On the convergence rate of a modified Fourier series [PREPRINT]

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Abstract The rate of convergence for an orthogonal series that is a slight modification of the Fourier series is proved. This series converges pointwise at a faster rate than the Fourier series for nonperiodic functions. We present the error as an asymptotic expansion, where the lowest term in this expansion is of asymptotic order two. Subtracting out the terms from this expansion allows us to increase the order of convergence. We also present a method for the efficient computation of the coefficients in the series.

1. Introduction

The standard Fourier series serves as the bedrock of much of modern applied mathematics. Its utility is ubiquitous: from signal analysis to solving differential equations. The discrete counterpart, in particular the Fast Fourier Transform (FFT), has revolutionized numerical analysis. The Fourier series over the interval $[-1,1]$, written here in terms of the real cosine and sine functions, is:

$$f(x) \sim \frac{c_0}{2} + \sum_{k=1}^{\infty} \left[c_k \cos \pi k x + d_k \sin \pi k x\right],$$

for

$$c_k = \langle f, \cos \pi k x \rangle = \int_{-1}^{1} f(x) \cos \pi k x \, \mathrm{d}x \qquad \text{and} \qquad d_k = \langle f, \sin \pi k x \rangle = \int_{-1}^{1} f(x) \sin \pi k x \, \mathrm{d}x.$$

Part of the Fourier series' importance rests on the fact that, when f is analytic and periodic, the series converges exponentially fast. Thus only a very small amount of terms need to be computed in order to obtain a tremendously accurate approximation. But as soon as f loses its periodicity, the convergence rate drops drastically to $\mathcal{O}(k^{-1})$. Though the order of convergence can be increased by subtracting out a polynomial, this suffers from the fact that the expansion no longer has an orthogonal basis.

In [**4**], it was noted that with a slight alteration to the Fourier series we obtain a series that converges for nonperiodic differentable functions, and whose coefficients decay like $\mathcal{O}(k^{-2})$. In Section 2, we give a brief overview of what is known about this modified Fourier series. It was demonstrated numerically that the n-term partial sums of the series appear to approximate the original function with an error of order $\mathcal{O}(n^{-2})$. In Section 3, we will prove this result, assuming that the function is sufficiently differentiable. The error of the partial sums is written as an asymptotic expansion, which means that subtracting out a simple correction term allows us to increase the order of convergence.

The usefulness of this series relies on the computation of the coefficients, which are highly oscillatory integrals. Though traditionally considered difficult to compute, modern advances have shattered this view. In fact, methods exist which actually increase with accuracy as the frequency of oscillations increases. This is similar in behaviour to an asymptotic expansion, though significantly more accurate. In Section 4, we give a brief overview of one of these methods, and show how it can be utilized for the modified Fourier series constructed in Section 2.

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2. Modified Fourier series

This section is based on material found in [**4**]. Consider for a moment the standard Fourier series. When f is sufficiently differentiable, we can determine the order of decay of the coefficients with a straightforward application of integration by parts:

$$c_k = -\frac{1}{\pi k}\int_{-1}^{1} f'(x)\sin \pi kx \,\mathrm{d}x = \frac{(-1)^k}{(\pi k)^2}\left[f'(1) - f'(-1)\right] - \frac{1}{(\pi k)^2}\int_{-1}^{1} f''(x)\cos \pi kx \,\mathrm{d}x = \mathcal{O}\big(k^{-2}\big)\,,$$
$$d_k = \frac{(-1)^{k+1}}{\pi k}\left[f(1) - f(-1)\right] + \frac{1}{\pi k}\int_{-1}^{1} f'(x)\cos \pi kx \,\mathrm{d}x = \mathcal{O}\big(k^{-1}\big)\,. \tag{2.1}$$

Interestingly, only the sine terms decay like $\mathcal{O}\big(k^{-1}\big)$; the cosine terms decay at the faster rate of $\mathcal{O}\big(k^{-2}\big)$. Indeed, if f is even, then all the sine terms are zero and the rate of convergence increases. In the construction of the modified Fourier series, we replace the sine terms with an alternate basis such that the orthogonal property is still maintained, whilst the order of decay of the coefficients is increased to $\mathcal{O}\big(k^{-2}\big)$.

Consider expansion in the series

$$f(x) \sim \frac{c_0}{2} + \sum_{k=1}^{\infty}\left[c_k \cos \pi kx + s_k \sin \pi\left(k - \tfrac{1}{2}\right)x\right],$$

where

$$c_k = \langle f, \cos \pi kx\rangle = \int_{-1}^{1} f(x)\cos \pi kx \,\mathrm{d}x \qquad \text{and} \qquad s_k = \left\langle f, \sin \pi\left(k - \tfrac{1}{2}\right)x\right\rangle = \int_{-1}^{1} f(x)\sin \pi\left(k - \tfrac{1}{2}\right)x \,\mathrm{d}x.$$

We will denote the n-term partial sum of this series as f_n:

$$f_n(x) = \frac{c_0}{2} + \sum_{k=1}^{n}\left[c_k \cos \pi kx + s_k \sin \pi\left(k - \tfrac{1}{2}\right)x\right].$$

Like the standard Fourier series, it can be shown that the basis functions of this series are orthogonal. This follows since they are the eigenfunctions of the differential equation

$$u'' + \alpha^2 u = 0, \qquad 0 = u'(1) = u'(-1).$$

Unlike the Fourier series, the new sine terms are not periodic with respect to the interval $[-1, 1]$, hence we can consider this series for approximating nonperiodic functions.

If $f \in C^{2s}[-1, 1]$ and $f^{(2s)}$ has bounded variation, then Theorem 13.2 in [**8**] tells us that

$$\int_{-1}^{1} f^{(2s)}(x)\mathrm{e}^{\mathrm{i}\pi kx}\,\mathrm{d}x = \mathcal{O}\big(k^{-1}\big)\,.$$

In this case the asymptotic expansion for the coefficients can be found by repeatedly integrating by parts in the same manner as in (2.1), giving us, for $k \neq 0$:

$$c_k = \sum_{j=1}^{s} \frac{(-1)^{k+j+1}}{(\pi k)^{2j}}\left[f^{(2j-1)}(1) - f^{(2j-1)}(-1)\right] + \mathcal{O}\big(k^{-2s-1}\big)\,,$$
$$s_k = \sum_{j=1}^{s} \frac{(-1)^{k+j}}{(\pi(k - 1/2))^{2j}}\left[f^{(2j-1)}(1) + f^{(2j-1)}(-1)\right] + \mathcal{O}\big(k^{-2s-1}\big)\,. \tag{2.2}$$

The cosine terms c_k are the exact same as in the standard Fourier series, hence still decay like $\mathcal{O}\big(k^{-2}\big)$. The new sine terms s_k, unlike the standard Fourier series sine terms d_k, also decay like $\mathcal{O}\big(k^{-2}\big)$. Since the coefficients go to zero at a faster rate, it stands to reason that the series itself will converge to the actual function more rapidly than the standard Fourier series. This will be proved to be the case in Section 3.

In order to prove the rate of convergence, we first need to know that the series does indeed converge pointwise. This is not trivial, but thankfully was proved in [**4**]:

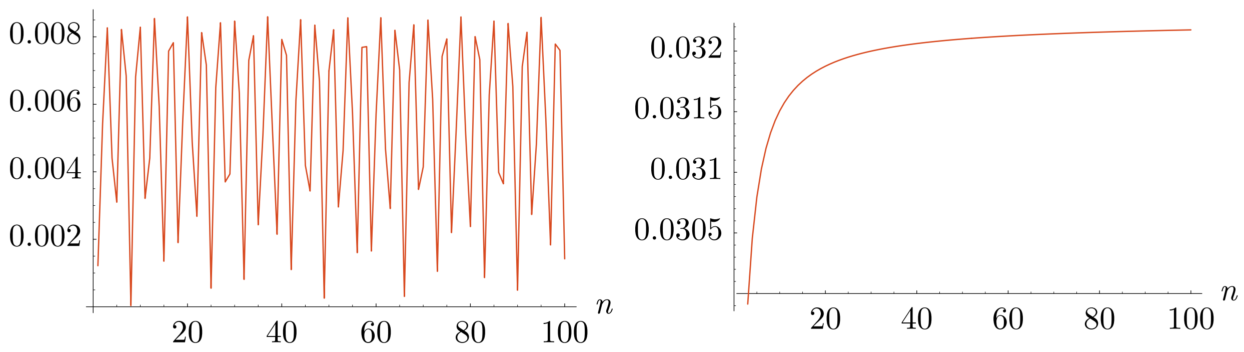

Figure 1: For $f(x) = \mathrm{Ai}\,(x)$, the error $\left|f(-1/\sqrt{2}) - f_n(-1/\sqrt{2})\right|$ scaled by n^2 (left graph) and the error $|f(1) - f_n(1)|$ scaled by n (right graph).

Theorem 2.1 [**4**] *The modified Fourier series forms an orthonormal basis of* $\mathrm{L}^2[-1,1]$.

Theorem 2.2 [**4**] *Suppose that f is a Riemann integrable function in $[-1,1]$ and that*

$$c_k, s_k = \mathcal{O}\big(n^{-1}\big)\,, \qquad n \to \infty.$$

If f is Lipshitz at $x \in (-1,1)$, then

$$f_n(x) \to f(x).$$

The convergence at the endpoints in [**4**] was proved only for the case when f is analytic. Thus we present an alternate proof that enlarges the class of functions such that pointwise convergence is known:

Theorem 2.3 *Suppose that $f \in C^2[-1,1]$ and f'' has bounded variation. Then*

$$f_n(\pm 1) \to f(\pm 1).$$

Proof: We know that the coefficients of the series decay like $\mathcal{O}\big(k^{-2}\big)$, thus

$$\sum_{k=1}^{\infty}(|c_k| + |s_k|) < \infty$$

and f_n converges uniformly to a continuous function. Two continuous functions that differ at a single point must differ in a neighbourhood of that point. The theorem thus follows from Theorem 2.2, since f is Lipshitz everywhere.

Q.E.D.

We can demonstrate the accuracy of this expansion numerically. Consider the Airy function $\mathrm{Ai}\,(x)$ [**1**]. Since the turning point of the Airy function is within $[-1,1]$, it behaves very differently on the negative segment of the interval than it does on the positive segment. Despite this difficulty, we can still approximate the function efficiently. This can be seen in Figure 1, which demonstrates the error of the n-term partial sum f_n at the points $x = -1/\sqrt{2}$ and $x = 1$. We intentially chose an irrational number to demonstrate generality, as the proof of the convergence rate relies on a special function that has a simpler form at rational points. As can be seen, the error does appear to decay like $\mathcal{O}\big(n^{-2}\big)$ at the chosen interior point, while at the reduced rate of $\mathcal{O}\big(n^{-1}\big)$ at the right endpoint.

As another example, consider the function $f(x) = 2/(7 + 20x + 20x^2)$. This function suffers from Runge's phenomenon [**10**], where the successive derivatives of f grow very large within the interval of approximation, which makes the function notoriously difficult to approximate. Figure 2 shows that the modified Fourier series is still very accurate, and compares its accuracy to that of the standard Fourier series. We see the error of the partial sums over the entire interval, for four choices of n. As can be seen,

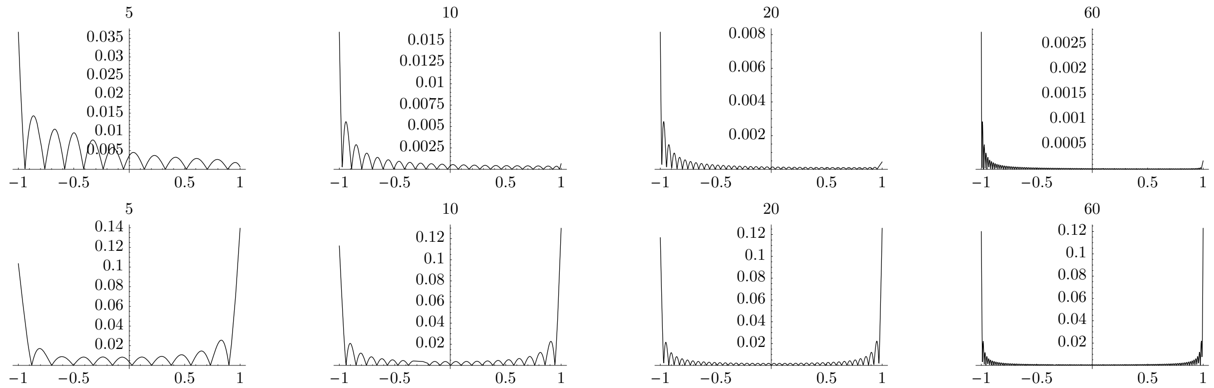

Figure 2: For $f(x) = 2/(7 + 20x + 20x^2)$, the error $|f(x) - f_n(x)|$ (top graphs), versus the error of the n-term partial sums of the Fourier series (bottom graphs), for $n = 5, 10, 20, 60$.

the modified Fourier series approximates inside the interior of the interval significantly better than it does at the endpoints. With as little as 60 terms we obtain an error of about 2×10^{-5} around zero, compared to an error of about 10^{-3} for the standard Fourier series. Furthermore, unlike the standard Fourier series, there is no Gibbs' phenomenon at the boundary points.

3. Convergence rate

Though [**4**] proves the convergence of the modified Fourier series, and demonstrates numerically that it converges at a rate of $\mathcal{O}(n^{-2})$, it fails to prove this convergence rate, even for the case where f is analytic. In this section, we present a proof for the case when $f \in C^2[-1,1]$ and f'' has bounded variation. Furthermore, we prove this result in such a way that, assuming sufficient differentiability of f, we know explicitly the error term that decays like $\mathcal{O}(n^{-2})$, meaning we can actually increase the order of approximation by subtracting out this term. In fact, if f is smooth, we can expand the error $f(x) - f_n(x)$ in terms of an asymptotic expansion for $n \to \infty$, meaning by subtracting out a correction term we can increase the order of approximation to be arbitrarily high. This expansion even holds valid at the endpoints of the interval.

The proof involves writing the error itself in terms of its modified Fourier series, replacing the coefficients of this series by the first s-terms of their asymptotic expansions, and using a known special function, in particular the Lerch transcendent function Φ [**11**], to rewrite the resulting infinite sum. The Lerch transcendent function is defined as

$$\Phi(z, s, m) = \sum_{k=0}^{\infty} \frac{z^k}{(k+m)^s}.$$

We begin by defining a couple key properties of the asymptotics of Φ:

Lemma 3.1 *Suppose that $s \geq 2$ and define*

$$\sigma_{x,s}(t) = \frac{t^{s-1}}{1 + \mathrm{e}^{\mathrm{i}\pi x - t}}.$$

If $-1 < x < 1$, then

$$\Phi\left(-\mathrm{e}^{\mathrm{i}\pi x}, s, m\right) \sim \frac{1}{\Gamma(s)} \sum_{k=s}^{\infty} \frac{\sigma_{x,s}^{(k-1)}(0)}{m^k}, \qquad m \to \infty.$$

If $x = \pm 1$, then

$$\Phi(1, s, m) \sim \frac{1}{\Gamma(s)} \sum_{k=s-1}^{\infty} \lim_{t \to 0} \frac{\sigma_{x,s}^{(k-1)}(t)}{m^k}, \qquad m \to \infty.$$

Proof: We first consider the case where $-1 < x < 1$. It is clear that $0 = \sigma_{x,s}(0) = \sigma'_{x,s}(0) = \cdots = \sigma_{x,s}^{(s-2)}(0)$. We know from [**11**] that, whenever $-\mathrm{e}^{\mathrm{i}\pi x} \neq 1$, the Lerch transcendent function has the integral representation

$$\Phi(z,s,m) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{1 - z\mathrm{e}^{-t}} \mathrm{e}^{-mt}\, dt.$$

Furthermore,

$$\int_0^\infty \sigma_{x,s}^{(k)}(t)\mathrm{e}^{-mt}\, dt = \frac{1}{m}\sigma_{x,s}^{(k)}(0) + \frac{1}{m}\int_0^\infty \sigma_{x,s}^{(k+1)}(t)\mathrm{e}^{-mt}\, dt.$$

The first part of the theorem thus follows from induction, since

$$\Phi\big(-\mathrm{e}^{\mathrm{i}\pi x}, s, m\big) = \int_0^\infty \sigma_{x,s}(t)\mathrm{e}^{-mt}\, dt.$$

For the case when $x = \pm 1$, hence $-\mathrm{e}^{\mathrm{i}\pi x} = 1$, the Lerch transcendent reduces to the Hurwitz zeta function: $\Phi(1,s,m) = \zeta(s,m)$ [**1**]. This has the same integral representation, however a singularity is now introduced at zero:

$$\zeta(s,m) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{1 - \mathrm{e}^{-t}} \mathrm{e}^{-mt}\, dt = \frac{1}{\Gamma(s)} \int_0^\infty \sigma_{\pm 1,s}(t)\mathrm{e}^{-mt}\, dt.$$

With a bit of care, however, we can integrate by parts in the same manner. For this choice of x

$$\sigma_{\pm 1,s}(t) = \frac{t^{s-1}}{1 - \mathrm{e}^{-t}},$$

hence, due to L'Hôpital's rule, only the first $s-3$ derivatives of $\sigma_{\pm 1,s}$ vanish at zero. But $\sigma_{\pm 1,s}$ is still C^∞ at zero, since

$$\sigma_{\pm 1,s}(t) = \frac{t^{s-1}}{1 - \mathrm{e}^{-t}} = \frac{t^{s-1}}{t(1 + \mathcal{O}(t^2))} = \frac{t^{s-2}}{1 + \mathcal{O}(t^2)}.$$

Thus integration by parts gives us its asymptotic expansion.

Q.E.D.

With the Lerch transcendent function in hand, we can successfully find the asymptotic expansion for the error of the modified Fourier series:

Theorem 3.2 *Suppose that $f \in C^{2s}[-1,1]$ and $f^{(2s)}$ has bounded variation. Then*

$$f(x) - f_n(x) = E_{s,n}(x) + \mathcal{O}\big(n^{-2s}\big),$$

for

$$\begin{aligned} E_{s,n}(x) = \sum_{j=1}^{s} \frac{(-1)^{j+n}}{2\pi^{2j}} \Bigg[& \Big(f^{(2j-1)}(1) - f^{(2j-1)}(-1)\Big) \Big\{ \mathrm{e}^{\mathrm{i}\pi(n+1)x}\Phi\big(-\mathrm{e}^{\mathrm{i}\pi x}, 2j, n+1\big) \\ & \qquad + \mathrm{e}^{-\mathrm{i}\pi(n+1)x}\Phi\big(-\mathrm{e}^{-\mathrm{i}\pi x}, 2j, n+1\big) \Big\} \\ & + \mathrm{i}\Big(f^{(2j-1)}(1) + f^{(2j-1)}(-1)\Big) \Big\{ \mathrm{e}^{\mathrm{i}\pi(n+\frac{1}{2})x}\Phi\big(-\mathrm{e}^{\mathrm{i}\pi x}, 2j, n+\tfrac{1}{2}\big) \\ & \qquad - \mathrm{e}^{-\mathrm{i}\pi(n+\frac{1}{2})x}\Phi\big(-\mathrm{e}^{-\mathrm{i}\pi x}, 2j, n+\tfrac{1}{2}\big) \Big\} \Bigg]. \end{aligned}$$

Proof:

Note that

$$f(x) - f_n(x) = \sum_{k=n+1}^{\infty} \left[c_k \cos \pi k x + s_k \sin \pi \left(k - \tfrac{1}{2}\right) x\right].$$

This follows from the pointwise convergence of the modified Fourier series. We define the following two constants for brevity:

$$A_j^C = \frac{f^{(2j-1)}(1) - f^{(2j-1)}(-1)}{2\pi^{2j}}, \qquad A_j^S = \frac{f^{(2j-1)}(1) + f^{(2j-1)}(-1)}{2\mathrm{i}\pi^{2j}}.$$

Substituting in the asymptotic expansion for the coefficients (2.2), the error of the partial sum is equal to

$$\begin{aligned} f(x) - f_n(x) = \sum_{j=1}^{s} (-1)^{j+1} \Bigg[& A_j^C \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k^{2j}} (\mathrm{e}^{\mathrm{i}\pi k x} + \mathrm{e}^{-\mathrm{i}\pi k x}) \\ & - A_j^S \sum_{k=n+1}^{\infty} \frac{(-1)^k}{\left(k - \frac{1}{2}\right)^{2j}} (\mathrm{e}^{\mathrm{i}\pi (k-\frac{1}{2})x} - \mathrm{e}^{-\mathrm{i}\pi (k-\frac{1}{2})x}) \Bigg] + \sum_{k=n+1}^{\infty} \mathcal{O}\big(k^{-2s-1}\big) \\ = \sum_{j=1}^{s} (-1)^{j+n} \Bigg[& A_j^C \left\{ \mathrm{e}^{\mathrm{i}\pi(n+1)x} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+n+1)^{2j}} \mathrm{e}^{\mathrm{i}\pi k x} + \mathrm{e}^{-\mathrm{i}\pi(n+1)x} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+n+1)^{2j}} \mathrm{e}^{-\mathrm{i}\pi k x} \right\} \\ & - A_j^S \left\{ \mathrm{e}^{\mathrm{i}\pi(n+\frac{1}{2})x} \sum_{k=0}^{\infty} \frac{(-1)^k}{\left(k+n+\frac{1}{2}\right)^{2j}} \mathrm{e}^{\mathrm{i}\pi k x} - \mathrm{e}^{-\mathrm{i}\pi(n+\frac{1}{2})x} \sum_{k=0}^{\infty} \frac{(-1)^k}{\left(k+n+\frac{1}{2}\right)^{2j}} \mathrm{e}^{-\mathrm{i}\pi k x} \right\} \Bigg] \\ & + \mathcal{O}\big(k^{-2s}\big) \end{aligned}$$

By replacing the infinite sums with their Lerch transcendent function equivalent, we obtain

$$\begin{aligned} f(x) - f_n(x) = \sum_{j=1}^{s} (-1)^{j+n} \Bigg[& A_j^C \left\{ \mathrm{e}^{\mathrm{i}\pi(n+1)x} \Phi\big(-\mathrm{e}^{\mathrm{i}\pi x}, 2j, n+1\big) + \mathrm{e}^{-\mathrm{i}\pi(n+1)x} \Phi\big(-\mathrm{e}^{-\mathrm{i}\pi x}, 2j, n+1\big) \right\} \\ & - A_j^S \left\{ \mathrm{e}^{\mathrm{i}\pi(n+\frac{1}{2})x} \Phi\big(-\mathrm{e}^{\mathrm{i}\pi x}, 2j, n+\tfrac{1}{2}\big) - \mathrm{e}^{-\mathrm{i}\pi(n+\frac{1}{2})x} \Phi\big(-\mathrm{e}^{-\mathrm{i}\pi x}, 2j, n+\tfrac{1}{2}\big) \right\} \Bigg] \\ & + \mathcal{O}\big(k^{-2s}\big). \end{aligned}$$

Q.E.D.

Remark: If $f \in C^{2s+1}[-1,1]$ in the preceding theorem, then the order of the error term increases to $\mathcal{O}\big(n^{-2s-1}\big)$. This follows since we can integrate by parts once more in the asymptotic expansion of the coefficients, meaning the s-term expansion now has an order of error $\mathcal{O}\big(n^{-2s-2}\big)$. We could have also written the cosine and sine functions as the real and imaginary parts of $\mathrm{e}^{\mathrm{i}x}$, which would have resulted in two less evaluations of the Lerch transcendent function.

We could substitute the Lerch transcendent functions in this theorem with the asymptotic expansion found in Lemma 3.1, which results in an asymptotic expansion for the error in terms of elementary functions. Unfortunately this expansion explodes near the endpoints, even though another expansion in terms of elementary functions is known at the endpoints. Thus, since the Lerch transcendent functions can be computed efficiently [**2**], it is preferred to leave the Lerch transcendent functions in their unexpanded form.

As a result of this theorem, we obtain a sufficient condition for obtaining a rate of convergence of order two:

Corollary 3.3 *Suppose that $f \in C^2[-1,1]$ and f'' has bounded variation. If $-1 < x < 1$, then*

$$f(x) - f_n(x) = \mathcal{O}\big(n^{-2}\big).$$

Otherwise,

$$f(\pm 1) - f_n(\pm 1) = \mathcal{O}\big(n^{-1}\big).$$

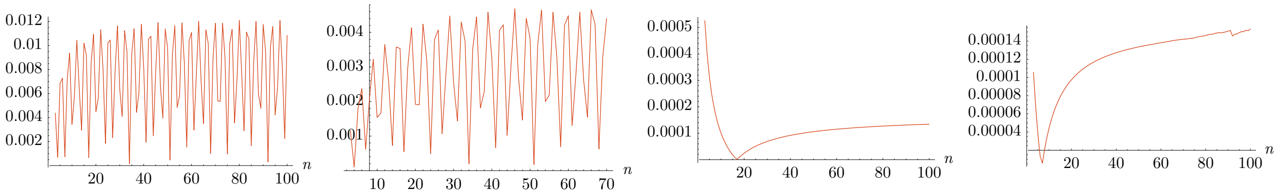

Figure 3: For $f(x) = \mathrm{Ai}\,(x)$, the error $\left|f(-1/\sqrt{2}) - f_n(-1/\sqrt{2}) - E_{1,n}(-1/\sqrt{2})\right|$ scaled by n^4 (first graph), $\left|f(-1/\sqrt{2}) - f_n(-1/\sqrt{2}) - E_{2,n}(-1/\sqrt{2})\right|$ scaled by n^6 (second graph), $\left|f(1) - f_n(1) - E_{1,n}(1)\right|$ scaled by n^3 (third graph) and $\left|f(1) - f_n(1) - E_{2,n}(1)\right|$ scaled by n^5 (last graph).

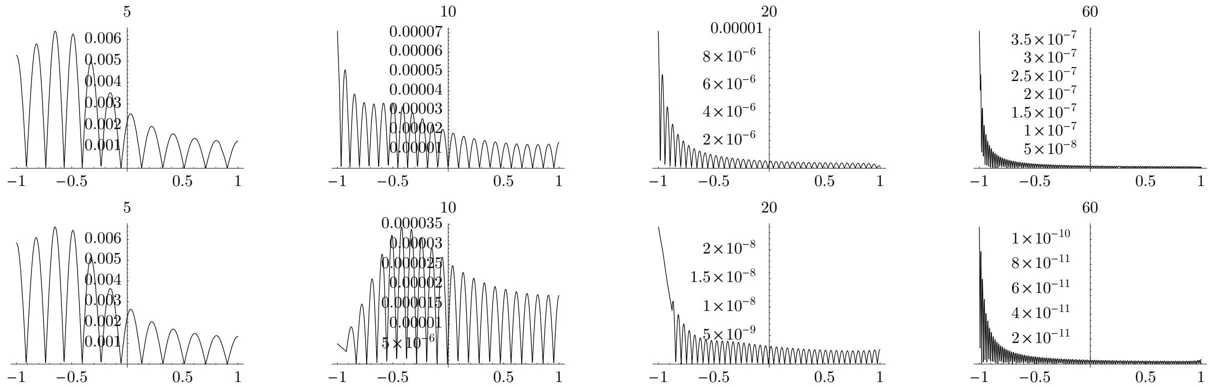

Figure 4: For $f(x) = 2/(7 + 20x + 20x^2)$, the error $\left|f(x) - f_n(x) - E_{1,n}(x)\right|$ (top graphs) and the error $\left|f(x) - f_n(x) - E_{2,n}(x)\right|$ (bottom graphs), for $n = 5, 10, 20, 30$.

Proof: This corollary follows immediately by combining Theorem 3.2 with the asymptotic behaviour of Φ found in Lemma 3.1.

Q.E.D.

As an example, consider again the function $\mathrm{Ai}\,(x)$. In Figure 3, we demonstrate that subtracting out the error term $E_{s,n}(x)$ does indeed increase the order of approximation. In the first two graphs of this figure, we see that this is true for the interior point $x = -1/\sqrt{2}$, where subtracting out $E_{1,n}(x)$ increases the order to n^4 and subtracting out $E_{2,n}(x)$ increases the order further to n^6. In the second graph we end the data at $n = 60$, as we have already reached machine precision. Since $\mathrm{Ai}\,(x)$ is analytic, we could continue this process to obtain arbitrarily high orders. The last two graphs show that we can employ the same technique at the boundary point $x = 1$, where the order is first increased to n^3, then to n^5.

We can see how subtracting out the first term in the error expansion affects the approximation over the entire interval. In Figure 4, we return to the example first presented in Figure 2, where $f(x) = 2/(7 + 20x + 20x^2)$. As can be seen, we obtain incredibly more accurate results. When we remove the first error term $E_{1,n}$, by $n = 10$ the approximation is about the same as $n = 60$ without the small alteration, and significantly more accurate at the boundary. Subtracting out the next term in the error expansion sees an equally impressive improvement of accuracy, where with $n = 30$ we have at least eight digits of accuracy throughout the interval, and nine digits within the interior.

4. Computation of series coefficients

Up until now we have avoided computing the coefficients c_k and s_k of the modified Fourier series; we merely assumed that they were given. When the function f is well-behaved, i.e., nonoscillatory, the first few coefficients are well-behaved integrals, meaning standard quadrature methods such as Gauss-Legendre quadrature will approximate these integrals efficiently. As k becomes large, the coefficients become highly oscillatory integrals, and the standard methods are no longer efficient. With a bit of finesse, we can successfully compute these integrals in an efficient manner. In fact, using the correct quadrature methods, the accuracy improves as the frequency of oscillations increases.

The most well-known solution for approximating highly oscillatory integrals is to use the asymptotic expansion, which we have already derived in (2.2). Using a partial sum of the expansion, we obtain an approximation which is more accurate when the frequency of oscillations is large. In fact, when f is sufficiently differentiable, we can obtain an approximation of arbitrarily high asymptotic order. Unfortunately, the asymptotic expansion does not in general converge for fixed values of k, meaning that the accuracy of this approximation is limited.

To combat this problem we will use Levin-type methods, which were developed in [**9**] based on results found in [**7**]. We consider integrals of the form

$$I_\omega[f] = \int_{-1}^{1} f(x)\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x,$$

whose real and imaginary parts can be used to compute c_k and s_k:

$$c_k = \mathrm{Re}\; I_{\pi k}[f], \qquad s_k = \mathrm{Im}\; I_{\pi(k-1/2)}[f].$$

Suppose that we have a function F such that

$$\mathcal{L}[F] = f, \qquad \text{for} \qquad \mathcal{L}[F] = F' + \mathrm{i}\omega F.$$

Since $\frac{\mathrm{d}}{\mathrm{d}x}[F(x)\mathrm{e}^{\mathrm{i}\omega x}] = \mathcal{L}[F]\,(x)\mathrm{e}^{\mathrm{i}\omega x}$, it follows that

$$I_\omega[f] = I_\omega[\mathcal{L}[F]] = \int_{-1}^{1} \frac{\mathrm{d}}{\mathrm{d}x}[F(x)\mathrm{e}^{\mathrm{i}\omega x}]\,\mathrm{d}x = F(1)\mathrm{e}^{\mathrm{i}\omega} - F(-1)\mathrm{e}^{-\mathrm{i}\omega}.$$

Finding such a function F exactly is as difficult as solving the integral explicitly, however, we can use collocation to approximate it. Assume we are given a sequence of nodes $\{x_1, \ldots, x_\nu\}$, multiplicities $\{m_1, \ldots, m_\nu\}$ and basis functions $\{\psi_1, \ldots, \psi_\nu\}$. Then, for $n = \sum_{k=1}^{\nu} m_k$, we determine the coefficients c_k for the approximating function $v(x) = \sum_{k=1}^{n} c_k \psi_k(x)$ by solving the system

$$\mathcal{L}[v]\,(x_k) = f(x_k), \ldots, \mathcal{L}[v]^{(m_k-1)}\,(x_k) = f^{(m_k-1)}(x_k), \qquad k = 1, \ldots, \nu.$$

It turns out that this decays at the same order as the partial sums of the asymptotic expansion:

Theorem 4.1 *Suppose that $f \in C^r[-1,1]$, $f^{(r)}$ has bounded variation and that $\{\psi_1, \ldots, \psi_\nu\}$ can interpolate at the given nodes and multiplicities. Then*

$$I_\omega[f] - Q^L_\omega[f] = \mathcal{O}\big(\omega^{-s-1}\big),$$

for $s = \min\{m_1, m_\nu, r\}$ and

$$Q^L_\omega[f] = v(1)\mathrm{e}^{\mathrm{i}\omega} - v(-1)\mathrm{e}^{-\mathrm{i}\omega}.$$

Proof: From the proof of Theorem 4.1 in [**9**] we know that $\mathcal{L}[v]$ and all its derivatives are $\mathcal{O}(1)$. Note that

$$\mathcal{L}[v]\,(\pm 1) = f(\pm 1), \cdots, \mathcal{L}[v]^{(s-1)}\,(\pm 1) = f^{(s-1)}(\pm 1).$$

Thus the first s terms in the asymptotic expansion of $I_\omega[f - \mathcal{L}[v]]$ are zero and

$$I_\omega[f] - Q^L_\omega[f] = I_\omega[f - \mathcal{L}[v]] = (-\mathrm{i}\omega)^{-s} \int_{-1}^{1} \left(f^{(s)} - \mathcal{L}[v]^{(s)}\right) \mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x = (-\mathrm{i}\omega)^{-s} \left(I_\omega[f^{(s)}] - I_\omega[\mathcal{L}[v]^{(s)}]\right).$$

The integral $I_\omega[f^{(s)}]$ behaves like $\mathcal{O}\big(\omega^{-1}\big)$ since $f^{(s)}$ has bounded variation, while integrating by parts once more reveals that $I_\omega[\mathcal{L}[v]^{(s)}]$ also behaves like $\mathcal{O}\big(\omega^{-1}\big)$.

Q.E.D.

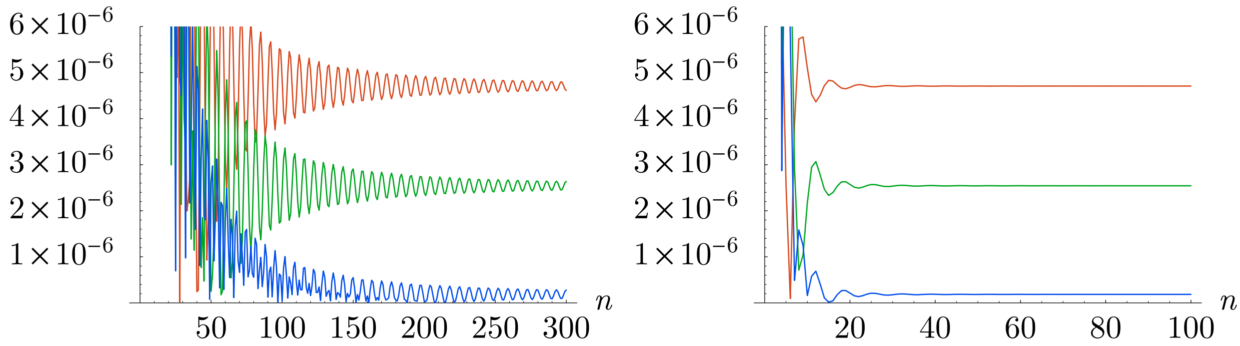

Figure 5: The error $\left|f(-1/\sqrt{2}) - f_n^L(-1/\sqrt{2})\right|$ (left graph) with endpoints for nodes and multiplicities both two (top), nodes $\left\{-1, -\sqrt{3}/2, 0, \sqrt{3}/2, 1\right\}$ and multiplicities $\{2, 1, 1, 1, 2\}$ (middle) and endpoints for nodes and multiplicities both three (bottom), versus the error $\left|f(-1/\sqrt{2}) - f_n^L(-1/\sqrt{2}) - E_{1,n}(-1/\sqrt{2})\right|$ with the same choices of Levin-type methods (right graph).

The conditions on this theorem are satisfied whenever $\{\psi_1, \ldots, \psi_\nu\}$ is a Chebyshev set, for example the standard polynomial basis. This theorem lets us obtain an approximation with the same asymptotic order as the asymptotic expansion, but significantly more accurate.

This leaves us with the slight problem of how to decide when to use oscillatory quadrature methods and when to use nonoscillatory quadrature methods. We avoid investigating this problem as it is too tangential to the subject of this paper. Instead, in our example, we compute the coefficients $c_0, \ldots, c_3$ and $s_1, \ldots, s_3$ to machine precision using Gauss-Legendre quadrature, and compute the other coefficients with three different Levin-type methods. A more sophisticated quadrature method exists, namely exotic quadrature [**3**], which reuses the information already obtained for a Levin-type method to approximate the nonoscillatory coefficients. For simplicity, we will use the standard polynomial basis $\psi_k = x^{k-1}$.

We denote the Levin-type approximation for the coefficients as c_k^L and s_k^L, and the partial sum of the modified Fourier series using these coefficients as f_n^L, where the choices for nodes and multiplicities are determined by context. We could use different nodes and multiplicities for different values of k, as less information is needed for large k in order to obtain accurate approximations. We however use the same Levin-type method for all k for simplicity. When we replace the exact value of the coefficients with an approximation, we incur an error that cannot be rectified, hence we no longer have convergence. Instead, we converge to the constant

$$f(x) - \lim_{n\to\infty} f_n^L(x) = \sum_{k=0}^{\infty} \left[(c_k - c_k^L) \cos \pi k x + (s_k - s_k^L) \sin \pi \left(k - \tfrac{1}{2}\right) x \right]. \tag{4.1}$$

This sum is finite since c_k^L and s_k^L are at least $\mathcal{O}\left(k^{-2}\right)$, assuming that the endpoints are in the collocation nodes. As the Levin-type method used becomes more accurate, this constant will decrease.

Returning to the Airy example of Figure 1 with $x = -1/\sqrt{2}$, we now show the results for three Levin-type methods: endpoints for nodes and multiplicities both two, nodes $\left\{-1, -\sqrt{3}/2, 0, \sqrt{3}/2, 1\right\}$ and multiplicities $\{2, 1, 1, 1, 2\}$ and endpoints for nodes and multiplicities both three. The choice of nodes in the second Levin-type method are determined by appending the endpoints of the interval to the standard Chebysev points [**10**]. Note that with or without the correcting term $E_{1,n}(x)$, we converge to the same value (4.1). With this correcting term, however, we converge to this value significantly faster. The last Levin-type method is of a higher order, so the terms in (4.1) go to zero at a quicker rate, and as a result the error is significantly smaller. In this case (4.1) is on the order of 10^{-7}.

Remark: See [**4**] for a discussion based on Filon-type methods [**5**], which are equivalent to Levin-type methods whenever a polynomial basis is used in both methods and the oscillatory kernel is of the form $e^{i\omega x}$.

5. Future work

The most immediate open question is whether we can determine a more general class of functions which converge at the rate of $\mathcal{O}(n^{-2})$. We know it cannot be increased to $\mathrm{L}^2[-1,1]$, since, as an example, the function $\operatorname{sgn} x$ does not have coefficients which decay like $\mathcal{O}(k^{-2})$:

$$s_k = \left\langle \operatorname{sgn}, \sin \pi \left(k - \tfrac{1}{2}\right) x \right\rangle = \int_0^1 \sin \pi \left(k - \tfrac{1}{2}\right) x \, \mathrm{d}x - \int_{-1}^0 \sin \pi \left(k - \tfrac{1}{2}\right) x \, \mathrm{d}x = -\frac{4}{\pi(1-2k)}.$$

Substituting this into the proof of Theorem 3.2 results in an infinite sum that reduces to the Lerch transcendent function $\Phi\left(-\mathrm{e}^{-\mathrm{i}\pi x}, 1, n + \tfrac{1}{2}\right)$. Since $s = 1$, it is known that this decays no faster than $\mathcal{O}(n^{-1})$. But it might be possible to impose conditions similar to bounded variation on a C^0 or C^1 function in order to guarantee the faster rate of convergence.

Consider again the eigenvalue problem from which we obtained the basis in our orthogonal series:

$$u'' + \alpha^2 u = 0, \qquad 0 = u'(1) = u'(-1).$$

In [**3**], a generalization of this eigenvalue problem was explored:

$$u^{(2q)} + (-1)^{q+1} \alpha^{2q} u = 0, \qquad 0 = u^{(i)}(-1) = u^{(i)}(1), \qquad i = q, q+1, \ldots, 2q-1.$$

The solutions of this problem, namely the polyharmonic eigenfunctions [**6**], lead to a new orthogonal series which is dense in $\mathrm{L}^2[-1,1]$, and whose coefficients decay like $\mathcal{O}(n^{-q-1})$ for sufficiently differentiable functions. When q is one we again obtain the modified Fourier series we have analysed. Choosing q larger than one, on the other hand, gives us a series whose coefficients decay at an even faster rate. It might be possible to prove the convergence rate of this series in the same manner that we did Theorem 3.2. Unfortunately, pointwise convergence of this series has not been proved, which would be necessary in order to rewrite the error as an infinite sum. Some initial results have been found for generalizing polyharmonic eigenfunctions for the approximation of multivariate functions. It may also be possible to generalize the results of this paper to prove the convergence rate in multivariate domains.

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