On the quadrature of multivariate highly oscillatory integrals over non-polytope domains

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October 3, 2005

Abstract

In this paper, we present a Levin-type method for approximating multivariate highly oscillatory integrals, subject to a non-resonance condition. Unlike existing methods, we do not require the knowledge of moments, which enables us to derive an approximation when the oscillator is complicated, and when the domain is neither a simplex nor a polytope. The accuracy of this method improves as the frequency of oscillations increases. A special case of this method has the property that the asymptotic order increases with each additional sample point.

1 Introduction

Let $\Omega \subset \mathbb{R}^d$ be a connected, open and bounded domain with piecewise smooth boundary. The subject of this paper is a numerical approximation of the multivariate integral

$$I_g[f, \Omega] = \int_\Omega f(\mathbf{x})\, \mathrm{e}^{\mathrm{i}\omega g(\mathbf{x})} \mathrm{d}V,$$

where ω is real and large. We focus on the situation where f and g are in $C^\infty[\Omega]$ and bounded. Furthermore, we assume that g has no critical points, i.e. $\nabla g \neq 0$ within the closure of Ω.

Traditional means of approximating $I_g[f, \Omega]$ fail in the face of high oscillations. Repeated univariate quadrature is completely impractical, as using Gaussian quadrature to approximate such integrals requires an exorbitant amount of sample points even in a single dimension, and the number of required sample points grows exponentially with each additional dimension. In addition, for a fixed number of sample points, both repeated univariate quadrature and Monte Carlo (Press et al., 1988) can easily be seen to have an error of order $\mathcal{O}(1)$ as $\omega \to \infty$, whereas the integral itself is typically of order $\mathcal{O}(\omega^{-d})$ (Stein, 1993). This implies that, for large ω, approximating the integral by zero is more accurate than using traditional quadrature techniques! The method of stationary phase (Olver, 1974) is also unsuitable for our needs, as it requires sophisticated mathematical analysis that depends on the choice of f and g. This is impractical for computational purposes.

In this paper we will derive a Levin-type method for approximating multivariate highly oscillatory integrals, subject to a non-resonance condition on the oscillator g and domain Ω. As in the univariate case, the accuracy actually improves when ω is large—in fact, it has an order of error $\mathcal{O}(\omega^{-s-d})$, where the integer s depends on the information we use about the function f. We also develop a multivariate version of the asymptotic basis, a choice of basis for a Levin-type method such that the order increases with each additional sample point and multiplicity. Finally, we investigate what goes wrong when the non-resonance condition does not hold.

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2 Univariate asymptotic expansion

We begin with an overview of methods for approximating $I_g[f, \Omega]$ when Ω is a one-dimensional domain; a problem which has received considerable attention in recent years (Iserles and Nørsett, 2004; Iserles and Nørsett, 2005a; Olver, 2005). The basic idea behind the recent research is to derive an asymptotic expansion, then use the asymptotic expansion to prove the order of error for methods which have the potential of being considerably more accurate than the partial sums of the asymptotic expansion. The derivation of the following theorem is irrelevant to the purposes of this paper, hence we omit the proof for the sake of succinctness and refer the reader to (Iserles and Nørsett, 2005a).

Theorem 1 *Let $\Omega = (a, b)$ and let f and g be smooth functions in the closure of Ω, such that $g' \neq 0$ in the closure of Ω. Define σ_k as*

$$\sigma_1[f] = \frac{f}{g'}, \qquad \sigma_{k+1}[f] = \frac{\sigma_k[f]'}{g'}, \qquad k \geq 1.$$

Then, for $\omega \to \infty$,

$$I_g[f, \Omega] \sim -\sum_{k=1}^{\infty} \frac{1}{(-\mathrm{i}\omega)^k} \left\{ \sigma_k[f](b)\, \mathrm{e}^{\mathrm{i}\omega g(b)} - \sigma_k[f](a)\, \mathrm{e}^{\mathrm{i}\omega g(a)} \right\}. \tag{2.1}$$

An immediate consequence is the following corollary, which will be used to find the order of error of a Levin-type method. In the following corollary, we use the mth order derivative operator $\mathcal{D}^m$, as defined in Appendix A, in order that its definition is consistent with the multivariate version; namely Corollary 2. Note that we allow f to depend on ω as a parameter.

Corollary 1 *Assume $\Omega = (a, b)$. Suppose that $\|\mathcal{D}^m f\|_\infty = \mathcal{O}(\omega^{-n})$ for every non-negative integer m. Furthermore, suppose that*

$$0 = \mathcal{D}^m f(a) = \mathcal{D}^m f(b)$$

for every non-negative integer $m \leq s - 1$. Then

$$I_g[f, \Omega] \sim \mathcal{O}\big(\omega^{-n-s-1}\big).$$

The proof of this corollary can be found in (Olver, 2005), though it follows almost immediately from the asymptotic expansion. The purpose for allowing f and its derivatives to depend on ω will become clear in Section 7. Until then it is safe to assume that $n = 0$, i.e. f and its derivatives are merely bounded for increasing ω.

3 Univariate Levin-type expansion

One immediate consequence of having an asymptotic expansion is that its partial sums provide a quick-and-dirty numerical approximation. Indeed, unlike traditional integration techniques, the accuracy of an asymptotic expansion improves as the frequency ω increases. Unfortunately, the problem with asymptotic expansions as numerical approximations is that there is a limit to how accurate the approximation can be for any fixed ω. To combat this issue we construct a Levin-type method, a generalization of a method developed in (Levin, 1997). The multivariate Levin-type method will continually 'push' the integral to the boundary until we arrive at univariate integrals, hence the following construction is central to the multivariate version.

The general idea behind the method, as developed by Levin, is that if we have a function F such that $\frac{\mathrm{d}}{\mathrm{d}x}\left[F\mathrm{e}^{\mathrm{i}\omega g}\right] = f\mathrm{e}^{\mathrm{i}\omega g}$ then computing $I_g[f,(a,b)]$ is trivial. We can rewrite this requirement as $\mathcal{L}[F] = f$, for the differential operator

$$\mathcal{L}[F] = F' + \mathrm{i}\omega g' F.$$

Finding such an F explicitly is in general not possible, however we can approximate F by a function $v = \sum c_k \psi_k$, where $\{\psi_0, \ldots, \psi_\nu\}$ is a set of basis functions, using collocation with the operator $\mathcal{L}[v]$. In other words, we solve the system

$$\mathcal{L}[v](x_k) = f(x_k)\,, \qquad k = 0, 1, \ldots, \nu,$$

for some set of nodes $\{x_0, \ldots, x_\nu\}$, in order to determine the coefficients $\{c_0, \ldots, c_\nu\}$. Since the number of nodes is arbitrary, this allows us to increase the accuracy by simply adding additional nodes.

In (Olver, 2005), the current author generalized this idea to obtain a Levin-type method, the major improvement being that we equate both the function values and derivatives of $\mathcal{L}[v]$ and f at the nodes $\{x_0, \ldots, x_\nu\}$, up to given multiplicities $\{m_0, \ldots, m_\nu\}$. This allows us to obtain an arbitrarily high order of error by taking suitably large multiplicities at the endpoints. We repeat the proof of the order of error of a Levin-type method as found in (Olver, 2005), since the proof for the multivariate case will be somewhat similar. The following lemma will be used for both the univariate and multivariate proofs.

Lemma 2 *Suppose that two sets of vectors in $\mathbb{R}^{n+1}$, $\{\mathbf{p}_0, \ldots, \mathbf{p}_n\}$ and $\{\mathbf{g}_0, \ldots, \mathbf{g}_n\}$, are independent of ω, and that $\{\mathbf{g}_0, \ldots, \mathbf{g}_n\}$ are linearly independent. Furthermore let*

$$A = [\mathbf{p}_0 + \mathrm{i}\omega\mathbf{g}_0, \cdots, \mathbf{p}_n + \mathrm{i}\omega\mathbf{g}_n]\,.$$

Then, for suffcently large ω, A is non-singular, and the solution $\mathbf{c} = [c_0, \cdots, c_n]^\top$ to the system $A\mathbf{c} = \mathbf{f}$, for any vector $\mathbf{f}$ independent of ω, satisfies $c_k = \mathcal{O}(\omega^{-1})$ for every integer $0 \le k \le n$.

Proof We know that

$$\det A = \det[\mathrm{i}\omega\mathbf{g}_0, \cdots, \mathrm{i}\omega\mathbf{g}_n] + \mathcal{O}(\omega^n) = (\mathrm{i}\omega)^{n+1}\det[\mathbf{g}_0, \cdots, \mathbf{g}_n] + \mathcal{O}(\omega^n)\,.$$

Since $\det[\mathbf{g}_0, \cdots, \mathbf{g}_n] \neq 0$, this is a polynomial of degree $n+1$, and sufficiently large ω causes the determinant to be nonzero. Furthermore $\frac{1}{\det A} = \mathcal{O}(\omega^{-n-1})$. Due to Cramer's rule, we know that $c_k = \frac{\det D_k}{\det A}$ where D_k is equal to A with the $(k+1)$th column replaced by $\mathbf{f}$. It is clear that $\det D_k = \mathcal{O}(\omega^n)$ as there are exactly n columns with ω terms. Thus the proof is complete. □

By combining this lemma with Corollary 1 we will obtain the proof of the order of error for a Levin-type method. We begin by defining the regularity condition. The regularity condition is satisfied if the functions $\{g'\psi_0, g'\psi_1, \ldots\}$ can interpolate f at the nodes $\{x_0, \ldots, x_\nu\}$ with multiplicities $\{m_0, \ldots, m_\nu\}$. Note that this condition depends on the choice of oscillator g, the nodes, the multiplicities and the basis.

Theorem 3 *Suppose g' is non-zero in the closure of $\Omega \subset \mathbb{R}$ and the regularity condition is satisfied. Let $v = \sum c_k\psi_k$, where $\mathbf{c} = [c_0, \cdots, c_n]^\top$ is determined by solving the system*

$$\mathcal{D}^m\mathcal{L}[v](x_k) = \mathcal{D}^m f(x_k)\,, \qquad m = 0, 1, \ldots, m_k - 1, \qquad k = 0, 1, \ldots, \nu \tag{3.2}$$

for the operator $\mathcal{L}[v] = v' + \mathrm{i}\omega g' v$, and where $n+1$ is the number of equations in this system. Then

$$I_g[f,\Omega] - Q_g^L[f,\Omega] \sim \mathcal{O}\big(\omega^{-s-1}\big)\,,$$

where $s = \min\{m_0, m_\nu\}$ and

$$Q_g^L[f,\Omega] \equiv v(b)\,\mathrm{e}^{\mathrm{i}\omega g(b)} - v(a)\,\mathrm{e}^{\mathrm{i}\omega g(a)}.$$

Proof Note that $Q_g^L[f,\Omega] = I_g[\mathcal{L}[v]\,,\Omega]$. Define the operator $\mathcal{P}[f]$, written in partitioned form, as

$$\mathcal{P}[f] = \begin{pmatrix}\rho_0[f]\\ \vdots \\ \rho_\nu[f]\end{pmatrix}, \qquad \text{for} \qquad \rho_k[f] = \begin{pmatrix} f(x_k) \\ \vdots \\ \mathcal{D}^{m_\nu - 1} f(x_k)\end{pmatrix}, \qquad k = 0,1,\ldots,\nu.$$

In other words, $\mathcal{P}[f]$ maps a function f to a vector whose rows consist of f evaluated at the nodes $\{x_0,\ldots,x_\nu\}$ with multiplicities $\{m_0,\ldots,m_\nu\}$. The system (3.2) can now be rewritten as $A\mathbf{c} = \mathbf{f}$ where

$$A = [\mathbf{p}_0 + \mathrm{i}\omega\mathbf{g}_0, \cdots, \mathbf{p}_n + \mathrm{i}\omega\mathbf{g}_n]\,, \qquad \mathbf{p}_j = \mathcal{P}\big[\psi_j'\big]\,, \qquad \mathbf{g}_j = \mathcal{P}\big[g'\psi_j\big]\,, \qquad \mathbf{f} = \mathcal{P}[f]\,.$$

Due to the regularity condition, we know that $\{\mathbf{g}_0,\ldots,\mathbf{g}_n\}$ are linearly independent. Thus Lemma 2 applies, hence we know that v and its derivatives are $\mathcal{O}\big(\omega^{-1}\big)$. Thus $\mathcal{L}[v]$ and its derivatives are $\mathcal{O}(1)$, meaning that

$$I_g[f,\Omega] - Q_g^L[f,\Omega] = I_g[f,\Omega] - I_g[\mathcal{L}[v]\,,\Omega] = I_g[f - \mathcal{L}[v]\,,\Omega] = \mathcal{O}\big(\omega^{-s-1}\big)\,,$$

by Corollary 1 with the function $f - \mathcal{L}[v]$. □

The regularity condition is actually quite weak: in fact (Olver, 2005) contains a proof that, if $\{\psi_0,\ldots,\psi_n\}$ is the standard polynomial basis or any other Chebyshev set (Powell, 1981), then the vectors $\{\mathbf{g}_0,\ldots,\mathbf{g}_n\}$ are guaranteed to be linearly independent. However, there is no equivalent to a Chebyshev set in higher dimensions (Cheney and Light, 2000). It should be mentioned that there exists another method of approximating these integrals, namely a Filon-type method (Iserles and Nørsett, 2005a). It works by interpolating the function f by a polynomial v, and integrating v directly; assuming that moments are explicitly computable. Though it is often more accurate than a Levin-type method, the requirement of knowing moments makes it much less practical for multivariate integrals.

For a simple example, consider the case of $f(x) = \cos x$ with oscillator $g(x) = \cos x - \sin x$ in $\Omega = (0,1)$, collocating only at the endpoints with multiplicities both one. Figure 1 demonstrates that $Q_g^L[f,\Omega]$ does, in fact, approximate $I_g[f,\Omega]$ with an order of error $\mathcal{O}\big(\omega^{-2}\big)$. This compares to the integral itself which goes to zero like $\mathcal{O}\big(\omega^{-1}\big)$. Had we added internal nodes, the approximation would be the same order but more accurate. Adding multiplicities to the endpoints would cause the order to increase. Further examples and comparisons can be found in (Olver, 2005).

4 Multivariate asymptotic expansion

With a firm concept of how to handle the univariate case, we now begin delving into how to approximate higher dimensional integrals. We closely mirror the univariate version: we first derive an asymptotic expansion, which we then use to prove the order of error for a Levin-type method. We

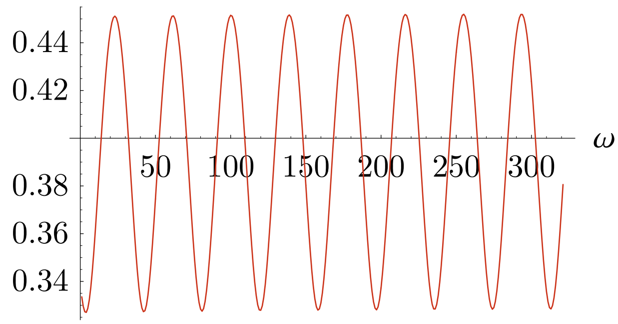

Figure 1: The error of $Q_g^L[f,(0,1)]$, scaled by ω^2, with only endpoints and multiplicities all one, for $I_g[f,(0,1)] = \int_0^1 \cos x \, \mathrm{e}^{\mathrm{i}\omega(\cos x - \sin x)} \mathrm{d}x$.

begin by investigating the case where the non-resonance condition is satisfied, which is somewhat similar in spirit to the condition that g' is nonzero within the interval of integration. The non-resonance condition is satisfied if, for every $\mathbf{x}$ on the boundary of Ω, $\nabla g(\mathbf{x})$ is not orthogonal to the boundary of Ω at $\mathbf{x}$. In addition, $\nabla g \neq 0$ in the closure of Ω, i.e. there are no critical points. Note that the non-resonance condition does not hold true if g is linear and Ω has a completely smooth boundary, such as a circle, since ∇g must be orthogonal to at least one point in $\partial\Omega$.

Based on results from (Iserles and Nørsett, 2005b), we derive the following asymptotic expansion, where $|\mathbf{m}|$ for $\mathbf{m} \in \mathbb{N}^d$ is the sum of the entries, as defined in Appendix A. We also use the notion of a vertex of Ω, for which the definition may not be immediately obvious. Specifically, we define the vertices of Ω as:

- If Ω consists of a single point in $\mathbb{R}^d$, then that point is a vertex of Ω.
- Otherwise, let $\{Z_\ell\}$ be an enumeration of the smooth components of the boundary of Ω, where each Z_ℓ is of one dimension less than Ω, and has a piecewise smooth boundary itself. Then $\mathbf{v} \in \partial\Omega$ is a vertex of Ω if and only if $\mathbf{v}$ is a vertex of some Z_ℓ.

In other words, the vertices are the endpoints of all the smooth one-dimensional edges in the boundary of Ω. In two-dimensions, these are the points where the boundary is not smooth.

Theorem 4 *Suppose that Ω has a piecewise smooth boundary, and that the non-resonance condition is satisfied. Then, for $\omega \to \infty$,*

$$I_g[f,\Omega] \sim \sum_{k=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{k+d}} \Theta_k[f],$$

where $\Theta_k[f]$ depends on $\mathcal{D}^{\mathbf{m}} f$ for all $|\mathbf{m}| \leq k$, evaluated at the vertices of Ω.

Proof Fix an integer $s \geq 1$. From (Iserles and Nørsett, 2005b) we know that, if a domain S is a polytope and g has no critical points in S, then

$$I_g[f,S] = Q_{g,s}^A[f,S] + \frac{1}{(-\mathrm{i}\omega)^s} I_g[\sigma_s, S],$$

where

$$Q^A_{g,s}[f,S] = -\sum_{k=0}^{s-1} \frac{1}{(-\mathrm{i}\omega)^{k+1}} \int_{\partial S} \mathbf{n}^\top \nabla g \frac{\sigma_k}{\|\nabla g\|^2} \mathrm{e}^{\mathrm{i}\omega g} \mathrm{d}S,$$

$\mathbf{n}$ is the outward facing unit normal and

$$\sigma_0 = f, \qquad \sigma_{k+1} = \nabla^\top \left[\frac{\sigma_k}{\|\nabla g\|^2} \nabla g \right], \qquad k \geq 0.$$

Let $\{S_0, S_1, \ldots\}$ be a sequence of polytope domains such that $\lim S_j = \Omega$, where each S_j is a tessellation of Ω. Because ∇g is continuous, there is an open set U containing the closure of Ω such that $\nabla g \neq 0$ in U. Assume that each $S_j \subset U$, which is true whenever a sufficiently fine grid is used.

Note that σ_k is bounded in U for all k, because there are no critical points. Hence, due to the boundedness of each integrand and the dominating convergence theorem, it is clear that

$$I_g[f, S_j] \to I_g[f, \Omega]\,,$$
$$\frac{1}{(-\mathrm{i}\omega)^s} I_g[\sigma_s, S_j] \to \frac{1}{(-\mathrm{i}\omega)^s} I_g[\sigma_s, \Omega]\,,$$
$$\int_{\partial S_j} \mathbf{n}^\top \nabla g \frac{\sigma_k}{\|\nabla g\|^2} \mathrm{e}^{\mathrm{i}\omega g} \mathrm{d}S \to \int_{\partial \Omega} \mathbf{n}^\top \nabla g \frac{\sigma_k}{\|\nabla g\|^2} \mathrm{e}^{\mathrm{i}\omega g} \mathrm{d}S.$$

It follows that $I_g[f,\Omega] = Q^A_{g,s}[f,\Omega] + \frac{1}{(-\mathrm{i}\omega)^s} I_g[\sigma_s,\Omega] = Q^A_{g,s}[f,\Omega] + \mathcal{O}\big(\omega^{-s-d}\big)$, using the fact that $I_g[\sigma_s,\Omega] = \mathcal{O}\big(\omega^{-d}\big)$ (Stein, 1993).

We now prove the theorem by expressing $Q^A_{g,s}[f,\Omega]$ in terms of its asymptotic expansion. Assume the theorem holds true for lower dimensions, where the univariate case follows from Theorem 1. Note that, for each ℓ, there exists a domain $\Omega_\ell \in \mathbb{R}^{d-1}$ and a smooth map $T_\ell : \Omega_\ell \to Z_\ell$ such that Z_ℓ is parameterized by Ω_ℓ using the mapping T_ℓ, where every vertex of Ω_ℓ corresponds to a vertex of Z_ℓ, and vice-versa. We can rewrite each surface integral in $Q^A_{g,s}[f,\Omega]$ as a sum of standard integrals:

$$\int_{\partial \Omega} \mathbf{n}^\top \nabla g \frac{\sigma_k}{\|\nabla g\|^2} \mathrm{e}^{\mathrm{i}\omega g} \mathrm{d}S = \sum_\ell \int_{Z_\ell} \mathbf{n}^\top \nabla g \frac{\sigma_k}{\|\nabla g\|^2} \mathrm{e}^{\mathrm{i}\omega g} \mathrm{d}S = \sum_\ell I_{g_\ell}[f_\ell, \Omega_\ell]\,, \tag{4.3}$$

where f_ℓ is a smooth function multiplied by $\sigma_k \circ T_\ell$, and $g_\ell = g \circ T_\ell$. It follows from the definition of the non-resonance condition that the function g_ℓ satisfies the non-resonance condition in Ω_ℓ. Thus, by assumption,

$$I_{g_\ell}[f_\ell, \Omega_\ell] \sim \sum_{i=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{i+d-1}} \Theta_i[f_\ell]\,,$$

where $\Theta_i[f_\ell]$ depends on $\mathcal{D}^{\mathbf{m}} f_\ell$ for $|\mathbf{m}| \leq i$ applied at the vertices of Ω_ℓ. But $\mathcal{D}^{\mathbf{m}} f_\ell$ depends on $\mathcal{D}^{\mathbf{m}}[\sigma_k \circ T_\ell]$ for $|\mathbf{m}| \leq i$ applied at the vertices of Ω_ℓ, which in turn depends on $\mathcal{D}^{\mathbf{m}} f$ for $|\mathbf{m}| \leq i+k$, now evaluated at the vertices of Z_ℓ, which are also vertices of Ω. The theorem follows from plugging these asymptotic expansions into the definition of $Q^A_{g,s}[f,\Omega]$. □

It is not necessary to find $\Theta_k[f]$ explicitly as we only use this asymptotic expansion for error analysis, not as a means of approximation. The following corollary serves the same purpose as Corollary 1: it will be used to prove the order of error for a multivariate Levin-type method.

Corollary 2 *Let V be the set of all vertices of a domain Ω. Suppose that $\|\mathcal{D}^{\mathbf{m}} f\|_\infty = \mathcal{O}(\omega^{-n})$ for all $\mathbf{m} \in \mathbb{N}^d$. Suppose further that*

$$0 = \mathcal{D}^{\mathbf{m}} f(\mathbf{v})$$

for all $\mathbf{v} \in V$ *and* $\mathbf{m} \in \mathbb{N}^d$ *such that* $|\mathbf{m}| \leq s-1$. *Then*

$$I_g[f,\Omega] \sim \mathcal{O}\left(\omega^{-n-s-d}\right).$$

Proof

We prove this corollary by induction on the dimension d, with the univariate case following from Corollary 1. We begin by showing that $Q^A_{g,s+d}[f,\Omega] = \mathcal{O}\big(\omega^{-n-s-d}\big)$. Since every σ_k depends on f and its partial derivatives of order less than or equal to k, it follows that $\|\sigma_k\|_\infty = \mathcal{O}(\omega^{-n})$. Furthermore, $0 = \mathcal{D}^{\mathbf{m}}\sigma_k(\mathbf{v})$ for all $\mathbf{v} \in V$ and every $|\mathbf{m}| \leq s-k-1$, where $0 \leq k \leq s-1$. Hence (4.3) is of order $\mathcal{O}\big(\omega^{-n-(s-k)-(d-1)}\big)$ for all $0 \leq k \leq s-1$. For $k \geq s$, we know that (4.3) is at least of order $\mathcal{O}\big(\omega^{-n-(d-1)}\big)$. Since each (4.3) is multiplied by $(-\mathrm{i}\omega)^{-k-1}$ in the construction of $Q^A_{g,s+d}[f,\Omega]$, it follows that $Q^A_{g,s+d}[f,\Omega] = \mathcal{O}\big(\omega^{-n-s-d}\big)$. Finally,

$$\left|I_g[f,\Omega] - Q^A_{g,s+d}[f,\Omega]\right| = \left|\frac{1}{(-\mathrm{i}\omega)^{-s-d}} I_g[\sigma_{s+d},\Omega]\right| = \mathcal{O}\left(\omega^{-s-n-d}\right),$$

since $\|\sigma_{s+d}\|_\infty = \mathcal{O}(\omega^{-n})$. Thus $I_g[f,\Omega] \sim \mathcal{O}\big(\omega^{-s-n-d}\big)$. □

As in the univariate case, until Section 7 we assume f and its derivatives in the preceding corollary are $\mathcal{O}(1)$, i.e. $n = 0$. In (Iserles and Nørsett, 2005b), a generalization of Filon-type methods for multivariate integrals was developed, where, as in the univariate case, the function f is interpolated by a polynomial v, and moments are assumed to be available. We will not investigate this method in depth, but mention it as a point of reference.

Remark In this section we used a weaker definition for the non-resonance condition than that which was found in (Iserles and Nørsett, 2005b). Also, for the cited result in Theorem 4, we only require that g has no critical points, whereas the original statement requires that the non-resonance condition holds. This is due to the proofs cited from that paper holding true for the weaker conditions, without any other alterations.

5 Multivariate Levin-type method

We now have the tools needed to construct a Levin-type method for integrating highly oscillatory functions over multidimensional domains. We begin by demonstrating how this can be accomplished over a two-dimensional simplex, followed by a generalization to higher dimensional domains, along with a proof of asymptotic order. Consider the simplex $S = S_2$, as drawn in Figure 2. In the construction of a multivariate Levin-type method we use the multivariate version of the fundamental theorem of calculus, namely the Stokes' theorem, to determine the collocation operator $\mathcal{L}[v]$. First write the integral as a differential form:

$$I_g[f,S] = \iint_S f\mathrm{e}^{\mathrm{i}\omega g}\,\mathrm{d}x\,\mathrm{d}y = \iint_S f\mathrm{e}^{\mathrm{i}\omega g}\mathrm{d}x \wedge \mathrm{d}y.$$

Note that if we have the 1-form $\rho = v\mathrm{e}^{\mathrm{i}\omega g}(\mathrm{d}x + \mathrm{d}y)$, where $v = \sum_{k=0}^{n} c_k\psi_k$ for some set of basis functions $\{\psi_0, \ldots, \psi_n\}$, then

$$\begin{aligned}\mathrm{d}\rho &= (v_x + \mathrm{i}\omega g_x v)\mathrm{e}^{\mathrm{i}\omega g}\mathrm{d}x \wedge \mathrm{d}y + (v_y + \mathrm{i}\omega g_y v)\mathrm{e}^{\mathrm{i}\omega g}\mathrm{d}y \wedge \mathrm{d}x \\ &= (v_x + \mathrm{i}\omega g_x v - v_y - \mathrm{i}\omega g_y v)\mathrm{e}^{\mathrm{i}\omega g}\mathrm{d}x \wedge \mathrm{d}y. \end{aligned} \tag{5.4}$$

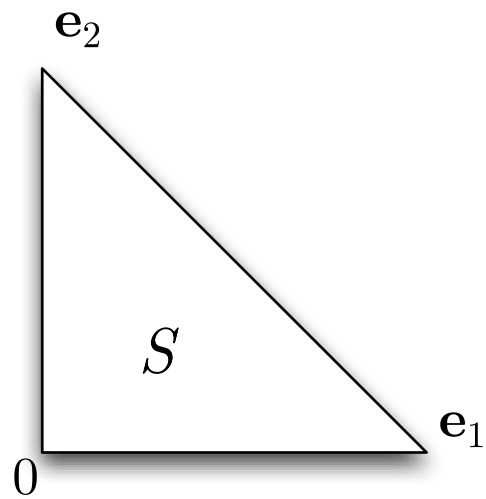

Figure 2: The two-dimensional simplex $S = S_2$, where $\mathbf{e}_1 = [1,0]^\top$ and $\mathbf{e}_2 = [0,1]^\top$.

Thus, in a manner similar to the univariate case, we collocate f using the linear operator $\mathcal{L}[v] = v_x + \mathrm{i}\omega g_x v - v_y - \mathrm{i}\omega g_y v$ at a given set of nodes $\{\mathbf{x}_0, \ldots, \mathbf{x}_\nu\}$. Using the Stokes' theorem, we 'push' the integral to the boundary of the simplex:

$$\iint_S f \mathrm{e}^{\mathrm{i}\omega g} dx \wedge dy \approx \iint_S \mathcal{L}[v]\, \mathrm{e}^{\mathrm{i}\omega g} dx \wedge dy = \iint_S \mathrm{d}\rho = \oint_{\partial S} \rho = \oint_{\partial S} v \mathrm{e}^{\mathrm{i}\omega g}(dx + dy). \tag{5.5}$$

We can now break up this line integral into three line integrals, integrating counter-clockwise. Since the integrand, in vector notation, is orthogonal to the diagonal edge of S, the integral for that particular boundary is zero. Therefore we need only integrate over the edges on the x and y axes. Hence we can rewrite (5.5) as

$$\int_1^0 v(0,y)\, \mathrm{e}^{\mathrm{i}\omega g(0,y)} \mathrm{d}y + \int_0^1 v(x,0)\, \mathrm{e}^{\mathrm{i}\omega g(x,0)} \mathrm{d}x = I_{g(\cdot,0)}[v(\cdot,0)\,, (0,1)] - I_{g(0,\cdot)}[v(0,\cdot)\,, (0,1)]\,.$$

As a result of the non-resonance condition, we know that the derivatives of $g(\cdot,0)$ and $g(0,\cdot)$ are nonzero within the interval of integration; in other words, the integrands of the preceding two univariate integrals do not have stationary points. Thus both of these integrals satisfy the conditions for a univariate Levin-type method: the regularity condition is satisfied whenever polynomials are used as basis functions in one-dimension. Hence we define

$$Q_g^L[f,S] = Q_{g(\cdot,0)}^L[v(\cdot,0)\,, (0,1)] - Q_{g(0,\cdot)}^L[v(0,\cdot)\,, (0,1)]\,.$$

We approach the general case in a similar manner. Suppose we are given nodes $\{\mathbf{x}_0, \ldots, \mathbf{x}_\nu\}$ in $\Omega \subset \mathbb{R}^d$, multiplicities $\{m_0, \ldots, m_\nu\}$ and, for all dimensions less than or equal to d, basis functions $\{\psi_0^d, \ldots, \psi_n^d\}$. Assume further that we are given a positive-oriented boundary of Ω defined as a set of functions $T_\ell : \Omega_\ell \to \mathbb{R}^d$, where $\Omega_\ell \subset \mathbb{R}^{d-1}$ and the ℓth boundary component is the image of T_ℓ. Furthermore, assume we have boundary parameterization for each Ω_ℓ, recursively down to the one-dimensional edges. We define $Q_g^L[f,\Omega]$ recursively as follows:

- If $\Omega \subset \mathbb{R}$, then $Q_g^L[f,\Omega]$ is equivalent to a univariate Levin-type method, as presented earlier in this paper.

- If $\Omega \subset \mathbb{R}^d$, then we begin by choosing a vector $\mathbf{t} = [t_1, \cdots, t_d]^\top \in \mathbb{R}^d$. The purpose and necessity of $\mathbf{t}$ will become clear further in the paper. Typically we let $t_k = \pm 1$, as $\mathbf{t}$ determines the orientation of the approximation, and we want each dimension to be of equal influence. Define the operators

$$\mathcal{J} = \sum_{k=1}^{d} (-1)^k t_k \mathcal{D}^{\mathbf{e}_k}, \qquad \mathcal{L}[v] = \mathcal{J}[v] + \mathrm{i}\omega \mathcal{J}[g]\, v,$$

where $\mathbf{e}_k$ is the kth standard unit vector in $\mathbb{R}^d$. Now, for $\mathbf{c} = [c_0, \cdots, c_n]^\top$ and $v = \sum c_k \psi_k^d$, solve for $\mathbf{c}$ using the following collocation system:

$$\mathcal{D}^{\mathbf{m}}\mathcal{L}[v](\mathbf{x}_k) = \mathcal{D}^{\mathbf{m}} f(\mathbf{x}_k)\,, \qquad 0 \le |\mathbf{m}| \le m_k - 1, \qquad k = 0, 1, \ldots, \nu. \tag{5.6}$$

We conclude by defining

$$Q_g^L[f, \Omega] = \sum Q_{g_\ell}^L[f_\ell, \Omega_\ell]\,, \tag{5.7}$$

where $g_\ell(\mathbf{x}) = g(T_\ell(\mathbf{x}))$ and $f_\ell(\mathbf{x}) = v(T_\ell(\mathbf{x}))\, \mathbf{t} \cdot \mathbf{J}_{T_\ell}^d(\mathbf{x})$, for $\mathbf{x} \in \Omega_\ell$ and $\mathbf{J}_{T_\ell}^d(\mathbf{x})$ the vector of Jacobian determinants associated with the surface differential $(d-1)$-form, as defined in Appendix A. For the proof of the order of error, we require that the set of nodes used for each $Q_{g_\ell}^L[f_\ell, \Omega_\ell]$ contains all vertices of Ω_ℓ, with multiplicities equal to the multiplicity of the vertex mapped to by T_ℓ. Typically we will simply use nodes $\left\{T_\ell^{-1}(\mathbf{x}_j)\right\}$ and multiplicities $\{m_j\}$ for all j such that $\mathbf{x}_j$ is in the range of T_ℓ—in other words every node that is on that particular boundary.

Observe that, since each f_ℓ is linear with respect to v and, by the law of superposition, v is linear with respect to f, we know that $Q_g^L[f, \Omega]$ is linear with respect to f. The multivariate regularity condition requires that the following two conditions hold:

- The functions $\left\{\mathcal{J}[g]\, \psi_0^d, \mathcal{J}[g]\, \psi_1^d, \ldots\right\}$ can interpolate f at the given nodes and multiplicities.
- The regularity condition is satisfied for each Levin-type method in the right-hand side of (5.7).

Note that this is where the vector $\mathbf{t}$ comes in, as it provides a degree of freedom to ensure that this condition is satisfied. For example, if $\mathbf{t} = [1, 1]^\top$ and $g(x, y) = x + y$, then $\mathcal{J}[g] = g_x - g_y = 0$, and the regularity condition can never be satisfied. This holds true even over a simplex reflected over the y-axis, which in fact satisfies the non-resonance condition.

We now show that, if the regularity and non-resonance conditions are satisfied, $Q_g^L[f, \Omega]$ approximates $I_g[f, \Omega]$ with an asymptotic order that depends on the multiplicities at the vertices of Ω.

Theorem 5 *Suppose that both the non-resonance condition and the regularity condition are satisfied. Suppose further that $\{\mathbf{x}_0, \ldots, \mathbf{x}_\nu\}$ contains all vertices of Ω, namely $\left\{\mathbf{x}_{i_0}, \ldots, \mathbf{x}_{i_\eta}\right\}$. Then, for sufficiently large ω, $Q_g^L[f, \Omega]$ is well defined and*

$$I_g[f, \Omega] - Q_g^L[f, \Omega] \sim \mathcal{O}\left(\omega^{-s-d}\right),$$

where $s = \min\left\{m_{i_0}, \ldots, m_{i_\eta}\right\}$.

Proof We begin by assuming that this theorem holds true for all dimensions less than d, with Theorem 3 providing the proof for the univariate case. We first show that

$$I_g[f, \Omega] - I_g[\mathcal{L}[v]\,, \Omega] = I_g[f - \mathcal{L}[v]\,, \Omega] = \mathcal{O}\left(\omega^{-s-d}\right). \tag{5.8}$$

In analogy to the univariate proof, we define an operator $\mathcal{P}[f]$ to be equal to f evaluated at the nodes $\{\mathbf{x}_0, \ldots, \mathbf{x}_\nu\}$ with multiplicities $\{m_0, \ldots, m_\nu\}$. We can write this explicitly in partitioned form:

$$\mathcal{P}[f] = \begin{pmatrix} \rho_0[f] \\ \vdots \\ \rho_\nu[f] \end{pmatrix}, \qquad \text{for} \qquad \rho_k[f] = \begin{pmatrix} \mathcal{D}^{\mathbf{p}_{k,1}} f(\mathbf{x}_k) \\ \vdots \\ \mathcal{D}^{\mathbf{p}_{k,n_k}} f(\mathbf{x}_k) \end{pmatrix}, \qquad k = 0, 1, \ldots, \nu,$$

where $\mathbf{p}_{k,1}, \ldots, \mathbf{p}_{k,n_k} \in \mathbb{N}^d$, $n_k = \frac{1}{2} m_k(m_k + 1)$, are the lexicographically ordered vectors such that $|\mathbf{p}_{k,i}| \leq m_k - 1$. The system (5.6) can now be writen as $A\mathbf{c} = \mathbf{f}$, for

$$A = [\mathbf{p}_0 + \mathrm{i}\omega\mathbf{g}_0, \cdots, \mathbf{p}_n + \mathrm{i}\omega\mathbf{g}_n], \qquad \mathbf{p}_j = \mathcal{P}\Big[\mathcal{J}\Big[\psi_j^d\Big]\Big], \qquad \mathbf{g}_j = \mathcal{P}\Big[\mathcal{J}[g]\,\psi_j^d\Big], \qquad \mathbf{f} = \mathcal{P}[f],$$

where $n + 1$ is now the number of equations in the system (5.6). By Lemma 2 and the regularity condition, which again implies that $\{\mathbf{g}_0, \ldots, \mathbf{g}_n\}$ are linearly independent, we know that $c_k = \mathcal{O}\big(\omega^{-1}\big)$ for all $0 \leq k \leq n$, hence $\mathcal{L}[v]$ is bounded for increasing ω. Thus we can use Corollary 2, since $f - \mathcal{L}[v]$ and its partial derivatives of order less than or equal to $s - 1$ equal zero at the vertices. This proves (5.8).

We now show that

$$Q_g^L[f, \Omega] - I_g[\mathcal{L}[v], \Omega] = \mathcal{O}\Big(\omega^{-s-d}\Big).$$

We begin by defining the $(d-1)$-form

$$\rho = v\mathrm{e}^{\mathrm{i}\omega g} \sum_{k=1}^{d} t_k \bigwedge_{\substack{i=1 \\ i \neq k}}^{d} \mathrm{d}x_i. \tag{5.9}$$

Similar to (5.4), differentiating ρ we obtain

$$\mathrm{d}\rho = \sum_{k=1}^{d} (-1)^k t_k \left(\mathcal{D}^{\mathbf{e}_k} v + \mathrm{i}\omega v \mathcal{D}^{\mathbf{e}_k} g\right) \mathrm{e}^{\mathrm{i}\omega g} \; \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d = \mathcal{L}[v]\, \mathrm{e}^{\mathrm{i}\omega g} \; \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d.$$

It follows that

$$I_g[\mathcal{L}[v], \Omega] = \int_\Omega \mathrm{d}\rho = \int_{\partial\Omega} \rho = \sum_\ell \int_{Z_\ell} \rho.$$

We now invoke the definition of the integral of a differential form:

$$\begin{aligned} \int_{Z_\ell} \rho &= \int_{\Omega_\ell} v(T_\ell(\mathbf{x}))\, \mathrm{e}^{\mathrm{i}\omega g(T_\ell(\mathbf{x}))} \mathbf{t} \cdot \mathbf{J}_{T_\ell}^d(\mathbf{x})\, \mathrm{d}\mathbf{x} \\ &= \sum_{j=0}^{n} c_j \int_{\Omega_\ell} \psi_j^d(T_\ell(\mathbf{x}))\, \mathrm{e}^{\mathrm{i}\omega g(T_\ell(\mathbf{x}))} \mathbf{t} \cdot \mathbf{J}_{T_\ell}^d(\mathbf{x})\, \mathrm{d}\mathbf{x} \\ &= \sum_{j=0}^{n} c_j I_{g_\ell}[f_{\ell,j}, \Omega_\ell], \end{aligned}$$

where $f_{\ell,j}(\mathbf{x}) = \psi_j^d(T_\ell(\mathbf{x}))\, \mathbf{t} \cdot \mathbf{J}_{T_\ell}^d(\mathbf{x})$. By assumption, each integral $I_{g_\ell}[f_{\ell,j}, \Omega_\ell]$ can be approximated by $Q_{g_\ell}^L[f_{\ell,j}, \Omega_\ell]$ with order of error $\mathcal{O}\big(\omega^{-s-d+1}\big)$, as long as the non-resonance and regularity conditions are satisfied. But both of these conditions are satisfied in Ω_ℓ, since both of these conditions are satisfied in Ω. Thus we obtain

$$\begin{aligned}
Q_g^L[f,\Omega] - I_g[\mathcal{L}[v],\Omega] &= \sum_\ell \left(Q_{g_\ell}^L[f_\ell,\Omega_\ell] - \int_{T_\ell} \rho \right) \\
&= \sum_\ell \sum_{j=0}^n c_j \left(Q_{g_\ell}^L[f_{\ell,j},\Omega_\ell] - I_{g_\ell}[f_{\ell,j},\Omega_\ell] \right) \\
&= \sum_\ell \sum_{j=0}^n \mathcal{O}(\omega^{-1})\, \mathcal{O}\Big(\omega^{-s-d+1}\Big) = \mathcal{O}\Big(\omega^{-s-d}\Big),
\end{aligned} \tag{5.10}$$

where we used the fact that $\sum_j c_j f_{\ell,j}$ is equal to f_ℓ, and the linearity of Q^L. Putting both parts together, we obtain

$$\begin{aligned}
I_g[f,\Omega] - Q_g^L[f,\Omega] &= (I_g[f,\Omega] - I_g[\mathcal{L}[v],\Omega]) - \left(Q_g^L[f,\Omega] - I_g[\mathcal{L}[v],\Omega]\right) \\
&= \mathcal{O}\Big(\omega^{-s-d}\Big) + \mathcal{O}\Big(\omega^{-s-d}\Big) = \mathcal{O}\Big(\omega^{-s-d}\Big).
\end{aligned}$$

□

Remark The differential form (5.9) could have been defined as

$$\rho = \mathrm{e}^{\mathrm{i}\omega g} \left(v_1 \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{d-1} + \cdots + v_d \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_d \right),$$

where each v_k is an independent function. This choice for ρ would result in a different definition for $\mathcal{L}[v]$. Though this provides more degrees of freedom, it comes at the cost of additional complexity, hence we will not investigate it further.

Admittedly the regularity condition seems strict, however in practice it usually holds. The following corollary states that, for simplicial domains and affine g, i.e. linear plus a constant, a Levin-type method is equivalent to a Filon-type method. This is the main problem domain where Filon-type methods works, so effectively Levin-type methods are an extension to Filon-type methods.

Corollary 3 *Define*

$$Q_g^F[f,\Omega] = I_g[u,\Omega],$$

where u is the Hermite interpolation polynomial of f at the nodes $\{\mathbf{x}_0,\ldots,\mathbf{x}_\nu\}$ with multiplicities $\{m_0,\ldots,m_\nu\}$. If g is affine, then $I_g[\mathcal{L}[v],\Omega] = Q_g^F[f,\Omega]$ whenever $\{\psi_0^d,\ldots,\psi_n^d\}$ is the standard polynomial basis and $\mathbf{t}$ is chosen so that $\mathcal{J}[g] \neq 0$. Furthermore, if Ω is the d-dimensional simplex S_d then $Q_g^L[f,S_d]$ is equivalent to $Q_g^F[f,S_d]$ when a sufficient number of sample points are taken.

Proof Note that solving a Levin-type method system is equivalent to interpolating with the basis $\tilde{\psi}_j = \mathcal{L}\Big[\psi_j^d\Big]$. We begin by showing that these two bases are equivalent. We assume that $\Big\{\tilde{\psi}_0,\ldots,\tilde{\psi}_{j-1}\Big\}$ has equivalent span to $\Big\{\psi_0^d,\ldots,\psi_{j-1}^d\Big\}$, which is true for the case $\mathcal{L}[1] = \mathrm{i}\omega\mathcal{J}[g] = C$, where $C \neq 0$ by hypothesis. Note that $\psi_j^d(x_1,\ldots,x_d) = x_1^{p_1}\ldots x_d^{p_d}$ for some nonnegative integers p_k. Then

$$\begin{aligned}
\tilde{\psi}_j = \mathcal{L}\Big[\psi_j^d\Big] &= \mathrm{i}\omega\mathcal{J}[g]\,\psi_j^d + \mathcal{J}\Big[\psi_j^d\Big] = C\psi_j^d + \sum_{k=1}^d (-1)^k t_k \mathcal{D}^{\mathbf{e}_k}\psi_j^d \\
&= C\psi_j^d + \sum_{k=1}^d (-1)^k t_k p_k x_1^{p_1}\ldots x_{k-1}^{p_{k-1}} x_k^{p_k-1} x_{k+1}^{p_{k+1}}\ldots x_d^{p_d}.
\end{aligned}$$

But the sum is a polynomial of degree less than the degree of ψ_j^d, hence it lies in the span of $\{\psi_0, \ldots, \psi_{j-1}\}$. Thus ψ_j^d lies in the span of $\left\{\tilde{\psi}_0, \ldots, \tilde{\psi}_j\right\}$. It follows that interpolation by each of these two bases is equivalent, or in other words $I_g[\mathcal{L}[v], \Omega] = Q_g^F[f, \Omega]$.

We prove the second part of the theorem by induction, where the case of $\Omega = S_1$ holds true by the definition $Q_g^L[f, S_1] = I_g[\mathcal{L}[v], S_1]$. Now assume it is true for each dimension less than d. Since g is affine and each boundary T_ℓ of the simplex is affine we know that each g_ℓ is affine. Furthermore we know that the Jacobian determinants of T_ℓ are constants, hence each f_ℓ is a polynomial. Thus $Q_{g_\ell}^L[f_\ell, \Omega_\ell] = Q_{g_\ell}^F[f_\ell, S_{d-1}] = I_{g_\ell}[f_\ell, S_{d-1}]$ as long as enough sample points are taken so that f_ℓ lies in the span of the interpolation basis. Hence $Q_g^L[f, S_d] = I_g[\mathcal{L}[v], S_d] = Q_g^F[f, S_d]$. $\square$

An important consequence of this corollary is that, in the two-dimensional case, a Levin-type method provides an approximation whenever the standard polynomial basis can interpolate f at the given nodes and multiplicities, assuming that g is affine and the non-resonance condition is satisfied in Ω.

6 Examples

Having developed the theory, we now demonstrate the effectiveness of the method in practice. As the only known efficient methods for solving these integrals are Filon-type methods, which are equivalent to Levin-type methods in many applicable cases, we present the results without comparison. We begin with the relatively simple domain of a simplex. We use the notation of (Rudin, 1964) with regards to simplices and boundaries. An oriented affine d-simplex $T = [\mathbf{p}_0, \cdots, \mathbf{p}_d]$ is defined by the function

$$T(x_1, \ldots, x_d) = \mathbf{p}_0 + \sum_{k=1}^{d} x_k(\mathbf{p}_k - \mathbf{p}_0),$$

where $x_1, \ldots, x_d$ are in S_d. The standard d-dimensional simplex can also be written as $[\mathbf{0}, \mathbf{e}_1, \cdots, \mathbf{e}_d]$, which is equivalent to the identity mapping $I : S_d \to S_d$. The formula for the positive oriented boundary of T is

$$\partial T = [\mathbf{p}_1, \cdots, \mathbf{p}_d] - [\mathbf{p}_0, \mathbf{p}_1, \cdots, \mathbf{p}_d] + \cdots + (-1)^d [\mathbf{p}_0, \cdots, \mathbf{p}_{d-1}],$$

where the addition is only formal. In other words the ℓth boundary component T_ℓ is equal to $(-1)^\ell$ times T with the the ℓth index removed, for $0 \le \ell \le d$. Thus we have all the information needed to construct a Levin-type method.

We now compute the error of $Q_g^L[f, S_d]$ numerically, using the standard d-dimensional polynomials as a basis and $\mathbf{t} = \mathbf{1} = [1, \cdots, 1]^\top$. We begin with $f(x, y, z, t) = x^2$, $g(x, y, z, t) = x - 2y + 3z - 4t$ and $Q_g^L[f, S_4]$ collocating only at the vertices with multiplicities all one. As expected, we obtain an error of order $\mathcal{O}(\omega^{-5})$, as seen in Figure 3. Because this Levin-type method is equivalent to a Filon-type method, it would have solved this integral exactly had we increased the number of node points so that $\psi_k^4(x, y, z, t) = x^2$ was included as a basis vector.

Now consider the more complicated function $f(x, y) = \frac{1}{x+1} + \frac{2}{y+1}$ with oscillator $g(x, y) = 2x - y$, approximated by $Q_g^L[f, S_2]$, again only sampling at the vertices with multiplicities all one. As expected we obtain an order of error of $\mathcal{O}(\omega^{-3})$. By adding an additional multiplicity to each vertex, as well as the sample point $\left[\frac{1}{3}, \frac{1}{3}\right]^\top$ with multiplicity one, we increase the order by one to $\mathcal{O}(\omega^{-4})$. Both of these cases can be seen in Figure 4. Note that the different scale factor

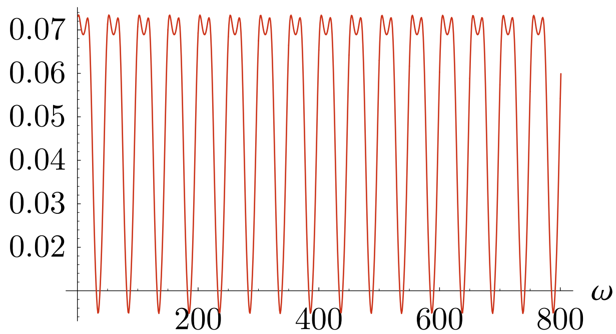

Figure 3: The error scaled by ω^5 of $Q_g^L[f, S_4]$ collocating only at the vertices with multiplicities all one, for $I_g[f, S_4] = \int_{S_4} x^2 \mathrm{e}^{\mathrm{i}\omega(x-2y+3z-4t)} \mathrm{d}V$.

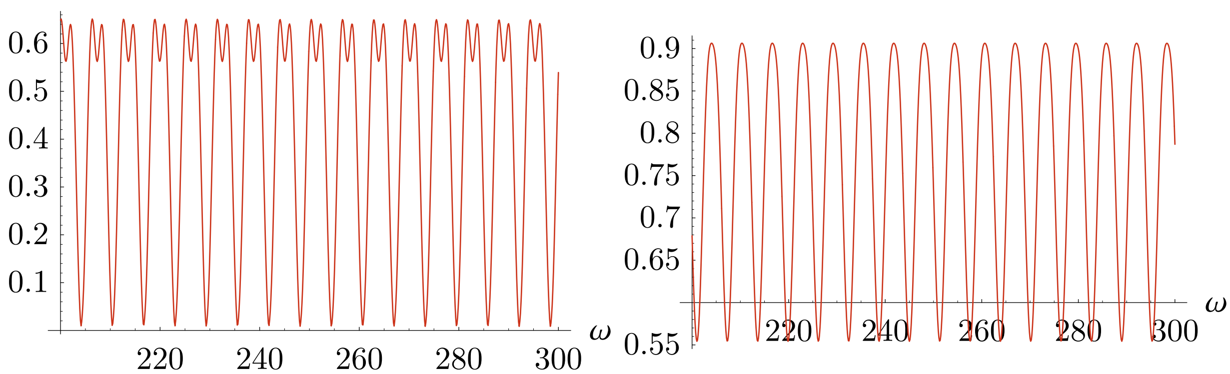

Figure 4: The error scaled by ω^3 of $Q_g^L[f, S_2]$ collocating only at the vertices with multiplicities all one (left), and the error scaled by ω^4 with multiplicities all two (right), for $I_g[f, S_2] = \int_{S_2} \left(\frac{1}{x+1} + \frac{2}{y+1}\right) \mathrm{e}^{\mathrm{i}\omega(2x-y)} \mathrm{d}V$.

means that the right-hand graph is in fact much more accurate, as it has about $\frac{1}{\omega}$th the error. Finally we demonstrate an integral over a three-dimensional simplex. Let $f(x, y) = x^2 - y + z^3$ and $g(x, y) = 3x + 4y - z$. Figure 5 shows the error of $Q_g^L[f, S_3]$, sampling only at the vertices, multiplied by ω^4.

Because Levin-type methods do not require moments, they allow us to integrate over more complicated domains that satisfy the non-resonance condition, without resorting to tessellation. For example, consider the quarter unit circle H, as depicted in Figure 6. We parameterize the boundary as $T_1(t) = [\cos(t), \sin(t)]^\top$ for $\Omega_1 = \left(0, \frac{\pi}{2}\right)$, $T_2(t) = [0, 1-t]^\top$ and $T_3(t) = [t, 0]^\top$ for $\Omega_2 = \Omega_3 = (0, 1)$. This results in the approximation

$$Q_g^L[f, H] = Q_{g_1}^L\left[f_1, \left(0, \frac{\pi}{2}\right)\right] + Q_{g_2}^L[f_2, (0, 1)] + Q_{g_3}^L[f_3, (0, 1)],$$

where $f_1(t) = (\cos t - \sin t)\, v(\cos t, \sin t)$, $g_1(t) = g(\cos t, \sin t)$, $f_2(t) = -v(0, 1-t)$, $g_2(t) = g(0, 1-t)$, $f_3(t) = v(t, 0)$ and $g_3(t) = g(t, 0)$. We used the fact that $\mathbf{t} \cdot \mathbf{J}_{T_1}^2(t) = \mathbf{1} \cdot T_1'(t) = \cos t - \sin t$, $\mathbf{t} \cdot \mathbf{J}_{T_2}^2 = -1$ and $\mathbf{t} \cdot \mathbf{J}_{T_3}^2 = 1$ for finding the formulas of f_ℓ and g_ℓ.

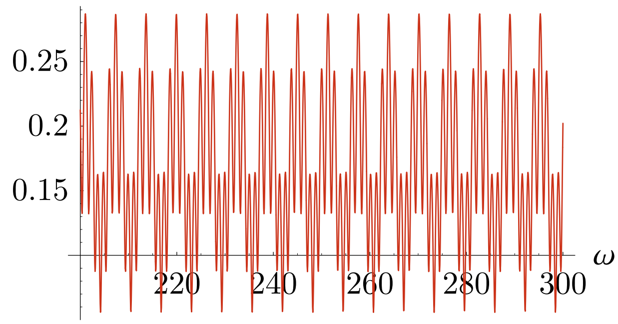

Figure 5: The error scaled by ω^4 of $Q_g^L[f, S_3]$ collocating only at the vertices with multiplicities all one, for $I_g[f, S_3] = \int_{S_3} (x^2 - y + z^3) e^{i\omega(3x+4y-z)} dV$.

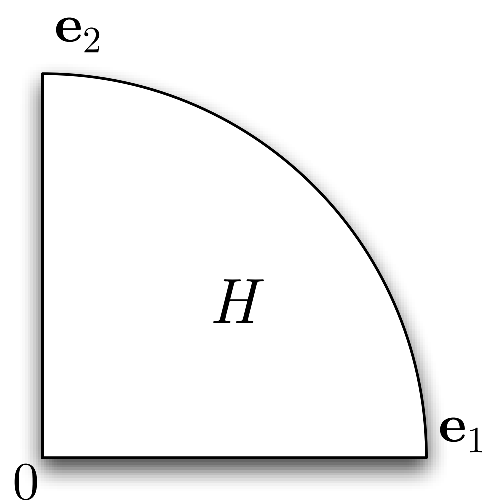

Figure 6: Diagram of a unit quarter circle H.

Let $f(x, y) = e^x \cos xy$, $g(x, y) = x^2 + x - y^2 - y$, and choose vertices for nodes with multiplicities all one. Note that g is nonlinear, in addition to the domain not being a simplex. Despite these difficulties, $Q_g^L[f, H]$ still attains an order of error $\mathcal{O}(\omega^{-3})$, as seen in the left hand side of Figure 7. If we increase the multiplicities at the vertices to two, adding an additional node at $\left[\frac{1}{3}, \frac{1}{3}\right]^\top$ with multiplicity one to ensure that we have ten equations in our system as required by polynomial interpolation, we obtain an error of order $\mathcal{O}(\omega^{-4})$. This can be seen in the right-hand side of Figure 7.

This example is significant since, due to the unavailability of moments, Filon-type methods fail to provide approximations in a quarter circle, let alone with nonlinear g. If g was linear, we could have tessellated H to obtain a polytope, but that would have resulted in an unnecessarily large number of calculations. However, with nonlinear g we do not even have this option, hence Filon-type methods are completely unsuitable.

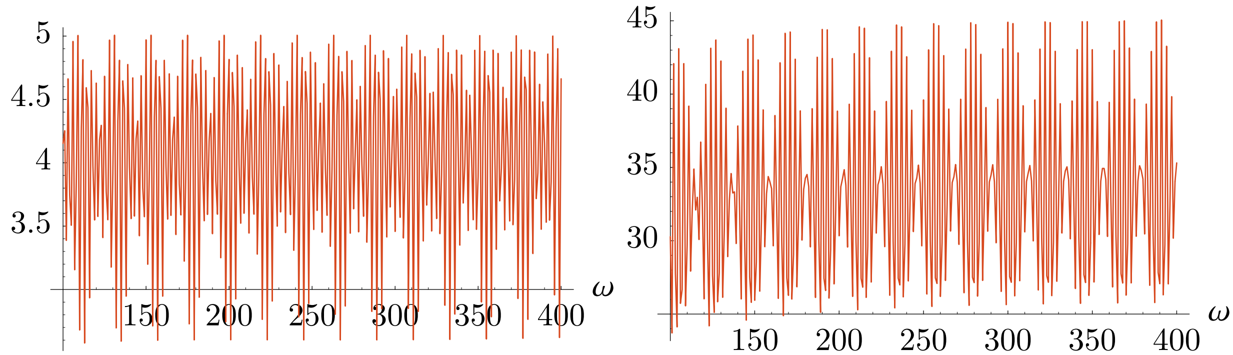

Figure 7: The error scaled by ω^3 of $Q_g^L[f, H]$ collocating only at the vertices with multiplicities all one (left), and the error scaled by ω^4 with multiplicities all two (right), for $I_g[f, H] = \int_H \mathrm{e}^x \cos xy \; \mathrm{e}^{\mathrm{i}\omega(x^2+x-y^2-y)} \mathrm{d}V$.

7 Asymptotic basis

In the univariate case, the asymptotic basis for a Levin-type method results in internal nodes, in addition to endpoints, increasing the order of error (Olver, 2005). The concept of an asymptotic basis generalizes to multidimensional integrals in a fairly straightforward manner. The idea is to choose the basis so that, using the notation of the proof of Theorem 5, each $\mathbf{g}_{k+1}$ is a multiple of $\mathbf{p}_k$ for $1 \le k \le n-1$ and $\mathbf{g}_1$ is a multiple of $\mathbf{f}$. This can be accomplished by choosing the basis

$$\psi_0^d = 1, \qquad \psi_1^d = \frac{f}{\mathcal{J}[g]}, \qquad \psi_{k+1}^d = \frac{\mathcal{J}\left[\psi_k^d\right]}{\mathcal{J}[g]}, \qquad k \ge 1,$$

where, as before, $\mathcal{J} = \sum_{k=1}^{d} (-1)^k t_k \mathcal{D}^{\mathbf{e}_k}$. Surprisingly, this increases the asymptotic order to $\mathcal{O}\big(\omega^{-\tilde{n}-s-d}\big)$, where s is again the minimum vertex multiplicity and $\tilde{n}+1$ is equal to the minimum of the number of equations in every collocation system solved for in the definition of Q^L, recursively down to the univariate integrals. It follows that if $\Omega \subset \mathbb{R}$, then $\tilde{n} = n$. As an example, if we are collocating on a two-dimensional simplex at only the three vertices with multiplicities all one, then the two-dimensional collocation system has three equations, while each one-dimensional collocation system has only two equations. Thus $\tilde{n}+1 = \min\{3, 2, 2, 2\} = 2$.

The following lemma is used extensively in the proof of the asymptotic order:

Lemma 6 *Let ψ_k^d be as defined. Then*

$$\det\left[\mathbf{g}_k, \mathbf{a}_k, \cdots, \mathbf{a}_{k+j}, B\right] = \det\left[\mathbf{g}_k, \mathbf{g}_{k+1}, \cdots, \mathbf{g}_{k+j+1}, B\right],$$

where B represents all remaining columns that render the matrices square and $\mathbf{a}_k = \mathbf{p}_k + \mathrm{i}\omega\mathbf{g}_k$, for $\mathbf{p}_k$ and $\mathbf{g}_k$ as defined previously in this paper.

Proof We know that $\mathbf{p}_k = \mathbf{g}_{k-1}$. Thus we can multiply the first column by $\mathrm{i}\omega$ and subtract it from the second to obtain

$$\det\left[\mathbf{g}_k, \mathbf{p}_k + \mathrm{i}\omega\mathbf{g}_k, \cdots, \mathbf{a}_{k+j}, B\right] = \det\left[\mathbf{g}_k, \mathbf{g}_{k+1}, \mathbf{a}_{k+1}, \cdots, \mathbf{a}_{k+j}, B\right].$$

The lemma follows by repeating this process on the remaining columns. □

This holds for any column interchange on both sides of the determinant. We now prove the theorem, in a manner which is very similar to the omitted univariate version.

Theorem 7 *Using the preceding definition of* ψ_k^d *and* $\tilde{n}$*, if the non-resonance condition and regularity condition hold, then*

$$I_g[f,\Omega] - Q_g^L[f,\Omega] \sim \mathcal{O}\Big(\omega^{-\tilde{n}-s-d}\Big).$$

Proof We begin by showing that $\mathcal{L}[v](x) - f(x) = \mathcal{O}(\omega^{-n})$. Note that

$$\begin{aligned}
\mathcal{L}[v] - f = \sum_{k=0}^{n} c_k \mathcal{L}\Big[\psi_k^d\Big] - f &= \sum_{k=0}^{n} c_k \Big\{\mathcal{J}\Big[\psi_k^d\Big] + \mathrm{i}\omega\psi_k^d \mathcal{J}[g]\Big\} - f \\
&= \mathrm{i}\omega c_0 \mathcal{J}[g] + \sum_{k=1}^{n} c_k \Big\{\mathcal{J}[g]\,\psi_{k+1}^d + \mathrm{i}\omega \mathcal{J}[g]\,\psi_k^d\Big\} - \mathcal{J}[g]\,\psi_1^d \\
&= \mathcal{J}[g]\left[\mathrm{i}\omega c_0 + (\mathrm{i}\omega c_1 - 1)\psi_1^d + \sum_{k=2}^{n}(c_{k-1} + \mathrm{i}\omega c_k)\psi_k^d + c_n\psi_{n+1}^d\right] \\
&= \frac{\mathcal{J}[g]}{\det A}\Bigg[\mathrm{i}\omega \det D_0 + (\mathrm{i}\omega \det D_1 - \det A)\psi_1^d \\
&\qquad + \sum_{k=2}^{n}(\det D_{k-1} + \mathrm{i}\omega \det D_k)\psi_k^d + \det D_n \psi_{n+1}^d\Bigg],
\end{aligned}$$

where again D_k is the matrix A with the $(k+1)$th column replaced by $\mathbf{f}$. We know that $(\det A)^{-1} = \mathcal{O}\big(\omega^{-n-1}\big)$, thus it remains to be shown that each term in the preceding equation is $\mathcal{O}(\omega)$. This boils down to showing that each of the following terms are $\mathcal{O}(\omega)$: $\mathrm{i}\omega \det D_0$, $\mathrm{i}\omega \det D_1 - \det A$, $\det D_{k-1} + \mathrm{i}\omega \det D_k$ for $2 \le k \le n$ and finally $\det D_n$. The first case follows directly from Lemma 6. The second case follows from Lemma 6 after rewriting the determinants as

$$\begin{aligned}
\mathrm{i}\omega \det D_1 - \det A &= \mathrm{i}\omega \det D_1 - \det\left[\mathbf{a}_0, \mathbf{p}_1 + \mathrm{i}\omega \mathbf{g}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\right] \\
&= \mathrm{i}\omega \det D_1 - \mathrm{i}\omega \det\left[\mathbf{a}_0, \mathbf{g}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\right] - \det\left[\mathbf{a}_0, \mathbf{p}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\right] \\
&= -\mathrm{i}\omega \det\left[\mathbf{g}_0, \mathbf{g}_2, \mathbf{a}_2, \cdots, \mathbf{a}_n\right],
\end{aligned}$$

where we used the facts that $\mathbf{a}_k = \mathbf{p}_k + \mathrm{i}\omega \mathbf{g}_k$, $\mathbf{a}_0 = \mathrm{i}\omega \mathbf{g}_0$ and $\mathbf{g}_1 = \mathbf{f}$. Similarly,

$$\begin{aligned}
\det D_{k-1} + \mathrm{i}\omega \det D_k &= \det\left[\mathbf{a}_0, \cdots, \mathbf{a}_{k-2}, \mathbf{g}_1, \mathbf{p}_k + \mathrm{i}\omega \mathbf{g}_k, \mathbf{a}_{k+1}, \cdots, \mathbf{a}_n\right] \\
&\quad + \mathrm{i}\omega \det\left[\mathbf{a}_0, \cdots, \mathbf{a}_{k-2}, \mathbf{p}_{k-1} + \mathrm{i}\omega \mathbf{g}_{k-1}, \mathbf{g}_1, \mathbf{a}_{k+1}, \cdots, \mathbf{a}_n\right] \\
&= \det\left[\mathbf{a}_0, \cdots, \mathbf{a}_{k-2}, \mathbf{g}_1, \mathbf{p}_k, \mathbf{a}_{k+1}, \cdots, \mathbf{a}_n\right] \\
&\quad + \mathrm{i}\omega \det\left[\mathbf{a}_0, \cdots, \mathbf{a}_{k-2}, \mathbf{g}_1, \mathbf{g}_k, \mathbf{a}_{k+1}, \cdots, \mathbf{a}_n\right] \\
&\quad + \mathrm{i}\omega \det\left[\mathbf{a}_0, \cdots, \mathbf{a}_{k-2}, \mathbf{g}_k, \mathbf{g}_1, \mathbf{a}_{k+1}, \cdots, \mathbf{a}_n\right] \\
&\quad - \omega^2 \det\left[\mathbf{a}_0, \cdots, \mathbf{a}_{k-2}, \mathbf{g}_{k-1}, \mathbf{g}_1, \mathbf{a}_{k+1}, \cdots, \mathbf{a}_n\right] \\
&= \det\left[\mathbf{a}_0, \cdots, \mathbf{a}_{k-2}, \mathbf{g}_1, \mathbf{p}_k, \mathbf{a}_{k+1}, \cdots, \mathbf{a}_n\right] \\
&\quad - \omega^2 \det\left[\mathbf{a}_0, \cdots, \mathbf{a}_{k-2}, \mathbf{g}_{k-1}, \mathbf{g}_1, \mathbf{a}_{k+1}, \cdots, \mathbf{a}_n\right].
\end{aligned}$$

Using Lemma 6 the first of these determinants is $\mathcal{O}(\omega)$, while the second determinant has two columns equal to $\mathbf{g}_{k-1}$, hence is equal to zero. The last determinant $\det D_n$ is also $\mathcal{O}(\omega)$, due to Lemma 6. Thus we have shown that $\mathcal{L}[v](x) - f(x) = \mathcal{O}(\omega^{-n})$.

Since $\mathcal{L}[v] - f$ is a linear combination of functions independent of ω, where the coefficents depend on ω, it follows that $\mathcal{D}^{\mathbf{m}}[\mathcal{L}[v] - f](x) = \mathcal{O}(\omega^{-n})$ as well. Hence, by Corollary 2, $I_g[f,\Omega] -$

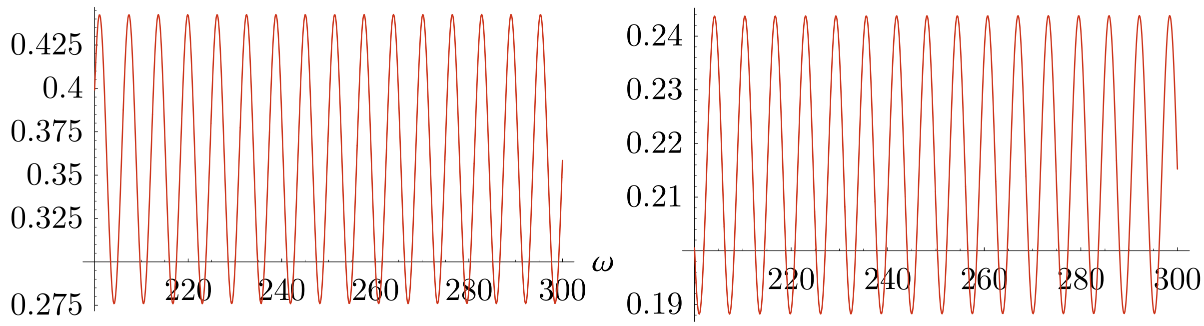

Figure 8: The error scaled by ω^4 of $Q_g^B[f, S_2]$ collocating only at the vertices with multiplicities all one (left), and the error scaled by ω^5 with vertices and boundary midpoints again with multiplicities all one (right), for $\int_{S_2} \left(\frac{1}{x+1} + \frac{2}{y+1} \right) \mathrm{e}^{\mathrm{i}\omega(2x-y)} \mathrm{d}V$.

$I_g[\mathcal{L}[v], \Omega] = \mathcal{O}\big(\omega^{-n-s-d}\big) = \mathcal{O}\big(\omega^{-\tilde{n}-s-d}\big)$. For the univariate case the lemma has been proved, since $Q_g^L[\mathcal{L}[v], \Omega] = I_g[\mathcal{L}[v], \Omega]$. By induction, $Q_{g_\ell}^L[f_{\ell,j}, \Omega_\ell] - I_{g_\ell}[f_{\ell,j}, \Omega_\ell] = \mathcal{O}\big(\omega^{-\tilde{n}-s-(d-1)}\big)$ in (5.10). It follows that

$$I_g[f, \Omega] - Q_g^L[f, \Omega] = (I_g[f, \Omega] - I_g[\mathcal{L}[v], \Omega]) - \big(Q_g^L[f, \Omega] - I_g[\mathcal{L}[v], \Omega]\big) = \mathcal{O}\left(\omega^{-\tilde{n}-s-d}\right).$$

□

We will use $Q_g^B[f, \Omega]$ to denote $Q_g^L[f, \Omega]$ with the asymptotic basis. There are important cases when this definition for $\{\psi_k^d\}$ does not lead to a basis. For example, if g is linear and $f(x, y) = f(y, x)$ then $\psi_2^d = 0$. Of course, whether or not $Q_g^B[f, \Omega]$ is well defined is completely determined by whether or not the regularity condition is satisfied, which can be easily determined using linear algebra.

We now demonstrate numerically that the asymptotic basis does in fact result in a higher order approximation. Recall the case where $f(x, y) = \frac{1}{x+1} + \frac{2}{y+1}$ with oscillator $g(x, y) = 2x - y$ over the simplex S_2. We now use $Q_g^B[f, S_2]$ in place of $Q_g^L[f, S_2]$, collocating only at the vertices. Since this results in each univariate boundary collocation having two node points, we know that $\tilde{n} = 1$. Hence we now scale the error by ω^4, i.e. we have increased the order by one, as seen in Figure 8. Since the initial two-dimensional system has three node points, adding the midpoint to the sample points of each univariate integral should increase the order again by one to $\mathcal{O}\big(\omega^{-5}\big)$. This can be seen in the right-hand side of Figure 8.

There is nothing special about a simplex or linear g: the asymptotic basis works just as well on other domains with nonlinear g, assuming that the regularity and non-resonance conditions are satisfies. Recall the example with $f(x, y) = \mathrm{e}^x \cos xy$ and $g(x, y) = x^2 + x - y^2 - y$ on the quarter circle H. As in the simplex case, $Q_g^B[f, H]$ collocating only at vertices with multiplicities all one results in an error of $\mathcal{O}\big(\omega^{-4}\big)$, as seen in the left-hand side of Figure 9. Note that increasing multiplicities not only increases s, but also $\tilde{n}$. If we increase the multiplicities to two, then $\tilde{n} = 3$ and $s = 2$, and the order increases to $\mathcal{O}\big(\omega^{-7}\big)$, as seen in the right-hand side of Figure 9. It should be emphasized that, though the scale is large in the graph, the error is being divided by $\omega^7 \geq 100^7 = 10^{14}$. As a result, the errors for the right-hand graph are in fact less than the

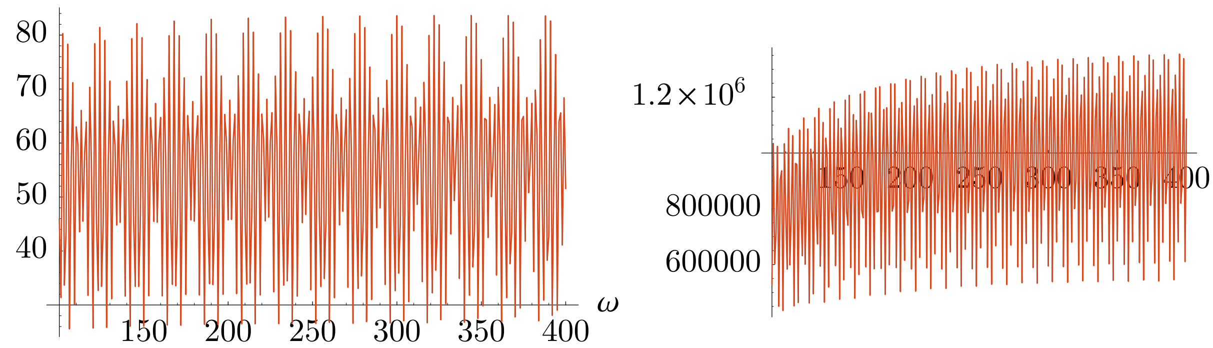

Figure 9: The error scaled by ω^4 of $Q_g^B[f,H]$ collocating only at the vertices with multiplicities all one (left), and the error scaled by ω^7 of $Q_g^B[f,H]$ collocating only at the vertices with multiplicities all two (left), for $I_g[f,H] = \int_H \mathrm{e}^x \cos xy\ \mathrm{e}^{\mathrm{i}\omega(x^2+x-y^2-y)} \mathrm{d}V$.

errors in the left-hand graph. Numerical evidence in (Olver, 2005) suggests that Q^B is typically more accurate for the same order when additional nodes are added; as opposed to increasing the multiplicities at the endpoints.

8 Points of resonance

Up until this point we have avoided discussing highly oscillatory integrals that do not satisfy the non-resonance condition. But we know that a large class of integrals fail this condition: for example if g is linear then any Ω with smooth boundary must have at least one point of resonance. In this section we investigate such integrals, and see where Levin-type methods fail.

Suppose that ∇g is orthogonal to the boundary of $\Omega \subset \mathbb{R}^d$ at a single point $\mathbf{u}$. Let us analyse what happens at this point when we push the integral to the boundary, as in a Levin-type method. If T is the map that defines the boundary of Ω then the statement of orthogonality is equivalent to

$$\nabla g(T(\xi))\, T'(\xi) = \mathbf{0},$$

where $\xi \in \mathbb{R}^{d-1}$, $\mathbf{u} = T(\xi)$ and T' is the derivative matrix of T, as defined in Appendix A. After pushing the integral to the boundary we now have the oscillator $\tilde{g} = g \circ T$. But it follows that

$$\nabla \tilde{g}(\xi) = (g \circ T)'(\xi) = \nabla g(T(\xi))\, T'(\xi) = \mathbf{0}.$$

In other words the resonance point has become a critical point. (Iserles and Nørsett, 2005b) states that a Filon-type method must sample at a critical point in order to obtain a higher asymptotic order than that of the integral, hence, by the same logic, a Levin-type method must also sample at a critical point. This means that the regularity condition can never be satisified, since $\mathcal{J}[\tilde{g}](\xi) = 0$, hence a Levin-type method cannot be used. Moreover, in general each $\tilde{g}$ is a fairly complicated functionand no moments are available, thus neither the asymptotic nor the Filon-type methods are feasible.

Perhaps a concrete example is in order. Consider the unit half-circle U, with $g(x,y) = y - x$, as seen in Figure 10. The boundary curve which exhibits the problem is defined for $\Omega_1 = (0,\pi)$ as $T_1(t) = [\cos t, \sin t]^\top$. We find that ∇g is orthogonal to the boundary at the point $T_1\left(\frac{3\pi}{4}\right) =$

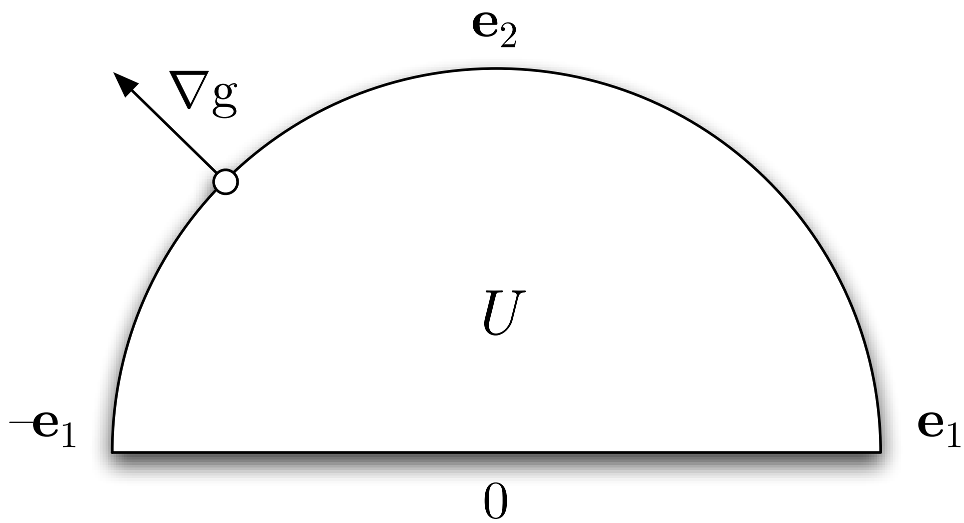

Figure 10: Depiction of a half circle boundary U, where the vector ∇g represents the direction of the gradient of $g(x,y) = y - x$, highlighting where it is orthogonal to the boundary of U.

$\left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]^\top$, since $\nabla g\big(T_1\big(\frac{3\pi}{4}\big)\big)\, T_1'\big(\frac{3\pi}{4}\big) = [-1,1]\left[-\sin\frac{3\pi}{4}, \cos\frac{3\pi}{4}\right]^\top = 0$. Combining Theorem 4 and (Iserles and Nørsett, 2005b), we know that in order to obtain an order of error of $\mathcal{O}\Big(\omega^{-s-\frac{3}{2}}\Big)$ our collocation points must include $[-1,0]^\top$ and $[1,0]^\top$ with multiplicity s, as well as the point of resonance $\left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]^\top$ with multiplicity $2s-1$. We assume that the resulting system is in fact solvable. When we push the integral to the boundary, we obtain two line integrals:

$$\int_U f\mathrm{e}^{\mathrm{i}\omega g} \approx \int_U \mathcal{L}[v]\,\mathrm{e}^{\mathrm{i}\omega g} = \int_{T_1} v\mathrm{e}^{\mathrm{i}\omega g}(\mathrm{d}x + \mathrm{d}y) + \int_{T_2} v\mathrm{e}^{\mathrm{i}\omega g}(\mathrm{d}x + \mathrm{d}y)$$
$$= I_{g_1}[f_1, (0,\pi)] + I_{g_2}[f_2, (-1,1)]$$

where T_2 corresponds to the boundary of U on the x-axis, $f_1(t) = (\cos t - \sin t)\, v(\cos t, \sin t)$, $g_1(t) = g(\cos t, \sin t) = \sin t - \cos t$, $f_2(t) = v(t,0)$ and $g_2(t) = g(t,0) = -t$. We see that $I_g[f,U] - I_{g_1}[f_1,(0,\pi)] - I_{g_2}[f_2,(-1,1)]$ does indeed appear to have an order of error $\mathcal{O}\big(\omega^{-5/2}\big)$ in Figure 11. It follows that, if we can approximate these univariate integrals with the appropriate error, then we can derive an equivalent to Theorem 5 for when the non-resonance condition is not satisfied.

Note that $I_{g_1}[f_1, (-1,1)]$ is a one-dimensional integral with oscillator $g_1(t) = \sin t - \cos t$. But $g_1'\big(\frac{1}{2}\big) = -\cos\frac{3\pi}{4} + \sin\frac{3\pi}{4} = 0$, meaning that we have a stationary point. Unfortunately none of the moments of g_1 are elementary, including the zeroth moment. Thus neither the univariate Filon-type method nor the asymptotic method from (Iserles and Nørsett, 2005a) are applicable. Furthermore, the univariate Levin-type method cannot satisfy the regularity condition, as we are required to sample at the stationary point. Thus we are left with the problem of what to do once the integral has been pushed to the boundary. This issue represents a work in progress, the results of which will be published elsewhere.

Acknowledgements

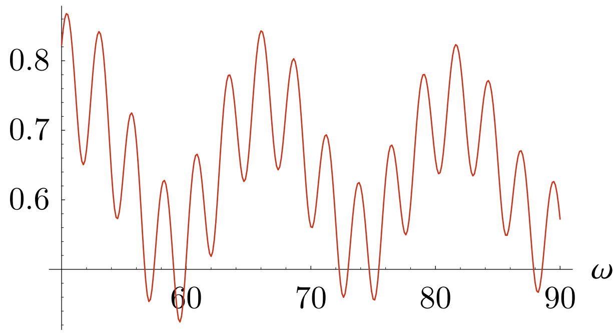

Figure 11: The error, scaled by $\omega^{\frac{5}{2}}$, of $I_g[\mathcal{L}[v], U]$ approximating $I_g[f, U] = \int_H \cos x \cos y \mathrm{e}^{\mathrm{i}\omega(y-x)} \mathrm{d}V$, where $\mathcal{L}[v]$ is determined by collocation at the two vertices and the resonance point, all with multiplicities one.

The author wishes to thank his PhD supervisor Arieh Iserles, as well as Alexei Shadrin, Alex Benton, Gates Cambridge Trust, University of Cambridge, and his parents Peter Olver and Chehrzad Shakiban.

References

Cheney, W. and Light, W. (2000). *A Course in Approximation Theory.* Brooks/Cole Publishing Company, Pacific Grove, CA.

Iserles, A. and Nørsett, S. P. (2004). On quadrature methods for highly oscillatory integrals and their implementation. *BIT*, 44:755–772.

Iserles, A. and Nørsett, S. P. (2005a). Efficient quadrature of highly oscillatory integrals using derivatives. *Proceedings Royal Soc. A.*, 461:1383–1399.

Iserles, A. and Nørsett, S. P. (2005b). Quadrature methods for multivariate highly oscillatory integrals using derivatives. Technical Report NA2005/02, DAMTP, University of Cambridge.

Levin, D. (1997). Analysis of a collocation method for integrating rapidly oscillatory functions. *J. Comput. Appl. Maths*, 78:131–138.

Olver, F. W. J. (1974). *Asymptotics and Special Functions.* Academic Press, New York, NY.

Olver, S. (2005). Moment-free numerical integration of highly oscillatory functions. *to appear in IMA Journal of Numerical Analysis.*

Powell, M. J. D. (1981). *Approximation Theory and Methods.* Cambridge University Press, Cambridge.

Press, W. H., Flannery, B. P., Teukolsky, S., and Vetterling, W. T. (1988). *Numerical recipes in C: the art of scientific computing.* Cambridge Unversity Press, Cambridge.

Rudin, W. (1964). *Principles of Mathematical Analysis.* McGraw-Hill, Inc., New York, NY.

Stein, E. (1993). *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*. Princeton University Press, Princeton, NJ.

Appendix A

We define the differential operator $\mathcal{D}^{\mathbf{m}}$ as follows:

- $\mathcal{D}^{\mathbf{0}}$ is the identity operator.
- $\mathcal{D}^m$ for non-negative integer $m \in \mathbb{N}$ is simply the mth derivative:

$$\mathcal{D}^m = \frac{\mathrm{d}^m}{\mathrm{d}x^m}.$$

- $\mathcal{D}^{\mathbf{m}}$ for $\mathbf{m} = [m_1, \cdots, m_d] \in \mathbb{N}^d$ is the partial derivative

$$\mathcal{D}^{\mathbf{m}} = \frac{\partial^{|\mathbf{m}|}}{\partial x_1^{m_1} \dots \partial x_d^{m_d}},$$

  where $|\mathbf{m}| = \|\mathbf{m}\|_1 = \sum_{k=1}^{d} m_k$. Note that the absolute-value signs are not needed since each m_k is nonnegative.

The bottom two definitions are equivalent in the scalar case if we regard the scalar k as a vector in $\mathbb{N}^1$. Furthermore, it is clear that $\mathcal{D}^{\mathbf{m}_1}\mathcal{D}^{\mathbf{m}_2} = \mathcal{D}^{\mathbf{m}_1+\mathbf{m}_2}$.

The definition of the determinant matrix of a map $T : \mathbb{R}^d \to \mathbb{R}^n$, with component functions $T_1, \dots, T_n$, is simply the $n \times d$ matrix

$$T' = \begin{pmatrix} \mathcal{D}^{\mathbf{e}_1} T_1 & \cdots & \mathcal{D}^{\mathbf{e}_d} T_1 \\ \vdots & \ddots & \vdots \\ \mathcal{D}^{\mathbf{e}_1} T_n & \cdots & \mathcal{D}^{\mathbf{e}_d} T_n \end{pmatrix}.$$

Note that $\nabla g = g'$ when g is a scalar-valued function. The chain rule states that $(g \circ T)'(\mathbf{x}) = g'(T(\mathbf{x}))T'(\mathbf{x})$.

The Jacobian determinant J_T of a function $T : \mathbb{R}^d \to \mathbb{R}^d$ is the determinant of its derivative matrix T'. For the case $T : \mathbb{R}^d \to \mathbb{R}^n$ with $n \geq d$ we define the Jacobian determinant of T for indices $i_1, \dots, i_d$ as $J_T^{i_1,\dots,i_d} = J_{\tilde{T}}$, where $\tilde{T} = [T_{i_1}, \cdots, T_{i_d}]^\top$. Finally, we define

$$\mathbf{J}_T^d(\mathbf{x}) = \left[J_T^{2,\dots,d}(\mathbf{x}), \cdots, J_T^{1,\dots,d-1}(\mathbf{x}) \right]^\top,$$

a vector of Jacobian determinants that are used in the definition of the integral of a $(d-1)$-form.