On the value of the max-norm of the orthogonal projector onto splines with multiple knots

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Abstract

The supremum over all knot sequences of the max-norm of the orthogonal spline projector is studied with respect to the order k of the splines and their smoothness. It is first bounded from below in terms of the max-norm of the orthogonal projector onto a space of incomplete polynomials. Then, for continuous and for differentiable splines, its order of growth is shown to be $\sqrt{k}$.

Key words: Orthogonal projectors, Splines

1 Introduction

In 2001, Shadrin [10] confirmed de Boor's long standing conjecture [1] that the max-norm of the orthogonal spline projector is bounded independently of the underlying knot sequence. However, the problem was not solved to complete satisfaction as the behavior of the max-norm supremum remains unclear. Shadrin conjectured that its actual value is $2k-1$, having shown that it cannot be smaller. Here the integer k represents the order of the splines, meaning that the splines are of degree at most $k-1$.

In this paper, we study the max-norm of the orthogonal projector onto splines of lower smoothness. For a knot sequence $\Delta = (-1 = t_0 < t_1 < \cdots < t_{N-1} < t_N = 1)$ and for integers k and m satisfying $0 \leq m \leq k-1$, we denote by

$$\mathcal{S}_{k,m}(\Delta) := \Big\{ s \in \mathcal{C}^{m-1}[-1,1] : s_{|(t_{i-1},t_i)} \text{ is a polynomial of order } k, i = 1, ..., N \Big\}$$

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the space of splines of order k satisfying m conditions of smoothness at each breakpoint $t_1, \dots, t_{N-1}$. Thus $\mathcal{S}_{k,0}(\Delta)$ is the space of piecewise polynomials, $\mathcal{S}_{k,1}(\Delta)$ is the space of continuous splines, and so on until $\mathcal{S}_{k,k-1}(\Delta)$ which is the usual space of splines with simple knots. The orthogonal projector $P_{\mathcal{S}_{k,m}(\Delta)}$ onto the space $\mathcal{S}_{k,m}(\Delta)$ is the only linear map from $L_2[-1,1]$ into $\mathcal{S}_{k,m}(\Delta)$ satisfying

$$\langle P_{\mathcal{S}_{k,m}(\Delta)}(f), s \rangle = \langle f, s \rangle, \qquad f \in L_2[-1,1], \quad s \in \mathcal{S}_{k,m}(\Delta),$$

where $\langle \cdot , \cdot \rangle$ is the usual inner product on $L_2[-1,1]$. We are interested in the norm of this projector when interpreted as a linear map from $L_\infty[-1,1]$ into $L_\infty[-1,1]$. Shadrin established the finiteness of

$$\Lambda_{k,m} := \sup_{\Delta} \left\| P_{\mathcal{S}_{k,m}(\Delta)} \right\|_\infty$$

by proving that $\Lambda_{k,k-1} = \max_m \Lambda_{k,m}$ is finite. His proof was based on the bound

$$\left\| P_{\mathcal{S}_{k,k-1}(\Delta)} \right\|_\infty \leq \left\| G_\Delta^{-1} \right\|_\infty$$

in terms of the ℓ_∞-norm of the inverse of the B-spline Gram matrix. But he also remarked that the order of the bound obtained as such cannot be better than $4^k/\sqrt{k}$, the order of $\|G_\delta^{-1}\|_\infty$ for the Bernstein knot sequence δ. Therefore, in order to get closer to the value $2k-1$, it is necessary to propose a new approach.

The approach we exploit in the second part of this paper originates from the known behavior of the quantity $\Lambda_{k,0}$. The orthogonal projector onto $\mathcal{S}_{k,0}(\Delta)$ has a local character, hence is deduced from the orthogonal projector onto the space $\mathcal{P}_k$ of polynomials of order k on the interval $[-1,1]$. In particular, for any knot sequence Δ, there holds $\|P_{\mathcal{S}_{k,0}(\Delta)}\|_\infty = \|P_{\mathcal{P}_k}\|_\infty$. Then, according to some properties of the orthogonal projector onto polynomials, see e.g. [5], we have

$$\left\| P_{\mathcal{S}_{k,0}(\Delta)} \right\|_\infty = \sup_{\|f\|_\infty \leq 1} |P_{\mathcal{P}_k}(f)(1)|, \qquad \text{so that} \quad \Lambda_{k,0} \asymp \sqrt{k}. \tag{1}$$

We will show that the behavior of $\Lambda_{k,m}$ is not radically changed if we increase the smoothness to $m=1$ and $m=2$, thus improving de Boor's estimate [2]

$$\Lambda_{k,1} \leq \left\| G_\delta^{-1} \right\|_\infty \asymp 4^k/\sqrt{k}.$$

Namely, we will prove that

$$\Lambda_{k,m} \leq \text{cst} \cdot \sqrt{k}, \qquad m = 1, 2.$$

On the other hand, the order of $\Lambda_{k,m}$ will be shown to be at least $\sqrt{k}$ for $m = 1, 2$. This is a consequence of a result which gives some insight into the inequality $\Lambda_{k,k-1} \geq 2k-1$. Indeed, for any m, we will indicate a connection, extending the one of (1), between $\Lambda_{k,m}$ and the orthogonal projector onto a certain space of incomplete polynomials. To be precise, we introduce the following space of polynomials on $[-1,1]$,

$$\mathcal{P}_{k,m} := \operatorname{span}\left\{(1+\bullet)^m, \ldots, (1+\bullet)^{k-1}\right\}, \tag{2}$$

and we denote by $\rho_{k,m}$ the value at the point 1 of the Lebesgue function of the orthogonal projector $P_{\mathcal{P}_{k,m}}$ onto the space $\mathcal{P}_{k,m}$, i.e.

$$\rho_{k,m} := \sup_{\|f\|_\infty \le 1} \left| P_{\mathcal{P}_{k,m}}(f)(1) \right|.$$

With this terminology, we prove below the inequality

$$\Lambda_{k,m} \ge \frac{k}{k-m}\,\rho_{k,m}. \tag{3}$$

This lower bound is of order $\sqrt{k}$ for small values of m and of order k for large values of m, which gives some support to the speculative guess $\Lambda_{k,m} \asymp k/\sqrt{k-m}$.

2 Bounding $\Lambda_{k,m}$ from below

In this section, we formulate a result which readily implies the lower estimate of (3). Let us introduce the quantity

$$\Upsilon_{k,m,N} := \sup_{\Delta=(-1=t_0<\cdots<t_N=1)} \left[\sup_{\|f\|_\infty \le 1} \left| P_{\mathcal{S}_{k,m}(\Delta)}(f)(1) \right| \right].$$

We aim to bound $\Upsilon_{k,m,N+1}$ from below in terms of $\Upsilon_{k,m,N}$, following an idea used for $m=k-1$ in [10] and which appeared first in [8] in the case $k=2$. Namely, we prove in subsections 2.1 and 2.2 that

$$\Upsilon_{k,m,N+1} \ge \frac{m}{k}\Upsilon_{k,m,N} + \rho_{k,m}. \tag{4}$$

In other words, we have

$$(\Upsilon_{k,m,N+1} - \sigma_{k,m}) \ge \frac{m}{k}\left(\Upsilon_{k,m,N} - \sigma_{k,m}\right), \qquad \text{where} \quad \sigma_{k,m} := \frac{k}{k-m}\,\rho_{k,m}.$$

In view of $\Upsilon_{k,m,1} = \rho_{k,0} = \sigma_{k,0}$, we infer

$$\Upsilon_{k,m,N} - \sigma_{k,m} \ge \left(\frac{m}{k}\right)^{N-1} (\sigma_{k,0} - \sigma_{k,m}) \underset{N\to\infty}{\longrightarrow} 0, \qquad \text{hence} \quad \sup_N \Upsilon_{k,m,N} \ge \sigma_{k,m}.$$

This translates into the following theorem.

Theorem 1 *There hold the inequalities*

$$\sup_{\Delta=(-1=t_0<\cdots<t_N=1)} \left\| P_{\mathcal{S}_{k,m}(\Delta)} \right\|_\infty \geq \Upsilon_{k,m,N} \geq \left[\left(\frac{m}{k}\right)^{N-1} \right] \sigma_{k,0} + \left[1 - \left(\frac{m}{k}\right)^{N-1} \right] \sigma_{k,m}.$$

In particular, one has

$$\sup_{\Delta} \left\| P_{\mathcal{S}_{k,m}(\Delta)} \right\|_\infty \; \geq \; \sigma_{k,m}.$$

We note that, in the case $k = 2$, Malyugin [7] established that these inequalities are all equalities.

2.1 Estimating $\Upsilon_{k,m,N+1}$ in terms of $\Upsilon_{k,m,N}$

In order to derive (4), let us fix a knot sequence

$$\Delta = (-1 = t_0 < t_1 < \cdots < t_{N-1} < t_N = 1),$$

and let us consider the refined knot sequence

$$\Delta_t := (-1 = t_0 < t_1 < \cdots < t_{N-1} < t < t_N = 1).$$

We have the splitting

$$\mathcal{S}_{k,m}(\Delta_t) = \mathcal{S}_{k,m}(\Delta) \oplus \mathcal{T}_{k,m,t}, \quad \text{where } \mathcal{T}_{k,m,t} := \operatorname{span}\left\{ (\bullet - t)_+^m, \ldots, (\bullet - t)_+^{k-1} \right\}.$$

Let P_t, P and Q_t denote the orthogonal projectors onto $\mathcal{S}_{k,m}(\Delta_t)$, $\mathcal{S}_{k,m}(\Delta)$ and $\mathcal{T}_{k,m,t}$ respectively, and let $\mathbf{1}$ denote the function constantly equal to 1. We are going to establish first that

$$\varepsilon_t := \sup_{\|f\|_\infty \leq 1} \| P_t(f) - P(f) - Q_t(f) + P(f)(1) Q_t(\mathbf{1}) \|_\infty \underset{t \to 1}{\longrightarrow} 0. \tag{5}$$

The following lemma is a kind of folklore.

Lemma 2 *The orthogonal projector P from a Hilbert space H onto a finite-dimensional subspace $V = V_1 \oplus V_2$ can be expressed in terms of the orthogonal projectors P_1 and P_2 onto V_1 and V_2 as*

$$P = (I - P_1 P_2)^{-1} P_1 (I - P_2) + (I - P_2 P_1)^{-1} P_2 (I - P_1).$$

PROOF. We remark first that the operator $I - P_1 P_2$ is invertible, because $\|P_1 P_2\| < 1$ for the operator norm subordinated to the Hilbert norm $\| \cdot \|$.

Indeed, for $v_2 \in V_2$, we have

$$\|v_2\|^2 = \|P_1 v_2\|^2 + \|v_2 - P_1 v_2\|^2 > \|P_1 v_2\|^2,$$

and due to the finite dimension of V_2, we derive that $\|P_{1|V_2}\| < 1$, hence that $\|P_1 P_2\| \le \|P_{1|V_2}\| \|P_2\| < 1$. Similar arguments prove that the operator $I - P_2 P_1$ is invertible. Then, for $h \in H$, we write $Ph =: v_1 + v_2$ for $v_1 \in V_1$ and $v_2 \in V_2$. We apply P_1 and $P_1 P_2$ to Ph, so that, in view of $P_1 P = P_1$ and $P_2 P = P_2$, we get

$$\begin{aligned} P_1 h &= \quad v_1 \quad + P_1 v_2, \\ P_1 P_2 h &= P_1 P_2 v_1 + P_1 v_2, \end{aligned} \qquad \text{thus} \quad P_1(I - P_2)h = (I - P_1 P_2) v_1.$$

We infer that $v_1 = (I - P_1 P_2)^{-1} P_1 (I - P_2) h$. The expression for v_2 is obtained by exchanging the indices. $\square$

In our situation, and in view of $(I - Q_t P)^{-1} = I + Q_t (I - P Q_t)^{-1} P$, Lemma 2 reads

$$\begin{aligned} P_t &= (I - PQ_t)^{-1} P (I - Q_t) + (I - Q_t P)^{-1} Q_t (I - P) \\ &= (I - PQ_t)^{-1} (P - PQ_t) + Q_t - Q_t P + Q_t (I - PQ_t)^{-1} P Q_t (I - P). \end{aligned} \tag{6}$$

We claim that, for the operator norm subordinated to the max-norm, one has

$$Q_t P - P(\bullet)(1) Q_t(\mathbf{1}) \longrightarrow 0, \qquad PQ_t \longrightarrow 0.$$

To justify this claim, we remark first that the orthogonal projector Q_t is obtained from the orthogonal projector $P_{\mathcal{P}_{k,m}}$ onto the space $\mathcal{P}_{k,m}$ introduced in (2) by a linear transformation between the intervals $[t, 1]$ and $[-1, 1]$. Namely, for $u \in [t, 1]$, we have

$$Q_t(f)(u) = P_{\mathcal{P}_{k,m}}(\widetilde{f}) \left(\frac{2u - 1 - t}{1 - t} \right), \qquad \widetilde{f}(x) := f\left(\frac{(1 - t)x + 1 + t}{2} \right).$$

Then, for $s \in \mathcal{S}_{k,m}(\Delta)$, $\|s\|_\infty \le 1$, we get, as $\|s'\|_\infty \le C$ for some constant C,

$$\begin{aligned} \|Q_t(s) - s(1) Q_t(\mathbf{1})\|_\infty &= \left\| P_{\mathcal{P}_{k,m}}(\widetilde{s} - s(1)\mathbf{1}) \right\|_\infty \\ &\le \left\| P_{\mathcal{P}_{k,m}} \right\|_\infty \|s - s(1)\mathbf{1}\|_{\infty,[t,1]} \le \left\| P_{\mathcal{P}_{k,m}} \right\|_\infty C\,(1 - t). \end{aligned}$$

This implies the first part of our claim. Next, fixing an orthonormal basis $(s_i)_{i=1}^L$ of $\mathcal{S}_{k,m}(\Delta)$, a function f vanishing on $[-1, t]$ and such that $\|f\|_\infty \le 1$ satisfies

$$\|Pf\|_\infty = \left\| \sum_{i=1}^L \langle s_i, f \rangle s_i \right\|_\infty \le \sum_{i=1}^L \int_t^1 |s_i(u)| du \cdot \|s_i\|_\infty =: \eta_t.$$

The second part of our claim follows from the facts that $\eta_t \to 0$ as $t \to 1$ and that the norm of Q_t is independent of t.

Now, looking at the limit of each term of (6) with respect to the operator norm, we derive (5) in the condensed form

$$P_t - P - Q_t + P(\bullet)(1)Q_t(\mathbf{1}) \underset{t\to 1}{\longrightarrow} 0.$$

From the definition of ε_t, one has in particular

$$\sup_{\|f\|_\infty \le 1} |P_t(f)(1) - [1 - Q_t(\mathbf{1})(1)]\, P(f)(1) - Q_t(f)(1)| \ \le \varepsilon_t. \tag{7}$$

Let us stress that the quantity $[1-Q_t(\mathbf{1})(1)]$ is independent of t, as it is simply $[1 - P_{\mathcal{P}_{k,m}}(\mathbf{1})(1)] =: \gamma_{k,m}$. For $f, g \in L_\infty[-1,1]$, $\|f\|_\infty \le 1$, $\|g\|_\infty \le 1$, and for $f_t \in L_\infty[-1,1]$ defined by

$$f_t(x) = \begin{cases} f(x) \ , & x \in [-1,t], \\ g(x) \ , & x \in [t,1], \end{cases}$$

we obtain from (7) the inequality

$$|P_t(f_t)(1) - \gamma_{k,m} P(f_t)(1) - Q_t(f_t)(1)| \le \varepsilon_t.$$

We note that $Q_t(f_t) = Q_t(g)$ and that $|P(f_t - f)(1)| \le \eta_t$ to get

$$\begin{aligned} \Upsilon_{k,m,N+1} \ge |P_t(f_t)(1)| &\ge |\gamma_{k,m} P(f_t)(1) + Q_t(f_t)(1)| - \varepsilon_t \\ &\ge |\gamma_{k,m} P(f)(1) + Q_t(g)(1)| - |\gamma_{k,m}|\, \eta_t - \varepsilon_t. \end{aligned}$$

As the functions f and g were arbitrary, we deduce that

$$\Upsilon_{k,m,N+1} \ge |\gamma_{k,m}| \sup_{\|f\|_\infty \le 1} |P(f)(1)| + \sup_{\|g\|_\infty \le 1} |Q_t(g)(1)| - |\gamma_{k,m}|\, \eta_t - \varepsilon_t.$$

The second supremum is simply the constant $\rho_{k,m}$. In this inequality, we now take first the limit as $t \to 1$ then the supremum over Δ to obtain (4) in the provisional form

$$\Upsilon_{k,m,N+1} \ge |\gamma_{k,m}|\, \Upsilon_{k,m,N} + \rho_{k,m}.$$

2.2 The orthogonal projector onto $\mathcal{P}_{k,m}$

To complete the proof of Theorem 1, we need the value of $\gamma_{k,m}$, thus the value of $P_{\mathcal{P}_{k,m}}(\mathbf{1})(1)$. For this purpose, we call upon a few important properties of Jacobi polynomials which can all be found in Szegö's monograph [12].

The Jacobi polynomials $P_n^{(\alpha,\beta)}$ are defined by Rodrigues' formula

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n\, n!}\frac{d^n}{dx^n}\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right]. \tag{8}$$

They are orthogonal on $[-1,1]$ with respect to the weight $(1-x)^\alpha(1+x)^\beta$, when $\alpha > -1$ and $\beta > -1$ to insure integrability. They obey the symmetry relation $P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$ and the differentiation formula

$$\frac{d}{dx}\left[P_n^{(\alpha,\beta)}(x)\right] = \frac{n+\alpha+\beta+1}{2} P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{9}$$

Their values at the point 1 are

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} = \frac{(n+\alpha)\cdots(\alpha+1)}{n!}. \tag{10}$$

These properties recalled, we can formulate the following lemma, which implies in particular that $\gamma_{k,m} = (-1)^{k-m}m/k$.

Lemma 3 *There hold the representation*

$$P_{\mathcal{P}_{k,m}}(f)(1) = 2^{-m-1}(k+m)\int_{-1}^{1}(1+x)^m P_{k-1-m}^{(1,2m)}(x)f(x)dx$$

and the equality

$$P_{\mathcal{P}_{k,m}}(\mathbf{1})(1) = 1-(-1)^{k-m}\frac{m}{k}.$$

PROOF. Let us introduce the polynomials $p_i \in \mathcal{P}_{k,m}$ defined by $p_i(x) := (1+x)^m P_i^{(0,2m)}(x)$. The orthogonality conditions

$$h_i^{(0,2m)}\cdot\delta_{i,j} := \int_{-1}^{1}(1+x)^{2m}P_i^{(0,2m)}(x)P_j^{(0,2m)}(x)dx = \int_{-1}^{1}p_i(x)p_j(x)dx$$

show that system $(p_i)_{i=0}^{k-1-m}$ is an orthogonal basis of $\mathcal{P}_{k,m}$. Therefore the orthogonal projector onto $\mathcal{P}_{k,m}$ admits the representation

$$P_{\mathcal{P}_{k,m}}(f) = \sum_{i=0}^{k-1-m}\frac{\langle p_i, f\rangle}{\|p_i\|_2^2}\,p_i.$$

For $y \in [-1,1]$, it reads

$$\begin{aligned}P_{\mathcal{P}_{k,m}}(f)(y) &= \sum_{i=0}^{k-1-m}\frac{1}{h_i^{(0,2m)}}\int_{-1}^{1}(1+x)^m P_i^{(0,2m)}(x)f(x)dx\cdot(1+y)^m P_i^{(0,2m)}(y)\\ &=: \int_{-1}^{1}(1+x)^m(1+y)^m K_{k-1-m}^{(0,2m)}(x,y)f(x)dx.\end{aligned}$$

According to [12, p 71], the kernel $K^{(0,2m)}_{k-1-m}(x,1)$ is $2^{-2m-1}(k+m)P^{(1,2m)}_{k-1-m}(x)$, hence the representation mentioned in the lemma. We then have

$$\begin{aligned}
P_{\mathcal{P}_{k,m}}(\mathbf{1})(1) &= 2^{-m-1}(k+m)\int_{-1}^{1}(1+x)^m P^{(1,2m)}_{k-1-m}(x)dx \\
&\underset{(9)}{=} 2^{-m}\int_{-1}^{1}(1+x)^m \frac{d}{dx}\left[P^{(0,2m-1)}_{k-m}(x)\right]dx \\
&= 2^{-m}\left(\left[(1+x)^m P^{(0,2m-1)}_{k-m}(x)\right]_{-1}^{1} - m\int_{-1}^{1}(1+x)^{m-1}P^{(0,2m-1)}_{k-m}(x)dx\right) \\
&\underset{(10)}{=} 1 - 2^{-m}m\int_{-1}^{1}(1+x)^{m-1}P^{(0,2m-1)}_{k-m}(x)dx.
\end{aligned}$$

The latter integral equals $(-1)^{k-m}2^m/k$, as the following calculation shows

$$\begin{aligned}
&\int_{-1}^{1}(1+x)^{m-1}P^{(0,2m-1)}_{k-m}(x)dx \\
&\quad\underset{(8)}{=} \frac{(-1)^{k-m}}{2^{k-m}(k-m)!}\int_{-1}^{1}(1+x)^{-m}\cdot\frac{d^{k-m}}{dx^{k-m}}\left[(1-x)^{k-m}(1+x)^{k+m-1}\right]dx \\
&\quad= \frac{1}{2^{k-m}(k-m)!}\int_{-1}^{1}\frac{d^{k-m}}{dx^{k-m}}\left[(1+x)^{-m}\right]\cdot(1-x)^{k-m}(1+x)^{k+m-1}dx \\
&\quad= \frac{1}{2^{k-m}(k-m)!}\,\frac{(-1)^{k-m}(k-1)!}{(m-1)!}\int_{-1}^{1}(1-x)^{k-m}(1+x)^{m-1}dx \\
&\quad= \frac{(-1)^{k-m}(k-1)!}{2^{k-m}(k-m)!(m-1)!}\,\frac{2^k(k-m)!(m-1)!}{k!} = (-1)^{k-m}\frac{2^m}{k}. \qquad \square
\end{aligned}$$

3 On the constant $\rho_{k,m}$

We now justify that the quantity $\Lambda_{k,m}$ is at least of order $\sqrt{k}$ for small values of m and at least of order k for large values of m. Precisely, the behavior of $\sigma_{k,m}$ is given below.

Proposition 4 *The lower bounds $\sigma_{k,m}$ for $\Lambda_{k,m}$ satisfy*

$$\begin{aligned}
\sigma_{k,k-1} &= 2k-1, \\
\sigma_{k,k-2} &\underset{k\to\infty}{\sim} c_{k-2}\,k, && c_{k-2} = 4e^{-1} \approx 1.4715, \\
\sigma_{k,k-3} &\underset{k\to\infty}{\sim} c_{k-3}\,k, && c_{k-3} \approx 1.2216, \\
\sigma_{k,m} &\underset{k\to\infty}{\sim} c\sqrt{k}, && c = 2\sqrt{2/\pi} \approx 1.5957, \quad \textit{if } m \textit{ is independent of } k.
\end{aligned}$$

This will follow at once when we establish the behavior of the constant $\rho_{k,m}$. According to Lemma 3, this constant can be expressed as

$$\rho_{k,m} = 2^{-m-1}(k+m)\int_{-1}^{1}(1+x)^m \left|P_{k-1-m}^{(1,2m)}(x)\right| dx. \tag{11}$$

To the best of our knowledge, whether $\rho_{k,m}$ equals the max-norm of the orthogonal projector onto $\mathcal{P}_{k,m}$ is an open question, although this is known for $m = 0$, is trivial for $m = k-1$ and can be shown for $m = k-2$. It also seems that there has been no attempt to evaluate the order of growth of $\rho_{k,m}$ *uniformly* in m. Nevertheless, for small and large values of m, such evaluations can be carried out.

Lemma 5 *One has*

$$\begin{aligned}
\rho_{k,k-1} &= 2 - 1/k,\\
\rho_{k,k-2} &\underset{k\to\infty}{\longrightarrow} 8e^{-1} \approx 2.9430,\\
\rho_{k,k-3} &\underset{k\to\infty}{\longrightarrow} 2 + 8(2+\sqrt{3})e^{(-3-\sqrt{3})/2} - 8(2-\sqrt{3})e^{(-3+\sqrt{3})/2} \approx 3.6649.
\end{aligned}$$

PROOF. The fact that $P_0^{(1,2k-2)}(x) = 1$ clearly yields the value of $\rho_{k,k-1}$. We then compute $P_1^{(1,2k-4)}(x) = \frac{1}{2}\left[(2k-1)(1+x) - 4k + 6\right]$ and we subsequently obtain

$$\rho_{k,k-2} = \frac{2}{k} + \frac{4(2k-3)}{k}\left(\frac{2k-3}{2k-1}\right)^{k-1} \underset{k\to\infty}{\longrightarrow} 8e^{-1}.$$

Finally, we find that $P_2^{(1,2k-6)}(x)$ equals

$$\frac{1}{4}\Big[(k-1)(2k-1)(1+x)^2 - 8(k-1)(k-2)(1+x) + 4(k-2)(2k-5)\Big].$$

The roots of this quadratic polynomial are

$$x_1 = \frac{2k-7-2\sqrt{\frac{3(k-2)}{k-1}}}{2k-1}, \qquad x_2 = \frac{2k-7+2\sqrt{\frac{3(k-2)}{k-1}}}{2k-1}.$$

After some calculations, we obtain the announced limit from the expression

$$\begin{aligned}
\rho_{k,k-3} = \frac{2k-3}{k} &+ \frac{4(2k-3)}{k}\left[(2-k)(1+x_1) + 2k - 5\right]\left(\frac{1+x_1}{2}\right)^{k-2}\\
&- \frac{4(2k-3)}{k}\left[(2-k)(1+x_2) + 2k - 5\right]\left(\frac{1+x_2}{2}\right)^{k-2}. \qquad \square
\end{aligned}$$

As for small values of m, the behavior of $\rho_{k,m}$ follows from a result of Szegö [11, p 84–86], whose first part was sharpened in [6].

Proposition 6 ([11]) *If $2\lambda-\alpha+3/2>0$, there is a constant $c_{\lambda,\mu}^{(\alpha,\beta)}$ such that*

$$\int_0^1 (1-x)^\lambda (1+x)^\mu \left|P_n^{(\alpha,\beta)}(x)\right| dx \underset{n\to\infty}{\sim} c_{\lambda,\mu}^{(\alpha,\beta)}\ n^{-\frac{1}{2}}.$$

If $2\lambda-\alpha+3/2<0$, there is a constant $d_{\lambda,\mu}^{(\alpha,\beta)}$ such that

$$\int_0^1 (1-x)^\lambda (1+x)^\mu \left|P_n^{(\alpha,\beta)}(x)\right| dx \underset{n\to\infty}{\sim} d_{\lambda,\mu}^{(\alpha,\beta)}\ n^{-2\lambda+\alpha-2}.$$

Only the formula for the constant $c_{\lambda,\mu}^{(\alpha,\beta)}$ is relevant to us, it is

$$c_{\lambda,\mu}^{(\alpha,\beta)} = \frac{2^{\lambda+\mu+2}}{\pi\sqrt{\pi}} \int_0^{\frac{\pi}{2}} (\sin\theta/2)^{2\lambda-\alpha+\frac{1}{2}} (\cos\theta/2)^{2\mu-\beta+\frac{1}{2}}\, d\theta.$$

Lemma 7 *If m is independent of k, one has*

$$\rho_{k,m} \underset{k\to\infty}{\sim} \frac{2\sqrt{2}}{\sqrt{\pi}}\sqrt{k}.$$

PROOF. We split the integral appearing in (11) in two and use the symmetry relation to obtain

$$\begin{aligned}&\int_{-1}^1 (1+x)^m \left|P_{k-1-m}^{(1,2m)}(x)\right| dx \\ &\qquad = \int_0^1 (1-x)^m \left|P_{k-1-m}^{(2m,1)}(x)\right| dx + \int_0^1 (1+x)^m \left|P_{k-1-m}^{(1,2m)}(x)\right| dx \\ &\qquad \underset{k\to\infty}{\sim} \left(c_{m,0}^{(2m,1)} + c_{0,m}^{(1,2m)}\right) k^{-\frac{1}{2}}.\end{aligned}$$

Substituting the values of the constants gives

$$\begin{aligned}&c_{m,0}^{(2m,1)} + c_{0,m}^{(1,2m)} \\ &\quad = \frac{2^{m+2}}{\pi\sqrt{\pi}}\left[\int_0^{\frac{\pi}{2}} (\sin\theta/2)^{\frac{1}{2}} (\cos\theta/2)^{-\frac{1}{2}}\, d\theta + \int_0^{\frac{\pi}{2}} (\sin\theta/2)^{-\frac{1}{2}} (\cos\theta/2)^{\frac{1}{2}}\, d\theta\right] \\ &\quad = \frac{2^{m+2}}{\pi\sqrt{\pi}}\left[\int_0^{\frac{\pi}{2}} (\sin\theta/2)^{\frac{1}{2}} (\cos\theta/2)^{-\frac{1}{2}}\, d\theta + \int_{\frac{\pi}{2}}^{\pi} (\cos\eta/2)^{-\frac{1}{2}} (\sin\eta/2)^{\frac{1}{2}}\, d\eta\right] \\ &\quad = \frac{2^{m+2}}{\pi\sqrt{\pi}} \int_0^{\pi} (\sin\theta/2)^{\frac{1}{2}} (\cos\theta/2)^{-\frac{1}{2}}\, d\theta.\end{aligned}$$

For $p, q > 0$, it is known that

$$\int_0^\pi (\sin\theta/2)^{2p-1} (\cos\theta/2)^{2q-1}\, d\theta = \int_0^1 u^{p-1}(1-u)^{q-1} du = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Thus, in view of $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, we derive that

$$c_{m,0}^{(2m,1)} + c_{0,m}^{(1,2m)} = \frac{2^{m+2}}{\pi\sqrt{\pi}} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)} = \frac{2^{m+2}\sqrt{2}}{\sqrt{\pi}},$$

and the conclusion follows. $\square$

Some numerical values of the constant $\rho_{k,m}$ are indicated in the table below.

$\rho_{k,m}$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$m=0$	1	1.6666	2.1757	2.6042	2.9815	3.3225	3.6360
$m=1$		1.5	2.1066	2.5693	2.9625	3.3120	3.6305
$m=2$			1.6666	2.3221	2.8	3.1959	3.5430
$m=3$				1.75	2.4493	2.9503	3.3586
$m=4$					1.8	2.5332	3.0560
$m=5$						1.8333	2.5927
$m=6$							1.8571

We observe that $\rho_{k,0}$ increases with k, a fact which has been proved in [9]. It also seems that $\rho_{k,m}$ increases with k for any fixed m. On the other hand, when k is fixed, the quantity $\rho_{k,m}$ does not decrease with m, e.g. we have $\rho_{10,0} \approx 4.4607 < \rho_{10,1} \approx 4.4619$. The tentative inequality $\rho_{2k,k} \leq \rho_{2k,0}$ may nevertheless hold and would account for the guess $\sigma_{k,m} \asymp k(k-m)^{-1/2}$ rather than the other seemingly natural one, namely $\sigma_{k,m} \asymp k^{(k+m)/2k}$. Indeed, we would have $\sigma_{2k,k} = 2k/k \cdot \rho_{2k,k} \leq 2\rho_{2k,0} \leq \text{cst} \cdot \sqrt{k}$, so that the order of $\sigma_{2k,k}$ could not be $k^{3/4}$.

We display at last some numerical values of the lower bound $\sigma_{k,m}$.

$\sigma_{k,m}$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$m=0$	1	1.6666	2.1757	2.6042	2.9815	3.3225	3.6360
$m=1$		3	3.16	3.4258	3.7031	3.9744	4.2356
$m=2$			5	4.6443	4.6666	4.7938	4.9603
$m=3$				7	6.1233	5.9006	5.8775
$m=4$					9	7.5996	7.1308
$m=5$						11	9.0745
$m=6$							13

For a fixed k, it seems that $\sigma_{k,m}$ increases with m. However, for a fixed m, it appears that $\sigma_{k,m}$ is not a monotonic function of k. The initial decrease of $\sigma_{k,m}$ could be explained by the facts that $\sigma_{m+1,m} = 2m+1$ and that $\sigma_{2m,m} \asymp \sqrt{m}$, if confirmed.

4 Bounding $\Lambda_{k,m}$ from above: description of the method

We present here the key steps of the arguments we will use to determine an upper bound for $\Lambda_{k,m}$. The idea of orthogonal splitting comes from Shadrin, who suggested it to us in a private communication.

4.1 Orthogonal splitting

The space $\mathcal{S}_{k,m}(\Delta)$, of dimension $kN - m(N-1)$, is a subspace of the space $\mathcal{S}_{k,0}(\Delta)$, of dimension kN, hence we can consider the orthogonal splitting

$$\mathcal{S}_{k,0}(\Delta) =: \mathcal{S}_{k,m}(\Delta) \overset{\perp}{\oplus} \mathcal{R}_{k,m}(\Delta), \qquad \text{with } \dim \mathcal{R}_{k,m}(\Delta) = m(N-1).$$

If $P_{\mathcal{S}_{k,0}(\Delta)}$, $P_{\mathcal{S}_{k,m}(\Delta)}$ and $P_{\mathcal{R}_{k,m}(\Delta)}$ represent the orthogonal projectors onto $\mathcal{S}_{k,0}(\Delta)$, $\mathcal{S}_{k,m}(\Delta)$ and $\mathcal{R}_{k,m}(\Delta)$ respectively, we have

$$P_{\mathcal{S}_{k,0}(\Delta)} = P_{\mathcal{S}_{k,m}(\Delta)} + P_{\mathcal{R}_{k,m}(\Delta)}, \text{ thus } \left\| P_{\mathcal{S}_{k,m}(\Delta)} \right\|_\infty \leq \left\| P_{\mathcal{S}_{k,0}(\Delta)} \right\|_\infty + \left\| P_{\mathcal{R}_{k,m}(\Delta)} \right\|_\infty .$$

We have already mentioned that $\|P_{\mathcal{S}_{k,0}(\Delta)}\|_\infty = \rho_{k,0}$ for any knot sequence Δ, therefore our task is to bound the norm $\|P_{\mathcal{R}_{k,m}(\Delta)}\|_\infty$.

In order to describe the space $\mathcal{R}_{k,m}(\Delta)$, we set

$$\begin{aligned}&(\underbrace{t_0 = \cdots = t_0}_{k} < \underbrace{t_1 = \cdots = t_1}_{k-m} < \cdots < \underbrace{t_{N-1} = \cdots = t_{N-1}}_{k-m} < \underbrace{t_N = \cdots = t_N}_{k}) \\ &\quad =: (\tau_1 \leq \cdots \leq \tau_{L+k}),\end{aligned}$$

so that $\mathcal{S}_{k,m}(\Delta)$ admits the basis of L_1-normalized B-splines $(M_i)_{i=1}^{L}$, where $M_i := M_{\tau_i,\ldots,\tau_{i+k}}$. Using the Peano representation of divided differences, we have

$$\begin{aligned} f \in \mathcal{R}_{k,m}(\Delta) &\iff f \in \mathcal{S}_{k,0}(\Delta),\ \int_{-1}^{1} M_i \cdot f = 0, \text{ all } i \\ &\iff f = F^{(k)},\ F \in \mathcal{S}_{2k,k}(\Delta),\ [\tau_i, \ldots, \tau_{i+k}]F = 0, \text{ all } i. \end{aligned}$$

It is then derived that

$$\mathcal{R}_{k,m}(\Delta) = \left\{ F^{(k)},\ F \in \mathcal{S}_{2k,k}(\Delta), \begin{array}{c} F \equiv 0\ \ k\text{-fold at } t_0, \\ F \equiv 0\ \ (k-m)\text{-fold at } t_i,\ \ i = 1, ..., N-1, \\ F \equiv 0\ \ k\text{-fold at } t_N \end{array} \right\}$$
$$= \mathcal{R}^1_{k,m}(\Delta) \oplus \mathcal{R}^2_{k,m}(\Delta) \oplus \cdots \oplus \mathcal{R}^{N-1}_{k,m}(\Delta),$$

where each space $\mathcal{R}^i_{k,m}(\Delta)$, supported on $[t_{i-1}, t_{i+1}]$ and of dimension m, is characterized by

$$f \in \mathcal{R}^i_{k,m}(\Delta) \iff f = F^{(k)} \text{ for some } F \in \mathcal{S}_{2k,k}(\Delta), \operatorname{supp} F = [t_{i-1}, t_{i+1}],$$
$$\text{and} \quad \begin{cases} F \equiv 0\ \ k\text{-fold at } t_{i-1}, \\ F \equiv 0\ \ (k-m)\text{-fold at } t_i, \\ F \equiv 0\ \ k\text{-fold at } t_{i+1}. \end{cases}$$

4.2 A Gram matrix

The max-norm of the orthogonal projector onto the space $\mathcal{R}_{k,m}(\Delta)$ will be bounded with the help of a Gram matrix. We reproduce here an idea that has been central to the theme of the orthogonal spline projector for some time.

Lemma 8 *Let $(\varphi_i)_{i=1}^{m(N-1)}$ and $(\widehat{\varphi}_j)_{j=1}^{m(N-1)}$ be bases of $\mathcal{R}_{k,m}(\Delta)$ and let $M := [\langle \varphi_i, \widehat{\varphi}_j \rangle]_{i,j=1}^{m(N-1)}$ be the Gram matrix with respect to these bases. If, for some constants κ, γ_1 and γ_∞, there hold*

$$\text{(i)}\ \left\| M^{-1} \right\|_\infty \le \kappa, \qquad \text{(ii)}\ \|\varphi_i\|_1 \le \gamma_1, \qquad \text{(iii)}\ \left\| \sum a_j \widehat{\varphi}_j \right\|_\infty \le \gamma_\infty \|a\|_\infty,$$

then the max-norm of the orthogonal projector onto $\mathcal{R}_{k,m}(\Delta)$ satisfies

$$\left\| P_{\mathcal{R}_{k,m}(\Delta)} \right\|_\infty \le \kappa \cdot \gamma_1 \cdot \gamma_\infty.$$

PROOF. Let P denote the projector $P_{\mathcal{R}_{k,m}(\Delta)}$. For $f \in L_\infty[-1,1]$, $\|f\|_\infty = 1$, let us write $P(f) = \sum_{j=1}^{m(N-1)} a_j \widehat{\varphi}_j$, so that $\|P(f)\|_\infty \le \gamma_\infty \|a\|_\infty$. The equalities

$$b_i := \langle \varphi_i, f \rangle = \langle \varphi_i, P(f) \rangle = \sum_j a_j \langle \varphi_i, \widehat{\varphi}_j \rangle = (Ma)_i$$

mean that $a = M^{-1}b$. Since $|b_i| \le \|\varphi_i\|_1$, we infer that $\|a\|_\infty \le \|M^{-1}\|_\infty \cdot \|b\|_\infty \le \kappa \cdot \gamma_1$. Hence we have $\|P(f)\|_\infty \le \kappa \cdot \gamma_1 \cdot \gamma_\infty$, which completes the proof, as the function f was arbitrary. □

Let us remark that the entries of the Gram matrix will be easily calculated by applying the following formula, obtained by integration by parts. One has, for $r_i := R_i^{(k)} \in \mathcal{R}^i_{k,m}(\Delta)$,

$$\langle r_i, s\rangle = \sum_{l=0}^{m-1} (-1)^l R_i^{(k-1-l)}(t_i) \left[s^{(l)}(t_i^-) - s^{(l)}(t_i^+) \right], \qquad s \in \mathcal{S}_{k,0}(\Delta). \tag{12}$$

4.3 Bounding the norm of the inverse of some matrices

If we combine bases of the spaces $\mathcal{R}^i_{k,m}(\Delta)$ to obtain L_1 and L_∞-normalized bases of $\mathcal{R}_{k,m}(\Delta)$, with respect to which we form the Gram matrix, we observe that the latter is block-tridiagonal, as a result of the disjointness of the supports of $\mathcal{R}^i_{k,m}(\Delta)$ and $\mathcal{R}^j_{k,m}(\Delta)$ when $|i-j| > 1$. However, we may permute the elements of the bases to obtain the Gram matrix in the form considered in the following lemma and to bound the ℓ_∞-norm of its inverse accordingly. Let us recall that a square matrix A is said to be of bandwidth d if $A_{i,j} = 0$ as soon as $|i-j| > d$.

Lemma 9 *Let B and C be two matrices such that BC and CB are of bandwidth d. If $\zeta := \max(\|BC\|_1, \|CB\|_1) < 1$, then, with $\xi := \max(\|B\|_\infty, \|C\|_\infty)$, the matrix*

$$N := \left[\begin{array}{c|c} I & B \\ \hline C & I \end{array}\right] \quad \text{has an inverse satisfying} \quad \left\|N^{-1}\right\|_\infty \le (1+\xi)\frac{1+(2d-1)\zeta}{(1-\zeta)^2}.$$

PROOF. First of all, let A be a matrix of bandwidth d satisfying $\|A\|_1 < 1$. For indices i and j, let $q := \left\lceil \frac{|i-j|}{d} \right\rceil$ represent the smallest integer not smaller than $\frac{|i-j|}{d}$. We borrow from Demko [3] the estimate

$$\left|(I-A)^{-1}_{i,j}\right| \le \frac{\|A\|_1{}^q}{1-\|A\|_1}.$$

Indeed, for any integer p the matrix A^p is of bandwidth pd and, as $|i-j| > (q-1)d$, we get

$$\left|(I-A)^{-1}_{i,j}\right| = \left|\sum_{p=0}^{\infty} {A^p}_{i,j}\right| = \left|\sum_{p=q}^{\infty} {A^p}_{i,j}\right| \le \sum_{p=q}^{\infty} |{A^p}_{i,j}| \le \sum_{p=q}^{\infty} \|A^p\|_1 \le \sum_{p=q}^{\infty} \|A\|_1^p,$$

hence the announced inequality. It then follows that

$$\left\|(I-A)^{-1}\right\|_\infty = \max_i \sum_j \left|(I-A)^{-1}_{i,j}\right|$$
$$\leq \frac{1}{1-\|A\|_1}\left[1+2d\sum_{q=1}^{\infty}\|A\|_1^q\right] = \frac{1+(2d-1)\|A\|_1}{(1-\|A\|_1)^2}. \qquad (13)$$

We now observe that

$$\left[\begin{array}{c|c} I & B \\ \hline C & I \end{array}\right]^{-1} = \left[\begin{array}{c|c} (I-BC)^{-1} & -B(I-CB)^{-1} \\ \hline -C(I-BC)^{-1} & (I-CB)^{-1} \end{array}\right].$$

The estimate of (13) for $A = BC$ and $A = CB$ implies the conclusion. □

5 Bounding $\Lambda_{k,m}$ from above: the case of continuous splines

We consider here the case $m = 1$, $k \geq 2$. We have already established that the order of growth of $\Lambda_{k,1} = \sup_\Delta \|P_{\mathcal{S}_{k,1}(\Delta)}\|_\infty$ is at least $\sqrt{k}$ and we prove in this section that it is in fact $\sqrt{k}$. We exploit the method we have just described to obtain the following theorem.

Theorem 10 *For any knot sequence* Δ*,*

$$\left\|P_{\mathcal{R}_{k,1}(\Delta)}\right\|_\infty \leq \frac{2k(k+1)}{(k-1)^2}\,\sigma_{k,0}\,, \qquad \left\|P_{\mathcal{S}_{k,1}(\Delta)}\right\|_\infty \leq \frac{3k^2+1}{(k-1)^2}\,\sigma_{k,0}.$$

First of all, we note that the space $\mathcal{R}^i_{k,1}(\Delta)$ is spanned by a single function f_i supported on $[t_{i-1}, t_{i+1}]$. The latter must be the k-th derivative of a piecewise polynomial F_i of order $2k$ that vanishes k-fold at t_{i-1} and at t_{i+1}, $(k-1)$-fold at t_i and whose $(k-1)$-st derivative is continuous at t_i. It is constructed from the following polynomial of order $2k$,

$$F(x) := \frac{(-1)^{k-1}}{2^{k-1}\,k!}(1-x)^{k-1}(1+x)^k,$$

which vanishes k-fold at -1 and $(k-1)$-fold at 1. The notations

$$h_i := t_i - t_{i-1}, \qquad \delta_i := \frac{1}{h_i}, \qquad i = 1, ..., N,$$

are to be used in the rest of the paper. We define the function F_i by

$$F_i(x) = \begin{cases} \left(\dfrac{h_i}{2}\right)^{k-1} F\left(\dfrac{2x - t_{i-1} - t_i}{h_i}\right) & , \ x \in (t_{i-1}, t_i), \\ \left(\dfrac{-h_{i+1}}{2}\right)^{k-1} F\left(\dfrac{t_i + t_{i+1} - 2x}{h_{i+1}}\right) & , \ x \in (t_i, t_{i+1}), \\ 0 & , \ x \notin (t_{i-1}, t_{i+1}). \end{cases}$$

We renormalize the function $f_i := F_i^{(k)}$ by setting $\widehat{f}_i := \dfrac{1}{4(\delta_i + \delta_{i+1})} f_i$, where

$$f_i(x) = \begin{cases} 2\delta_i \, F^{(k)}\left(\dfrac{2x - t_{i-1} - t_i}{h_i}\right) & , \ x \in (t_{i-1}, t_i), \\ -2\delta_{i+1} F^{(k)}\left(\dfrac{t_i + t_{i+1} - 2x}{h_{i+1}}\right) & , \ x \in (t_i, t_{i+1}), \\ 0 & , \ x \notin (t_{i-1}, t_{i+1}). \end{cases}$$

At this point, let us recall the connection [12, p 64] between the Jacobi polynomials $P_n^{(-l,\beta)}$ and $P_{n-l}^{(l,\beta)}$,

$$\binom{n}{l} P_n^{(-l,\beta)}(x) = \binom{n+\beta}{l} \left(\frac{x-1}{2}\right)^l P_{n-l}^{(l,\beta)}(x), \qquad l = 1, ..., n, \tag{14}$$

which accounts for the following expression for $F^{(k)}$,

$$F^{(k)}(x) \underset{(8)}{=} -2(1-x)^{-1} P_k^{(-1,0)}(x) \underset{(14)}{=} P_{k-1}^{(1,0)}(x).$$

We are now going to establish that the bases $(f_i)_{i=1}^{N-1}$ and $(\widehat{f}_j)_{j=1}^{N-1}$ of $\mathcal{R}_{k,1}(\Delta)$ satisfy the three conditions of Lemma 8.

5.1 Condition (i)

First we determine the inner products $\langle f_i, \widehat{f}_j \rangle$, non-zero only for $|i-j| \le 1$. This requires the values of the successive derivatives of F_i at t_{i-1}, at t_i and at t_{i+1}, which are derived from the values of the successive derivatives of F at -1 and at 1. These are obtained from (9) and (10), namely they are

$$\begin{aligned} & & F^{(k-1)}(1) &= \frac{2}{k}, \\ F^{(k)}(-1) &= (-1)^{k-1}, & F^{(k)}(1) &= k, \\ F^{(k+1)}(-1) &= (-1)^k \frac{k^2-1}{2}, & F^{(k+1)}(1) &= \frac{k(k^2-1)}{4}. \end{aligned}$$

Equation (12) for $r_i = f_i$ reads

$$\langle f_i, s\rangle = F_i^{(k-1)}(t_i)\left[s(t_i^-) - s(t_i^+)\right] = \frac{2}{k}\left[s(t_i^-) - s(t_i^+)\right], \qquad s \in \mathcal{S}_{k,0}(\Delta).$$

We compute the differences

$$\begin{aligned} f_i(t_i^-) - f_i(t_i^+) &= 2\delta_i F^{(k)}(1) + 2\delta_{i+1}F^{(k)}(1) = 2k(\delta_i + \delta_{i+1}), \\ f_i(t_{i-1}^-) - f_i(t_{i-1}^+) &= \quad 0 \qquad\quad - 2\delta_i F^{(k)}(-1) = 2(-1)^k\delta_i. \end{aligned}$$

As a result, we obtain

$$\langle f_i, \widehat{f}_i\rangle = 1, \quad \langle f_{i-1}, \widehat{f}_i\rangle = \frac{(-1)^k}{k}\frac{\delta_i}{\delta_i + \delta_{i+1}}, \qquad \text{then } \langle f_{i+1}, \widehat{f}_i\rangle = \frac{(-1)^k}{k}\frac{\delta_{i+1}}{\delta_i + \delta_{i+1}}.$$

The Gram matrix with respect to the bases $(f_i)_{i=1}^{N-1}$ and $(\widehat{f}_j)_{j=1}^{N-1}$ therefore has the form

$$M = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} \widehat{f}_1 & \widehat{f}_2 & \widehat{f}_3 & \widehat{f}_4 & \cdots \end{array} \\ \left[\begin{array}{ccccc} 1 & \frac{(-1)^k}{k}\alpha_2 & 0 & 0 & \cdots \\ \frac{(-1)^k}{k}\beta_1 & 1 & \frac{(-1)^k}{k}\alpha_3 & 0 & \cdots \\ 0 & \frac{(-1)^k}{k}\beta_2 & 1 & \ddots & \\ 0 & 0 & \frac{(-1)^k}{k}\beta_3 & 1 & \\ \vdots & \vdots & 0 & \ddots & \ddots \end{array}\right] \end{array}$$

where $\alpha_i := \dfrac{\delta_i}{\delta_i + \delta_{i+1}} \geq 0$ and $\beta_i := \dfrac{\delta_{i+1}}{\delta_i + \delta_{i+1}} \geq 0$ satisfy $\alpha_i + \beta_i = 1$. To bound the ℓ_∞-norm of the inverse of this matrix, we could use (13) directly. However, a result of Kershaw [4] about scaled transposes of such matrices provide estimates for the entries of M^{-1} which, when summed, yield the more accurate bound

$$\left\|M^{-1}\right\|_\infty \leq \frac{k^2}{(k-1)^2}.$$

5.2 Condition (ii)

From the expression for f_i, we get $\|f_i\|_1 = 2\|F^{(k)}\|_1 = 2\|P_{k-1}^{(1,0)}\|_1$. Therefore, according to (11), we have

$$\|f_i\|_1 = \frac{4}{k}\sigma_{k,0}.$$

Let us start by establishing the following lemma.

Lemma 11 *For any $\eta, \nu \in \mathbb{R}$, one has*

$$\max_{x\in[-1,1]} \left|\eta P_{k-l}^{(l,0)}(x) + \nu P_{k-l}^{(l,0)}(-x)\right| = \max_{x\in\{-1,1\}} \left|\eta P_{k-l}^{(l,0)}(x) + \nu P_{k-l}^{(l,0)}(-x)\right|.$$

PROOF. Without loss of generality, we can assume that $\eta \geq |\nu|$. First of all, the identity

$$P_{k-l}^{(l,0)}(x) = \sum_{j=0}^{l} \binom{l}{j} \left(\frac{1+x}{2}\right)^j P_{k-l-j}^{(j,j)}(x)$$

is easily derived using (8), (9) and (14). Indeed, we have

$$\begin{aligned}
P_{k-l}^{(l,0)}(x) &= 2^l(-1)^l(1-x)^{-l}P_k^{(-l,0)}(x) \\
&= \frac{(k-l)!}{k!} \frac{(-1)^{k-l}}{2^{k-l}\,(k-l)!} \frac{d^k}{dx^k}\left[(1+x)^l \cdot (1-x)^{k-l}(1+x)^{k-l}\right] \\
&= \frac{(k-l)!}{k!} \sum_{j=0}^{l} \binom{k}{j} \frac{d^j}{dx^j}\left[(1+x)^l\right] \cdot \frac{d^{l-j}}{dx^{l-j}}\left[P_{k-l}^{(0,0)}(x)\right] \\
&= \sum_{j=0}^{l} \frac{(k-l)!}{k!} \frac{k!}{(k-j)!\,j!} \frac{l!}{(l-j)!} \frac{(k-j)!}{(k-l)!} \left(\frac{1+x}{2}\right)^{l-j} P_{k-2l+j}^{(l-j,l-j)}(x) \\
&= \sum_{j=0}^{l} \binom{l}{j} \left(\frac{1+x}{2}\right)^j P_{k-l-j}^{(j,j)}(x).
\end{aligned}$$

This identity and the symmetry relation yield

$$\begin{aligned}
&\eta P_{k-l}^{(l,0)}(x) + \nu P_{k-l}^{(l,0)}(-x) \\
&\qquad = \sum_{j=0}^{l} \binom{l}{j} \left[\eta \left(\frac{1+x}{2}\right)^j + (-1)^{k-l-j}\nu \left(\frac{1-x}{2}\right)^j\right] P_{k-l-j}^{(j,j)}(x).
\end{aligned}$$

Every term in the previous sum is maximized in absolute value at $x = 1$. Indeed, according to [12, Theorem 7.32.1], there holds $\left|P_{k-l-j}^{(j,j)}(x)\right| \leq P_{k-l-j}^{(j,j)}(1)$. Besides, for $j \geq 1$, we have

$$\left|\eta \left(\frac{1+x}{2}\right)^j + (-1)^{k-l-j}\nu \left(\frac{1-x}{2}\right)^j\right| \leq \eta \left[\left(\frac{1+x}{2}\right)^j + \left(\frac{1-x}{2}\right)^j\right] \leq \eta,$$

and for $j = 0$, we have $\left|\eta + (-1)^{k-l}\nu\right| = \eta + (-1)^{k-l}\nu$. These facts imply that

$$\left|\eta P_{k-l}^{(l,0)}(x) + \nu P_{k-l}^{(l,0)}(-x)\right| \leq \eta P_{k-l}^{(l,0)}(1) + \nu P_{k-l}^{(l,0)}(-1). \qquad \square$$

Let us now bound the max-norm of $r := \sum a_j \widehat{f}_j$ in terms of $\|a\|_\infty$. This max-norm is achieved on $[t_l, t_{l+1}]$, say, and since $r_{|[t_l,t_{l+1}]} = a_l \widehat{f}_l + a_{l+1}\widehat{f}_{l+1}$, Lemma 11 guarantees that this max-norm is achieved at one of the endpoints of $[t_l, t_{l+1}]$, say at t_l. Thus we have

$$\|r\|_\infty \le \left[\left|\widehat{f}_l(t_l^+)\right| + \left|\widehat{f}_{l+1}(t_l^+)\right|\right] \|a\|_\infty \le \left[\frac{1}{2}\left|F^{(k)}(1)\right| + \frac{1}{2}\left|F^{(k)}(-1)\right|\right] \|a\|_\infty,$$

that is

$$\left\|\sum a_j \widehat{f}_j\right\|_\infty \le \frac{k+1}{2}\|a\|_\infty.$$

5.4 Conclusion

The estimates obtained from conditions (i), (ii) and (iii) yield

$$\left\|P_{\mathcal{R}_{k,1}(\Delta)}\right\|_\infty \le \frac{k^2}{(k-1)^2}\cdot\frac{4}{k}\sigma_{k,0}\cdot\frac{k+1}{2} = \frac{2k(k+1)}{(k-1)^2}\sigma_{k,0}. \tag{15}$$

To conclude, we derive the bound

$$\left\|P_{\mathcal{S}_{k,1}(\Delta)}\right\|_\infty \le \left\|P_{\mathcal{S}_{k,0}(\Delta)}\right\|_\infty + \left\|P_{\mathcal{R}_{k,1}(\Delta)}\right\|_\infty \le \sigma_{k,0} + \frac{2k(k+1)}{(k-1)^2}\sigma_{k,0} = \frac{3k^2+1}{(k-1)^2}\sigma_{k,0}.$$

This upper bound is much better than the bound $\|G_\delta^{-1}\|_\infty$, already mentioned in the introduction, which was given by de Boor in [2], at least asymptotically. In fact, this becomes true as soon as $k = 4$, as the following table shows. The values of $\|G_\delta^{-1}\|_\infty$ are taken from [10].

k	2	3	4	5	6	7	8
$\frac{3k^2+1}{(k-1)^2}\,\sigma_{k,0}$	21.666	15.230	14.178	14.162	14.486	14.948	15.470
$\|G_\delta^{-1}\|_\infty$	3	13	41.666	171	583.8	2 364.2	8 373.857

Let us finally note that the estimate of (15) is fairly precise in the sense that it is possible to obtain $\sup_\Delta \|P_{\mathcal{R}_{k,1}(\Delta)}\|_\infty \ge 2\sigma_{k,0}$ simply by considering $P_{\mathcal{R}_{k,1}(\Delta)}(\bullet)(t_1^-)$ when $N = 2$, $t_1 \to 0$. This implies

$$\sup_\Delta \left\|P_{\mathcal{S}_{k,1}(\Delta)}\right\|_\infty \ge \sup_\Delta \left\|P_{\mathcal{R}_{k,1}(\Delta)}\right\|_\infty - \left\|P_{\mathcal{S}_{k,0}(\Delta)}\right\|_\infty \ge \sigma_{k,0}.$$

If, as we believe, the lower bound $\sigma_{k,m}$ is the actual value of $\Lambda_{k,m}$, the previous inequality reads $\sigma_{k,1} \ge \sigma_{k,0}$. This is in accordance with the expected monotonicity of $\sigma_{k,m}$ and can be proved as follow. First, we readily check that

$$\mathcal{P}_{k,m} = \mathcal{P}_{k,m+1} \overset{\perp}{\oplus} \operatorname{span}\left[(1+\bullet)^m P_{k-1-m}^{(0,2m+1)}\right].$$

From the representations of the Lebesgue functions at the point 1 of the orthogonal projectors onto these spaces, we obtain, for some constant C, the identity

$$2^{-m-1}(k+m)(1+x)^m P_{k-1-m}^{(1,2m)}(x) = 2^{-m-2}(k+m+1)(1+x)^{m+1} P_{k-2-m}^{(1,2m+2)}(x) \\ + C(1+x)^m P_{k-1-m}^{(0,2m+1)}(x).$$

The value of the constant C is $2^{-m-1}(2m+1)$, as seen from the choice $x=1$. With $m=0$, we get

$$\frac{k}{2} P_{k-1}^{(1,0)}(x) = \frac{k+1}{4}(1+x) P_{k-2}^{(1,2)}(x) + \frac{1}{2} P_{k-1}^{(0,1)}(x).$$

The inequality $\sigma_{k,0} \le \sigma_{k,1}$ is then deduced from

$$\begin{aligned} \sigma_{k,0} = \rho_{k,0} &= \frac{k}{2}\int_{-1}^{1} |P_{k-1}^{(1,0)}(x)| dx \\ &\le \frac{k+1}{4}\int_{-1}^{1}(1+x)|P_{k-2}^{(1,2)}(x)|dx + \frac{1}{2}\int_{-1}^{1}|P_{k-1}^{(0,1)}(x)|dx \\ &= \rho_{k,1} + \frac{1}{k}\rho_{k,0} = \frac{k-1}{k}\sigma_{k,1} + \frac{1}{k}\sigma_{k,0}. \end{aligned}$$

6 Bounding $\Lambda_{k,m}$ from above: the case of differentiable splines

We consider here the case $m=2$, $k \ge 3$, for which the order of growth of $\Lambda_{k,2} = \sup_\Delta \|P_{\mathcal{S}_{k,2}(\Delta)}\|_\infty$ is also shown to be $\sqrt{k}$. This section is dedicated to the proof of the following proposition, where the notation $u_n \lesssim v_n$ for two sequences (u_n) and (v_n) means that there exists a sequence (w_n) such that $u_n \le w_n$, $n \in \mathbb{N}$, and $w_n \underset{n\to\infty}{\sim} v_n$.

Proposition 12 *For any knot sequence Δ,*

$$\left\|P_{\mathcal{R}_{k,2}(\Delta)}\right\|_\infty \lesssim \frac{36\sqrt{2}}{\sqrt{\pi}}\sqrt{k}, \qquad \left\|P_{\mathcal{S}_{k,2}(\Delta)}\right\|_\infty \lesssim \frac{38\sqrt{2}}{\sqrt{\pi}}\sqrt{k}.$$

The function f_i previously defined is an element of the 2-dimensional space $\mathcal{R}^i_{k,2}(\Delta)$. In this space, we consider an element g_i orthogonal to f_i. It must be the k-th derivative of a piecewise polynomial G_i of order $2k$ supported on $[t_{i-1}, t_{i+1}]$. The function G_i must vanish k-fold at t_{i-1} and at t_{i+1}, $(k-2)$-fold at t_i and its $(k-2)$-nd and $(k-1)$-st derivatives must be continuous at t_i. It is then guaranteed that $g_i = G_i^{(k)}$ belongs to $\mathcal{R}^i_{k,2}(\Delta)$. To be orthogonal to f_i, the function g_i must further be continuous at t_i . Let us introduce the polynomial G of order $2k$,

$$G(x) := \frac{(-1)^k}{2^{k-2}k!}(1-x)^{k-2}(1+x)^k,$$

which vanishes k-fold at -1 and $(k-2)$-fold at 1. Let us remark that

$$G^{(k)}(x) \underset{(8)}{=} 4(1-x)^{-2}P_k^{(-2,0)}(x) \underset{(14)}{=} P_{k-2}^{(2,0)}(x).$$

We now define the auxiliary function H_i by

$$H_i(x) := \begin{cases} \left(\delta_{i+1} + \frac{k-1}{k+1}\delta_i\right)\left(\frac{h_i}{2}\right)^{k-1} F\left(\frac{2x-t_{i-1}-t_i}{h_i}\right) \\ \qquad - \frac{1}{k+1}\left(\frac{h_i}{2}\right)^{k-2} G\left(\frac{2x-t_{i-1}-t_i}{h_i}\right) & , x \in (t_{i-1}, t_i), \\ -\left(\delta_i + \frac{k-1}{k+1}\delta_{i+1}\right)\left(\frac{-h_{i+1}}{2}\right)^{k-1} F\left(\frac{t_i+t_{i+1}-2x}{h_{i+1}}\right) \\ \qquad - \frac{1}{k+1}\left(\frac{-h_{i+1}}{2}\right)^{k-2} G\left(\frac{t_i+t_{i+1}-2x}{h_{i+1}}\right) & , x \in (t_i, t_{i+1}), \\ 0 & , x \notin (t_{i-1}, t_{i+1}), \end{cases}$$

and we set, for some positive constants λ and μ to be chosen later,

$$G_i := \frac{\lambda}{\delta_i + \delta_{i+1}} H_i, \qquad g_i := G_i^{(k)} \qquad \text{and} \qquad \widehat{g}_i := \frac{\mu}{\delta_i + \delta_{i+1}} g_i.$$

First of all, we have to verify that g_i defined in this way is indeed an element of $\mathcal{R}^i_{k,2}(\Delta)$ orthogonal to f_i, i.e. we have to establish the continuity at t_i of the $(k-2)$-nd, $(k-1)$-st and k-th derivatives of G_i, or equivalently of H_i. The values of the successive derivatives of G at -1 and at 1, obtained from (9) and (10), are needed. They are

$$\begin{aligned} G^{(k)}(-1) &= (-1)^k, \\ G^{(k+1)}(-1) &= (-1)^{k-1}\frac{(k-2)(k+1)}{2}, \end{aligned} \qquad \begin{aligned} G^{(k-2)}(1) &= \frac{4}{k(k-1)}, \\ G^{(k-1)}(1) &= 2, \\ G^{(k)}(1) &= \frac{k(k-1)}{2}, \\ G^{(k+1)}(1) &= \frac{k(k-2)(k^2-1)}{12}. \end{aligned}$$

As $F^{(k-2)}(1) = 0$, the continuity of $H_i^{(k-2)}$ at t_i is readily checked. We have

$$H_i^{(k-2)}(t_i^-) = H_i^{(k-2)}(t_i^+) = -\frac{1}{k+1}G^{(k-2)}(1) = -\frac{4}{k(k^2-1)}.$$

As for the continuity of $H_i^{(k-1)}$ at t_i, it follows from

$$H_i^{(k-1)}(t_i^-) = \left(\delta_{i+1} + \frac{k-1}{k+1}\delta_i\right) \cdot \frac{2}{k} - \frac{1}{k+1} \cdot 2\delta_i \cdot 2 = \frac{2}{k}(\delta_{i+1} - \delta_i),$$
$$H_i^{(k-1)}(t_i^+) = -\left(\delta_i + \frac{k-1}{k+1}\delta_{i+1}\right) \cdot \frac{2}{k} - \frac{1}{k+1} \cdot (-2\delta_{i+1}) \cdot 2 = \frac{2}{k}(\delta_{i+1} - \delta_i).$$

Finally, the continuity of $H_i^{(k)}$ at t_i is a consequence of

$$H_i^{(k)}(t_i^-) = \left(\delta_{i+1} + \frac{k-1}{k+1}\delta_i\right) \cdot 2\delta_i \cdot k - \frac{1}{k+1} \cdot 4\delta_i^2 \cdot \frac{k(k-1)}{2} = 2k\delta_i\delta_{i+1},$$
$$\begin{aligned} H_i^{(k)}(t_i^+) &= -\left(\delta_i + \frac{k-1}{k+1}\delta_{i+1}\right) \cdot (-2\delta_{i+1}) \cdot k - \frac{1}{k+1} \cdot 4\delta_{i+1}^2 \cdot \frac{k(k-1)}{2} \\ &= 2k\delta_i\delta_{i+1}. \end{aligned}$$

This justifies the definition of g_i. We are now going to establish that the bases $(f_i, g_i)_{i=1}^{N-1}$ and $(\widehat{f}_i, \widehat{g}_i)_{i=1}^{N-1}$ of $\mathcal{R}_{k,2}(\Delta)$ satisfy the three conditions of Lemma 8.

6.1 Condition (i)

First we determine the entries of the Gram matrix. The values of $H_i^{(k+1)}(t_i^-)$ and $H_i^{(k+1)}(t_i^+)$ are required, they are

$$\begin{aligned} H_i^{(k+1)}(t_i^-) &= \left(\delta_{i+1} + \frac{k-1}{k+1}\delta_i\right) \cdot 4\delta_i^2 \cdot \frac{k(k^2-1)}{4} \\ &\quad - \frac{1}{k+1} \cdot 8\delta_i^3 \cdot \frac{k(k-2)(k^2-1)}{12} \;=\; \frac{k(k^2-1)}{3}[\delta_i^3 + 3\delta_i^2\delta_{i+1}], \\ H_i^{(k+1)}(t_i^+) &= -\left(\delta_i + \frac{k-1}{k+1}\delta_{i+1}\right) \cdot 4\delta_{i+1}^2 \cdot \frac{k(k^2-1)}{4} \\ &\quad - \frac{1}{k+1} \cdot (-8\delta_{i+1}^3) \cdot \frac{k(k-2)(k^2-1)}{12} \;=\; -\frac{k(k^2-1)}{3}[\delta_{i+1}^3 + 3\delta_i\delta_{i+1}^2]. \end{aligned}$$

Equation (12) yields, in view of the continuity of $H_i^{(k)}$ at t_i,

$$\begin{aligned} \langle g_i, \widehat{g}_i \rangle &= \frac{\lambda^2\mu}{(\delta_i + \delta_{i+1})^3} \cdot \left(-H_i^{(k-2)}(t_i)\right) \cdot \left[H_i^{(k+1)}(t_i^-) - H_i^{(k+1)}(t_i^+)\right] \\ &= \frac{\lambda^2\mu}{(\delta_i + \delta_{i+1})^3} \cdot \frac{4}{k(k^2-1)} \cdot \frac{k(k^2-1)}{3}(\delta_i + \delta_{i+1})^3 = \frac{4\lambda^2\mu}{3}. \end{aligned}$$

We impose from now on $4\lambda^2\mu = 3$, so that $\langle g_i, \widehat{g}_i \rangle = 1$. Consequently, after a reordering of the bases, the Gram matrix has the form

$$
M \;=\; \begin{array}{c|c|c|}
 & \widehat{f}_1\ \widehat{g}_1\ \widehat{f}_3\ \widehat{g}_3\ \ldots & \widehat{f}_2\ \widehat{g}_2\ \widehat{f}_4\ \widehat{g}_4\ \ldots \\
\hline
f_1,\ g_1,\ f_3,\ g_3,\ \vdots & I & B \\
\hline
f_2,\ g_2,\ f_4,\ g_4,\ \vdots & C & I \\
\hline
\end{array} .
$$

The matrices B and C are respectively lower and upper bidiagonal by blocks of size 2×2. Their entries are given in Lemma 13 below and their ℓ_1-norms satisfy $\max(\|B\|_1, \|C\|_1) = \max_i \max(\Phi_i, \Psi_i)$, where

$$
\begin{aligned}
\Phi_i &:= |\langle f_{i-1}, \widehat{f}_i\rangle| + |\langle g_{i-1}, \widehat{f}_i\rangle| + |\langle f_{i+1}, \widehat{f}_i\rangle| + |\langle g_{i+1}, \widehat{f}_i\rangle|, \\
\Psi_i &:= |\langle f_{i-1}, \widehat{g}_i\rangle| + |\langle g_{i-1}, \widehat{g}_i\rangle| + |\langle f_{i+1}, \widehat{g}_i\rangle| + |\langle g_{i+1}, \widehat{g}_i\rangle|.
\end{aligned}
$$

Lemma 13 *With* $\alpha_i = \dfrac{\delta_i}{\delta_i + \delta_{i+1}}$ *and* $\beta_i = \dfrac{\delta_{i+1}}{\delta_i + \delta_{i+1}}$, *one has*

$$
\begin{aligned}
\langle f_{i-1}, \widehat{f}_i\rangle &= \frac{(-1)^k}{k}\alpha_i, & \langle f_{i+1}, \widehat{f}_i\rangle &= \frac{(-1)^k}{k}\beta_i, \\
\langle g_{i-1}, \widehat{f}_i\rangle &= \lambda\frac{(-1)^{k-1}}{k}\alpha_i, & \langle g_{i+1}, \widehat{f}_i\rangle &= \lambda\frac{(-1)^k}{k}\beta_i, \\
\langle f_{i-1}, \widehat{g}_i\rangle &= \frac{3}{\lambda}\frac{(-1)^k}{k}\alpha_i, & \langle f_{i+1}, \widehat{g}_i\rangle &= \frac{3}{\lambda}\frac{(-1)^{k-1}}{k}\beta_i, \\
|\langle g_{i-1}, \widehat{g}_i\rangle| &\leq \frac{3}{k}\alpha_i, & |\langle g_{i+1}, \widehat{g}_i\rangle| &\leq \frac{3}{k}\beta_i.
\end{aligned}
$$

PROOF. 1) The inner products $\langle f_{i-1}, \widehat{f}_i\rangle$ and $\langle f_{i+1}, \widehat{f}_i\rangle$ have been computed in the previous section.

2) We now calculate

$$
\begin{aligned}
\langle f_i, g_{i-1}\rangle &= \frac{\lambda}{\delta_{i-1}+\delta_i}\cdot\frac{2}{k}\cdot\left[H_{i-1}^{(k)}(t_i^-) - H_{i-1}^{(k)}(t_i^+)\right] = \frac{\lambda}{\delta_{i-1}+\delta_i}\cdot\frac{2}{k}\cdot \\
&\quad \left[-\left(\delta_{i-1}+\frac{k-1}{k+1}\delta_i\right)\cdot(-2\delta_i)\cdot(-1)^{k-1} - \frac{1}{k+1}\cdot 4\delta_i^2\cdot(-1)^k\right] \\
&= 4\lambda\frac{(-1)^{k-1}}{k}\delta_i, \\
\langle f_i, g_{i+1}\rangle &= \frac{\lambda}{\delta_{i+1}+\delta_{i+2}}\cdot\frac{2}{k}\left[H_{i+1}^{(k)}(t_i^-) - H_{i+1}^{(k)}(t_i^+)\right] = \frac{\lambda}{\delta_{i+1}+\delta_{i+2}}\cdot\frac{2}{k}\cdot \\
&\quad \left[-\left(\delta_{i+2}+\frac{k-1}{k+1}\delta_{i+1}\right)\cdot 2\delta_{i+1}\cdot(-1)^{k-1} + \frac{1}{k+1}\cdot 4\delta_{i+1}^2\cdot(-1)^k\right] \\
&= 4\lambda\frac{(-1)^{k}}{k}\delta_{i+1}.
\end{aligned}
$$

The values of the inner products $\langle g_{i-1}, \widehat{f}_i\rangle$, $\langle g_{i+1}, \widehat{f}_i\rangle$, $\langle f_{i+1}, \widehat{g}_i\rangle$ and $\langle f_{i-1}, \widehat{g}_i\rangle$ are easily deduced, keeping in mind that $4\lambda^2\mu = 3$.

3) As for the inner products $\langle g_{i-1}, \widehat{g}_i\rangle$ and $\langle g_{i+1}, \widehat{g}_i\rangle$, we determine first the value of $H_{i-1}^{(k+1)}(t_i^-)$. We have

$$
\begin{aligned}
H_{i-1}^{(k+1)}(t_i^-) &= -\left(\delta_{i-1}+\frac{k-1}{k+1}\delta_i\right)\cdot 4\delta_i^2\cdot(-1)^k\frac{k^2-1}{2} \\
&\qquad - \frac{1}{k+1}\cdot(-8\delta_i^3)\cdot(-1)^{k-1}\frac{(k-2)(k+1)}{2} \\
&= 2(-1)^{k-1}(k^2-1)(\delta_{i-1}+\delta_i)\delta_i^2 + 4(-1)^k\delta_i^3.
\end{aligned}
$$

Let us note that the value of $H_{i-1}^{(k)}(t_i^-)$ has just been determined in stage 2) when we computed $\langle f_i, g_{i-1}\rangle$. Then, according to (12), we obtain

$$
\begin{aligned}
\langle g_i, g_{i-1}\rangle &= \frac{\lambda^2}{(\delta_{i-1}+\delta_i)(\delta_i+\delta_{i+1})}\cdot\Big\{H_i^{(k-1)}(t_i)\cdot\left[H_{i-1}^{(k)}(t_i^-) - H_{i-1}^{(k)}(t_i^+)\right] \\
&\qquad - H_i^{(k-2)}(t_i)\cdot\left[H_{i-1}^{(k+1)}(t_i^-) - H_{i-1}^{(k+1)}(t_i^+)\right]\Big\} \\
&= \frac{\lambda^2}{(\delta_{i-1}+\delta_i)(\delta_i+\delta_{i+1})}\cdot\Big\{\frac{2}{k}(\delta_{i+1}-\delta_i)\cdot 2(-1)^{k-1}\delta_i(\delta_{i-1}+\delta_i) \\
&\qquad + \frac{4}{k(k^2-1)}\cdot\left(2(-1)^{k-1}(k^2-1)(\delta_{i-1}+\delta_i)\delta_i^2 + 4(-1)^k\delta_i^3\right)\Big\} \\
&= \frac{\lambda^2}{(\delta_{i-1}+\delta_i)(\delta_i+\delta_{i+1})}\cdot\frac{4(-1)^{k-1}}{k}\cdot\left[(\delta_{i-1}+\delta_i)(\delta_i+\delta_{i+1})\delta_i - \frac{4}{k^2-1}\delta_i^3\right] \\
&= 4\lambda^2\frac{(-1)^{k-1}}{k}\left[1-\frac{4\beta_{i-1}\alpha_i}{k^2-1}\right]\delta_i.
\end{aligned}
$$

Remembering that $4\lambda^2\mu = 3$, it now follows that

$$\langle g_{i-1}, \widehat{g}_i \rangle = 3\frac{(-1)^{k-1}}{k}\left[1 - \frac{4\beta_{i-1}\alpha_i}{k^2-1}\right]\alpha_i$$

$$\text{and that } \langle g_{i+1}, \widehat{g}_i \rangle = 3\frac{(-1)^{k-1}}{k}\left[1 - \frac{4\beta_i\alpha_{i+1}}{k^2-1}\right]\beta_i.$$

To complete the proof, we just have to remark that the two expressions in square brackets are not greater than 1 in absolute value. □

We infer from Lemma 13 that $\Phi_i \leq \frac{1+\lambda}{k}$ and $\Psi_i \leq \frac{\frac{3}{\lambda}+3}{k}$, so that

$$\max\left(\|B\|_1, \|C\|_1\right) \leq \frac{1}{k}\max(1+\lambda, \frac{3}{\lambda}+3).$$

The latter is minimized for $1+\lambda = 3/\lambda + 3$, i.e. for $\lambda = 3$. In view of Lemma 9, the ℓ_∞-norm of M^{-1} can be bounded provided that $k > 4$. Precisely, since BC and CB are of bandwidth 3 and since $\max\left(\|B\|_\infty, \|C\|_\infty\right) \leq \frac{12}{k}$, we have

$$\left\|M^{-1}\right\|_\infty \leq \frac{k(k+12)(k^2+80)}{(k^2-16)^2}. \tag{16}$$

6.2 Condition (ii)

From the expression of H_i, we obtain

$$\begin{aligned}
\|g_i\|_1 &= \frac{3}{\delta_i + \delta_{i+1}}\left\|\left(\delta_{i+1} + \frac{k-1}{k+1}\delta_i\right)F^{(k)} - \frac{2\delta_i}{k+1}G^{(k)}\right\|_1 \\
&\quad + \frac{3}{\delta_i + \delta_{i+1}}\left\|-\left(\delta_i + \frac{k-1}{k+1}\delta_{i+1}\right)F^{(k)} + \frac{2\delta_{i+1}}{k+1}G^{(k)}\right\|_1 \\
&= 3\left\|F^{(k)} - \frac{2\alpha_i}{k+1}\left(F^{(k)} + G^{(k)}\right)\right\|_1 + 3\left\|F^{(k)} - \frac{2(1-\alpha_i)}{k+1}\left(F^{(k)} + G^{(k)}\right)\right\|_1 \\
&\leq 3\left\|F^{(k)}\right\|_1 + 3\left\|F^{(k)} - \frac{2}{k+1}\left(F^{(k)} + G^{(k)}\right)\right\|_1,
\end{aligned}$$

the last inequality holding due to the convexity with respect to $\alpha_i \in [0,1]$ of the function involved. We remark that, according to Proposition 6, the quantity $\|G^{(k)}\|_1 = \|P_{k-2}^{(2,0)}\|_1$ tends to a constant as k tends to infinity. This accounts for the rough estimate

$$\|g_i\|_1 \leq \frac{6k}{k+1}\left\|F^{(k)}\right\|_1 + \frac{6}{k+1}\left\|G^{(k)}\right\|_1 = \frac{12}{k+1}\sigma_{k,0} + \frac{6}{k+1}\left\|G^{(k)}\right\|_1 \lesssim \frac{24\sqrt{2}}{\sqrt{\pi}\sqrt{k}}.$$

The same estimate holds for $\|f_i\|_1$, as can be inferred from subsection 5.2.

6.3 Condition (iii)

Let us now consider the max-norm of $r := \sum a_j \widehat{f}_j + \sum b_j \widehat{g}_j$, which we want to bound in terms of $\max_j(|a_j|, |b_j|)$. The function r achieves its max-norm on $[t_l, t_{l+1}]$, say, where the form of $r(x)$, $x \in (t_l, t_{l+1})$, is

$$\eta P_{k-1}^{(1,0)}(u) + \nu P_{k-1}^{(1,0)}(-u) + \eta' P_{k-2}^{(2,0)}(u) + \nu' P_{k-2}^{(2,0)}(-u), \quad u := \frac{2x - t_l - t_{l+1}}{h_{l+1}}.$$

Such a function of u does not necessarily achieve its max-norm at $u = \pm 1$, e.g. $\eta = \nu = 2$ and $\eta' = \nu' = -1$ provides a counter-example when $k = 5$. However, the separate contributions $C_1(u) = \eta P_{k-1}^{(1,0)}(u) + \nu P_{k-1}^{(1,0)}(-u)$ and $C_2(u) = \eta' P_{k-2}^{(2,0)}(u) + \nu' P_{k-2}^{(2,0)}(-u)$ do. The first contribution is

$$\begin{aligned} C_1(u) = {} & \frac{-a_l \delta_{l+1}}{2(\delta_l + \delta_{l+1})} F^{(k)}(-u) + \frac{a_{l+1}\delta_{l+1}}{2(\delta_{l+1} + \delta_{l+2})} F^{(k)}(u) \\ & + \frac{b_l \left(\delta_l + \frac{k-1}{k+1}\delta_{l+1}\right)\delta_{l+1}}{2(\delta_l + \delta_{l+1})^2} F^{(k)}(-u) + \frac{b_{l+1}\left(\delta_{l+2} + \frac{k-1}{k+1}\delta_{l+1}\right)\delta_{l+1}}{2(\delta_{l+1} + \delta_{l+2})^2} F^{(k)}(u). \end{aligned}$$

The max-norm of C_1 is achieved at 1, say, and we have

$$\begin{aligned} |C_1(u)| \le |C_1(1)| \le {} & \Bigg[\frac{\delta_{l+1}}{2(\delta_l + \delta_{l+1})} + \frac{\delta_{l+1}}{2(\delta_{l+1} + \delta_{l+2})} k \\ & + \frac{\left(\delta_l + \frac{k-1}{k+1}\delta_{l+1}\right)\delta_{l+1}}{2(\delta_l + \delta_{l+1})^2} + \frac{\left(\delta_{l+2} + \frac{k-1}{k+1}\delta_{l+1}\right)\delta_{l+1}}{2(\delta_{l+1} + \delta_{l+2})^2} k \Bigg] \max_j(|a_j|, |b_j|) \\ = {} & \left[\frac{\left(\delta_l + \frac{k}{k+1}\delta_{l+1}\right)\delta_{l+1}}{(\delta_l + \delta_{l+1})^2} + \frac{\left(\delta_{l+2} + \frac{k}{k+1}\delta_{l+1}\right)\delta_{l+1}}{(\delta_{l+1} + \delta_{l+2})^2} k \right] \max_j(|a_j|, |b_j|). \end{aligned}$$

We use the fact that, for $t \ge 0$, one has $[t + k/(k+1)]/(t+1)^2 \le k/(k+1)$ with $t = \delta_l/\delta_{l+1}$ and $t = \delta_{l+2}/\delta_{l+1}$ to obtain $|C_1(u)| \le k \max_j(|a_j|, |b_j|)$.

As for the second contribution, we get

$$\begin{aligned} |C_2(u)| & = \left| -\frac{b_l\, \delta_{l+1}^2}{(k+1)(\delta_l + \delta_{l+1})^2} G^{(k)}(-u) - \frac{b_{l+1}\, \delta_{l+1}^2}{(k+1)(\delta_{l+1} + \delta_{l+2})^2} G^{(k)}(u) \right| \\ & \le \frac{1}{k+1}\left(1 + \frac{k(k-1)}{2}\right) \max_j(|a_j|, |b_j|) = \frac{k^2 - k + 2}{2(k+1)} \max_j(|a_j|, |b_j|). \end{aligned}$$

Putting these two contributions together, we deduce that

$$\left\| \sum a_j \widehat{f}_j + \sum b_j \widehat{g}_j \right\|_\infty \le \frac{3k^2 + k + 2}{2(k+1)} \max_j(|a_j|, |b_j|) \underset{k \to \infty}{\sim} \frac{3k}{2} \max_j(|a_j|, |b_j|).$$

The estimates obtained from conditions (i), (ii) and (iii) yield

$$\left\|P_{\mathcal{R}_{k,2}(\Delta)}\right\|_\infty \lesssim 1 \cdot \frac{24\sqrt{2}}{\sqrt{\pi}\sqrt{k}} \cdot \frac{3k}{2} = \frac{36\sqrt{2}}{\sqrt{\pi}}\sqrt{k}, \quad \text{thus } \left\|P_{\mathcal{S}_{k,2}(\Delta)}\right\|_\infty \lesssim \frac{38\sqrt{2}}{\sqrt{\pi}}\sqrt{k}.$$

In contrast with the case of continuous splines, the numerical values of our upper bound are unsatisfactory, e.g. we obtain roughly 1574 for $k = 6$. When k is small, this is partly due to the poor estimate of (16). One way to improve it would be to consider bases of $\mathcal{R}_{k,2}(\Delta)$ better suited to the evaluation of the inverse of the Gram matrix, providing in particular a bound also valid for $k = 3$ and $k = 4$.

Let us finally remark that if we consider $P_{\mathcal{R}_{k,2}(\Delta)}(\bullet)(t_1^-)$ in the case $N = 2$, $t_1 \to 0$, we can again show that $\sup_\Delta \|P_{\mathcal{R}_{k,2}(\Delta)}\|_\infty \geq 2\sigma_{k,0}$, hence that $\sup_\Delta \|P_{\mathcal{S}_{k,2}(\Delta)}\|_\infty \geq \sigma_{k,0}$. If the lower bound $\sigma_{k,m}$ is indeed the value of $\Lambda_{k,m}$, this reads $\sigma_{k,2} \geq \sigma_{k,0}$, in accordance with the expected monotonicity of $\sigma_{k,m}$.

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