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ON THE COMPUTATION OF HIGHLY OSCILLATORY MULTIVARIATE INTEGRALS WITH CRITICAL POINTS *

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Abstract.

We consider two types of highly oscillatory bivariate integrals with a nondegenerate critical point. In each case we produce an asymptotic expansion and two kinds of quadrature algorithms: an asymptotic method and a Filon-type method. Our results emphasize the crucial role played by the behaviour at the critical point and by the geometry of the boundary of the underlying domain.

1 Introduction

Germund Dahlquist's work has frequently emphasized the importance of numerical quadrature, both as an end in itself and because of its centrality to a wide range of other computational issues: cf. for example [1].

Recent years have witnessed major resurgence of interest in the theory and computation of highly oscillatory integrals. This is justified by a wide range of applications, e.g. to the numerical computation of highly oscillatory differential equations and to computational electromagnetism. On the face of it, high oscillation renders computation more challenging and expensive. Perhaps surprisingly, once the mathematical mechanism underpinning high oscillation is understood, the computation of many highly oscillatory integrals becomes exceedingly precise and affordable. Indeed, high oscillation often renders computation much easier!

As things stand, four broad families of methods are available and reasonably well understood: asymptotic methods [5], Filon-type methods [5], Levin-type methods [9] and methods based on the technique of stationary phase [3]. All such methods address themselves to the computation of

$$I[f] = \int_a^b f(x) \mathrm{e}^{\mathrm{i}\omega g(x)} \mathrm{d}x, \qquad (1.1)$$

where f and g are given, sufficiently smooth functions, while $|\omega| \gg 1$.

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The quadrature of (1.1) is particularly easy when the *oscillator* g has no *stationary points:* $g'(x) \neq 0$ for $x \in [a,b]$. The key to quadrature methods in that case is the *asymptotic expansion*

$$(1.2)\qquad I[f] \sim \sum_{n=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{n+1}} \left[\frac{\mathrm{e}^{\mathrm{i}\omega g(a)}}{g'(a)}\sigma_n(a) - \frac{\mathrm{e}^{\mathrm{i}\omega g(b)}}{g'(b)}\sigma_n(b) \right], \qquad |\omega| \gg 1,$$

where

$$\sigma_0(x) = f(x), \qquad \sigma_{n+1}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sigma_n(x)}{g'(x)}, \quad n \in \mathbb{Z}_+.$$

Note that each $\sigma_n(x)$ is a linear combination of $f^{(k)}(x)$, $k = 0,\ldots,n$. Thus, truncating (1.2), we obtain the *asymptotic method*

$$(1.3)\qquad Q_s^{\mathsf{A}}[f] = \sum_{n=0}^{s-1} \frac{1}{(-\mathrm{i}\omega)^{n+1}} \left[\frac{\mathrm{e}^{\mathrm{i}\omega g(a)}}{g'(a)}\sigma_n(a) - \frac{\mathrm{e}^{\mathrm{i}\omega g(b)}}{g'(b)}\sigma_n(b) \right]$$

which matches $I[f]$ up to ω^{-s-1} while requiring just $2s$ data: the values of $f^{(k)}$, $k = 0,\ldots,s-1$ at the endpoints. We say that Q_s^{A} is of *asymptotic order* s.

Suppose next that we have a linearly independent set of functions $\phi_1, \phi_2, \ldots, \phi_r$ which form a *Chebyshev set* in $[a,b]$. In other words, given any distinct $a = c_1 < c_2 < \ldots < c_\nu = b$ and multiplicities $m_1, m_2, \ldots, m_\nu$, such that $\sum m_i = r$, there exists a unique linear combination

$$\phi(x) = \sum_{i=1}^{r} \alpha_i \phi_i(x)$$

such that

$$\phi^{(k)}(c_j) = f^{(k)}(c_j), \qquad k = 0,\ldots,m_j - 1, \quad j = 1,\ldots,\nu.$$

In all standard applications it is perfectly satisfactory to use a polynomial basis, $\phi_k(x) = x^{k-1}$.

Assuming that the *moments* $I[\phi_i]$ can be explicitly calculated for all $i = 1,\ldots,r$, the *Filon-type method* is

$$(1.4)\qquad Q_s^{\mathsf{F}}[f] = I[\phi] = \sum_{i=1}^{r} \alpha_i I[\phi_i],$$

where $s = \min\{n_1, n_\nu\}$. The asymptotic order of (1.4) is s, matching that of (1.3). This follows at once from the observation that $Q_s^{\mathsf{F}}[f] - I[f] = I[\phi - f]$ and the substitution of $\phi - f$ into the asymptotic expansion (1.2). As an interesting aside, we note that the classical textbook of Dahlquist and Björck describes a local version of a Filon method [2, p. 297].

Filon-type methods enjoy a number of advantages in comparison with (1.3). Although their 'minimalist' version, namely $\nu = 2$, $c_1 = a$, $c_2 = b$, $m_1 = m_2 = s$, use exactly the same data as the asymptotic method, the error is typically much smaller, corresponding to the error in (1.3) once it is applied to the approximation error $\phi - f$, rather than to f itself. The error can be further decreased by adding extra quadrature points in (a,b). Finally, using well-chosen finite differences, we can attain the same asymptotic order without the need to compute derivatives at all [4]. The disadvantage of Filon is that we must be able to compute the moments.

We note in passing that two other methods afford the possibility of computing $I[f]$ to arbitrarily high asymptotic order, using roughly the same information as (1.3) or (1.4): Levin-type methods [9] and methods based on a numerical implementation of the technique of stationary phase of Huybrechs and Vandewalle [3]. Neither requires the computation of moments, but each exhibits its own disadvantages. Levin-type methods cannot be extended to cater for stationary points, while the Huybrechs–Vandewalle method imposes further conditions on f in the complex plane. These two methods will play no further role in this paper, yet they are very valuable and promising. The reader is referred to [7] for a review.

The numerical treatment of (1.1) has been generalized in two distinct directions: firstly, allowing for stationary points and, secondly, venturing into a multivariate setting. An asymptotic expansion proved itself invariably crucial to progress and, once known, it readily leads to both asymptotic and Filon-type methods.

We commence assuming for simplicity that (1.1) possesses just a single *non-degenerate* stationary point in $\xi \in (a,b)$. In other words, $g'(\xi)=0$, $g''(\xi)\neq 0$ and $g'(x)\neq 0$ for $x\in[a,b]\setminus\{\xi\}$. (Once we understand this case, the general case is fairly straightforward.) The asymptotic expansion (1.2) generalizes to

$$
\begin{aligned}
I[f] \;&\sim\; \mu_0(\omega)\sum_{n=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^n}\sigma_n(\xi)\\
&\quad+\sum_{n=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{n+1}}\left\{\frac{\mathrm{e}^{\mathrm{i}\omega g(b)}}{g'(b)}[\sigma_n(b)-\sigma_n(\xi)]-\frac{\mathrm{e}^{\mathrm{i}\omega g(a)}}{g'(a)}[\sigma_n(a)-\sigma_n(\xi)]\right\},
\end{aligned}
\tag{1.5}
$$

where

$$\mu_0(\omega)=\int_a^b \mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x$$

is the zeroth moment of the oscillator, while

$$\sigma_0(x)=f(x),\qquad \sigma_{n+1}(x)=\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sigma_n(x)-\sigma_n(\xi)}{g'(x)},\quad n\in\mathbb{Z}_+.$$

Note that $\sigma_n(x)$ depends on $f^{(k)}(x)$, $k=0,\ldots,n$ for $x\neq\xi$, but $\sigma_n(\xi)$ is a linear combination of $f^{(k)}(\xi)$, $k=0,\ldots,2n$. Moreover, according to the *van der Corput theorem* [10], it is true that $\mu_0(\omega)\sim O(\omega^{-\frac{1}{2}})$. We obtain the asymptotic method

$$
\begin{aligned}
Q_s^{\mathsf{A}}[f] \;&=\; \mu_0(\omega)\sum_{n=0}^{s-1}\frac{1}{(-\mathrm{i}\omega)^n}\sigma_n(\xi)\\
&\quad+\sum_{n=0}^{s-1}\frac{1}{(-\mathrm{i}\omega)^{n+1}}\left\{\frac{\mathrm{e}^{\mathrm{i}\omega g(b)}}{g'(b)}[\sigma_n(b)-\sigma_n(\xi)]-\frac{\mathrm{e}^{\mathrm{i}\omega g(a)}}{g'(a)}[\sigma_n(a)-\sigma_n(\xi)]\right\},
\end{aligned}
$$

of asymptotic order $s-\frac{1}{2}$. Likewise, interpolating f and its first $s-1$ derivatives at the endpoints and $f^{(k)}$, $k=0,\ldots,2s-2$ at ξ, we obtain a Filon-type method of the same asymptotic order. The error of the latter can be further reduced by interpolation at additional points.

The generalization of (1.3) and (1.4) to a multivariate setting in [6] follows a different path. Again, the key is an asymptotic expansion. The paper in question presents, in effect, two methods of analysis but, taking on board subsequent observations in [8], it is enough to follow the approach of a *Stokes-type theorem.* To set the scene, we replace (1.1) with

$$I[f]=\int_\Omega f(\boldsymbol{x})\mathrm{e}^{\mathrm{i}\omega g(\boldsymbol{x})}\mathrm{d}V, \tag{1.6}$$

where $\Omega\subset\mathbb{R}^d$ is a bounded open domain with piecewise-smooth boundary, while $f,g:\Omega\to\mathbb{R}$ are suitably smooth. The multivariate equivalent of a stationary point is called a *critical point:* $\boldsymbol{\xi}\in\mathrm{cl}\,\Omega$ such that $\boldsymbol{\nabla} g(\boldsymbol{\xi})=\boldsymbol{0}$ [10]. We assume for the time being that there are no critical points in the closure of Ω. In that case, it is possible to prove that

$$\begin{aligned} I[f] \;=\; & \frac{1}{\mathrm{i}\omega}\int_{\partial\Omega}\boldsymbol{n}(\boldsymbol{x})^\top\boldsymbol{\nabla} g(\boldsymbol{x})\frac{f(\boldsymbol{x})}{\|\boldsymbol{\nabla} g(\boldsymbol{x})\|^2}\mathrm{e}^{\mathrm{i}\omega g(\boldsymbol{x})}\mathrm{d}S \\ & -\frac{1}{\mathrm{i}\omega}\int_\Omega \boldsymbol{\nabla}^\top\left[\frac{f(\boldsymbol{x})}{\|\boldsymbol{\nabla} g(\boldsymbol{x})\|^2}\boldsymbol{\nabla} g(\boldsymbol{x})\right]\mathrm{e}^{\mathrm{i}\omega g(\boldsymbol{x})}\mathrm{d}V, \end{aligned} \tag{1.7}$$

where the norm is Euclidean and $\boldsymbol{n}$ is the unit outward normal at the boundary. The proof of (1.7) in [6] follows several steps. First we prove it when Ω is the regular simplex with vertices at the origin and at the d unit vectors. This is generalized to arbitrary simplices using an affine mapping and, subsequently, to polytopes, tessellating them with a simplicial complex. Finally, as observed in [8], the formula (1.7) extends to non-polytope domains using the dominating convergence theorem.

Next, we iterate (1.7) to obtain the Stokes-type asymptotic expansion

$$I[f]\sim-\sum_{n=0}^\infty\frac{1}{(-\mathrm{i}\omega)^{n+1}}\int_{\partial\Omega}\boldsymbol{n}(\boldsymbol{x})^\top\boldsymbol{\nabla} g(\boldsymbol{x})\frac{\sigma_m(\boldsymbol{x})}{\|\boldsymbol{\nabla} g(\boldsymbol{x})\|^2}\mathrm{e}^{\mathrm{i}\omega g(\boldsymbol{x})}\mathrm{d}S, \tag{1.8}$$

where

$$\sigma_0(\boldsymbol{x})=f(\boldsymbol{x}),\qquad \sigma_{n+1}(\boldsymbol{x})=\boldsymbol{\nabla}^\top\left[\frac{\sigma_n(\boldsymbol{x})}{\|\boldsymbol{\nabla} g(\boldsymbol{x})\|^2}\boldsymbol{\nabla} g(\boldsymbol{x})\right],\quad n\in\mathbb{Z}_+.$$

The way forward is seemingly straightforward: consider the oriented boundary of Ω as a domain in $\mathbb{R}^{d-1}$ and continue with this procedure. This, unfortunately, is careless and not always valid. The problem is that, although critical points are absent in $\mathbb{R}^d$, they may occur in lower dimensions. To avoid this, we need to impose the *nonresonance condition:* the vector $\boldsymbol{\nabla} g(\boldsymbol{x})$, $\boldsymbol{x}\in\partial\Omega$, is nowhere orthogonal to the boundary. Note that if the boundary of Ω is smooth, the nonresonance condition must necessarily fail. In this instance 'corners' are good!

Once the nonresonance condition holds, we may progressively pass to lower-dimensional integrals, stopping once we reach zero-dimensional sets: specifically, the points where $\partial\Omega$ is nondifferentiable. This results in an asymptotic expansion of the form

$$I[f] \sim \sum_{n=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{n+d}} \theta_n, \tag{1.9}$$

where each θ_n is a linear functional which depends on $\partial^{|\boldsymbol{m}|} f / \partial \boldsymbol{x}^{\boldsymbol{m}}$, $|\boldsymbol{m}| = 0, \ldots, n$, at the points of C^1 discontinuity of $\partial\Omega$. We can now obtain an asymptotic quadrature truncating (1.9), while a Filon-type method follows by interpolating f and its derivatives at the 'vertices', e.g. with finite-element functions.

The subject of this paper is to address the presence of critical points in a multivariate setting. Thus, we are interested in computing (1.6) whilst assuming that there exists $\boldsymbol{\xi} \in \Omega$ such that $\boldsymbol{\nabla} g(\boldsymbol{\xi}) = \mathbf{0}$, $\det \boldsymbol{\nabla}\boldsymbol{\nabla}^\top g(\boldsymbol{x}) \neq 0$ (the second condition, usually termed *nondegeneracy* [10], corresponds to $g''(\xi) \neq 0$ in univariate setting). We assume that $\boldsymbol{\nabla} g(\boldsymbol{x}) \neq \mathbf{0}$ for $\boldsymbol{x} \in \mathrm{cl}\,\Omega \setminus \{\boldsymbol{\xi}\}$.

Of course, the subject in its totality is well beyond the scope of a single paper, therefore we present here just an initial foray into this very broad topic. Our difficulty is compounded by the absence of a comprehensive body of theory on the asymptotic behaviour of $I[f]$ in this situation, along the lines of the method of stationary phase and the van der Corput lemma in a single variable [10]. Full asymptotic expansion, of the kind so central in the derivation of our quadrature methods, is simply not available.

The most obvious line of attack is to generalize the Stokes-type formula (1.8) to the current setting, employing similar technique as that we have already used in [5] in the univariate case. Thus, assume that $\boldsymbol{\xi} \in \Omega$ is a unique, nondegenerate critical point. Writing $f(\boldsymbol{x}) = f(\boldsymbol{\xi}) + [f(\boldsymbol{x}) - f(\boldsymbol{\xi})]$, we have

$$I[f] = f(\boldsymbol{\xi}) I[1] + I[\tilde{f}], \qquad \tilde{f}(\boldsymbol{x}) = f(\boldsymbol{x}) - f(\boldsymbol{\xi}).$$

In the univariate case, $\tilde{f}$ vanishes at the critical point, hence $\tilde{f}/g'$ has a removable singularity there. In the multivariate case we require that the singularity of

$$h(\boldsymbol{x}) = \frac{\tilde{f}(\boldsymbol{x})}{\|\boldsymbol{\nabla} g(\boldsymbol{x})\|^2} \boldsymbol{\nabla} g(\boldsymbol{x})$$

at $\boldsymbol{\xi}$ is removable. Unfortunately, this need not the case, as can be easily seen taking $g(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{x}\|^2$. Worse: suppose that we consider

$$I[f] = f(\boldsymbol{\xi}) I[1] + \sum_{k=1}^{d} \frac{\partial f(\boldsymbol{x})}{\partial x_k} I[x_k - \xi_k] + I[\tilde{f}],$$

where $\tilde{f}(\boldsymbol{x}) = f(\boldsymbol{x}) - f(\boldsymbol{\xi}) - \boldsymbol{\nabla} f(\boldsymbol{\xi})^\top (\boldsymbol{x} - \boldsymbol{\xi})$. Even then the singularity of h (with the new definition of $\tilde{f}$) at $\boldsymbol{\xi}$ is not removable. Generalizing (1.7) to cater for critical points might be possible but at present it is not clear how to do so.

Instead, in the current paper we adopt a fairly minimalist plan of action and address just two bivariate problems, the *separable oscillator*

$$I[f] = \int_{-1}^{1} \int_{-1}^{1} f(x, y) \mathrm{e}^{\mathrm{i}\omega[g_1(x) + g_2(y)]} \mathrm{d}y \mathrm{d}x, \tag{1.10}$$

where $g_1'(x_0) = g_2'(y_0) = 0$ for some $x_0, y_0 \in (-1,1)$, $g_1''(x_0) + g_2''(y_0) \neq 0$, $g_1, g_2' \neq 0$ in $[-1,1]^2 \setminus \{(x_0,y_0)\}$, and

$$I[f] = \int_{x^2+y^2<1} f(x,y) \mathrm{e}^{\mathrm{i}\omega(x^2+y^2)} \mathrm{d}y \mathrm{d}x. \tag{1.11}$$

Note that for (1.11) every point on the boundary is a point of resonance. We wish to explore the delicate connection between the nature of the oscillator, the geometry of the boundary, the asymptotic expansion and numerical quadrature.

Multivariate highly oscillatory quadrature presents a wide range of challenges, mostly unexplored. Note thus that the oscillators in (1.10) and (1.11) are fairly similar (take $g_1(x) = x^2$, $g_2(y) = y^2$ in (1.10)!), yet underlying asymptotic expansions and Filon-type methods require completely different information. Moreover, neither integral provides a clue to the asymptotic behaviour of, say,

$$\int_\Omega f(x,y) \mathrm{e}^{\mathrm{i}\omega(x^2+y^2)} \mathrm{d}V$$

for general $\Omega \subset \mathbb{R}^2$ although note that if Ω is a square, $\mathbf{0} \in \Omega$, then we can easily generalize our analysis using rotation and affine translation. We expect to address the problem in greater generality in forthcoming papers.

2 The separable oscillator

We assume that both f and g_1, g_2 are smooth functions. As the first step in our asymptotic analysis of (1.10), we define the differential operators

$$\mathcal{D}_x \phi(x,y) = \frac{\partial}{\partial x} \frac{\phi(x,y) - \phi(x_0,y)}{g_1'(x)}, \qquad \mathcal{D}_y \phi(x,y) = \frac{\partial}{\partial y} \frac{\phi(x,y) - \phi(x,y_0)}{g_2'(y)}$$

Proposition 2.1. *The operators $\mathcal{D}_x$ and $\mathcal{D}_y$ commute.*

Proof. By straightforward calculation. Letting

$$\tilde{\phi}(x,y) = \phi(x,y) - \phi(x,y_0) - \phi(x_0,y) + \phi(x_0,y_0)$$

we calculate

$$\begin{aligned} \mathcal{D}_x \mathcal{D}_y \phi \; = \; & \frac{g_1''(x) g_2''(y)}{{g_1'}^2(x) {g_2'}^2(y)} [\phi(x,y) - \phi(x,y_0) - \phi(x_0,y) + \phi(x_0,y_0)] \\ & - \frac{g_1''(x)}{{g_1'}^2(x) g_2'(y)} [\phi_y(x,y) - \phi_y(x_0,y)] - \frac{g_2''(y)}{g_1'(x) {g_2'}^2(y)} [\phi_x(x,y) \\ & - \phi_x(x,y_0)] + \frac{1}{g_1'(x) g_2'(y)} \phi_{xy}(x,y) \\ = \; & \frac{1}{g_1'(x) g_2'(y)} \left[\frac{g_1''(x) g_2''(y)}{g_1'(x) g_2'(y)} - \frac{g_1''(x)}{g_1'(x)} \frac{\partial}{\partial y} - \frac{g_2''(y)}{g_2'(y)} \frac{\partial}{\partial x} + \frac{\partial^2}{\partial x \partial y} \right] \tilde{\phi}(x,y). \end{aligned}$$

Since this expression is symmetric in x and y, we immediately deduce that $\mathcal{D}_x \mathcal{D}_y = \mathcal{D}_y \mathcal{D}_x$. ☐

We define

$$f_{m,n}(x,y) = \mathcal{D}_x^m \mathcal{D}_y^n f(x,y), \qquad m,n \in \mathbb{Z}_+,$$

and note that, by virtue of Proposition 2.1, $f_{m,n}$ is independent of the order of $\mathcal{D}_x$ and $\mathcal{D}_y$. It is easy to prove by induction that each $f_{m,n}$ in the punctured square $[-1,1]^2 \setminus \{(x_0,y_0)\}$ is a linear combination of $\partial^{i+j} f(x,y)/\partial x^i/\partial y^j$ for $i = 0,\ldots,m$ and $j = 0,\ldots,n$, while at (x_0,y_0) it is a linear combination of $\partial^{i+j} f(x_0,y_0)/\partial x^i/\partial y^j$ for $i = 0,\ldots,2m$ and $j = 0,\ldots,2n$.

Finally, let

$$\mu_0(\omega) = \int_{-1}^{1} \mathrm{e}^{\mathrm{i}\omega g_1(x)} \mathrm{d}x, \qquad \nu_0(\omega) = \int_{-1}^{1} \mathrm{e}^{\mathrm{i}\omega g_2(y)} \mathrm{d}y.$$

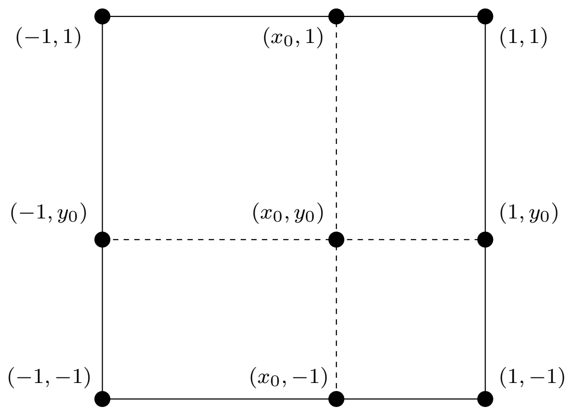
We have all the necessary machinery to expand $I[f]$ asymptotically, nesting univariate expansions (1.5). The detailed expansion below is long, fairly tedious and requires great attention to detail.

$$\begin{aligned}
& I[f] \\
= & \int_{-1}^{1} \left[\int_{-1}^{1} f(x,y) \mathrm{e}^{\mathrm{i}\omega g_2(y)} \mathrm{d}y \right] \mathrm{e}^{\mathrm{i}\omega g_1(x)} \mathrm{d}x \\
\sim & \int_{-1}^{1} \left\{ \nu_0(\omega) \sum_{n=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^n} f_{0,n}(x,y_0) \right. \\
& - \frac{\mathrm{e}^{\mathrm{i}\omega g_2(1)}}{g_2'(1)} \sum_{n=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{n+1}} [f_{0,n}(x,1) - f_{0,n}(x,y_0)] \\
& \left. + \frac{\mathrm{e}^{\mathrm{i}\omega g_2(-1)}}{g_2'(-1)} [f_{0,n}(x,-1) - f_{0,n}(x,y_0)] \right\} \mathrm{e}^{\mathrm{i}\omega g_1(x)} \mathrm{d}x \\
= & \ \nu_0(\omega) \sum_{n=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^n} \int_{-1}^{1} f_{0,n}(x,y_0) \mathrm{e}^{\mathrm{i}\omega g_1(x)} \mathrm{d}x \\
& - \frac{\mathrm{e}^{\mathrm{i}\omega g_2(1)}}{g_2'(1)} \sum_{n=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{n+1}} \int_{-1}^{1} [f_{0,n}(x,1) - f_{0,n}(x,y_0)] \mathrm{e}^{\mathrm{i}\omega g_1(x)} \mathrm{d}x \\
& + \frac{\mathrm{e}^{\mathrm{i}\omega g_2(-1)}}{g_2'(-1)} \sum_{n=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{n+1}} \int_{-1}^{1} [f_{0,n}(x,-1) - f_{0,n}(x,y_0)] \mathrm{e}^{\mathrm{i}\omega g_1(x)} \mathrm{d}x \\
\sim & \ \nu_0(\omega) \sum_{n=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^n} \left\{ \mu_0(\omega) \sum_{m=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^m} f_{m,n}(x_0,y_0) \right. \\
& - \frac{\mathrm{e}^{\mathrm{i}\omega g_1(1)}}{g_1'(1)} \sum_{m=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{m+1}} [f_{m,n}(1,y_0) - f_{m,n}(x_0,y_0)] \\
& \left. + \frac{\mathrm{e}^{\mathrm{i}\omega g_1(-1)}}{g_1'(-1)} \sum_{m=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{m+1}} [f_{m,n}(-1,y_0) - f_{m,n}(x_0,y_0)] \right\} \\
& - \frac{\mathrm{e}^{\mathrm{i}\omega g_2(1)}}{g_2'(1)} \sum_{n=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{n+1}} \left\{ \mu_0(\omega) \sum_{m=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{m+1}} [f_{m,n}(x_0,1) - f_{m,n}(x_0,y_0)] \right.
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\mathrm{e}^{\mathrm{i}\omega g_1(1)}}{g_1'(1)}\sum_{m=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{m+1}}[f_{m,n}(1,1)-f_{m,n}(1,y_0)-f_{m,n}(x_0,1)\\
&\qquad\qquad +f_{m,n}(x_0,y_0)]\\
&+\frac{\mathrm{e}^{\mathrm{i}\omega g_1(-1)}}{g_1'(-1)}\sum_{m=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{m+1}}[f_{m,n}(-1,1)-f_{m,n}(-1,y_0)-f_{m,n}(x_0,1)\\
&\qquad\qquad +f_{m,n}(x_0,y_0)]\Bigg\}\\
&+\frac{\mathrm{e}^{\mathrm{i}\omega g_2(-1)}}{g_2'(-1)}\sum_{n=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{n+1}}\Bigg\{\mu_0(\omega)\sum_{m=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{m+1}}[f_{m,n}(x_0,-1)\\
&\qquad\qquad -f_{m,n}(x_0,y_0)]\\
&-\frac{\mathrm{e}^{\mathrm{i}\omega g_1(1)}}{g_1'(1)}\sum_{m=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{m+1}}[f_{m,n}(1,-1)-f_{m,n}(1,y_0)-f_{m,n}(x_0,-1)\\
&\qquad\qquad +f_{m,n}(x_0,y_0)]\\
&+\frac{\mathrm{e}^{\mathrm{i}\omega g_1(-1)}}{g_1'(-1)}\sum_{m=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{m+1}}[f_{m,n}(-1,-1)-f_{m,n}(-1,y_0)-f_{m,n}(x_0,-1)\\
&\qquad\qquad +f_{m,n}(x_0,y_0)]\Bigg\}\\
=\;&\mu_0(\omega)\nu_0(\omega)\sum_{m=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{m}}\sum_{n=0}^{m}f_{m-n,n}(x_0,y_0)\\
&-\mu_0(\omega)\frac{\mathrm{e}^{\mathrm{i}\omega g_2(1)}}{g_2'(1)}\sum_{m=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{m+1}}\sum_{n=0}^{m}[f_{m-n,n}(x_0,1)-f_{m-n,n}(x_0,y_0)]\\
&+\mu_0(\omega)\frac{\mathrm{e}^{\mathrm{i}\omega g_2(-1)}}{g_2'(-1)}\sum_{m=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{m+1}}\sum_{n=0}^{m}[f_{m-n,n}(x_0,-1)-f_{m-n,n}(x_0,y_0)]\\
&-\nu_0(\omega)\frac{\mathrm{e}^{\mathrm{i}\omega g_1(1)}}{g_1'(1)}\sum_{m=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{m+1}}\sum_{n=0}^{m}[f_{m-n,m}(1,y_0)-f_{m-n,n}(x_0,y_0)]\\
&+\nu_0(\omega)\frac{\mathrm{e}^{\mathrm{i}\omega g_1(-1)}}{g_1'(-1)}\sum_{m=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{m+1}}\sum_{n=0}^{m}[f_{m-n,m}(-1,y_0)-f_{m-n,n}(x_0,y_0)]\\
&+\frac{\mathrm{e}^{\mathrm{i}\omega[g_1(1)+g_2(1)]}}{g_1'(1)g_2'(1)}\sum_{m=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{m+2}}\sum_{n=0}^{m}[f_{m-n,n}(1,1)-f_{m-n,n}(x_0,1)\\
&\qquad\qquad -f_{m-n,n}(1,y_0)+f_{m-n,n}(x_0,y_0)]\\
&-\frac{\mathrm{e}^{\mathrm{i}\omega[g_1(-1)+g_2(1)]}}{g_1'(-1)g_2'(1)}\sum_{m=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{m+2}}\sum_{n=0}^{m}[f_{m-n,n}(-1,1)-f_{m-n,n}(x_0,1)\\
&\qquad\qquad -f_{m-n,n}(-1,y_0)+f_{m-n,n}(x_0,y_0)]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\mathrm{e}^{\mathrm{i}\omega[g_1(1)+g_2(-1)]}}{g_1'(1)g_2'(-1)} \sum_{m=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{m+2}} \sum_{n=0}^{m} [f_{m-n,n}(1,-1) - f_{m-n,n}(x_0,-1) \\
&\qquad - f_{m-n,n}(1,y_0) + f_{m-n,n}(x_0,y_0)] \\
&+ \frac{\mathrm{e}^{\mathrm{i}\omega[g_1(-1)+g_2(-1)]}}{g_1'(-1)g_2'(-1)} \sum_{m=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{m+2}} \sum_{n=0}^{m} [f_{m-n,n}(-1,-1) - f_{m-n,n}(x_0,-1) \\
&\qquad - f_{m-n,n}(-1,y_0) + f_{m-n,n}(x_0,y_0)].
\end{aligned}
$$

The asymptotic expansion of $I[f]$ depends on *nine* points,



This makes sense: the vertices impact on the expansion even when there are no critical points, (x_0, y_0) is the critical point, while $(x_0, \pm 1)$ and $(\pm 1, y_0)$ are resonance points of our oscillator.

In line with the van der Corput theorem [10], we have $\mu_0(\omega), \nu_0(\omega) = O(\omega^{-\frac{1}{2}})$. Therefore we deduce that

$$I[f] = O(\omega^{-1}), \qquad |\omega| \gg 1.$$

We can truncate the asymptotic expansion to produce an asymptotic method. Thus, for example

$$
\begin{aligned}
Q_1^{\mathsf{A}}[f] \;=\; & \mu_0(\omega)\nu_0(\omega) \left\{ f(x_0,y_0) + \frac{1}{-\mathrm{i}\omega}[f_{1,0}(x_0,y_0) + f_{0,1}(x_0,y_0)] \right\} \\
& - \frac{1}{-\mathrm{i}\omega} \left\{ \mu_0(\omega) \frac{\mathrm{e}^{\mathrm{i}\omega g_2(1)}}{g_2'(1)} [f(x_0,1) - f(x_0,y_0)] \right. \\
& - \mu_0(\omega) \frac{\mathrm{e}^{\mathrm{i}\omega g_2(-1)}}{g_2'(-1)} [f(x_0,-1) - f(x_0,y_0)] \\
& + \nu_0(\omega) \frac{\mathrm{e}^{\mathrm{i}\omega g_1(1)}}{g_1'(1)} [f(1,y_0) - f(x_0,y_0)] \\
& \left. - \nu_0(\omega) \frac{\mathrm{e}^{\mathrm{i}\omega g_1(-1)}}{g_1'(-1)} [f(-1,y_0) - f(x_0,y_0)] \right\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{(-\mathrm{i}\omega)^2}\left\{\frac{\mathrm{e}^{\mathrm{i}\omega[g_1(1)+g_2(1)]}}{g_1'(1)g_2'(1)}[f(1,1)-f(x_0,1)-f(1,y_0)+f(x_0,y_0)]\right.\\
&-\frac{\mathrm{e}^{\mathrm{i}\omega[g_1(-1)+g_2(1)]}}{g_1'(-1)g_2'(1)}[f(-1,1)-f(x_0,1)-f(-1,y_0)+f(x_0,y_0)]\\
&-\frac{\mathrm{e}^{\mathrm{i}\omega[g_1(1)+g_2(-1)]}}{g_1'(1)g_2'(-1)}[f(1,-1)-f(x_0,-1)-f(1,y_0)+f(x_0,y_0)]\\
&\left.+\frac{\mathrm{e}^{\mathrm{i}\omega[g_1(-1)+g_2(-1)]}}{g_1'(-1)g_2'(-1)}[f(-1,1)-f(x_0,-1)-f(-1,y_0)+f(x_0,y_0)]\right\}
\end{aligned}
$$

yields asymptotic order $\frac{3}{2}$: in other words, $Q_1^{\mathsf{A}}[f] \sim I[f] + O(\omega^{-\frac{5}{2}})$. All this, as well as our subsequent discussion of a Filon-type method, can be immediately generalized to higher asymptotic orders.

The asymptotic expansion Q_1^{A} used function values at the eight points along the boundary, as well as $f, f_x, f_y, f_{xx}, f_{xy}$ and f_{yy} at (x_0, y_0), altogether thirteen data items. Once we wish to generalize a Filon-type method to this setting, we need to construct a smooth interpolating function ψ, say, that satisfies all these conditions, whereby

$$Q_1^{\mathsf{F}}[f] = I[\psi].$$

It is clear that $Q_1^{\mathsf{F}}[f] = I[f] + O(\omega^{-\frac{5}{2}})$.

A bivariate quadratic, with fifteen degrees of freedom, might seem a reasonable choice of ψ, with two degrees of freedom to spare. Unfortunately, once we form the relevant 13×15 matrix, it is of rank 13 for $x_0, y_0 \neq 0$, but just rank 12 when one of $\{x_0, y_0\}$ vanishes and rank 10 for $x_0 = y_0 = 0$. A more careful examination of the interpolation conditions makes it clear, however, that we need less conditions at the critical point. Since

$$
\begin{aligned}
f_{1,0}(x_0,y_0) &= -\tfrac{1}{2}\frac{g_1'''(x_0)}{g_1''^2(x_0)}f_x(x_0,y_0) + \tfrac{1}{2}\frac{1}{g_1''(x_0)}f_{xx}(x_0,y_0),\\
f_{0,1}(x_0,y_0) &= -\tfrac{1}{2}\frac{g_2'''(y_0)}{g_2''^2(y_0)}f_y(x_0,y_0) + \tfrac{1}{2}\frac{1}{g_2''(y_0)}f_{yy}(x_0,y_0),
\end{aligned}
$$

we need to satisfy at (x_0, y_0) just three (rather than six) interpolation conditions,

$$
\begin{aligned}
\psi(x_0,y_0)-f(x_0,y_0) &= 0,\\
\frac{g_1'''(x_0)}{g_1''^2(x_0)}[\psi_x(x_0,y_0)-f_x(x_0,y_0)] + \frac{1}{g_1''(x_0)}[\psi_{xx}(x_0,y_0)-f_{xx}(x_0,y_0)] &= 0,\\
\frac{g_2'''(y_0)}{g_2''^2(y_0)}[\psi_y(x_0,y_0)-f_y(x_0,y_0)] + \frac{1}{g_2''(y_0)}[\psi_{yy}(x_0,y_0)-f_{yy}(x_0,y_0)] &= 0.
\end{aligned}
$$

Altogether, we have eleven conditions, which can be satisfied by a biquadratic.

In Fig. 2.1 we display the errors, scaled by $\omega^{\frac{5}{2}}$, in the quadrature of

$$\int_{-1}^{1}\int_{-1}^{1}(1+x)\mathrm{e}^{x-y}\mathrm{e}^{\mathrm{i}\omega(x^2+y^2)}\mathrm{d}y\mathrm{d}x$$

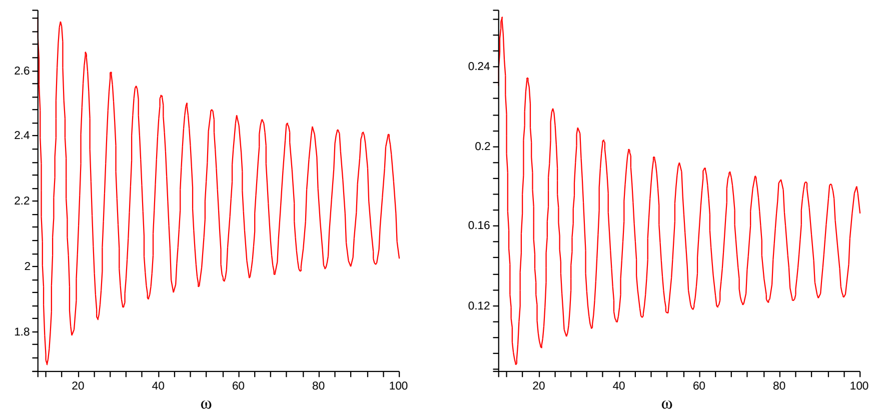


Figure 2.1: Scaled errors $\omega^{\frac{5}{2}}|Q_1^{\mathsf{A}}[f]-I[f]|$ (on the left) and $\omega^{\frac{5}{2}}|Q_1^{\mathsf{F}}[f]-I[f]|$ for $g(x,y)=x^2+y^2$ and $f(x,y)=(1+x)\mathrm{e}^{x-y}$.

by Q_1^{A} and Q_1^{F}. Both methods behave as predicted by the theory. Note that the error of the Filon-type method is roughly a tenth of that of the asymptotic method, although they use exactly the same information. This is typical behaviour, cf. [4, 6].

3 Critical point in a disc

Converting (1.11) to polar coordinates, we have

$$I[f]=\int_{x^2+y^2<1} f(x,y)\mathrm{e}^{\mathrm{i}\omega(x^2+y^2)}\mathrm{d}y\mathrm{d}x=2\pi\int_0^1 rF(r)\mathrm{e}^{\mathrm{i}\omega r^2}\mathrm{d}r, \tag{3.1}$$

where

$$F(r)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(r\cos\theta,r\sin\theta)\mathrm{d}\theta.$$

We expand the univariate integral in (3.1) asymptotically, similarly to (1.5). Let

$$\mu_0(\omega)=\int_0^1 \mathrm{e}^{\mathrm{i}\omega r^2}\mathrm{d}r=\frac{\pi^{\frac{1}{2}}\mathrm{erf}((-\mathrm{i}\omega)^{\frac{1}{2}})}{2(-\mathrm{i}\omega)^{\frac{1}{2}}}.$$

Then

$$\frac{1}{2\pi}I[f]=\frac{1}{2\mathrm{i}\omega}\int_0^1 F(r)\frac{\mathrm{d}\mathrm{e}^{\mathrm{i}\omega r^2}}{\mathrm{d}r}=\frac{1}{2\mathrm{i}\omega}[F(1)\mathrm{e}^{\mathrm{i}\omega}-F(0)]-\frac{1}{2\mathrm{i}\omega}\int_0^1 F'(r)\mathrm{e}^{\mathrm{i}\omega r^2}\mathrm{d}r.$$

The integral on the right is now expanded asymptotically similarly to (1.5),

$$\frac{1}{2\pi}I[f] \;\sim\; \frac{1}{2\mathrm{i}\omega}[F(1)\mathrm{e}^{\mathrm{i}\omega}-F(0)]+\mu_0(\omega)\sum_{n=0}^{\infty}\frac{1}{(-2\mathrm{i}\omega)^{n+1}}F_n'(0)$$

$$-\sum_{n=0}^{\infty}\frac{1}{(-2\mathrm{i}\omega)^{n+2}}\{[F_n(1)-F_n(0)]\mathrm{e}^{\mathrm{i}\omega}-F_n'(0)\}, \tag{3.2}$$

where

$$F_0(r)=F'(r), \qquad F_{n+1}(r)=\frac{\mathrm{d}}{\mathrm{d}r}\frac{F_n(r)-F_n(0)}{r}, \quad n\in\mathbb{Z}_+.$$

The asymptotic expansion (3.2) can be truncated, whereby we obtain an asymptotic method. Likewise, interpolating the function F and its derivatives at 0 and 1 (and perhaps additional points) results in a Filon-type method, whereby (3.2) is crucial to the proof of its asymptotic order. However, all this tacitly assumes that the integral F can be evaluated explicitly. Unless this is true, we need to reformulate (3.2) so that it is expressed in terms of the function f.

Proposition 3.1. *Suppose that f is analytic and*

$$f(x,y)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\phi_{i,j}}{i!j!}x^iy^j, \qquad \textit{where} \qquad \phi_{i,j}=\frac{\partial^{i+j}f(0,0)}{\partial x^i\partial y^j}, \quad i,j\in\mathbb{Z}_+.$$

Then

$$F(r)=\sum_{j=0}^{\infty}\frac{r^{2j}}{4^j j!^2}\Delta^j f(0,0). \tag{3.3}$$

Proof. Integrating term-by-term,

$$F(r)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\phi_{i,j}}{i!j!}r^{i+j}\tau_{i,j}$$

where

$$\tau_{i,j}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^i\theta\sin^j\theta\mathrm{d}\theta, \qquad i,j\in\mathbb{Z}_+.$$

Clearly, $\tau_{i,j}=0$ if either i or j is odd. The value of $\tau_{2i,2j}$ might well be known but for completeness herewith its derivation. Given $n\ge 0$, let

$$T_n(x,y)=\sum_{m=0}^{2n}\binom{2n}{m}\tau_{2n-m,m}x^{2n-m}y^m=\frac{1}{2\pi}\int_{-\pi}^{\pi}(x\cos\theta+y\sin\theta)^{2n}\mathrm{d}\theta.$$

Assuming that $x^2+y^2>0$, we let

$$\psi=\sin^{-1}\frac{x}{\sqrt{x^2+y^2}}.$$

Then

$$T_n(x,y) \;=\; \frac{(x^2+y^2)^n}{2\pi}\int_{-\pi}^{\pi}\left(\frac{x}{\sqrt{x^2+y^2}}\cos\theta+\frac{y}{\sqrt{x^2+y^2}}\cos\theta\right)^{2n}\mathrm{d}\theta$$

$$\begin{aligned}
&= \frac{(x^2+y^2)^n}{2\pi}\int_{-\pi}^{\pi}(\sin\psi\cos\theta+\sin\theta\cos\psi)^{2n}\mathrm{d}\theta\\
&= \frac{(x^2+y^2)^n}{2\pi}\int_{-\pi}^{\pi}\sin^{2n}(\theta+\psi)\mathrm{d}\theta = \frac{(x^2+y^2)^n}{2\pi}\int_{-\pi}^{\pi}\sin^{2n}\theta\mathrm{d}\theta\\
&= \frac{1}{4^n}\binom{2n}{n}(x^2+y^2)^n.
\end{aligned}$$

Therefore,

$$\tau_{2i,2j} = \frac{1}{4^{i+j}}\frac{(2i)!(2j)!}{i!j!(i+j)!}, \qquad i,j\in\mathbb{Z}_+.$$

Substituting the values of $\tau_{i,j}$ in the expansion of F, we have

$$F(r) = \sum_{j=0}^{\infty}\frac{r^{2j}}{4^j j!^2}\sum_{i=0}^{j}\binom{j}{i}\phi_{2i,2j-2i} = \sum_{j=0}^{\infty}\frac{r^{2j}}{4^j j!^2}\Delta^j f(0,0),$$

as postulated in (3.3). ∎

Given the expansion

$$F(r) = \sum_{m=0}^{\infty}\frac{F^{(m)}(0)}{m!}r^m,$$

it follows by an easy induction that

$$F_n(r) = 2^n\sum_{m=0}^{\infty}\frac{(\frac{m+1}{2})_n}{(m+2n)!}F^{(m+2n+1)}(0)r^n, \qquad n\in\mathbb{Z}_+,$$

where $(z)_n$ is the *Pochhammer symbol:* $(z)_0=1$ and $(z)_n=(z+n-1)(z)_{n-1}$, $n\in\mathbb{N}$. Therefore, it follows from (3.3) that $F_n(0)=0$,

$$F_n'(0) = \frac{1}{2^{n+1}(n+1)!}\Delta^{n+1}f(0,0), \qquad n\in\mathbb{Z}_+$$

and

$$\begin{aligned}
F_n(1) &= 2^n\sum_{j=0}^{\infty}\frac{(j+1)_n}{4^{n+j+1}(n+j+1)!}\Delta^{n+j+1}f(0,0)\\
&= \frac{1}{2^{n+1}}\sum_{j=0}^{\infty}\frac{1}{4^{j+1}j!(n+j+1)!}\Delta^{n+j+1}f(0,0).
\end{aligned}$$

Substituting the values of F_n and μ_0 into the asymptotic expansion of $I[f]$, we obtain after easy manipulation the series

$$\begin{aligned}
I[f] \sim\; & -4\pi\sum_{n=0}^{\infty}\frac{1}{(-4\mathrm{i}\omega)^{n+1}}\left[\mathrm{e}^{\mathrm{i}\omega}\sum_{j=0}^{\infty}\frac{\Delta^{n+j}f(0,0)}{4^j j!(n+j)!}-\frac{\Delta^n f(0,0)}{n!}\right]\\
&+\frac{\pi^{\frac{1}{2}}\mathrm{erf}((-\mathrm{i}\omega)^{\frac{1}{2}})}{(-\mathrm{i}\omega)^{\frac{1}{2}}}\sum_{n=1}^{\infty}\frac{1}{(-4\mathrm{i}\omega)^n}\frac{\Delta^n f(0,0)}{n!}. \qquad (3.4)
\end{aligned}$$

The entire information necessary for the computation of (3.4) consists of the values of $\Delta^j f(0,0)$ for $j \in \mathbb{Z}_+$, but the obvious disadvantage of this representation is that it requires the summation of infinite series. Now, even if the series

$$u_n[f] = \frac{1}{2^n} \sum_{j=0}^{\infty} \frac{\Delta^{n+j} f(0,0)}{4^j j!(n+j)!}, \qquad n \in \mathbb{Z}_+,$$

converges rapidly, hence can be truncated, this procedure is fairly expensive in terms of derivative calculations at the origin and the outcome is an asymptotic expansion where leading terms are reproduced only approximately.

Note however that $u_0[f] = (2\pi)^{-1} \int_{-\pi}^{\pi} f(\cos\theta, \sin\theta)\mathrm{d}\theta$, an integral along the perimeter of the disc. Were we able to represent $u_n[f]$ for $n \in \mathbb{N}$ as integrals on the unit circle, we could have computed them all to great accuracy using values of f therein, perhaps using the fast Fourier transform. Unfortunately, this is impossible. For suppose that, given $n \in \mathbb{N}$, there exists a kernel K such that

$$u_n[f] = \int_{-\pi}^{\pi} K(\theta) f(\cos\theta, \sin\theta)\mathrm{d}\theta$$

(note that $K \equiv (2\pi)^{-1}$ for $n = 0$). Taking $\tilde{f}(x,y) = x^2 + y^2$, we readily obtain

$$\int_{-\pi}^{\pi} K(\theta)\mathrm{d}\theta = u_n[\tilde{f}] = 1.$$

It is trivial to prove, changing variables $\theta \to -\theta$, that $K(-\pi) = K(\pi)$. We can thus extend K outside $[-\pi, \pi]$ by periodicity. Given $\psi \in [-\pi, \pi]$, we define

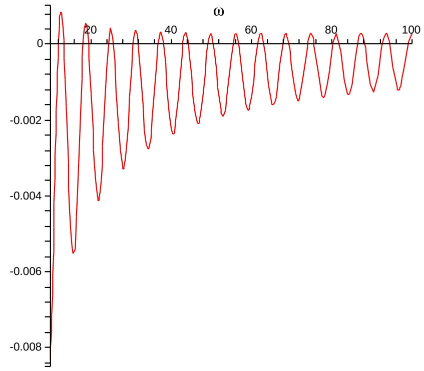
$$f_\psi(x,y) = f(x\cos\psi - y\sin\psi, x\sin\psi + y\cos\psi),$$

hence $f_\psi(\cos\theta, \sin\theta) = f(\cos(\theta+\psi), \sin(\theta+\psi))$. The Laplacian and all its powers being radially invariant, it is thus clear that $u_n[f_\psi] = u_n[f]$. We now change variables affinely and use the periodicity of K,

$$\begin{aligned} u_n[f] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} u_n[f_\psi]\mathrm{d}\psi = \frac{1}{2\pi} \int_{-\pi}^{\pi} K(\theta) f(\cos(\theta+\psi), \sin(\theta+\psi))\mathrm{d}\theta\mathrm{d}\psi \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} K(\theta-\psi)\mathrm{d}\psi f(\cos\theta, \sin\theta)\mathrm{d}\theta \\ &= \frac{1}{2\pi} \left[\int_{-\pi}^{\pi} K(\psi)\mathrm{d}\psi\right] \left[\int_{-\pi}^{\pi} f(\cos\theta, \sin\theta)\mathrm{d}\theta\right] = u_0[f]. \end{aligned}$$

Therefore, unless $u_n[f] = u_0[f]$, we cannot represent u_n as an integral along the boundary.

This disappointing result restricts the efficacy of asymptotic and Filon-type methods in this setting. Perhaps the simplest approach, a “Filon-type method” of sorts, is to truncate the Taylor expansion of f, evaluate exactly (e.g., using the quantities $\tau_{k,l}$) the function F and substitute it into (3.2). In that case, although the leading terms in the expansion do not vanish, they are very small in magnitude and we can expect small error.



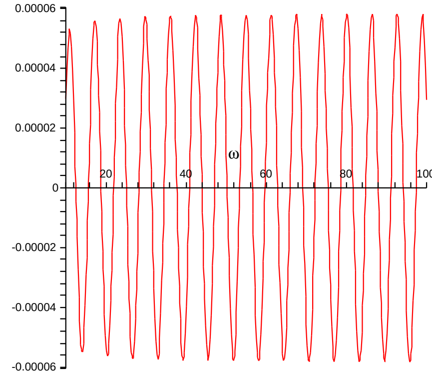


Figure 3.1: Scaled errors $\omega\mathrm{Re}(Q_{1,4}[f]-I[f])$ (on the left) and $\omega\mathrm{Re}(Q_{2,6}[f]-I[f])|$ for $f(x,y)=(4+x-y)^{-1}$.

As an example, consider $f(x,y)=(4+x-y)^{-1}$. Although the exact value of $I[f]$ is unknown, we have used high-order asymptotic quadrature in our numerical experiments. Using symbolic algebra, it is easy to expand

$$F(r)=\tfrac{1}{4}+\tfrac{1}{64}r^2+\tfrac{3}{2048}r^4+\tfrac{5}{32768}r^6+\tfrac{35}{2097152}r^8+\tfrac{63}{33554432}r^{10}+O(r^{12})$$

– the rate of decay is, indeed, rapid. We denote by $F^{[q]}$ the expansion of F truncated for $O(r^{q+1})$ and let

$$\begin{aligned} Q_{s,q}[f] &= \frac{\pi}{\mathrm{i}\omega}[F^{[q]}(1)\mathrm{e}^{\mathrm{i}\omega}-F^{[q]}(0)]+2\pi\mu_0(\omega)\sum_{n=0}^{s-1}\frac{1}{(-2\mathrm{i}\omega)^{n+1}}F_n^{[q]\prime}(0) \\ &\quad -2\pi\sum_{n=0}^{s-1}\frac{1}{(-2\mathrm{i}\omega)^{n+2}}\{[F_n^{[q]}(1)-F_n^{[q]}(0)]\mathrm{e}^{\mathrm{i}\omega}-F_n^{[q]\prime}(0)\}, \end{aligned}$$

where $F_n^{[q]}$ is derived from $F^{[n]}$ in the same manner as F_n from F. Fig. 3.1 displays the real parts of the error, scaled by ω. Note that there are two components to the error, implicit in the presence of two indices in $Q_{s,q}$. For moderate ω is it a good strategy to take more terms in the asymptotic expansion, rather than approximating F well. However, as evident from Fig. 3.2, once $|\omega|$ becomes very large and the contribution of the $O(\omega^{-2})$ terms vanishingly small, there is not much to choose between $Q_{1,q}$ and $Q_{2,q}$ and the error depends solely on the quality of the approximation of F.

4 Conclusion

This preliminary investigation of the asymptotic behaviour and numerical quadrature of multivariate highly oscillatory integrals with critical points highlights two issues. Firstly, it underscores the importance of asymptotic expansions

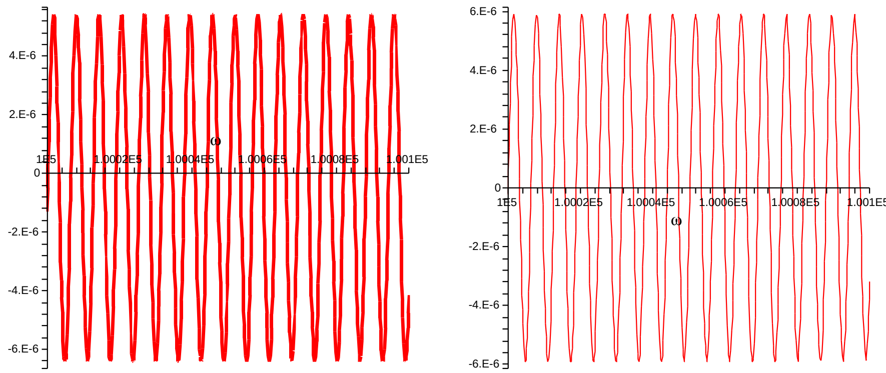


Figure 3.2: Scaled errors $\omega\mathrm{Re}(Q_{1,8}[f]-I[f])$ (on the left) and $\omega\mathrm{Re}(Q_{2,8}[f]-I[f])|$ for $f(x,y)=(4+x-y)^{-1}$ for very large ω.

in the design of quadrature formulae. Secondly, it emphasizes the subtle interplay between the nature of the critical point and the geometry of the boundary. Thus, given the oscillator $g(x,y)=x^2+y^2$, we can approximate the integral with great ease and precision in $[-1,1]^2$ but its quadrature in the unit disc is far from satisfactory.

More complicated oscillators and domains Ω with complicated geometries are bound to present further challenges. In particular, critical points need not be isolated: consider, for example, $g(x,y)=(x-y)^2$ in the triangle $\{(x,y)\,:\,0\leq 1,\ 0\leq x\leq 1-x\}$. More comprehensive analysis of the problem in hand is, clearly, a matter for further research.

Acknowledgments

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