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QUADRATURE METHODS FOR HIGHLY OSCILLATORY LINEAR AND NONLINEAR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS: PART I*

MARIANNA KHANAMIRYAN[1]

[1] *Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom.*
email: M.Khanamiryan@damtp.cam.ac.uk

Dedicated to the memory of Rudolf Khanamiryan.

Abstract.

This work presents methods of efficient numerical approximation for linear and nonlinear systems of highly oscillatory ordinary differential equations. We show how an appropriate choice of quadrature rule improves the accuracy of approximation as the frequency of oscillation grows. We present *asymptotic* and *Filon*-type methods to solve highly oscillatory linear systems of *ODE*s, and *WRF* method, representing a special combination of *Filon*-type methods and *waveform* relaxation methods, for nonlinear systems. Numerical examples support this paper.

AMS subject classification (2000): 65L05, 34E05, 34C15.

Key words: Highly oscillatory *ODE*s, asymptotic methods, Filon quadrature, waveform relaxation methods.

1 Introduction.

Let the system of highly oscillatory nonlinear *ODE*s be presented in the vector form

$$\boldsymbol{y}' = A_\omega \boldsymbol{y} + \boldsymbol{f}(t, \boldsymbol{y}), \quad \boldsymbol{y}(0) = \boldsymbol{y}_0 \in \mathbb{R}^d, \quad t \geq 0,$$

where A_ω is a constant non-singular $d \times d$ matrix with large eigenvalues, $\sigma(A_\omega) \subset \mathrm{i}\mathbb{R}$, $\|A_\omega\| \gg 1$, $\omega \gg 1$ is a real parameter and $\boldsymbol{f} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a smooth vector-valued function. Using the variation of constants formula we can write the implicit solution of the system in the form

$$\boldsymbol{y}(t) = \mathrm{e}^{tA_\omega}\boldsymbol{y}_0 + \int_0^t \mathrm{e}^{(t-\tau)A_\omega}\boldsymbol{f}(\tau, \boldsymbol{y}(\tau))\mathrm{d}\tau = \mathrm{e}^{tA_\omega}\boldsymbol{y}_0 + I[\boldsymbol{f}]. \tag{1.1}$$

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In essence, the matrix A_ω need not depend on the parameter ω explicitly. Rather, the largest absolute values of the matrix eigenvalues determine the frequency of oscillation in $I[\boldsymbol{f}]$, and the norm of the matrix grows with an increase in frequency. However, for numerical purposes, we have expressed this information in ω, as a way of showing improvement in the accuracy of approximation as ω grows. Expanding the integral $I[\boldsymbol{f}]$ into its asymptotic series we demonstrate the dependence of the error of approximation on the powers of the matrix inverse, having the implication that the error depends on the inverse powers of ω. By this we mean that the larger the eigenvalues of the matrix A_ω, the larger the norm of the matrix, and the better our approximation. One can obtain an arbitrary high order of approximation by adding higher order derivatives in $I[\boldsymbol{f}]$. Amazingly, both the *Filon*-type method and the *WRF* method work with end points only, and as such do not require further subdivision of the integration interval. Evolving ideas of the stationary phase approximation [24], our methods require explicit availability of more terms in the asymptotic expansion, leading to the assumptions of non singularity of the matrix A_ω and smoothness of the function $\boldsymbol{f}$. These requirements are crucial for the *Filon*-type method and hence also for *WRF* methods which employ *Filon* quadrature rule for discretization, Theorems 1.3 and 3.1, except a more comprehensive Theorem 4.1 for an arbitrary matrix and quadrature.

A. Iserles and S. Nørsett have shown in [10] that the standard numerical approach based on *Gauss–Christoffel quadrature* fails to approximate highly oscillatory integrals since the error of approximation is $\mathcal{O}(1)$ for $\omega \to \infty$. Instead the authors developed the *asymptotic* and the *Filon*-type methods, which share the feature that accuracy improves as ω increases.

In getting to grips with our underlying task of solving highly oscillatory nonlinear *ODE*s, we advance gradually, first considering some special cases of the problem. We commence from linear equation with the same properties as before,

$$\boldsymbol{y}' = A_\omega \boldsymbol{y} + \boldsymbol{f}(t), \quad \boldsymbol{y}(0) = \boldsymbol{y}_0 \in \mathbb{R}^d, \quad \boldsymbol{f} : \mathbb{R} \to \mathbb{R}^d, \quad t \geq 0,$$

having the exact solution

$$\boldsymbol{y}(t) = \mathrm{e}^{tA_\omega}\boldsymbol{y}_0 + \int_0^t \mathrm{e}^{(t-\tau)A_\omega}\boldsymbol{f}(\tau)\mathrm{d}\tau = \mathrm{e}^{tA_\omega}\boldsymbol{y}_0 + I[\boldsymbol{f}]. \tag{1.2}$$

Before beginning to present our methods, we briefly state the two important theorems from [10], describing the quadrature methods used to approximate highly oscillatory integrals of the form

$$I[f] = \int_a^b f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x, \tag{1.3}$$

where $f, g \in C^\infty$ are smooth, g is strictly monotone in $[a, b], a \leq x \leq b$ and the frequency is $\omega \gg 1$. Later in this section we will explain the link between the present work and [10].

LEMMA 1.1 (A. ISERLES & S. NØRSETT [10]). *Let $f, g \in C^\infty$, $g'(x) \neq 0$ on $[a,b]$ and*

$$\sigma_0[f](x) = f(x),$$
$$\sigma_{k+1}[f](x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sigma_k[f](x)}{g'(x)}, \quad k = 0, 1, \ldots .$$

Then, for $\omega \gg 1$,

$$I[f] \sim -\sum_{m=1}^{\infty} \frac{1}{(-\mathrm{i}w)^m}\left[\frac{\mathrm{e}^{\mathrm{i}\omega g(1)}}{g'(1)}\sigma_{m-1}[f](1) - \frac{\mathrm{e}^{\mathrm{i}\omega g(0)}}{g'(0)}\sigma_{m-1}[f](0)\right].$$

The *asymptotic* method is defined as follows,

$$Q_s^A[f] = -\sum_{m=1}^{s} \frac{1}{(-\mathrm{i}\omega)^m}\left[\frac{\mathrm{e}^{\mathrm{i}\omega g(1)}}{g'(1)}\sigma_{m-1}[f](1) - \frac{\mathrm{e}^{\mathrm{i}\omega g(0)}}{g'(0)}\sigma_{m-1}[f](0)\right].$$

THEOREM 1.2 (A. ISERLES & S. NØRSETT [10]). *For every smooth f and g, such that $g'(x) \neq 0$ on $[a,b]$, it is true that*

$$Q_s^A[f] - I[f] \sim \mathcal{O}(\omega^{-s-1}), \quad \omega \to \infty.$$

A precursor of a *Filon*-type method was first pioneered in the work of L. N. G. Filon in 1928, modified by E. A. Flinn [4], and developed by A. Iserles and S. Nørsett in [10] and later by A. Iserles, S. Nørsett and S. Olver in [11]. The method replaces f in (1.3) by its Hermite interpolant,

$$v(x) = \sum_{l=1}^{\nu}\sum_{j=0}^{\theta_l - 1} \alpha_{l,j}(x) f^{(j)}(c_l),$$

which satisfies $v^{(j)}(c_l) = f^{(j)}(c_l)$, at node points $a = c_1 < c_2 < \cdots < c_\nu = b$, with $\theta_1, \theta_2, \ldots, \theta_\nu \geq 1$ associated multiplicities, $j = 0, 1, \ldots, \theta_l - 1, l = 1, 2, \ldots, \nu$ and $r = \sum_{l=1}^{\nu} \theta_l - 1$ being the order of approximation polynomial. For the *Filon*-type method, by definition,

$$Q_s^F[f] = I[v] = \int_a^b v(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x = \sum_{l=1}^{\nu}\sum_{j=0}^{\theta_l - 1} b_{l,j}(w) f^{(j)}(c_l),$$

where

$$b_{l,j} = \int_0^1 \alpha_{l,j}(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x, \quad j = 0, 1, \ldots, \theta_l - 1, \quad l = 1, 2, \ldots, \nu.$$

THEOREM 1.3 (A. ISERLES & S. NØRSETT [10]). *Suppose that $\theta_1, \theta_\nu \geq s$. For every smooth f and g, $g'(x) \neq 0$ on $[a,b]$, it is true that*

$$I[f] - Q_s^F[f] \sim \mathcal{O}(\omega^{-s-1}), \quad \omega \to \infty.$$

EXAMPLE 1.1. Here we consider the *asymptotic* and the *Filon*-type method with first derivatives for (1.3) over the interval $[0,1]$ for the case $g(x)=x$.

$$
\begin{aligned}
Q_2^A &= \frac{\mathrm{e}^{\mathrm{i}\omega}f(1)-f(0)}{\mathrm{i}\omega}+\frac{\mathrm{e}^{\mathrm{i}\omega}f'(1)-f'(0)}{\omega^2},\\
Q_2^F &= \left(-\frac{1}{\mathrm{i}\omega}-6\frac{1+\mathrm{e}^{\mathrm{i}\omega}}{\mathrm{i}\omega^3}+12\frac{1-\mathrm{e}^{\mathrm{i}\omega}}{\omega^4}\right)f(0)\\
&\quad+\left(-\frac{\mathrm{e}^{\mathrm{i}\omega}}{\mathrm{i}\omega}+6\frac{1+\mathrm{e}^{\mathrm{i}\omega}}{\mathrm{i}\omega^3}-12\frac{1-\mathrm{e}^{\mathrm{i}\omega}}{\omega^4}\right)f(1)\\
&\quad+\left(-\frac{1}{\omega^2}-2\frac{2+\mathrm{e}^{\mathrm{i}\omega}}{\mathrm{i}\omega^3}+6\frac{1-\mathrm{e}^{\mathrm{i}\omega}}{\omega^4}\right)f'(0)\\
&\quad+\left(\frac{\mathrm{e}^{\mathrm{i}\omega}}{\omega^2}-2\frac{1+\mathrm{e}^{\mathrm{i}\omega}}{\mathrm{i}\omega^3}+6\frac{1-\mathrm{e}^{\mathrm{i}\omega}}{\omega^4}\right)f'(1).
\end{aligned}
$$

In Figure 1.1 we present numerical results on the *asymptotic* and *Filon*-type methods with function values and its first derivatives at the end points only, $c_1=0, c_2=1$, for the integral

$$
I[f]=\int_0^1 \cos(x)\mathrm{e}^{\mathrm{i}\omega x}\mathrm{d}x, \quad 100\le\omega\le 200.
$$

Both methods have the same asymptotic order and use exactly the same information; however as we can see from Figure 1.1, the *Filon*-type method yields a greater measure of accuracy than the *asymptotic* method. We would like to emphasize that the *Filon*-type method works for small values of ω either, and with *Gaussian* points it is equivalent to the *Gauss–Christoffel* quadrature for $\omega\to 0$, using the same information.

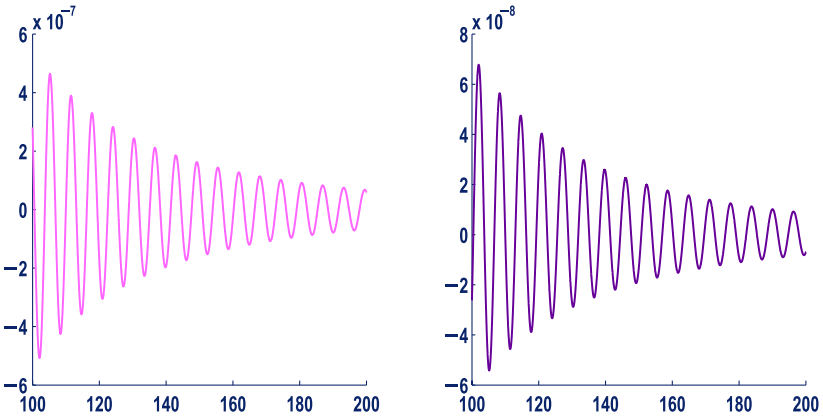

Figure 1.1: The error of the *asymptotic* method Q_2^A (right) and the *Filon*-type method Q_2^F (left) for $f(x)=\cos(x)$, $g(x)=x$, $\theta_1=\theta_2=2$, $100\le\omega\le 200$.

In [10] it was shown by the authors that adding more internal points leads to the decay of the leading error constant, resulting in a marked improvement in the accuracy of approximation. This addition does not affect asymptotic order but rather its contributory end points. We would like to mention here that these results remain valid for vector-valued functions and vector-valued polynomials as in $I[\boldsymbol{f}]$, and hence our methods are valid for both large and small eigenvalues.

Note that while replacing function f by a polynomial, the *Filon*-type method requires computation of the moments $\int_a^b x^m \mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x, m \geq 0$, which may not always be available. As a consequence, since $I[f]$ appear to be elements of the vector-valued integral $I[\boldsymbol{f}]$, the latter may also not always be available once $\boldsymbol{f}$ is replaced by a vector-valued polynomial in (1.2).

The current author used both MATLAB numerical package and its symbolic toolbox linked to a MAPLE kernel to perform numerical evaluations for this paper.

Numerical approximation of highly oscillatory integrals is a an advanced topic of research and involves wide spectrum of different approaches including moment-free approximation. This includes *Levin*-type methods, [16, 17], invented by D. Levin and further extended by S. Olver, [23]. The second alternative method the author refers to is *numerical steepest descent* method by D. Huybrechs and S. Vandewalle, [8, 9]. There is a very relevant work using exponential integrators for oscillatory equations by V. Grimm and M. Hochbruck, [7]. For asymptotic methods for integrals we refer to [3, 22, 27].

This paper entails the extension of the introduced points from [10] in a further approximation of highly oscillatory integrals $I[\boldsymbol{f}]$ with a matrix-valued kernel and a vector-valued function, with the results of approximation being used to solve linear and nonlinear systems of highly oscillatory *ODE*s. We should mention here that all norms in the paper are L^∞ norms.

Having introduced *asymptotic* and *Filon*-type methods for the family of integrals (1.3), we now explain the link between the present work and [10]. Take for simplicity a spectral decomposition of the matrix $A_\omega = PDP^{-1}$, having a pure imaginary spectrum $\sigma(A_\omega) = \{\mathrm{i}\omega_k\}_{k=1}^d$, $\omega_k \in \mathbb{R}$,

$$A_\omega = P \begin{pmatrix} \mathrm{i}\omega_1 & 0 & \dots & 0 \\ 0 & \mathrm{i}\omega_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \mathrm{i}\omega_d \end{pmatrix} P^{-1},$$

therefore

$$A_\omega^{-1} = P \begin{pmatrix} \frac{1}{\mathrm{i}\omega_1} & 0 & \dots & 0 \\ 0 & \frac{1}{\mathrm{i}\omega_2} & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \frac{1}{\mathrm{i}\omega_d} \end{pmatrix} P^{-1},$$

and

$$\mathrm{e}^{A_\omega} = P \begin{pmatrix} \mathrm{e}^{\mathrm{i}\omega_1} & 0 & \dots & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\omega_2} & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \mathrm{e}^{\mathrm{i}\omega_d} \end{pmatrix} P^{-1}.$$

This suggests that to approximate a highly oscillatory integral $I[\boldsymbol{f}]$ with a matrix-valued kernel and a vector-valued function in (1.2), we need to approximate a highly oscillatory integral of the kind (1.3) in $I[\boldsymbol{f}]$. Needless to mention the asymptotic order of our approximation to $I[\boldsymbol{f}]$ depends on the inverse powers of the eigenvalues, thus we wish the error will decay as the eigenvalues grow. For smaller eigenvalues the method is comparable with the classical methods, while in the case of zero eigenvalues the method will be equivalent to the polynomial approximation of the integrable function.

2 The asymptotic method.

Consider a vector-valued integral over a compact interval $[a, b]$

$$I[\boldsymbol{f}] = \int_a^b X_\omega(t)\boldsymbol{f}(t)\mathrm{d}t, \quad X'_\omega = A_\omega X_\omega, \tag{2.1}$$

where the matrix kernel X_ω satisfies a linear differential equation (2.1) with a constant non-singular matrix A_ω of large eigenvalues, $\|A_\omega^{-1}\| \ll 1$, $\sigma(A_\omega) \subset \mathrm{i}\mathbb{R}$, ω is a real parameter and $\boldsymbol{f} \in \mathbb{R}^d$ is a smooth vector-valued function: $\boldsymbol{f} \in \mathbb{C}^\infty[a, b]$. The fact that the matrix X_ω satisfies linear matrix *ODE* (2.1) allows us to integrate (2.1) by parts,

$$\begin{aligned} I[\boldsymbol{f}] &= A_\omega^{-1}[X_w(b)\boldsymbol{f}(b) - X_\omega(a)\boldsymbol{f}(a)] \\ &\quad - A_\omega^{-1}\int_a^b X_\omega(t)\boldsymbol{f}'(t)\mathrm{d}t = Q_1^A - A_\omega^{-1}I[\boldsymbol{f}']. \end{aligned}$$

We define *asymptotic* method Q_s^A as

$$Q_s^A[\boldsymbol{f}] = -\sum_{m=1}^{s}(-A_\omega)^{-m}\left[X_\omega(b)\boldsymbol{f}^{(m-1)}(b) - X_\omega(a)\boldsymbol{f}^{(m-1)}(a)\right],$$

representing s-partial sum of the asymptotic expansion for $I[\boldsymbol{f}]$.

At this point of discussion it will be appropriate if we introduce some notation on matrix and function asymptotics from [23]. We say that $f = \mathcal{O}(\tilde{f})$ for an arbitrary function f and non-negative constant $\tilde{f}$, which depend on a real parameter ω, if the norm of f and its derivatives are all of order $O(\tilde{f})$ as $\omega \to \infty$, namely $\|f^{(m)}\| = O(\tilde{f})$ for $m = 0, 1, \dots$. For arbitrary two $n \times m$ matrices $A(x) = (a_{ij}(x))$ and $\tilde{A} = (\tilde{a}_{ij})$, $\tilde{a}_{ij} \geq 0$, depending on a real parameter ω, we can thus posit $A(x) = \mathcal{O}(\tilde{A})$, if $a_{ij}(x) = \mathcal{O}(\tilde{a}_{ij})$ element-wise as $\omega \to \infty$. We

may also say that $f = \mathcal{O}(1)$, if f and its derivatives remain bounded on $[a, b]$, as $\omega \to \infty$. Let $\mathbf{1} = \{1_{ij}\}$ stand for a matrix with all entries one. This allows us to write $A(x) = \mathcal{O}(\mathbf{1})$, if $a_{ij}(x) = \mathcal{O}(1)$ element-wise as $\omega \to \infty$. And finally, if $A = \mathcal{O}(\tilde{A})$ and $B = \mathcal{O}(\tilde{B})$, then the integration and multiplication properties are $\int_a^b A(x)\mathrm{d}x = O(\tilde{A})$ and $AB = \mathcal{O}(\tilde{A}\tilde{B})$.

LEMMA 2.1. *Let*

$$I[\boldsymbol{f}] = \int_a^b X_\omega(t)\boldsymbol{f}(t)\mathrm{d}t, \quad X'_\omega = A_\omega X_\omega,$$

where the matrix kernel X_ω satisfies linear matrix ODE as above, A_ω is a constant non-singular matrix, $\|A_\omega^{-1}\| \ll 1$ and $\boldsymbol{f} : \mathbb{R} \to \mathbb{R}^d$ is a smooth vector-valued function. Then, for $\omega \gg 1$,

$$I[\boldsymbol{f}] \sim -\sum_{m=1}^{\infty} (-A_\omega)^{-m}\big[X_\omega(b)\boldsymbol{f}^{(m-1)}(b) - X_\omega(a)\boldsymbol{f}^{(m-1)}(a)\big].$$

For $\psi = \max\{\|\boldsymbol{f}^{(s)}\|, \|\boldsymbol{f}^{(s+1)}\|\}$,

$$Q_s^A[\boldsymbol{f}] - I[\boldsymbol{f}] \sim \mathcal{O}\big(\big\|A_\omega^{-s-1}\big\|\|X_\omega\|\psi\big), \quad \text{as } \omega \to \infty.$$

If $X_\omega = \mathcal{O}(\hat{X}_\omega)$ and $\boldsymbol{f} = \mathcal{O}(\tilde{\boldsymbol{f}})$, then

$$Q_s^A[\boldsymbol{f}] - I[\boldsymbol{f}] = \mathcal{O}\big(A_\omega^{-s-1}\hat{X}_\omega\tilde{\boldsymbol{f}}\big), \quad \text{as } \omega \to \infty,$$

element wise.

PROOF. By induction,

$$Q_s^A[\boldsymbol{f}] = I[\boldsymbol{f}] - (-A_\omega)^{-s}\int_a^b X_\omega(t)\boldsymbol{f}^{(s)}(t)\mathrm{d}t = I[\boldsymbol{f}] - (-A_\omega)^{-s}I[\boldsymbol{f}^{(s)}].$$

Indeed, for $s = 0$ the identity $Q_s^A = I[\boldsymbol{f}]$. Suppose that the equality holds for some $s \geq 1$, we now prove it for $s + 1$. This follows from

$$I[\boldsymbol{f}^s] = \int_a^b X_\omega(t)\boldsymbol{f}^{(s)}(t)\mathrm{d}t = A_\omega^{-1}\big[X_\omega(b)\boldsymbol{f}^{(s)}(b) - X_\omega(a)\boldsymbol{f}^{(s)}(a)\big] - A_\omega^{-1}\int_a^b X_\omega(t)\boldsymbol{f}^{(s+1)}(t)\mathrm{d}t.$$

For L^∞ norms,

$$I[\boldsymbol{f}^{(s)}] \sim \mathcal{O}\big(\big\|A_\omega^{-1}\big\|\|X_\omega\|\|\boldsymbol{f}^{(s)}\|\big) + \mathcal{O}\big(\big\|A_\omega^{-1}\big\|\|X_\omega\|\|\boldsymbol{f}^{(s+1)}\|\big) = \mathcal{O}\big(\big\|A_\omega^{-1}\big\|\|X_\omega\|\psi\big),$$

therefore

$$Q_s^A[\boldsymbol{f}] - I[\boldsymbol{f}] \sim \mathcal{O}\big(\big\|A_\omega^{-s-1}\big\|\|X_\omega\|\psi\big).$$

If $X_\omega = \mathcal{O}(\hat{X}_\omega)$ and $\boldsymbol{f} = \mathcal{O}(\tilde{\boldsymbol{f}})$ element-wise, then

$$I[\boldsymbol{f}^{(s)}] = \mathcal{O}\big(A_\omega^{-1}\hat{X}_\omega\tilde{\boldsymbol{f}}\big) + \mathcal{O}\big(A_\omega^{-1}\hat{X}_\omega\tilde{\boldsymbol{f}}\big) = \mathcal{O}\big(A_\omega^{-1}\hat{X}_\omega\tilde{\boldsymbol{f}}\big),$$

yielding the further result

$$Q_s^A[\boldsymbol{f}] - I[\boldsymbol{f}] = \mathcal{O}\big(A_\omega^{-s-1}\hat{X}_\omega\tilde{\boldsymbol{f}}\big).$$

□

Corollary 2.2. *If*

$$\boldsymbol{f}^{(i)}(a) = \boldsymbol{f}^{(i)}(b) = 0, \quad \textit{for } i = 0, \dots, s-1,$$

then in L^∞ *norm,*

$$I[\boldsymbol{f}] \sim \mathcal{O}\big(\big\|A_\omega^{-s-1}\big\|\|X_\omega\|\psi\big),$$

and

$$I[\boldsymbol{f}] = \mathcal{O}\big(A_\omega^{-s-1}\hat{X}_\omega\tilde{\boldsymbol{f}}\big)$$

element-wise.

Proof. Follows immediately from Lemma 2.1. □

Corollary 2.3. *If* $X_\omega = \mathcal{O}(\mathbf{1})$ *and* $\boldsymbol{f} = \mathcal{O}(\mathbf{1})$*, then*

$$Q_s^A[\boldsymbol{f}] - I[\boldsymbol{f}] = \mathcal{O}\big(A_\omega^{-s-1}\mathbf{1}\big).$$

Proof. Follows from the notation on matrix asymptotics and Lemma 2.1. □

The point of departure in construction of our numerical solvers for the systems of ordinary differential equations (1.2) is the initial-value integrator

$$\boldsymbol{y}_{n+1} = \mathrm{e}^{hA_\omega}\boldsymbol{y}_n + \int_0^h \mathrm{e}^{(h-\tau)A_\omega}\boldsymbol{f}(t_n+\tau)\mathrm{d}\tau. \tag{2.2}$$

Example 2.1. Let

$$I_h[\boldsymbol{f}] = \int_0^h \mathrm{e}^{A_\omega(h-t)}\boldsymbol{f}(t)\mathrm{d}t.$$

The *asymptotic* method for $s = 2$ with end points only is

$$Q_2^A[\boldsymbol{f}] = -A_\omega^{-1}(\boldsymbol{f}(h) - \mathrm{e}^{A_\omega h}\boldsymbol{f}(0)) - A_\omega^{-2}(\boldsymbol{f}'(h) - \mathrm{e}^{A_\omega h}\boldsymbol{f}'(0)).$$

In the sequel we provide some applications of the *asymptotic* method to solve highly oscillatory linear systems (1.2). Figure 2.1 captures how the accuracy of the method increases with ω, as long as the step size h is fixed and the characteristic frequency $h\omega \gg 1$. The method remains accurate for magnitudes

of $\omega = 10^4$ and $h\omega = 10^3$. This allows us to work with larger step-sizes, taking into account that it is the ω that reduces the error small rather than step-size.

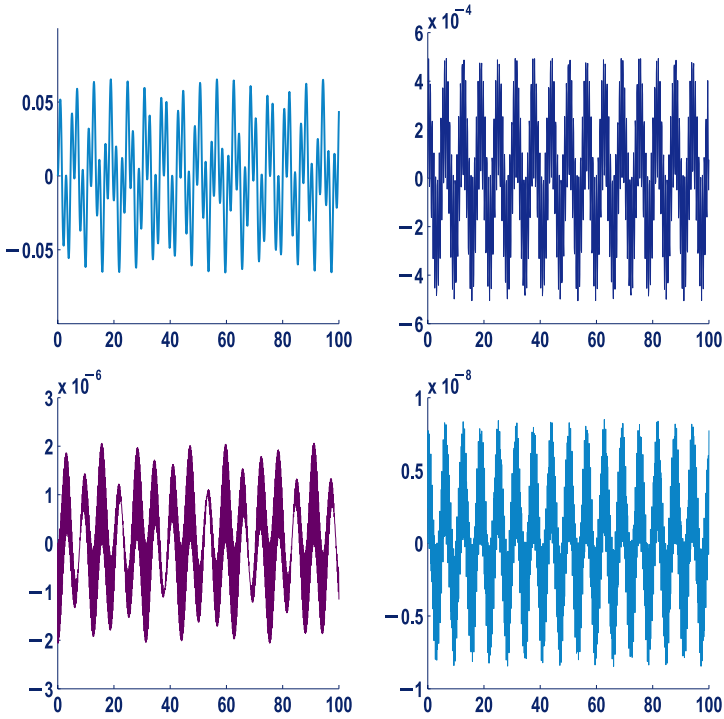

Figure 2.1: Global error of the *asymptotic* method Q_2^A with end points only for the equation $y''(t) = -\omega y(t) - \cos(t)$, $0 \le t \le 100$, with $[1, 0]^\intercal$ initial conditions and step-size $h = \frac{1}{10}$ for $\omega = 10$ (left figure, top), $\omega = 10^2$ (right figure, top), $\omega = 10^3$ (left figure, bottom), $\omega = 10^4$ (right figure, bottom).

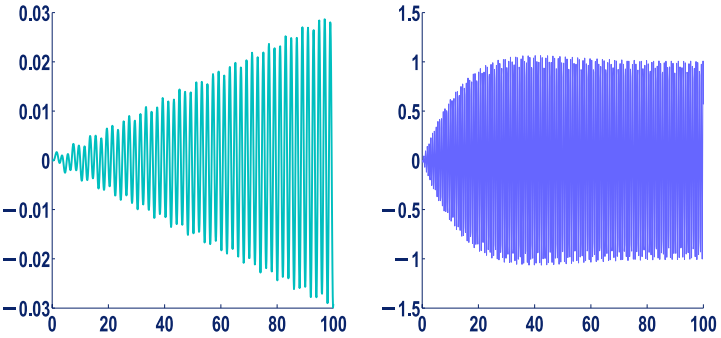

Figure 2.2: Global error of the fourth order Runge–Kutta method for the equation $y''(t) = -\omega y(t) - \cos(t)$, $0 \le t \le 100$, with $[1, 0]^\intercal$ initial conditions, step-size $h = \frac{1}{10}$, for $\omega = 10$ (left) and $\omega = 10^2$ (right).

Our method may be compared with the fourth order Runge–Kutta method presented in Figure 2.2 for the same equation and same step size, and MATLAB ode15s and ode113 solvers in Figure 4.1. For a fixed step-size $h = \frac{1}{10}$ the error of the fourth order Runge–Kutta method increases with ω. Due to stability of the Runge–Kutta method the error remains bounded as in the right Figure 2.2. However the method is accurate only for small values of t around the origin, whilst on a large time scale the approximation has nothing to do with exact solution for increasing ω.

3 Filon-type method.

In this section we extend the *Filon*-type method [10] to solve systems of ordinary differential equations. We interpolate a vector-valued function $\boldsymbol{f}$ in (2.1) by a r-degree vector-valued polynomial $\boldsymbol{v}$

$$\boldsymbol{v}(t) = \sum_{l=1}^{\nu} \sum_{j=0}^{\theta_l - 1} \alpha_{l,j}(t) \boldsymbol{f}^{(j)}(t_l), \tag{3.1}$$

such that $\boldsymbol{v}^{(j)}(t_l) = \boldsymbol{f}^{(j)}(t_l)$ at node points $a = t_1 < t_2 < \cdots < t_\nu = b$, $\theta_1, \theta_2, \ldots, \theta_\nu \geq 1$ being the associated multiplicities, $j = 0, 1, \ldots, \theta_l - 1$ and $l = 1, 2, \ldots, \nu$. We define the *Filon*-type method as follows,

$$Q_s^F[\boldsymbol{f}] = \int_a^b X_\omega(t)\boldsymbol{v}(t)\mathrm{d}t = \sum_{l=1}^{\nu} \sum_{j=0}^{\theta_l - 1} \beta_{l,j} \boldsymbol{f}^{(j)}(t_l),$$

where $\beta_{l,j} = \int_a^b X_\omega(t)\alpha_{l,j}(t)\mathrm{d}t$.

THEOREM 3.1. *Let*

$$I[\boldsymbol{f}] = \int_a^b X_\omega(t)\boldsymbol{f}(t)\mathrm{d}t, \quad X'_\omega = A_\omega X_\omega,$$

where A_ω is a constant non-singular matrix of a pure imaginary spectrum, $\sigma(A_\omega) \subset \mathrm{i}\mathbb{R}$, $\|A_\omega^{-1}\| \ll 1$, $\theta_1, \theta_\nu \geq s$ and $\boldsymbol{f} : \mathbb{R} \to \mathbb{R}^d$ is a smooth vector-valued function. Then, for $\psi = \max\{\|\boldsymbol{f}^{(s)}\|, \|\boldsymbol{f}^{(s+1)}\|\}$,

$$Q_s^F[\boldsymbol{f}] - I[\boldsymbol{f}] \sim \mathcal{O}\big(\big\|A_\omega^{-s-1}\big\|\|X_\omega\|\psi\big), \quad \textit{as } \omega \to \infty.$$

If $X_\omega = \mathcal{O}(\hat{X}_\omega)$ and $\boldsymbol{f} = \mathcal{O}(\tilde{\boldsymbol{f}})$, then element-wise

$$Q_s^F[\boldsymbol{f}] - I[\boldsymbol{f}] = \mathcal{O}\big(A_\omega^{-s-1}\hat{X}_\omega\tilde{\boldsymbol{f}}\big), \quad \textit{as } \omega \to \infty.$$

PROOF. The proof is equivalent to that for the classical *Filon*-type method. As a consequence of Corollary 2.2, replacing $\boldsymbol{f}$ in the asymptotic method with $\boldsymbol{f} - \boldsymbol{v}$, implies that $[\boldsymbol{f} - \boldsymbol{v}]^{(j)}(a) = [\boldsymbol{f} - \boldsymbol{v}]^{(j)}(b) = 0$, for $j = 0, 1, \ldots, s-1$. □

COROLLARY 3.2. *If* $X_\omega = \mathcal{O}(\mathbf{1})$ *and* $\boldsymbol{f} = \mathcal{O}(\mathbf{1})$, *then*

$$Q_s^F[\boldsymbol{f}] - I[\boldsymbol{f}] = \mathcal{O}\big(A_\omega^{-s-1}\mathbf{1}\big).$$

PROOF. The statement follows from the notation on matrix asymptotics and Corollary 2.3. □

Employing the initial-value integrator (2.2), we present the *Filon*-type method for the systems of highly oscillatory *ODE*s (1.2),

$$\begin{aligned}\boldsymbol{y}_{n+1} &= \mathrm{e}^{hA_\omega}\boldsymbol{y}_n + \int_0^h \mathrm{e}^{(h-\tau)A_\omega}\boldsymbol{v}(t_n+\tau)\mathrm{d}\tau \\ &= \mathrm{e}^{hA_\omega}\boldsymbol{y}_n + Q_s^F[\boldsymbol{f}]. \end{aligned} \tag{3.2}$$

EXAMPLE 3.1. Take the same integral $I_h[\boldsymbol{f}]$ as in the Example 2.1. For $s=2$, $t_1=0, t_2=h$ and $\boldsymbol{f}=[f_1,f_2]^\intercal$ we derive the *Filon*-type method,

$$\begin{aligned} Q_s^F[\boldsymbol{f}] = I_h[\boldsymbol{v}] &= \int_0^h \mathrm{e}^{A_\omega(h-t)}\boldsymbol{v}(t)\mathrm{d}t \\ &= \Big[\int_0^h \mathrm{e}^{A_\omega(h-t)}v_1(t)\mathrm{d}t\Big]\boldsymbol{f}(0) + \Big[\int_0^h \mathrm{e}^{A_\omega(h-t)}v_2(t)\mathrm{d}t\Big]\boldsymbol{f}(h) \\ &\quad + \Big[\int_0^h \mathrm{e}^{A_\omega(h-t)}v_3(t)\mathrm{d}t\Big]\boldsymbol{f}'(0) + \Big[\int_0^h \mathrm{e}^{A_\omega(h-t)}v_4(t)\mathrm{d}t\Big]\boldsymbol{f}'(h). \end{aligned}$$

We note by passing that the computational cost is relatively cheap. The algorithm requires only some linear algebra once we have precomputed moments in $I_h[\boldsymbol{v}]$.

THEOREM 3.3. *Let* $\theta_1, \theta_\nu \geq s$, $r = \sum_{l=1}^{\nu}\theta_l - 1$. *Then* r *is the numerical order of the Filon-type method applied to the linear system* (1.2),

$$\boldsymbol{y}(t_n) - \boldsymbol{y}_n = \mathcal{O}(h^{r+1}).$$

PROOF. Suppose that $\boldsymbol{f} = \boldsymbol{v} + \boldsymbol{p}$, where $\boldsymbol{v}$ is an r-degree vector-valued polynomial approximation (e.g. Hermite, 3.1) to $\boldsymbol{f}$, with an approximation error

$$\boldsymbol{p} = \frac{p_r}{r!}\boldsymbol{f}^{(r)}(\xi), \quad p_r = \prod_{l=1}^{\nu}(t-t_l)^{\theta_l}.$$

We can now derive the local error of our numerical solver,

$$I[\boldsymbol{p}] = \int_0^h \mathrm{e}^{(h-\tau)A_\omega}\boldsymbol{p}(t_n+\tau)\mathrm{d}\tau = \int_0^h \mathrm{e}^{(h-\tau)A_\omega}\frac{p_r(\tau)}{r!}\boldsymbol{f}^{(r)}(\xi)\mathrm{d}\tau,$$

where $p_r(\tau) = \mathcal{O}(\tau^r)$.

Recall that

$$I[\boldsymbol{p}] = I[\boldsymbol{f}] - I[\boldsymbol{v}] = I[\boldsymbol{f}] - Q_s^F[\boldsymbol{f}] = \mathcal{O}(h^{r+1}), \tag{3.3}$$

which proves the order of the method. □

THEOREM 3.4. *The numerical solution* (3.2) *is convergent.*

PROOF. Presenting matrix A_ω in its Jordan normal form $J = P^{-1}A_\omega P$, we let

$$C_\omega = \big\{\|P\|\|P^{-1}\| : P^{-1}A_\omega P = J\big\},$$

and consider the following bounds for matrix exponential,

$$\|\mathrm{e}^{tA_\omega}\| = \|P\mathrm{e}^{tJ}P^{-1}\| \le C_\omega\|\mathrm{e}^{tJ}\|.$$

We would like to remind our reader that $\sigma(A_\omega) \subset \mathrm{i}\mathbb{R}$, which means that the norm of the matrix exponential is always bounded, $\mathrm{e}^{tJ} = \mathcal{O}(\mathbf{1})$. In other words,

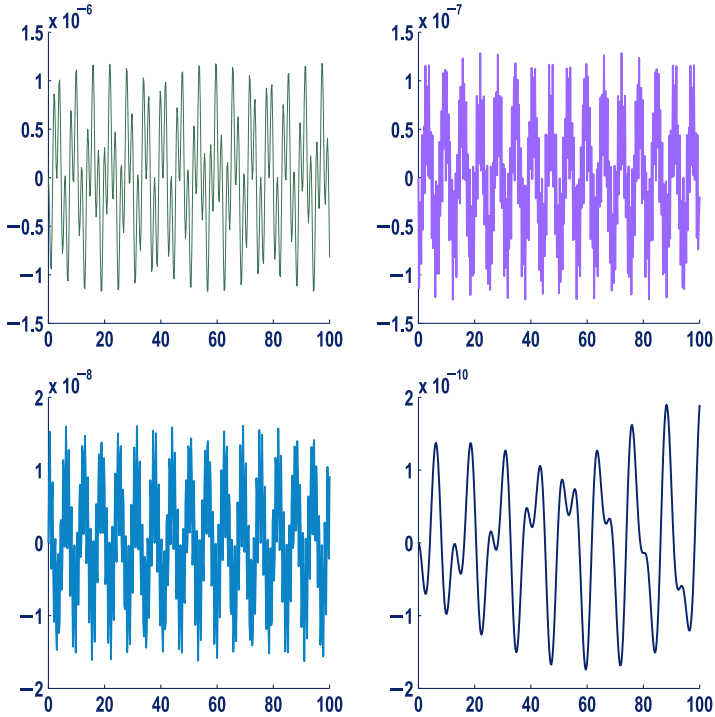

Figure 3.1: Global error of the *Filon*-type method Q_2^F with end points only and multiplicities all 2, for the equation $y''(t) = -\omega y(t) - \cos(t)$, $0 \le t \le 100$, with $[1, 0]^\intercal$ initial conditions and step-size $h = \frac{1}{4}$ for $\omega = 10$ (top figure, left) and $\omega = 10^2$ (top figure, right), $\omega = 10^3$ (left, bottom figure) and $\omega = 10^4$ (right, bottom figure).

for a fixed value of parameter ω, the convergence of the numerical scheme to the exact solution follows from the estimate of the residual term,

$$\|I[\boldsymbol{p}]\| \leq \frac{C_\omega h}{r!}\|p_r(\tau)\|\|\boldsymbol{f}^{(r)}\|,$$

as step-size h tends to zero. □

Note that the method requires information only about the function values and its derivatives at the end points. Figure 3.1 offers some numerical examples, where a step size h is fixed and ω increases. The examples demonstrate a gain in accuracy with the increase of ω. Comparison will note that the *Filon*-type method performs better than the *asymptotic* method, although both methods are of the same asymptotic order. It is evident now that for a larger step-size $h = \frac{1}{4}$ than that assumed with the asymptotic method ($h = \frac{1}{10}$) the accuracy of approximation improves as $h\omega \gg 1$.

We can compare our solutions with MATLAB solvers, presented in Figure 4.1. To achieve better results with MATLAB we set it to AbsTol = ReTol = 10^{-8}. Accuracy decreases for the solver ode15s while remaining the same for ode113. Both methods work with variable step size, but taking an average for $\omega = 100$ it is $h \approx \frac{1}{186}$ for ode15s and $h \approx \frac{1}{60}$ for ode113, reducing to the exceptionally small values for $\omega = 10^4$ of $h \approx 5 \times 10^{-4}$ in ode15s and $h \approx 10^{-3}$ in ode113, which is in no way comparable with $h = \frac{1}{4}$ of the *Filon*-type method. The logarithmic error in Figure 3.2 describes both numerical and asymptotic analysis of the method for increasing ω.

These considerations leave us at a point where the connections between [10] and the present work are evident. It follows from the spectral decomposition of the matrix A_ω that the factor $\mathrm{e}^{\mathrm{i}\lambda_k} f$ appears in the fundamental matrix in the solutions of both linear and nonlinear systems of highly oscillatory *ODE*s, and provides valid reasons to extend described methods for the given setting.

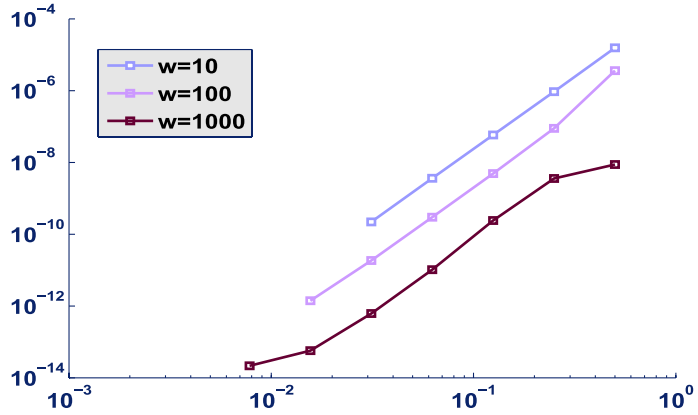

Figure 3.2: Logarithmic error (y-axis) and the step-size (x-axis) of the *Filon*-type method Q_2^F, with endpoints only and multiplicities all 2, for the equation $y''(t) = -\omega y(t) - \cos(t)$, with initial values in $[1, 0]^\intercal$.

4 WRF method.

In approximation of highly oscillatory nonlinear systems of *ODE*s we use the implicit representation of the solution (1.1) and the initial value integrator

$$\boldsymbol{y}_{n+1} = \mathrm{e}^{hA_\omega}\boldsymbol{y}_n + \int_0^h \mathrm{e}^{(h-\tau)A_\omega}\boldsymbol{f}(t_n+\tau, \boldsymbol{y}(t_n+\tau))\mathrm{d}\tau.$$

Our method further develops some of the implications of the *Filon* quadrature and *waveform relaxation* methods in this setting.

Waveform relaxation (*WR*) methods, a family of iterative techniques designed for analyzing dynamical systems, have been studied by a number of authors and we mention here some of them, [1, 2, 5, 6, 12, 13, 18, 19, 20, 21, 25, 26].

Assuming that $\boldsymbol{f} : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$ satisfies the Lipschitz condition, the classical *waveform Picard* algorithm states

$$\boldsymbol{y}^{[s]}(t) = X(t)\boldsymbol{y}_0 + \int_0^t X(t-\tau)\boldsymbol{f}(\tau, \boldsymbol{y}^{[s-1]}(\tau))\mathrm{d}\tau.$$

It is desirable to apply *Filon* quadrature with its nice properties for large ω to nonlinear dynamical systems to solve integral

$$I[\boldsymbol{f}] = \int_a^b X_\omega(t)\boldsymbol{f}(t, \boldsymbol{y}(t))\mathrm{d}t, \quad \text{where } X'_\omega = A_\omega X_\omega \quad \text{and} \quad \left\|A_\omega^{-1}\right\| \ll 1.$$

Formally, the asymptotic expansion for the integral $I[\boldsymbol{f}]$ as above looks as follows,

$$I[\boldsymbol{f}] \sim \sum_{m=1}^{\infty}(-1)^{m-1}A^{-m}\left[X_w(b)\boldsymbol{f}^{(m-1)}(b, \boldsymbol{y}(b)) - X_w(a)\boldsymbol{f}^{(m-1)}(a, \boldsymbol{y}(a))\right],$$

where the function values and its derivatives at the end point $\boldsymbol{f}^{(m)}(b, \boldsymbol{y}(b))$ are not available anymore for $m = 0, 1, 2, \ldots$. To overcome this issue we introduce *waveform methods*. Note that if the solution $\boldsymbol{y}$ is oscillatory, then the function $\boldsymbol{f}(t, \boldsymbol{y})$ is also likely to be oscillatory, causing *waveform relaxation* methods, if used by itself, to be inefficient. Thus, for efficiency, we discretize $I[\boldsymbol{f}]$ according to the rules of the *Filon* quadrature, choosing a vector-valued polynomial (e.g. Hermite), which agrees with our function values and its derivatives $\boldsymbol{f}^{(m)}$ at the node points. Our *WRF* method iterates $\boldsymbol{y}$ in (1.1) with a *waveform* method, solving $I[\boldsymbol{f}]$ at each step with *Filon* quadrature,

$$\begin{aligned}
\boldsymbol{y}_{n+1}^{[0]} &= \boldsymbol{y}_n^{[s]},\\
\boldsymbol{y}_{n+1}^{[1]} &= \mathrm{e}^{A_\omega h}\boldsymbol{y}_n^{[s]} + \int_0^h \mathrm{e}^{(h-\tau)A_\omega}\boldsymbol{f}\big(t_n+\tau, \boldsymbol{y}_{n+1}^{[0]}\big)\mathrm{d}\tau,\\
&\vdots\\
\boldsymbol{y}_{n+1}^{[s]} &= \mathrm{e}^{A_\omega h}\boldsymbol{y}_n^{[s]} + \int_0^h \mathrm{e}^{A_\omega(h-\tau)}\boldsymbol{v}\big(t_n+\tau, \boldsymbol{y}_{n+1}^{[s-1]}(t_n+\tau)\big)\mathrm{d}\tau,
\end{aligned}$$

where $\boldsymbol{v}$ is a polynomial approximation to $\boldsymbol{f}$ as before. The method takes the initial value constant $\boldsymbol{y}_{n+1}^{[0]}$ at a first step of iteration to obtain the first value of $\boldsymbol{y}_{n+1}^{[1]}$. Having values at two endpoints we can now evaluate the derivatives at those points and construct a polynomial, which agrees with function values and derivatives at the end points. In order to obtain any desirably high order method higher order derivatives of the function $\boldsymbol{f}$ are required. In that respect we recommend to differentiate the equation for the system $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{f}(t, \boldsymbol{y})$ itself to obtain higher order derivatives of the solution vector $\boldsymbol{y}$, and use the results in construction of approximation polynomial. The more derivatives at the end points are used in the polynomial approximation the more terms are cancelled in the asymptotic expansion to $I[\boldsymbol{f}]$, leading to higher order methods and hence better accuracy.

We highlight here that the *WRF* method employs end points only, otherwise adding internal points would have led to increasingly fine discretization of the interval. The reason for this is that approximation at internal points themselves requires further partition of the subinterval and this iteration is endless. From the point of view of the function asymptotics, the fundamental point here is that the performance of the *Filon* quadrature is determined by the values at the end points of the integration interval only, making addition of internal points less valuable, Theorem 1.3, Theorem 3.1.

Computational cost of the *WRF* method is comparable to that of the *Filon*-type method. Having precomputed the vector-valued moments, all that remains are only some linear algebra operations.

THEOREM 4.1. *Suppose that r is the numerical order of a waveform relaxation method and s is the numerical order of the quadrature discretization applied to a nonlinear system of ODEs*

$$\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{f}(t, \boldsymbol{y}), \quad \boldsymbol{y}(t_0) = \boldsymbol{y}_0, \quad t \geq 0,$$

of arbitrary matrix A and $\boldsymbol{f} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying the Lipschitz condition. Then,

$$\boldsymbol{y}(h) - \boldsymbol{y}_h = \mathcal{O}(h^{q+1}),$$

where $q = \min\{r, s\}$.

PROOF. Let $\boldsymbol{y}(h)$ be an exact solution, $\boldsymbol{y}_h^{[r]}$ its numerical solution by a *waveform* method, and $\boldsymbol{y}_h$ the final numerical solution after discretization. Then,

$$\begin{aligned} \|\boldsymbol{y}(h) - \boldsymbol{y}_h\| &\leq \left\|\boldsymbol{y}(h) - \boldsymbol{y}_h^{[r]}\right\| + \left\|\boldsymbol{y}_h^{[r]} - \boldsymbol{y}_h\right\| \\ &= \mathcal{O}(h^{r+1}) + \mathcal{O}(h^{s+1}) = \mathcal{O}(h^{q+1}) \end{aligned} \tag{4.1}$$

where $q = \min\{r, s\}$. □

COROLLARY 4.2. *The numerical order of the WRF method is the minimum over both the Filon quadrature and the waveform method applied to solve nonlinear system* (1.1).

PROOF. It follows immediately from Theorem 4.1. □

Thus, once we have chosen the quadrature rule of a given order, we iterate the equation until obtaining the order of the quadrature and further iteration is pointless.

THEOREM 4.3. *WRF method is convergent.*

PROOF. Follows from (4.1) as $h \to 0$. □

In the Table 4.1 we demonstrate the accuracy of the *WRF* method for increasing ω in just four iterations. Due to non-linearity, the accuracy improves slightly slower, since we have chosen to iterate oscillatory equations. On the other hand, taking into account rapid oscillation of the solution for large ω, applying only four iteration combined with *Filon* quadrature is a very little exercise to achieve up to ten digits accuracy as it is described the Table 4.1. Figure 4.2 demonstrates the logarithmic error of the *WRF* method for different values of ω. *Waveform* methods do not contribute to function asymptotics making it harder to parallel the two methods, namely *Filon*-type method and *WRF* method, as it is shown in Figures 4.2 and 3.2. We have seen that the *Filon*-type methods have the same asymptotic order as the *asymptotic* method, where the error depends on the inverse powers of ω. In the case of asymptotic behavior of *WRF* method the bottom line is that one iterates the asymptotic expansion, but the advantage of applying the right quadrature rule means that for increasing ω the error amazingly remains very close to that for smaller ω. *Waveform* methods do not preserve the nice asymptotic features of the *Filon* quadrature, but the latter even in the presence of iteration captures up to 10^{-12} accuracy as ω increases.

Table 4.1: Approximation error of the *WRF* method for the nonlinear *ODE* $y'' = -\omega y - 3y^3$, compared with the results of MATLAB ode45 solver set to RelTol $= 10^{-12}$, AbsTol $= 10^{-16}$.

h	$\omega = 10$	h	$\omega = 100$	h	$\omega = 1000$
2.50_{-01}	1.04_{-03}	1.00_{-01}	-8.32_{-04}	4.00_{-02}	-8.90_{-04}
1.00_{-01}	2.25_{-05}	5.00_{-02}	-5.33_{-05}	2.00_{-02}	-5.16_{-05}
5.00_{-02}	1.40_{-06}	2.50_{-02}	-3.35_{-06}	1.00_{-02}	-3.22_{-06}
4.00_{-02}	5.74_{-07}	1.00_{-02}	-8.59_{-08}	3.33_{-02}	-3.98_{-08}
1.25_{-02}	4.83_{-09}	5.00_{-03}	-5.36_{-09}	1.25_{-03}	-6.63_{-10}

Finally, for methods introduced in current paper there wasn't any restrictions on the phase of the solution, whilst for discretization of $I[\boldsymbol{f}]$ with *Filon* quadrature some special functions can be considered as the most obvious choice.

The analysis of the *asymptotic*, *Filon*-type and *WRF* methods for a time-dependant matrix using Magnus expansions as well as some alternative choices of the quadrature methods can be found in [14, 15].

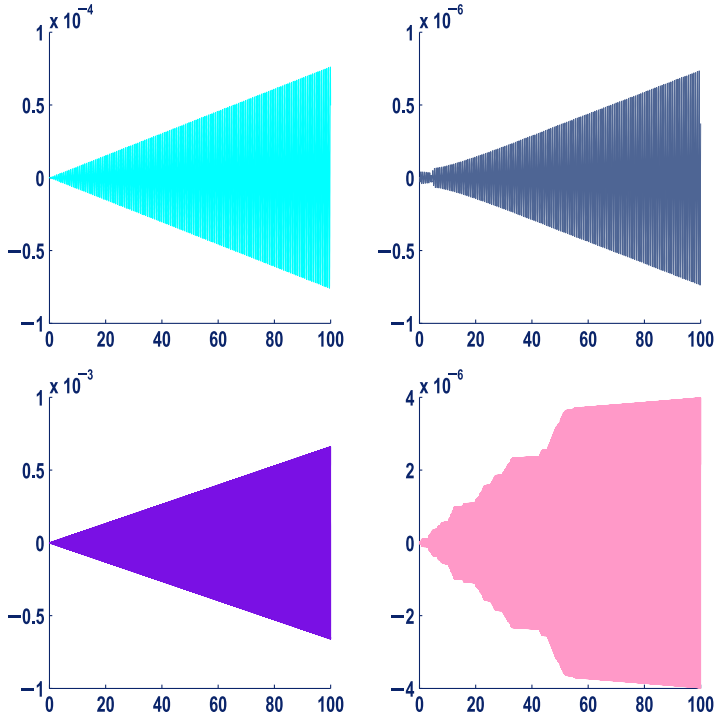

Figure 4.1: Global error of MATLAB ode15s routine (top figure, left) and ode113 routine (top figure, right) set to RelTol $= 1e-08$, AbsTol $= 1e-08$, for the equation $y''(t) = -\omega y(t) - \cos(t)$, $0 \leq t \leq 100$, with $[1,0]^{\intercal}$ initial conditions and $\omega = 10^2$; the same solvers with the same properties and $\omega = 10^4$ (two figures in the bottom respectively).

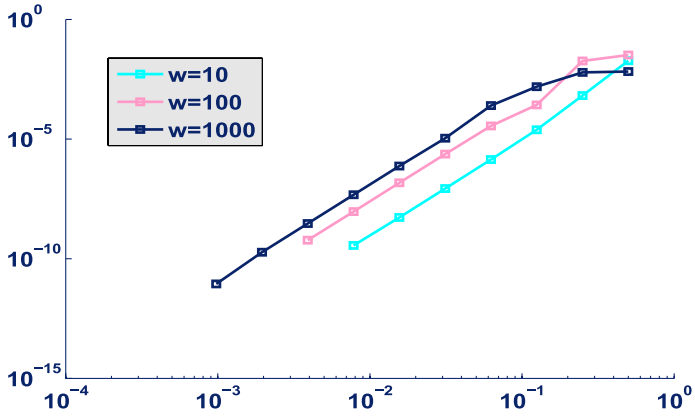

Figure 4.2: Logarithmic error (y-axis) and the step-size (x-axis) of the WRF method for the equation $y'' = -\omega y - 3y^3$, initial values in $[1,1]^{\intercal}$, with endpoints only and multiplicities all 2.

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