[f]$ while still not requiring the knowledge of moments. On the other hand, adding the same node points and multiplicities to $Q^F[f]$ results in an error indistinguishable from zero in comparison to the original $Q^L[f]$. It should be emphasized that even $Q^L[f]$ with only endpoints is still a very effective method, as all the values in this graph are divided by $\omega^3 \geq 200^3 = 8 \cdot 10^6$.

It comes as no surprise that the hierarchy of accuracy between the asymptotic, Filon-type, and Levin-type methods depends on the choice of f and g. After all, we have already seen that the Filon-type method can be less accurate than the asymptotic method when f exhibits Runge's phenomenon. Further in this paper, we will see an example where the Levin-type method with polynomial basis is significantly more accurate than the Filon-type method, and oscillates between more accurate and less accurate than the asymptotic method, for increasing ω.
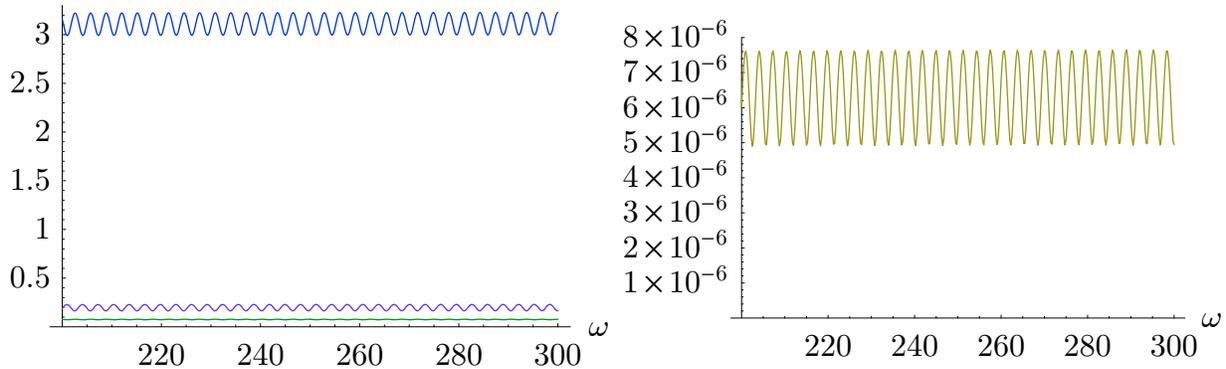
Figure 5: The error scaled by ω^3 of $Q^L[f]$ (left figure, top) and $Q^F[f]$ both with only endpoints and multiplicities two (left figure, bottom) compared to $Q^L[f]$ (left figure, middle) and $Q^F[f]$ (right figure) both with nodes $\left\{0, \frac{1}{4}, \frac{2}{3}, 1\right\}$ and multiplicities $\{2, 2, 1, 2\}$ for $I[f] = \int_0^1 \cos(x)\,\mathrm{e}^{\mathrm{i}\omega(x^2+x)}\mathrm{d}x$.

5 Choosing a basis

It is important to note that, for the Levin-type method, there is no particular reason to use polynomials for $\{\psi_k\}$. Not only can we greatly improve the accuracy of the approximation by choosing the basis wisely, but surprisingly we can even obtain higher asymptotic order. The idea is to choose $\{\psi_k\}$ so that $L[v]$ is qualitatively similar in shape to f within the interval of integration. Pretend for a moment that $Q_s^A[f]$ is equal to $I[f] = \left[F(x)\,\mathrm{e}^{\mathrm{i}\omega g(x)}\right]_a^b$. Then $F(x) = -\sum_{k=1}^s \frac{1}{(\mathrm{i}\omega)^k}\sigma_k[f](x)$. This suggests that a reasonable choice for $\{\psi_k\}$ is to define $\psi_0 = 1$ and $\psi_k = \sigma_k[f]$ for $k \geq 1$. Provided that this choice for $\{\psi_k\}$ satisfies the hypotheses of the Levin-type method, it turns out that we obtain an error of order $\mathcal{O}\left(\omega^{-n-s-1}\right)$ for $n = \sum_{k=0}^\nu m_k - 1$, i.e. where $n+1$ is the number of equations in the Levin-type method system. This is a very significant improvement since, unlike in the case of the Filon-type method as well as the Levin-type method with polynomial basis, node points within the interval increase the order and adding an additional multiplicity to each endpoint increases the order by three.

To prove the order of the error we rely heavily on Cramer's rule. As a result we need to do several determinant manipulations. Hence we derive the following lemma, which we use repeatedly in the proof of Theorem 6.

Lemma 5 *Let $\psi_0 = 1, \psi_1 = \frac{f}{g'}$, and $\psi_{k+1} = \frac{\psi_k'}{g'}$ for integer $k \geq 1$. If $k \geq 1$ then*

$$\det\left[\mathbf{g}_k, \mathbf{a}_k, \cdots, \mathbf{a}_{k+j}, B\right] = \det\left[\mathbf{g}_k, \mathbf{g}_{k+1}, \cdots, \mathbf{g}_{k+j+1}, B\right] \tag{5.3}$$

where $\mathbf{g}_k$ and $\mathbf{a}_k$ were defined in Theorem 3 and B represents any additional columns that render the matrix square.

Proof Note that $\psi_k' = g'\psi_{k+1}$, for $k \geq 1$. Hence we can rewrite $\mathbf{p}_k$ as

$$\begin{aligned}
\mathbf{p}_k &= \left[\psi_k'(x_0), \cdots, \psi_k^{(m_0)}(x_0), \cdots, \psi_k'(x_\nu), \cdots, \psi_k^{(m_\nu)}(x_\nu)\right]^\top \\
&= \left[(g'\psi_{k+1})(x_0), \cdots, (g'\psi_{k+1})^{(m_0-1)}(x_0), \cdots, (g'\psi_{k+1})(x_\nu), \cdots, (g'\psi_{k+1})^{(m_\nu-1)}(x_\nu)\right]^\top \\
&= \mathbf{g}_{k+1}
\end{aligned}$$

9

Recall that $\mathbf{a}_k = \mathbf{p}_k + i\omega\mathbf{g}_k$. Thus multiplying the first column of the determinant on the left hand side of (5.3) by $i\omega$ and subtracting it from the second results in a second column equal to $\mathbf{a}_k - i\omega\mathbf{g}_k = \mathbf{p}_k = \mathbf{g}_{k+1}$. Clearly we can repeat this process on the remaining columns, hence the lemma follows by an inductive argument. $\qquad\square$

Note that Lemma 5 holds for any column interchange that occurs on both sides of the equality. Using this lemma we can prove the following theorem:

Theorem 6 *Let $\{\psi_k\}$ be defined as in the preceding lemma, and suppose that $\{x_k\}_0^\nu$, $\{m_k\}_0^\nu$, and $\{\psi_k\}_0^n$ satisfy the conditions for Theorem 3. Then, if $m_0, m_\nu \geq s$,*

$$I[f] - Q^L[f] \sim \mathcal{O}\left(\omega^{-n-s-1}\right),$$

where as before $n = \sum_{k=0}^\nu m_k - 1$.

Proof

Provided that we can show that the functions in the set $\left\{L[v] - f, \cdots, L[v]^{(s+1)} - f^{(s+1)}\right\}$ are of order $\mathcal{O}(\omega^{-n})$, the theorem will follow from Corollary 1. If we fix $0 \leq j \leq s+1$, then

$$L[v]^{(j)} - f^{(j)} = \sum_{k=0}^n c_k L[\psi_k]^{(j)} - f^{(j)} = \sum_{k=0}^n c_k \left(\psi_k^{(j+1)} + i\omega \left(g'\psi_k\right)^{(j)}\right) - f^{(j)}$$

$$= i\omega c_0 g^{(j+1)} + \sum_{k=1}^n c_k \left[\left(g'\psi_{k+1}\right)^{(j)} + i\omega \left(g'\psi_k\right)^{(j)}\right] - \left(g'\psi_1\right)^{(j)}$$

$$= i\omega c_0 g^{(j+1)} + (i\omega c_1 - 1)\left(g'\psi_1\right)^{(j)} + \sum_{k=2}^n (c_{k-1} + i\omega c_k)\left(g'\psi_k\right)^{(j)} + c_n \left(g'\psi_{n+1}\right)^{(j)}$$

$$= \frac{1}{\det A}\left[i\omega \det D_0\, g^{(j+1)} + (i\omega \det D_1 - \det A)\left(g'\psi_1\right)^{(j)}\right.$$

$$\left. + \sum_{k=2}^n (\det D_{k-1} + i\omega \det D_k)\left(g'\psi_k\right)^{(j)} + \det D_n \left(g'\psi_{n+1}\right)^{(j)}\right].$$

Recall that we showed in the proof of Theorem 3 that $\frac{1}{\det A} = \mathcal{O}\left(\omega^{-n-1}\right)$. Hence it is sufficient to show that the numerator of the preceding fraction is $\mathcal{O}(\omega)$. There are four types of terms we need to handle: $i\omega \det D_0$, $i\omega \det D_1 - \det A$, $\det D_{k-1} + i\omega \det D_k$ for integer $2 \leq k \leq n$, and $\det D_n$. The first of these cases follows immediately from Lemma 5. For the second case note that

$$\det A - i\omega \det D_1 = \det [\mathbf{a}_0, \mathbf{p}_1 + i\omega\mathbf{g}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n] - i\omega \det D_1$$

$$= \det [\mathbf{a}_0, \mathbf{p}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n] + i\omega \det [\mathbf{a}_0, \mathbf{g}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n] - i\omega \det D_1$$

$$= i\omega \det [\mathbf{g}_0, \mathbf{g}_2, \mathbf{a}_2, \cdots, \mathbf{a}_n],$$

where we use the fact that $\mathbf{a}_0 = i\omega\mathbf{g}_0$ and $\mathbf{g}_1 = \mathbf{f}$, since $g'(x_k)\psi_1(x_k) = f(x_k)$. After applying Lemma 5 to this determinant, it is clear that this case is also $\mathcal{O}(\omega)$.

The third case is handled in a very similar manner. Like before we begin by rewriting determinants:

$$\det D_{k-1} + i\omega \det D_k = \det [\mathbf{a}_0, \cdots, \mathbf{a}_{k-2}, \mathbf{g}_1, \mathbf{p}_k + i\omega\mathbf{g}_k, \mathbf{a}_{k+1}, \cdots, \mathbf{a}_n]$$

$$+ i\omega \det [\mathbf{a}_0, \cdots, \mathbf{a}_{k-2}, \mathbf{p}_{k-1} + i\omega\mathbf{g}_{k-1}, \mathbf{g}_1, \mathbf{a}_{k+1}, \cdots, \mathbf{a}_n]$$

$$= \det [\mathbf{a}_0, \cdots, \mathbf{a}_{k-2}, \mathbf{g}_1, \mathbf{g}_{k+1}, \mathbf{a}_{k+1}, \cdots, \mathbf{a}_n]$$
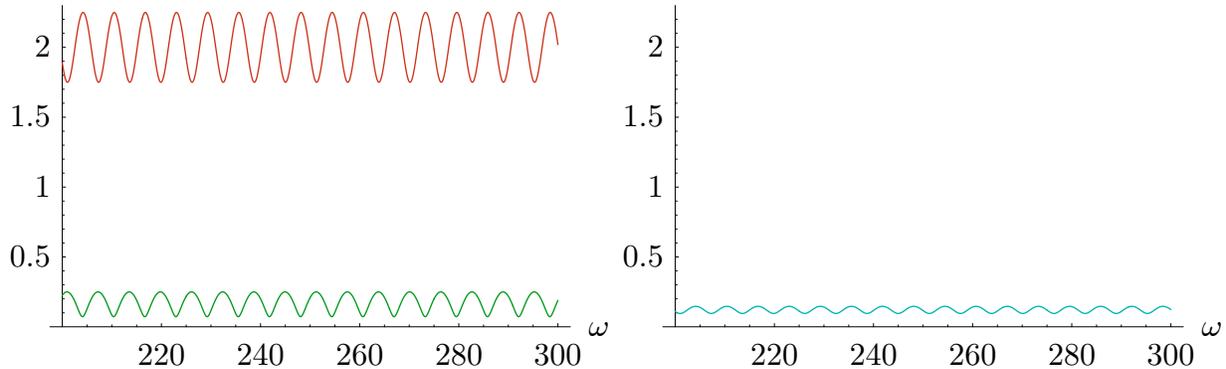
Figure 6: The error scaled by ω^4 of $Q_3^A[f]$ (left figure, top), $Q^F[f]$ with endpoints for nodes and multiplicities 2 (left figure, bottom), and $Q^B[f]$ in with nodes $\{0, \frac{1}{2}, 1\}$ and multiplicities one (right figure) for $I[f] = \int_0^1 \log(x+1)\, e^{i\omega x} dx$.

$$
\begin{aligned}
&+ i\omega \det\left[\mathbf{a}_0, \cdots, \mathbf{a}_{k-2}, \mathbf{g}_1, \mathbf{g}_k, \mathbf{a}_{k+1}, \cdots, \mathbf{a}_n\right] \\
&+ i\omega \det\left[\mathbf{a}_0, \cdots, \mathbf{a}_{k-2}, \mathbf{g}_k, \mathbf{g}_1, \mathbf{a}_{k+1}, \cdots, \mathbf{a}_n\right] \\
&+ (i\omega)^2 \det\left[\mathbf{a}_0, \cdots, \mathbf{a}_{k-2}, \mathbf{g}_{k-1}, \mathbf{g}_1, \mathbf{a}_{k+1}, \cdots, \mathbf{a}_n\right] \\
&= \det\left[\mathbf{a}_0, \cdots, \mathbf{a}_{k-2}, \mathbf{g}_1, \mathbf{g}_{k+1}, \mathbf{a}_{k+1}, \cdots, \mathbf{a}_n\right] \\
&+ (i\omega)^2 \det\left[\mathbf{a}_0, \cdots, \mathbf{a}_{k-2}, \mathbf{g}_{k-1}, \mathbf{g}_1, \mathbf{a}_{k+1}, \cdots, \mathbf{a}_n\right].
\end{aligned}
$$

After using Lemma 5 twice the first of these determinants is clearly $\mathcal{O}(\omega)$. But using the lemma on the second determinant results in two columns being equal to $\mathbf{g}_{k-1}$, hence the determinant is zero. The fourth and final case, much like the first case, is $\mathcal{O}(\omega)$ due to Lemma 5. Hence we have shown that $L[v]^{(j)} - f^{(j)}$ is of order $\mathcal{O}(\omega^{-n})$ for all $0 \le j \le s+1$, and the proof is complete. □

To emphasize the distinction, we denote the Levin-type method with the standard polynomial basis as $Q^L[f]$ and the Levin-type method with the basis of the preceding theorem, which we call the asymptotic basis, as $Q^B[f]$. Clearly, when the same node points and multiplicities are used and ω is sufficiently large, $Q^B[f]$ is a substantive improvement over $Q^L[f]$ and $Q^F[f]$. Of course, it also requires $f^{(k)}$ for k up to $n+s-1$, where the Filon-type method only requires $f^{(k)}$ for k up to $s-1$. Thus in some sense it is more appropriate to compare $Q^B[f]$ with other methods of the same order.

Consider the Fourier oscillator and let $f(x) = \log(x+1)$. We compare methods of order $\mathcal{O}(\omega^{-4})$, hence fix $s = 3$. This includes $Q_3^A[f]$, $Q^F[f]$ (which is equivalent to $Q^L[f]$) with nodes $\{0, 1\}$ and multiplicities $\{2, 2\}$, and $Q^B[f]$ using nodes $\{0, \frac{1}{2}, 1\}$ and multiplicities all one. With this set up we obtain Figure 6. The results are decent, with $Q^B[f]$ being slightly more accurate than $Q^F[f]$ on average.

The problem with $Q_s^A[f]$ and $Q^F[f]$ is that in general as $s \to \infty$ these methods diverge. Hence another worthwhile comparison is to see how $Q^B[f]$ compares to these two methods for fixed ω and increasing s. Thus fix $\omega = 50$, chosen purposely relatively small since the larger ω, the longer it takes for increasing s to cause the approximation to diverge. This choice results in Figure 7, where we take the base-10 logarithms of the error. This figure clearly shows the benefit of using $Q^B[f]$ for this particular case. Though at lower orders the error of $Q^F[f]$ and $Q^B[f]$ are very similar, at higher orders they differ by orders of magnitude. For $\mathcal{O}(\omega^{-9})$, the error of $Q^B[f]$ is slightly better than 10^{-16} while the error of $Q^F[f]$ is slightly better than 10^{-12}, and the error of $Q^A[f]$ is not
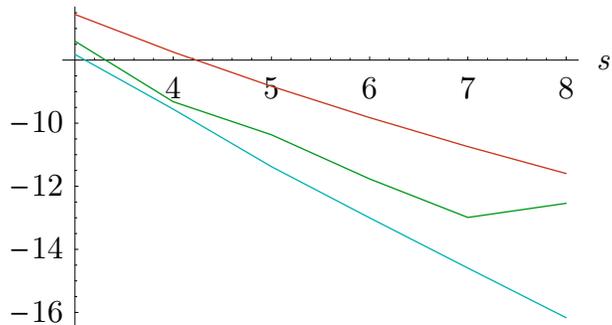
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Figure 7: The base-10 logarithm of the error of $Q_s^A[f]$ (top), $Q^F[f]$ with endpoints for nodes and multiplicities s (middle), and $Q^B[f]$ with nodes $\{k/(s-1)\}_{k=0}^{s-1}$ and multiplicities all one (bottom) for $I[f] = \int_0^1 \log(x+1)\, e^{i\omega x} dx$.
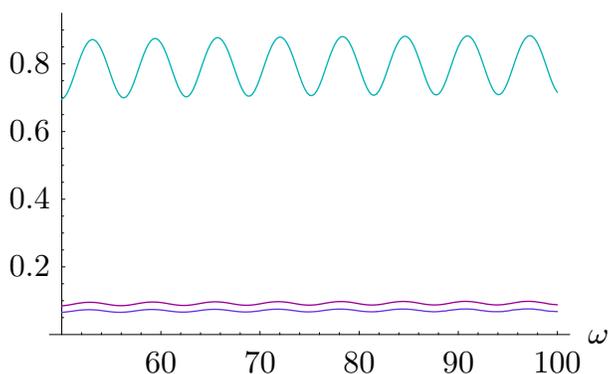

Figure 8: The error scaled by ω^6 of $Q^B[f]$ with nodes $\left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$ and multiplicities all one (bottom), nodes $\left\{0, \frac{1}{2}, 1\right\}$ and multiplicities $\{1, 3, 1\}$ (middle), and nodes $\{0, 1\}$ and multiplicities both equal to two (top) for $I[f] = \int_0^1 \log(x+1)\, e^{i\omega x} dx$.

even 10^{-12}. At around 10^{-16} we reach IEEE machine precision, hence it would be meaningless to extend this graph to higher orders.

We can also compare $Q^B[f]$ with itself under different choices of node points. Though we retain the same f and g, we compare different methods of order $\mathcal{O}(\omega^{-6})$ to increase the number of possible node choices. We consider three choices of nodes and multiplicities: nodes $\left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$ and multiplicities all one, nodes $\left\{0, \frac{1}{2}, 1\right\}$ and multiplicities $\{1, 3, 1\}$, and nodes $\{0, 1\}$ and multiplicities both equal to two. This results in Figure 8. We take relatively mild values for ω as any value significantly larger and the accuracy reaches IEEE machine precision. It is not entirely suprising that the more concentrated the sampling the less accurate the approximation. Though they are not displayed in the preceding figure, for comparison $Q_5^A[f]$ performed horribly, oscillating between 23 and 25, while $Q^F[f]$ with nodes $\{0, 1\}$ and multiplicities five performed roughly in the middle of the pack, oscillating between 0.2 and 0.8.
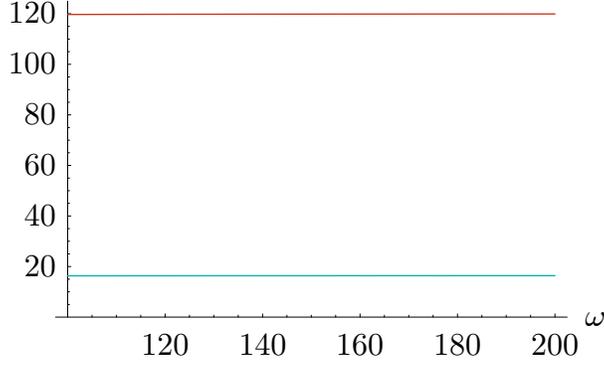
Figure 9: The error scaled by ω^6 of $Q_5^A[f]$ (top) and $Q^B[f]$ with nodes $\{1, 5, 10, 20, \infty\}$ and multiplicities all one (bottom) for $I[f] = \int_0^1 \frac{1}{x} e^{i\omega x} dx$ compared to $E_1(i\omega)$.
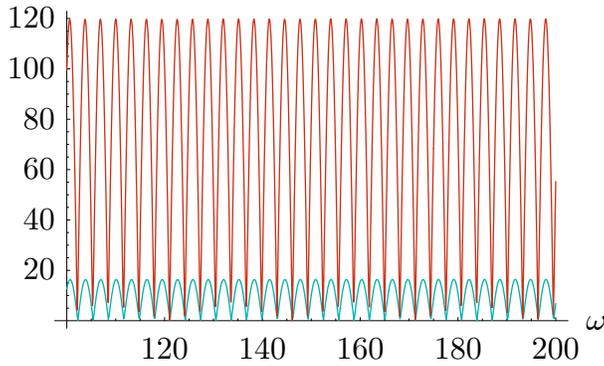

Figure 10: The error scaled by ω^6 of $\operatorname{Re} Q_5^A[f]$ (top) and $\operatorname{Re} Q^B[f]$ with nodes $\{1, 5, 10, 20, \infty\}$ and multiplicities all one (bottom) for $I[f] = \int_0^1 \frac{1}{x} e^{i\omega x} dx$ compared to $-\operatorname{Ci}(\omega)$.

Now consider the case of

$$E_1(-i\omega) = \int_1^\infty \frac{e^{i\omega x}}{x} dx$$

where E_1 is the exponential integral as defined in (Abramowitz and Stegun, 1964). This function is important since we can derive the sine and cosine integrals from its real and imaginary parts. Note that though we never explicitly handled the case of $b = \infty$, all the proofs up to this point are valid for this situation as long as we can integrate by parts. Thus we can use the asymptotic method with $\sigma_k[f](x) = f^{(k-1)}(x) = (-1)^{k-1}(k-1)! x^{-k}$, meaning that $\sigma_k[f](\infty) = 0$ and $\sigma_k[f](1) = (-1)^{k-1}(k-1)!$ for all k. Since $\sigma_k[f](\infty)$ is always zero, we have the added benefit that it is only necessary to evaluate f and its derivatives at one of the endpoints to obtain the desired order. Thus we derive the following asymptotic expansion:

$$E_1(-i\omega) \sim e^{i\omega} \sum_{k=1}^\infty \frac{(-1)^{k-1}(k-1)!}{(-i\omega)^k}.$$

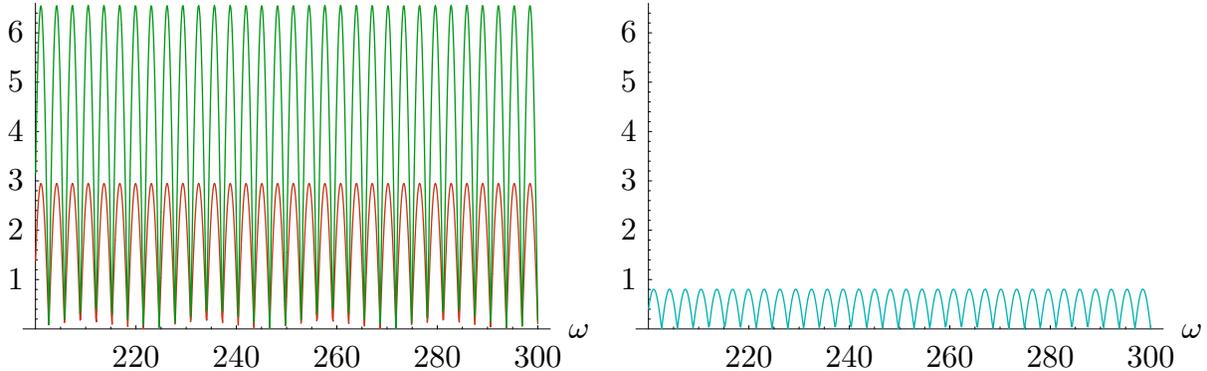It should come as no surprise that this is equivalent to the expansion in (Abramowitz and Stegun, 1964).

Figure 11: The error scaled by ω^4 of $Q_3^A[f]$ (left figure, bottom), $Q^F[f]$ with only endpoints and multiplicities two (left figure, top), and $Q^B[f]$ with end points and two Chebyshev nodes, all with multiplicity one (right figure) for $I[f] = \int_0^1 \frac{1}{1+10x^2}e^{i\omega x}dx$.

Clearly neither Filon-type method nor Levin-type method with polynomial basis can handle this situation since polynomials diverge at ∞. On the other hand, we can use the asymptotic basis with the Levin-type method to derive an approximation. Consider the case of arbitrarily chosen nodes $\{1, 5, 10, 20, \infty\}$ with multiplicities all one. This has order of error $\mathcal{O}(\omega^{-6})$, thus we compare it to the asymptotic method with $s = 5$ in Figure 9. Even with arbitrarily chosen nodes, $Q^B[f]$ is substantially more accurate than simply using the asymptotic expansion; in this case it has less than a tenth of the error on average. We can also compare the real parts of each approximation to $-\text{Ci}(\omega)$, where Ci is the cosine integral as defined in (Abramowitz and Stegun, 1964). This results in Figure 10.

We now consider again the function that suffered from Runge's phenomenon. Since $Q^B[f]$ is not polynomial interpolation, there is a good chance that Runge's phenomenon will not affect us in the same way. In fact, numerical tests show that $Q^B[f]$ has significantly less error than its polynomial counterparts. Direct computation shows that $\det A$ is a polynomial in ω of degree n, not of degree $n + 1$. Fortunately, the proof of Theorem 6 holds as is, except that $Q^B[f]$ now has error of order $\mathcal{O}(\omega^{-n-s})$. Again we compare methods of similar order in Figure 11, which shows that $Q^B[f]$ is the best of the three methods tried.

Another situation somewhat similar to Runge's phenomenon is when f increases much too fast to be accurately approximated by polynomials. Let $f(x) = e^{10x}$ and $g(x) = x^2 + x$. Note that this appears to be a ludicrously difficult example — not only do we have high oscillations but f exceeds $22,000$ in the interval of integration! Amazingly, we will see that the methods described within this paper are still very accurate, especially the Levin-type method with asymptotic basis. We compare $Q^B[f]$ which has only endpoints for nodes and multiplicities all one to $Q_2^A[f]$ and $Q^F[f]$ with only endpoints for nodes and multiplicities both two in Figure 12. We omit the proof that the vectors $\{\mathbf{g}_0, \cdots, \mathbf{g}_n\}$ associated with $Q^B[f]$ are linearly independent, as it is a simple exercise in linear algebra.

In this example $Q^F[f]$ produces a tremendously bad approximation, due to the difficulty in interpolating an exponential by a polynomial. As seen in Table 1, the actual error for $\omega = 200$ is about 0.042. On the other hand, $Q_2^A[f]$ performed significantly better than the Filon-type method, though still not spectacularly, with an error of approximately 0.0083 for $\omega = 200$. The star of this show is clearly $Q^B[f]$, where the actual error for $\omega = 200$ is about 0.000585; less than a tenth of the error of $Q_2^A[f]$.
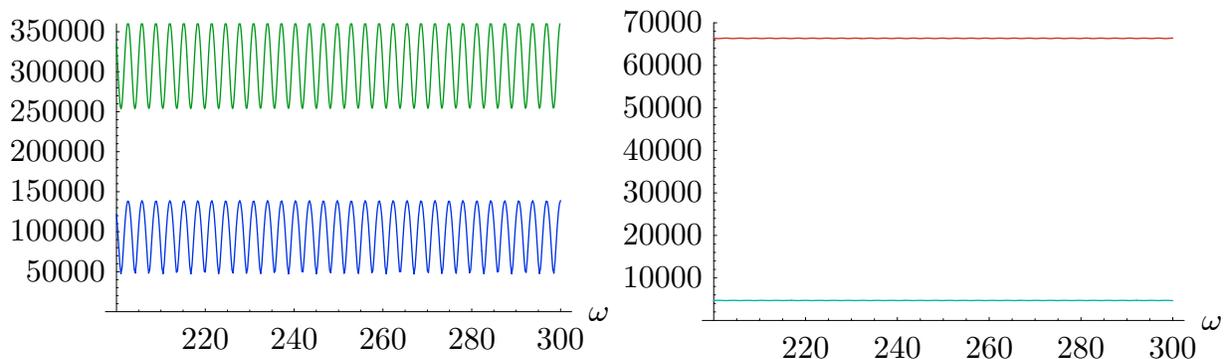
14

Figure 12: The error scaled by ω^3 of $Q^F[f]$ with endpoints and multiplicities both two (left figure, top), $Q^L[f]$ with endpoints and multiplicities both two (left figure, bottom), $Q_2^A[f]$ (right figure, top), and $Q^B[f]$ with endpoints and multiplicities all zero (right figure, bottom) for $I[f] = \int_0^1 e^{10x} e^{i\omega(x^2+x)} dx$.

s	$Q_s^A[f]$	$Q^F[f]$	$Q^L[f]$	$Q^B[f]$
2	0.0083	0.042	0.015	0.00059
3	0.00011	0.0016	0.00043	$2.8 \cdot 10^{-6}$
5	$1.7 \cdot 10^{-8}$	$1.3 \cdot 10^{-6}$	$3 \cdot 10^{-7}$	$9.9 \cdot 10^{-12}$

Table 1: The absolute value of the errors for $\omega = 200$ of the following methods of order $\mathcal{O}(\omega^{-s-1})$: $Q_s^A[f]$, $Q^F[f]$ and $Q^L[f]$ with endpoints and multiplicities both s, and $Q^B[f]$ with nodes $\{k/(s-1)\}_{k=0}^{s-1}$ and multiplicities all one for $I[f] = \int_0^1 e^{10x} e^{200i(x^2+x)} dx$.

Adding additional nodes to $Q^B[f]$ increases the accuracy further. For example, again with $\omega = 200$, adding a single node at the midpoint decreases the error to $2.79 \cdot 10^{-6}$ while adding nodes at $\frac{1}{4}$, the midpoint, and $\frac{3}{4}$ further decreases the error to the astoundingly small $9.93 \cdot 10^{-12}$. This example demonstrates just how powerful these quadrature techniques are compared to Gaussian quadrature: even with $100,000$ panels Gaussian quadrature had an error of 0.11, not even close to the accuracy of the Filon-type method, to say nothing of $Q^B[f]$.

Acknowledgements

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