The spectral problem for a class of highly oscillatory Fredholm integral operators

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Abstract

Let $\mathcal{F}_\omega$ be a linear, complex-symmetric Fredholm integral operator with highly oscillatory kernel $K_0(x,y)\mathrm{e}^{\mathrm{i}\omega|x-y|}$. We study the spectral problem for $\mathcal{F}_\omega$ for large ω and investigate the asymptotic properties of solutions $f = f(x;\omega)$ to the associated Fredholm integral equation $f = \mu\mathcal{F}_\omega f + a$ as $\omega \to \infty$. Possible extensions of these results to highly oscillatory Fredholm integral operators with more general highly oscillating kernels are also discussed.

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1 Introduction

The last few years have witnessed substantive effort towards the understanding of highly oscillatory phenomena, and in particular of highly oscillatory integrals [10, 12, 16]. Using tools of both asymptotic and numerical analysis, it has emerged that the presence of high oscillation is no hindrance to the understanding or computation of mathematical objects. Once we understand the mathematical mechanism underlying rapidly oscillating phenomena and employ tools of asymptotic analysis, often in computational setting, there is no need to fear high frequency!

This paper is the beginning of a longer commitment to extend this approach from highly oscillatory integrals to integral equations with highly oscillatory kernels. Such equations occur in a number of applications, in particular in electromagnetics, acoustic scattering and laser engineering, and the current level of their understanding is exceedingly poor. The range of issues that we wish eventually to bring under scrutiny include spectral problems for Fredholm operators, and also the solution of Fredholm and Volterra equations, all with highly oscillatory kernels.

In the current paper our aim is to analyse the spectral properties of the Fredholm integral operator $\mathcal{F}_\omega :\ C(I) \to C(I)$ defined by

$$(\mathcal{F}_\omega f)(x) := \int_{-1}^{1} K_\omega(x,y) f(y)\ \mathrm{d}y, \quad x \in I := [-1,1], \tag{1.1}$$

and possessing the highly oscillatory kernel

$$K_\omega(x,y) := K_0(x,y)\mathrm{e}^{\mathrm{i}\omega g(x,y)}, \quad \omega \gg 1, \tag{1.2}$$

with

$$g(x,y) = |x-y|, \quad (x,y) \in D := I^2\ . \tag{1.3}$$

The real-valued kernel $K_0(x,y)$ is assumed to be smooth on D and independent of ω. We observe that the integral operator $\mathcal{F}_\omega$ is compact and, for kernels K_0 satisfying $K_0(x,y) = K_0(y,x)$ on D, *complex-symmetric* (but *not* Hermitian). Thus for any given $\omega > 0$ the spectrum $\sigma(\mathcal{F}_\omega)$ is at most countable and has at most one limit point (see, e.g. Rudin [18] or Atkinsom [1, pp. 17–18]). We shall show in Section 3 that $\mathcal{F}_\omega$ possesses infinitely many discrete eigenvalues that lie on an ellipse-like curve in the complex plane, with the origin being the limit point of $\sigma(\mathcal{F}_\omega)$.

We are also interested in refining existing results on the asymptotic behaviour of functions $f(x) = f(x,\omega)$, as $\omega \to \infty$, that solve Fredholm integral equations of the second kind associated with $\mathcal{F}_\omega$,

$$f(x) = \mu(\mathcal{F}_\omega f)(x) + a(x), \quad x \in I, \tag{1.4}$$

when $\mu^{-1} \in \mathbf{C}$ is not in the spectrum $\sigma(\mathcal{F})$ of the Fredholm integral operator (1.1) and $a \in C^\infty(I)$ idoes not depend on ω. It is known (Ursell [19]) that the solution $f(x;\omega)$ of (1.4) is highly oscillatory, too.

Finally, we wish to employ our explicit knowledge of spectral properties to design exceedingly fast spectral solver of problems of the form (1.4).

Our study is the first step towards the understanding of the spectral properties of complex-symmetric Fredholm integral operators that are more general that the one given in (1.1) and (1.2); they correspond to the oscillators

$$g(x,y) = (x-y)^2, \quad (x,y) \in I^2\ , \tag{1.5}$$

and

$$g(x,y) = xy\ , \quad (x,y) \in I^2. \tag{1.6}$$

The study of the spectra of such integral operators is motivated by applications, especially in laser theory; see for example the papers by Fox and Li [8], Hochstadt [9], Cochran and Hinds [7], Landau [14, 15] and Berry [2, 3, 4] for the case (1.5) (we will refer to equation (1.3) with this function $g(x,y)$ as the *Fox–Li equation*, and Cochran and Hinds [7, p. 777] for (1.6). The analysis of the spectra of these more general Fredholm integral operators appears to be fairly challenging (see also Section 6) and we plan to return to it in future papers.

It is appropriate to recall a quote from Cochran's 1972 book [6, p. 279] that remains valid to this day:

The analysis of integral equations with general complex-symmetric kernels remains, at present, an art form in which each separate equation appears to necessitate treatment based almost solely on its own individual features and peculiarities.

Our ambition is to contribute to converting art into science in this context.

Remark 1.1 *The spectral problem corresponding to the function $g(x, y) = x - y$ in (1.2) is, for kernels $K_0(x, y)$ of the form*

$$K_0(x,y) = \sum_{j=1}^{r} A_j(x)B_j(y), \quad A_j, B_j \in C(I) \ \ (1 \le j \le r), \tag{1.7}$$

with linearly independent sets $\{A_j\}$ and $\{B_j\}$, trivial, since $\mathcal{F}_\omega$ is now a finite-rank operator. Setting

$$z_j := \int_{-1}^{1} B_j(y)\mathrm{e}^{-\mathrm{i}\omega y}\phi(y)\,\mathrm{d}y \quad (1 \le j \le r),$$

and introducing the vector $\mathbf{z} := (\ z_1, \dots, z_r\ )^\top \in \mathbb{R}^r$ and the matrix

$$\mathbf{C}_r := [\ C_{k,j}\ ] \in \mathbb{R}^{r\times r}, \quad C_{k,j} := \int_{-1}^{1} A_j(x)B_k(x)\,\mathrm{d}x \quad (= \langle A_j, B_k\rangle\ ), \tag{1.8}$$

the eigenvalue problem

$$(\mathcal{F}_\omega\phi)(x) = \lambda\phi(x), \quad x \in I, \tag{1.9}$$

for the Fredholm operator $\mathcal{F}_\omega$ with kernel $K_0(x, y)$ given by (1.7) and $g(x, y) = x - y$ reduces to the algebraic eigenvalue problem $\mathbf{C}_r\mathbf{z} = \lambda\mathbf{z}$. Hence, the eigenvalues $\lambda_1, \dots, \lambda_r$ of this finite-rank Fredholm integral operator do not depend on ω.

On the other hand, the solution $f \in C(I)$ of the Fredholm equation (1.4) with kernel (1.2), (1.7) (and $\mu^{-1} \ne \lambda_j$ $(j = 1, \dots, r)$) is given by

$$f(x) = \mu\sum_{j=1}^{r} A_j(x)f_j + a(x), \quad x \in I,$$

where $\mathbf{f} := (\ f_1, \dots, f_r\ )^\top$ is the (unique) solution of the algebraic system

$$(\mathbf{I}_r - \mu\mathbf{C}_r)\mathbf{f} = \mathbf{b}, \quad \textit{with} \quad b_j = b_j(\omega) := \int_{-1}^{1} a(x)B_j(x)\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \ \ (1 \le j \le r).$$

The matrix $\mathbf{C}_r \in \mathbb{R}^{r\times r}$ is the same as the one given in (1.8) and is independent of ω. Thus, the solution f of (1.4) with finite-rank kernel (1.7) does depend on ω and, by the Riemann–Lebesgue Lemma, has the property that $f(\,\cdot\,,\omega) \to 0$, uniformly on I, as $\omega \to \infty$. It will be seen in Section 4.1 that this asymptotic property remains true for the integral equation (1.4) with general highly oscillatory Fredholm operator (1.2–3); see also Ursell [19].)

In Section 2 we show that the eigenvalue problem (1.9) corresponding to the general Fredholm integral operator (1.1–3) is equivalent to a Sturm–Liouville eigenvalue problem with complex-valued Robin boundary conditions. This result is used in Section 3 to establish the existence of an infinite (but countable) spectrum $\{\lambda_m\}$ and the asymptotic behaviour of these eigenvalues for large ω and large m. In Section 4 we touch upon the asymptotic behaviour of the solution of the second-kind Fredholm integral equation (1.4) (with $\mu^{-1} \notin \sigma(\mathcal{F}_\omega)$), as $\omega \to \infty$. Section 5 deals briefly with current work on the extension of these asymptotic results to the more general complex-symmetric Fredhholm integral operators associated with (1.5) and (1.6).

2 The spectral problem

Recall that the spectral problem for the Fredholm integral operator $\mathcal{F}_\omega$ is described by

$$(\mathcal{F}_\omega \phi)(x) = \lambda \phi(x), \quad x \in I, \tag{2.1}$$

with $\lambda \in \mathbf{C}$ and $\phi \in C(I)$ ($\phi \not\equiv 0$). Let $\sigma(\mathcal{F}_\omega)$ denote the spectrum of $\mathcal{F}_\omega$, and assume that ϕ_m is an eigenfunction corresponding to $\lambda_m \in \sigma(\mathcal{F}_\omega)$. Unless stated otherwise, we will assume that $K_0(x, y) \equiv 1$ in (1.2), without any essential loss of insight.

Lemma 2.1 *The spectral problem for the complex-symmetric Fredholm integral operator $\mathcal{F}_\omega$ with $K_0(x, y) \equiv 1$ is equivalent to the Sturm–Liouville eigenvalue problem*

$$\phi_m''(x) + \nu_m^2 \phi_m(x) = 0, \quad x \in I, \tag{2.2}$$

with the complex-valued Robin boundary conditions

$$\mathrm{i}\omega\phi_m(-1) + \phi_m'(-1) = 0, \quad \mathrm{i}\omega\phi_m(1) - \phi_m'(1) = 0. \tag{2.3}$$

Here we have set

$$\nu_m = \nu_m(\omega) := \left(\omega^2 - \frac{2\mathrm{i}\omega}{\lambda_m}\right)^{1/2}, \tag{2.4}$$

implying that

$$\lambda_m = \frac{2\mathrm{i}\omega}{\omega^2 - \nu_m^2} \quad (= \mu_m^{-1}). \tag{2.5}$$

Proof Setting $F_m(x) := (\mathcal{F}_\omega \phi_m)(x)$ and observing that, for $x \in I$,

$$F_m(x) = \mathrm{e}^{\mathrm{i}\omega x} \int_{-1}^{x} \mathrm{e}^{-\mathrm{i}\omega y} \phi_m(y) \, \mathrm{d}y + \mathrm{e}^{-\mathrm{i}\omega x} \int_{x}^{1} \mathrm{e}^{\mathrm{i}\omega y} \phi_m(y) \, \mathrm{d}y,$$

we obtain

$$\frac{\mathrm{d}F_m(x)}{\mathrm{d}x} = \mathrm{i}\omega \left[\mathrm{e}^{\mathrm{i}\omega x} \int_{-1}^{x} \phi_m(y) \mathrm{e}^{-\mathrm{i}\omega y} \, \mathrm{d}y - \mathrm{e}^{-\mathrm{i}\omega x} \int_{x}^{1} \phi_m(y) \mathrm{e}^{\mathrm{i}\omega y} \, \mathrm{d}y \right] \tag{2.6}$$

and

$$\begin{aligned} \frac{\mathrm{d}^2 F_m(x)}{\mathrm{d}x^2} &= (\mathrm{i}\omega)^2 \left[\mathrm{e}^{\mathrm{i}\omega x} \int_{-1}^{x} \phi_m(y) \mathrm{e}^{-\mathrm{i}\omega y} + \mathrm{e}^{-\mathrm{i}\omega x} \int_{x}^{1} \phi_m(y) \mathrm{e}^{\mathrm{i}\omega y} \, \mathrm{d}y \right] + 2\mathrm{i}\omega\phi_m(x) \quad (2.7) \\ &= (\mathrm{i}\omega)^2 F_m(x) + 2\mathrm{i}\omega\phi_m(x). \end{aligned}$$

Since there holds $(\mathcal{F}_\omega \phi_m)(x) = \lambda_m \phi_m(x)$ for $x \in I$, it follows that

$$\frac{\mathrm{d}^2 (\mathcal{F}_\omega \phi_m)(x)}{\mathrm{d}x^2} = \lambda_m \phi_m''(x) = (\mathrm{i}\omega)^2 (\mathcal{F}_\omega \phi_m)(x) + 2\mathrm{i}\omega\phi_m(x), \quad x \in I,$$

or, for $\lambda_m \neq 0$,

$$\phi_m''(x) + \left[-\frac{2\mathrm{i}\omega}{\lambda_m} - (\mathrm{i}\omega)^2 \right] \phi_m(x) =: \phi_m''(x) + \nu_m^2 \phi_m(x) = 0; \tag{2.8}$$

cf. (2.4) and (2.5). The boundary conditions (2.3) follow from (2.6) and the eigenvalue equation: since $F_m'(-1) = -\mathrm{i}\omega F_m(-1)$, and

$$\frac{\mathrm{d}}{\mathrm{d}x} (\mathcal{F}_\omega \phi_m)(-1) = \lambda_m \phi_m'(-1), \qquad (\mathcal{F}_\omega \phi_m)(-1) = \lambda_m \phi_m(-1),$$

we find $\phi_m'(-1) = -\mathrm{i}\omega\phi_m(-1)$. The boundary condition at $x = 1$ is obtained in an analogous manner. This completes the proof of Lemma 2.1. □

Lemma 2.2 *The eigenfunction ϕ_m corresponding to an eigenvalue λ_m of $\mathcal{F}_\omega$ has (up to renormalization by a nonzero constant) the form*

$$\phi_m(x) = (\nu_m - \omega)\mathrm{e}^{i\nu_m(1+x)} + (\nu_m + \omega)\mathrm{e}^{-i\nu_m(1+x)}, \quad x \in I, \tag{2.9}$$

with ν_m given by (2.4).

Proof Since the general solution of (2.2) can be written in the form

$$\phi_m(x) = \alpha_m \mathrm{e}^{i\nu_m(1+x)} + \beta_m \mathrm{e}^{-i\nu_m(1+x)}, \tag{2.10}$$

the coefficients α_m and β_m are determined by the boundary conditions (2.3): we obtain nontrivial solutions if, and only if, the determinant of the matrix in the homogeneous linear system

$$\begin{bmatrix} i(\nu_m + \omega) & i(\nu_m - \omega) \\ i(\nu_m - \omega)\mathrm{e}^{2i\nu_m} & -i(\nu_m + \omega)\mathrm{e}^{-2i\nu_m} \end{bmatrix} \begin{bmatrix} \alpha_m \\ \beta_m \end{bmatrix} = 0 \tag{2.11}$$

vanishes; that is, if and only if the condition

$$(\nu_m - \omega)\mathrm{e}^{i\nu_m} = \pm(\nu_m + \omega)\mathrm{e}^{-i\nu_m} \tag{2.12}$$

holds. The assertion (2.9) of Lemma 2.2 then readily follows from (2.11). □

The condition (2.12) implies that we have two branches of solutions: setting $\theta := \nu_m$ (cf. (2.4) and (2.5)), these branches are given by

$$\theta \tan(\theta) = -\mathrm{i}\omega \qquad \text{and} \qquad \theta \cot(\theta) = \mathrm{i}\omega, \tag{2.13}$$

respectively, with $\theta = \sqrt{\omega^2 - 2\mathrm{i}\omega\mu}$ (where we will temporarily suppress the subscript m in $\theta = \theta_m$ and $\mu = \mu_m$). This observation will be the starting point for our analysis, in Section 3, of the nature and, especially, the asymptotic behaviour of the spectrum of $\mathcal{F}_\omega$ when $\omega \to \infty$.

3 The spectrum of $\mathcal{F}_\omega$

Subject to the assumptions stated in Section 1, the linear Fredholm integral operator $\mathcal{F}_\omega$ defined in (1.1–3) is compact. Hence, its spectrum $\sigma(\mathcal{F}_\omega) = \{\lambda_m\}$, that is, the set of (complex) numbers $\lambda := \mu^{-1}$ ($\mu \neq 0$) for which the integral equation

$$(\mathcal{F}_\omega \phi)(x) = \lambda\phi(x), \quad x \in [-1, 1],$$

has nontrivial solutions $\phi \in C(I)$, is countable. As the following theorem shows, $\sigma(\mathcal{F}_\omega)$ is in fact an infinite set, implying that $\lim_{m\to\infty} \lambda_m = 0$.

Theorem 3.1 *Let $\mathcal{F}_\omega$ be the Fredholm integral operator defined in (1.1–2), and assume that $K_0(t, s) \equiv 1$ and $\omega \gg 1$. The spectrum of this integral operator consists of infinitely many discrete eigenvalues λ_m which all lie in the right complex half-plane and converge to the origin. More precisely,*

(i) For fixed $\omega > 0$ and small m we have

$$\mathrm{Re}\,\lambda_m \sim \frac{(m\pi)^2}{\omega^4} - \frac{2(m\pi)^2 + 5/24(m\pi)^4}{\omega^6} + \mathcal{O}(\omega^{-8}), \tag{3.1}$$

$$\mathrm{Im}\,\lambda_m \sim -\frac{2}{\omega} + \frac{(m\pi)^2}{2\omega^3} - \frac{3/2(m\pi)^2 + 1/8(m\pi)^4}{\omega^5} + \mathcal{O}(\omega^{-7}). \tag{3.2}$$

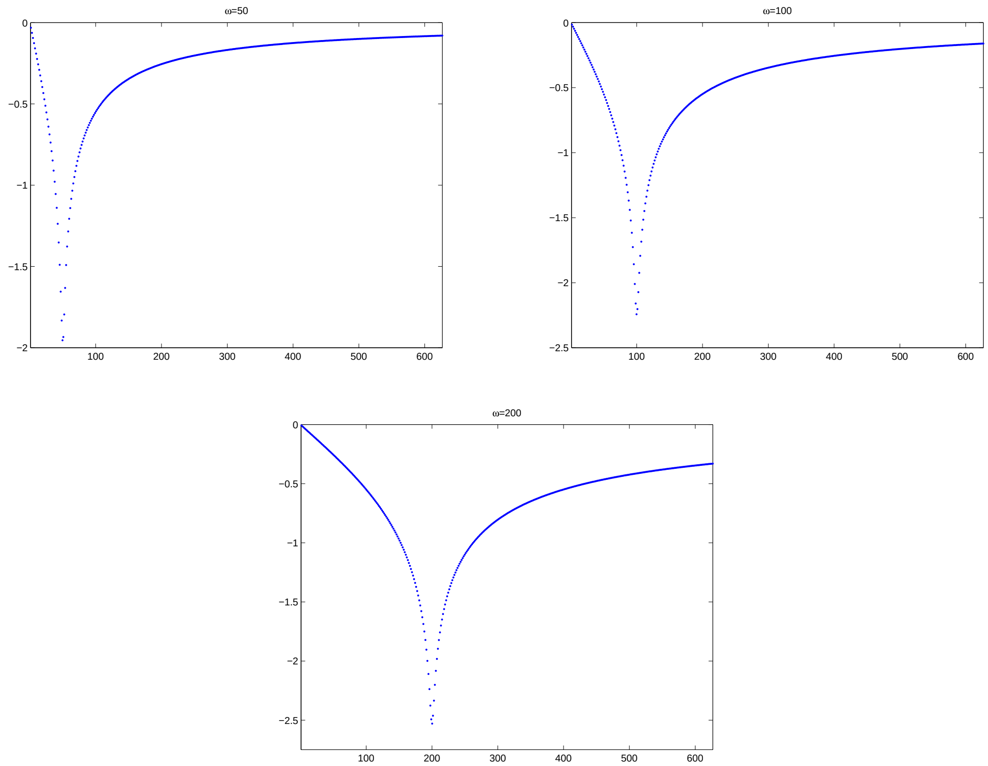

Figure 3.1: The values of θ_m for $\omega = 50, 100, 200$.

(ii) *For fixed $\omega > 0$ and $m \gg 1$ the real and imaginary parts of λ_m behave asymptotically like*

$$\mathrm{Re}\,\lambda_m \sim \frac{64\omega^2}{(m\pi)^4} + 1792\omega^4\left[\frac{1}{3}\frac{1}{(m\pi)^6} - \frac{4}{(m\pi)^8}\right] \tag{3.3}$$

$$+4096\omega^6\left[\frac{13}{15}\frac{1}{(m\pi)^8} - \frac{26}{(m\pi)^{10}} + \frac{152}{(m\pi)^{12}}\right] + \cdots, \tag{3.4}$$

$$\mathrm{Im}\,\lambda_m \sim -\frac{8\omega}{(m\pi)^2} - 32\omega^3\left[\frac{1}{(m\pi)^4} - \frac{20}{(m\pi)^6}\right] \tag{3.5}$$

$$-128\omega^5\left[\frac{1}{(m\pi)^6} - \frac{22}{3}\frac{1}{(m\pi)^8} + \frac{2016}{3}\frac{1}{(m\pi)^{10}}\right] + \cdots. \tag{3.6}$$

Also,

$$|\lambda_m| \sim \frac{8\omega}{\pi^2}m^{-2}. \tag{3.7}$$

(iii) *For $\omega \gg 1$ and fixed m we have*

$$\left|\lambda_m - \frac{1}{2}\right| \sim \frac{1}{4} + \frac{4}{\omega^2} - \frac{3m^2\pi^2}{\omega^4} + \mathcal{O}(\omega^{-6}), \tag{3.8}$$

while for $m \gg 1$ and fixed $\omega > 0$,

$$\left|\lambda_m - \frac{1}{2}\right| \sim \frac{1}{4} - \frac{128}{3}\frac{\omega^4}{m^6\pi^6} + \mathcal{O}(m^{-8}). \tag{3.9}$$

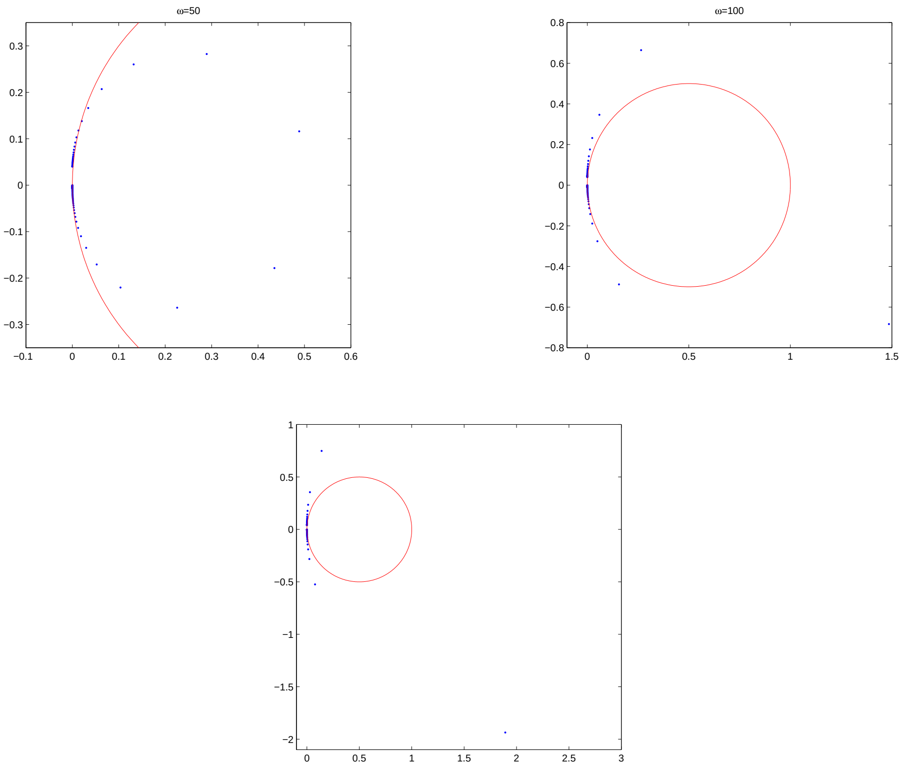

Figure 3.2: The eigenvalues λ_m for $\omega = 50, 100, 200$.

Remark 3.1 *In Fig. 3.1 we exhibit the solutions θ_m of the transcendental equations (2.13): the solution of the two equations interlace. It is apparent that the solution lie on a curve composed of two fairly regular regimes. The first regime corresponds to small m, when the asymptotic behaviour is governed by the oscillation die to ω – this is part (iii) of the theorem. In the second regime, on the right, ω is fixed while m becomes large (i.e., part (ii)) and dictates the asymptotics.*

As Fig. 3.2 shows, for small m the eigenvalues λ_m of $\mathcal{F}_\omega$ 'emerge' from the origin into the bottom-right quadrant and then asymptotically lie on a curve resembling an ellipse; part (iii) of Theorem 3.1 reveals that they depart from a circle only in the intermediate regime.

Proof Recalling the remark made at the end of the previous section, we start by rewriting the equations in (2.13) in the form

$$\cot(\theta) + \frac{\theta}{\mathrm{i}\omega} = 0 \qquad \text{and} \qquad \tan(\theta) - \frac{\theta}{\mathrm{i}\omega} = 0 \qquad (\omega > 0), \tag{3.10}$$

respectively.

Case 1: We first consider the branch corresponding to the equation

$$\cot(\theta) + \frac{\theta}{\mathrm{i}\omega} = 0 \tag{3.11}$$

in (3,10) (or (2.13)). Recall that $\theta := \sqrt{\omega^2 - 2\mathrm{i}\omega\mu}$ (where $\mu = \mu_m$). Since the roots of (3.11) corresponding to $\omega = \infty$ are given by $(m + 1/2)\pi,\ m \in \mathbb{N}$, we find that for $\omega \gg m\pi$, its roots θ_{2m+1} can be expressed in the asymptotic form

$$\theta_{2m+1} = G((m + 1/2)\pi), \quad m \in \mathbb{N}\,, \tag{3.12}$$

where for $|T| \ll \omega$ straightforward substitution of an expansion into MAPLE and comparison of coefficients results in

$$\begin{aligned} G(T) \quad &\sim \quad T + \frac{1}{\mathrm{i}\omega}T + \frac{1}{(\mathrm{i}\omega)^2}T + \frac{1}{(\mathrm{i}\omega)^3}\left(T - \frac{1}{3}T^3\right) + \frac{1}{(\mathrm{i}\omega)^4}[T - \frac{4}{3}T^3] \\ &\quad + \frac{1}{(\mathrm{i}\omega)^5}\left(T - \frac{10}{3}T^3 + \frac{1}{5}T^5\right) + \frac{1}{(\mathrm{i}\omega)^6}\left(T - \frac{20}{3}(T^3 + \frac{23}{15}T^5\right) \\ &\quad + \frac{1}{(\mathrm{i}\omega)^7}\left(T - \frac{35}{3}T^3 + \frac{98}{15}T^5 - \frac{1}{7}T^7\right) + \frac{1}{(\mathrm{i}\omega)^8}\left(T - \frac{56}{3}T^3 + \frac{308}{15}T^5 - \frac{176}{105}T^7\right) \\ &\quad + \mathcal{O}(\omega^{-9})\,. \end{aligned} \tag{3.13}$$

Case II: For the second equation in (3.10),

$$\tan(\theta) - \frac{\theta}{\mathrm{i}\omega} = 0, \tag{3.14}$$

we derive in an analogous fashion the asymptotic result ($|T| \ll \omega$)

$$\theta_{2m} = G((2m)\pi/2), \quad m \geq 1, \tag{3.15}$$

with $G(\,\cdot\,)$ as in (3.13).

Setting, for convenience,

$$F(T) := -\frac{(\mathrm{i}\omega)^2 + G(T)}{\mathrm{i}\omega},$$

we derive

$$\begin{aligned} F(T) \quad &\sim \quad \frac{1}{2}\mathrm{i}\omega - \frac{T^2}{2\mathrm{i}\omega} - \frac{T^2}{(\mathrm{i}\omega)^2} - \frac{3T^2}{2(\mathrm{i}\omega)^3} - \frac{2T^2 - T^4/3}{(\mathrm{i}\omega)^4} - \frac{5T^2/2 - 5T^4/3}{(\mathrm{i}\omega)^5} \\ &\qquad - \frac{3T^2 - 5T^4 + T^6/5}{(\mathrm{i}\omega)^6} + \mathcal{O}(\omega^{-7}). \end{aligned}$$

This allows us to obtain

$$\mu_m = F(m\pi/2), \quad m \in \mathbb{N}\,;$$

moreover,

$$\begin{aligned} \mathrm{Re}\,F(T) \quad &\sim \quad \frac{1}{\omega^2}T^2 + \frac{1}{\omega^4}[-2T^2 + T^4/3] + \frac{1}{\omega^6}[3T^2 - 5T^4 + T^6/5] + \mathcal{O}(\omega^{-8}), \\ \mathrm{Im}\,F(T) \quad &\sim \quad -\frac{1}{2}\omega + \frac{1}{2}\frac{1}{\omega}T^2 - \frac{3}{2}\frac{1}{\omega^3}T^2 + \frac{1}{\omega^5}[5T^2/2 - 5T^4/2] + \mathcal{O}(\omega^{-7}). \end{aligned}$$

The above analysis is of course valid only for $T = m\pi \ll \omega$. In this case, we deduce that all the $\{\mu_m\}$ lie on the complex curve corresponding to F and that there is precisely one Im μ_m in each interval $[a_m, b_m)$ with the endpoints

$$a_m := -\frac{1}{2}\omega + \frac{(m\pi)^2}{8\omega} \quad \text{and} \quad b_m := -\frac{1}{2}\omega + \frac{((m+1)\pi)^2}{8\omega} \quad (m \in \mathbb{N}).$$

In order to obtain the analogous result when $T = m\pi \gg \omega$, we will just consider the equation (3.11), as (3.14) admits a similar analysis. Thus, again resorting to MAPLE, we confirm the following asymptotic expansion for $\theta = \theta_{2m+1}$:

$$\begin{aligned} \theta_{2m+1} \quad &\sim \quad m\pi - \frac{\mathrm{i}\omega}{m\pi} - \frac{1}{m\pi}\left(\frac{\mathrm{i}\omega}{m\pi}\right)^2 + \left[\frac{1}{3} - \frac{2}{(m\pi)^2}\right]\left(\frac{\mathrm{i}\omega}{m\pi}\right)^3 \\ &\quad + \left[\frac{4}{3}\frac{1}{m\pi} - \frac{5}{(m\pi)^3}\right]\left(\frac{\mathrm{i}\omega}{m\pi}\right)^4 + \left[\frac{1}{5} - \frac{5}{(m\pi)^2} + \frac{14}{(m\pi)^4}\right]\left(\frac{\mathrm{i}\omega}{m\pi}\right)^5 \\ &\quad - \left[\frac{23}{15}\frac{1}{m\pi} - \frac{56}{3}\frac{1}{(m\pi)^3} + \frac{42}{(m\pi)^5}\right]\left(\frac{\mathrm{i}\omega}{m\pi}\right)^6 \\ &\quad + \left[\frac{1}{7} - \frac{392}{45}\frac{1}{(m\pi)^2} + \frac{70}{(m\pi)^4} - \frac{132}{(m\pi)^6}\right]\left(\frac{\mathrm{i}\omega}{m\pi}\right)^7 + \mathcal{O}\left(\left(\frac{\omega}{m\pi}\right)^8\right). \end{aligned}$$

It is now a matter of straightforward calculations to verify the assertions in (i)–(iii) of Theorem 3.1, recalling (see end of Section 2) that

$$\mu_m = \frac{\omega^2 - \theta_m^2}{2\mathrm{i}\omega}, \quad m \in \mathbb{N}.$$

This concludes the proof. $\square$

4 Behaviour of solutions for large ω

4.1 The result of Ursell (1969)

The asymptotic behaviour of solutions to the Fredholm integral equation (1.4) associated with the complex-symmetric Fredholm integral operator (1.1)-(1.3),

$$f(x) = \mu(\mathcal{F}_\omega f)(x) + a(x), \quad x \in I = [-1, 1], \tag{4.1}$$

as $\omega \to \infty$, was studied by Ursell ([19]) in 1969. We briefly summarise the result most relevant in the context of the present paper.

Theorem 4.1 *Suppose that the kernel $K_0(x, y)$ in (1.2) is continuous on $I \times I$, and let $g(x, y) = |x - y|$. If $a \in C(I)$ is independent of ω, then the solution $f = f(x; \omega)$ of (1.4) (with $\mu^{-1} \notin \sigma(\mathcal{F}_\omega)$) satisfies*

$$f(x; \omega) - a(x) = o(1) \quad \text{as} \ \ \omega \to \infty.$$

If K_0 and a are continuously differentiable on their respective domains, then we have

$$f(x; \omega) - a(x) = \mathcal{O}(1/\omega) \quad \text{as} \ \ \omega \to \infty. \tag{4.2}$$

These results hold uniformly in I and for $|\mu| \leq \mu^*$, where μ^* is an arbitrary fixed positive number.

Ursell's proof is based on the 'splitting' of the given Fredholm integral equation (4.1) into a pair of complementary second-kind Volterra integral equations and the analysis of the resolvent kernels underlying the representation of their solutions.

Remark 4.1 *It was shown in [19, p. 450] that the asymptotic result (4.1) is in general not true for highly oscillatory kernels of the form $K_\omega(x, y) = K_0(x, y)\cos(\omega(x - y))$. As an example, consider the integral equation*

$$f(x) = \int_{-1}^{1} \cos(\omega(x - y)) f(y) \, \mathrm{d}y + 1, \quad x \in I,$$

corresponding to a Fredholm integral operator of rank 2. Its (unique) solution is given by

$$f(x; \omega) = 1 - \frac{2\cos(\omega x)}{\cos(\omega)}.$$

4.2 More precise asymptotic results

We will now refine Ursell's asymptotic result given in Theorem 4.1 for arbitrarily smooth data K_0 and a in (1.4), by using an approach different from the one employed in his proof. Assume, without loss of generality, that $\mu^{-1} \notin \sigma(\mathcal{F}_\omega)$ is such that the Neumann series associated with the Fredholm integral equation (4.1) converges uniformly on I. For given $a \in C(I)$ the (unique) solution $f = f(x;\omega)$ of (4.1) is then given by

$$f(x;\omega) = a(x) + \mathcal{R}_\omega[a](x) := a(x) + \sum_{r=1}^{\infty} \mu^r \mathcal{F}_\omega^r[a](x), \quad x \in I, \tag{4.3}$$

where $\mathcal{F}_\omega^r := \mathcal{F}_\omega \circ \mathcal{F}_\omega^{r-1}$ (compare, e.g., [6] or [1]). For the subsequent analysis we assume that $a \in C^\infty(I)$, and we define, for $x \in I = [-1,1]$,

$$\mathcal{K}[a](x) := \mathrm{e}^{\mathrm{i}\omega(1+x)} \sum_{m=0}^{\infty} \frac{1}{(\mathrm{i}\omega)^{m+1}} a^{(m)}(-1) + \mathrm{e}^{\mathrm{i}\omega(1-x)} \sum_{m=0}^{\infty} \frac{(-1)^m}{(\mathrm{i}\omega)^{m+1}} a^{(m)}(1). \tag{4.4}$$

Proposition 4.1 *For every $r \geq 1$ it is true that*

$$\begin{aligned} \mathcal{F}_\omega^r[a](x) \quad &\sim \quad \mathcal{F}_\omega^{r-1}[\mathcal{K}[a]](x) + \sum_{l=1}^{r-1} (-2)^l \sum_{m=0}^{\infty} \frac{1}{(\mathrm{i}\omega)^{2m+1}} \binom{m+l-1}{m} \mathcal{F}_\omega^{r-1-l}[\mathcal{K}[a^{(2m)}]](x) \\ &+ (-2)^r \sum_{m=0}^{\infty} \frac{1}{(\mathrm{i}\omega)^{2m+r}} \binom{m+r-1}{m} a^{(2m)}(x). \end{aligned} \tag{4.5}$$

Proof By induction on r. (4.5) is certainly true for $r = 1$. Assuming that it is true for r, we obtain at once

$$\begin{aligned} \mathcal{F}_\omega^{r+1}[a] \quad &\sim \quad \mathcal{F}_\omega^r[\mathcal{K}[a]] + \sum_{l=1}^{r-1} (-2)^l \sum_{m=0}^{\infty} \frac{1}{(\mathrm{i}\omega)^{2m+l}} \binom{m+r-1}{m} \mathcal{F}_\omega^{r-l}[\mathcal{K}[a^{(2m)}]] \\ &+ (-2)^r \sum_{m=0}^{\infty} \frac{1}{(\mathrm{i}\omega)^{2m+r}} \binom{m+r-1}{m} \mathcal{F}_\omega[a^{(2m)}] \end{aligned}$$

However,

$$\begin{aligned} &\sum_{m=0}^{\infty} \frac{1}{(\mathrm{i}\omega)^{2m+r}} \binom{m+r-1}{m} \mathcal{F}_\omega[a^{(2m)}] \\ \sim \quad &\sum_{m=0}^{\infty} \frac{1}{(\mathrm{i}\omega)^{2m+r}} \binom{m+r-1}{m} \mathcal{K}[a] \\ &- 2 \sum_{m=0}^{\infty} \frac{1}{(\mathrm{i}\omega)^{2m+r}} \binom{m+r-1}{m} \sum_{n=0}^{\infty} \frac{1}{(\mathrm{i}\omega)^{2n+1}} a^{(2(m+n))}(y) \\ \sim \quad &\frac{1}{(\mathrm{i}\omega)^{2m+r}} \binom{m+r-1}{m} \mathcal{K}[a] - 2 \sum_{m=0}^{\infty} \frac{1}{(\mathrm{i}\omega)^{2m+r+1}} d_{r,m} a^{(2m)}(y), \end{aligned}$$

where

$$d_{r,m} = \sum_{n=0}^{m} \binom{n+r-1}{n}.$$

It is trivial, though, to prove by induction that $d_{r,m} = \binom{m+r}{m}$, since $d_{r,m} = d_{r-1,m} + d_{r,m-1}$. Substitution into the asymptotic expansion completes the proof. □

We now determine the asymptotic expansions of the terms $\mathcal{F}_\omega^{r-1-l}[\mathcal{K}[a]]$ in (4.5). Define the linear operator

$$\mathcal{M}_r := \mathcal{F}_\omega^r[\mathcal{K}[a]](x), \quad x \in I, \quad r \geq 0.$$

Since $\mathcal{F}_\omega$ is a linear operator, $\mathcal{M}_r$ is linear; moreover, for $r=0$ we have

$$\mathcal{M}_0 \sim \mathrm{e}^{\mathrm{i}\omega(1+x)} \sum_{m=0}^{\infty} \frac{1}{(\mathrm{i}\omega)^{m+1}} a^{(m)}(-1) + \mathrm{e}^{\mathrm{i}\omega(1-x)} \sum_{m=0}^{\infty} \frac{(-1)^m}{(\mathrm{i}\omega)^{m+1}} a^{(m)}(1).$$

The result in Proposition 4.2 then follows by induction.

Proposition 4.2 *It is true that*

$$\mathcal{M}_r[a](x) \sim \mathcal{F}_\omega^r[\mathrm{e}^{\mathrm{i}\omega(1+x)}] \sum_{m=0}^{\infty} \frac{1}{(\mathrm{i}\omega)^{m+1}} a^{(m)}(-1) + \mathcal{F}_\omega^r[\mathrm{e}^{\mathrm{i}\omega(1-x)}] \sum_{m=0}^{\infty} \frac{(-1)^m}{(\mathrm{i}\omega)^{m+1}} a^{(m)}(1). \quad (4.6)$$

We thus need to investigate $\mathcal{F}_\omega^r[\mathrm{e}^{\mathrm{i}\omega(1\pm x)}]$ for $r \geq 1$. Our contention is that

$$\mathcal{F}_\omega^r[\mathrm{e}^{\mathrm{i}\omega(1\pm x)}] = \sum_{m=0}^{r} \frac{1}{(\mathrm{i}\omega)^m} \sum_{k=0}^{m} (-1)^{m-k} \frac{\beta_{r,m,k}}{(r-m)!} [(2k+1) \pm (-1)^k x]^{r-m} \mathrm{e}^{\mathrm{i}\omega[(2k+1)\pm(-1)^k x]}, \quad (4.7)$$

with appropriate coefficients $\{\beta_{r,m,k}\}$ given below.

The proof is based on an elementary (but somewhat messy) induction argument, starting by setting $\beta_{0,0,0} = 1$ and by observing that (4.7) is certainly true for $r = 0$.

Long, although fairly straightforward, inductive argument demonstrates that for all $\alpha \in \mathbb{C}$ and $s \in \mathbb{Z}_+$

$$\begin{aligned} \mathcal{K}_\omega[(\alpha+y)^s \mathrm{e}^{\mathrm{i}\omega(\alpha+y)}] &= \frac{1}{s+1}[(\alpha+y)^{s+1} - (\alpha-1)^{s+1}] \mathrm{e}^{\mathrm{i}\omega(\alpha+y)} \\ &\quad + \sum_{k=0}^{s} \frac{(-1)^k}{(2\mathrm{i}\omega)^{k+1}} \frac{s!}{(s-k)!} [(\alpha+1)^{s-k} \mathrm{e}^{\mathrm{i}\omega(2+\alpha-y)} - (\alpha+y)^{s-k} \mathrm{e}^{\mathrm{i}\omega(\alpha+y)}] \end{aligned} \quad (4.8)$$

and

$$\begin{aligned} \mathcal{K}_\omega[(\alpha-y)^s \mathrm{e}^{\mathrm{i}\omega(\alpha-y)}] &= \frac{1}{s+1}[(\alpha-y)^{s+1} - (\alpha-1)^{s+1}] \mathrm{e}^{\mathrm{i}\omega(\alpha-y)} \\ &\quad + \sum_{k=0}^{s} \frac{(-1)^k}{(2\mathrm{i}\omega)^{k+1}} \frac{s!}{(s-k)!} [(\alpha+1)^{s-k} \mathrm{e}^{\mathrm{i}\omega(2+\alpha+y)} - (\alpha-y)^{s-k} \mathrm{e}^{\mathrm{i}\omega(\alpha-y)}]. \end{aligned}$$

Letting $\beta_{0,0,0} = 1$, this is certainly true for $r = 0$. By induction, using (4.8),

$$\begin{aligned} &\mathcal{K}_\omega^{r+1}[\mathrm{e}^{\mathrm{i}\omega(1+y)}] \\ &= \sum_{m=0}^{r} \frac{1}{(\mathrm{i}\omega)^m} \sum_{k=0}^{m} (-1)^{m-k} \frac{\beta_{r,m,k}}{(r-m)!} \mathcal{K}_\omega[[(2k+1) + (-1)^k y]^{r-m} \mathrm{e}^{\mathrm{i}\omega[(2k+1)+(-1)^k y]}] \\ &= \sum_{m=0}^{r} \frac{1}{(\mathrm{i}\omega)^m} \sum_{k=0}^{m} (-1)^{m-k} \frac{\beta_{r,m,k}}{(r-m)!} \\ &\quad \times \left(\frac{1}{r-m+1} \{[(2k+1) + (-1)^k y]^{r+1-m} - (2k)^{r+1-m}\} \mathrm{e}^{\mathrm{i}\omega[(2k+1)+(-1)^k y]} \right. \\ &\quad + \sum_{j=0}^{r-m} \frac{(-1)^j}{(2\mathrm{i}\omega)^{j+1}} \frac{(r-m)!}{(r-m-j)!} \Big\{ (2k+2)^{r-m-j} \mathrm{e}^{\mathrm{i}\omega[(2k+3)-(-1)^k y]} \\ &\qquad \left. - [(2k+1) + (-1)^k y]^{r-m-j} \mathrm{e}^{\mathrm{i}\omega[(2k+1)+(-1)^k y]} \Big\} \right) \\ &= \sum_{m=0}^{r} \frac{1}{(\mathrm{i}\omega)^m} \sum_{k=0}^{m} (-1)^{m-k} \frac{\beta_{r,m,k}}{(r-m+1)!} [(2k+1) + (-1)^k y]^{r+1-m} \mathrm{e}^{\mathrm{i}\omega[(2k+1)+(-1)^k y]} \end{aligned}$$

$$
\begin{aligned}
&- \sum_{m=0}^{r} \frac{1}{(\mathrm{i}\omega)^m} \sum_{k=1}^{m} (-1)^{m-k} \frac{\beta_{r,m,k}}{(r-m+1)!} (2k)^{r+1-m} \mathrm{e}^{\mathrm{i}\omega[(2k+1)+(-1)^k y]} \\
&+ \sum_{m=1}^{r+1} \frac{1}{(\mathrm{i}\omega)^m} \sum_{k=1}^{m} (-1)^{m-k} (2k)^{r+1-m} \sum_{j=0}^{m-k} \frac{1}{2^{j+1}} \frac{\beta_{r,m-j-1,k-1}}{(r-m+1)!} \mathrm{e}^{\mathrm{i}\omega[(2k+1)+(-1)^k y]} \\
&+ \sum_{m=1}^{r+1} \frac{1}{(\mathrm{i}\omega)^m} \sum_{k=0}^{m-1} (-1)^{m-k} \sum_{j=0}^{m-k-1} \frac{1}{2^{j+1}} \frac{\beta_{r,m-j-1,k}}{(r-m+1)!} [(2k+1)+(-1)^k y]^{r+1-m} \\
&\qquad \times \mathrm{e}^{\mathrm{i}\omega[(2k+1)+(-1)^k y]}.
\end{aligned}
$$

Now, let

$$
\beta_{r,m,k} = \sum_{j=1}^{m-k+1} \frac{1}{2^j} \beta_{r,m-j,k-1}, \qquad m = 0, \ldots, r, \quad k = 0, \ldots, m. \tag{4.9}
$$

Then the second and the third sums cancel, except for the $m = r+1$ term in the third sum. Thus,

$$
\beta_{r+1,m,k} = \sum_{j=0}^{m-k} \frac{1}{2^j} \beta_{r,m-j,k}, \qquad m = 1, \ldots, r \quad k = 0, \ldots, m
$$

and

$$
\beta_{r+1,r+1,k} = \sum_{j=1}^{r+2-k} \frac{1}{2^j} \beta_{r,r-j+1,k-1} + \sum_{j=1}^{r+1-k} \frac{1}{2^j} \beta_{r,r-j+1,k}, \qquad k = 0, \ldots, r+1.
$$

(We let $\beta_{r,m,k} = 0$ outside the range $0 \le m \le r$, $0 \le k \le m$.) However, substituting (4.9), we observe that

$$
\begin{aligned}
\beta_{r+1,m,k} &= \beta_{r,m,k} + \beta_{r,m,k+1}, \qquad m = 0, \ldots, r, \quad k = 0, \ldots, m, & (4.10) \\
\beta_{r+1,r+1,k} &= \frac{1}{2}(\beta_{r,r,k-1} + 2\beta_{r,r,k} + \beta_{r,r,k+1}), \qquad k = 0, \ldots, r+1. & (4.11)
\end{aligned}
$$

However, once we *assume* the last two equations, (4.9) follows by induction on r: suppose that it is true for r. Then, using (4.10) twice,

$$
\begin{aligned}
\sum_{j=1}^{m-k+1} \frac{1}{2^j} \beta_{r+1,m-j,k-1} &= \sum_{j=1}^{m-k+1} \frac{1}{2^j} \beta_{r,m-j,k-1} + \sum_{j=1}^{m-k} \frac{1}{2^j} \beta_{r,m-j,k} \\
&= \beta_{r,m-j,k} + \beta_{r,m-j,k+1} = \beta_{r+1,m,k}.
\end{aligned}
$$

Likewise, it follows from (4.10) and (4.11) that

$$
\begin{aligned}
\sum_{j=1}^{r-k+2} \frac{1}{2^j} \beta_{r+1,r+1-j,k-1} &= \sum_{j=1}^{r-k+2} \frac{1}{2^j} \beta_{r,r+1-j,k-1} + \sum_{j=1}^{r-k+1} \frac{1}{2^j} \beta_{r,r+1-j,k} \\
&= \frac{1}{2}\beta_{r,r,k-1} + \frac{1}{2} \sum_{j=1}^{r-k+1} \beta_{r,r-j,k-1} + \frac{1}{2}\beta_{r,r,k} + \frac{1}{2} \sum_{j=1}^{r-k} \beta_{r,r-j,k} \\
&= \frac{1}{2}\beta_{r,r,k-1} + \beta_{r,r,k} + \frac{1}{2}\beta_{r,r,k+1} = \beta_{r+1,r+1,k}.
\end{aligned}
$$

Therefore, (4.9) is equivalent to (4.10–11).

It would have been pleasing to identify the coefficients $\beta_{r,m,k}$ explicitly. This, however, seems to be hopeless. The first few terms are

$$
\beta_{r,r,r} = \frac{1}{2^r} = \frac{1}{2^r} \binom{r}{0},
$$

$$\begin{aligned}
\beta_{r,r,r-1} &= \frac{2r}{2^r} = \frac{1}{2^r}\left[2\binom{r-1}{1} + 2\binom{r-1}{0}\right], \\
\beta_{r,r,r-2} &= \frac{(r-1)(2r+1)}{2^r} = \frac{1}{2^r}\left[4\binom{r-2}{2} + 9\binom{r-2}{1} + 5\binom{r-2}{0}\right], \\
\beta_{r,r,r-3} &= \frac{2}{3}\frac{(r-2)r(2r+1)}{2^r} \\
&= \frac{1}{2^r}\left[8\binom{r-3}{3} + 28\binom{r-3}{2} + 34\binom{r-3}{1} + 14\binom{r-3}{0}\right]
\end{aligned}$$

and complexity grows rapidly.

Alternatively, we let

$$B_r(x,y) = \sum_{m=0}^{r}\sum_{k=0}^{m} \beta_{r,m,k} x^{r-m} y^k, \qquad r \in \mathbb{Z}_+.$$

Then

$$\begin{aligned}
B_{r+1}(x,y) &= \sum_{m=0}^{r+1}\sum_{k=0}^{m} \beta_{r+1,m,k} x^{r+1-m} y^k \\
&= \sum_{k=0}^{r+1} \beta_{r+1,r+1,k} y^k + \sum_{m=0}^{r}\sum_{k=0}^{m} \beta_{r+1,m,k} x^{r+1-m} y^k \\
&= \sum_{k=0}^{r+1} (\frac{1}{2}\beta_{r,r,k-1} + \beta_{r,r,k} + \frac{1}{2}\beta_{r,r,k=1}) y^k \\
&\quad + \sum_{m=0}^{r}\sum_{k=0}^{m} (\beta_{r,m,k} + \beta_{r,m,k+1}) x^{r+1-m} y^k \\
&= \frac{1}{2}\left(\frac{1}{y} + 2 + y\right) B_r(0,y) - \frac{1}{2y} B_r(0,0) + x\left(1 + \frac{1}{y}\right) B_r(x,y) - \frac{x}{y} B_r(x,0) \\
&= \frac{1}{y}[x(1+y)B_r(x,y) + \frac{1}{2}(1+y)^2 B_r(0,y) - xB_r(x,0) - \frac{1}{2}B_r(0,0)].
\end{aligned}$$

Now set

$$\mathcal{B}(t,x,y) = \sum_{r=0}^{\infty} \frac{1}{r!} B_r(x,y) t^r.$$

Then

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{B}(t,x,y) &= \sum_{r=0}^{\infty} \frac{1}{r!} B_{r+1}(x,y) t^r \\
&= \frac{1}{y}\left[x(1+y)\mathcal{B}(t,x,y) + \frac{1}{2}(1+y)^2\mathcal{B}(t,0,y) - x\mathcal{B}(t,x,0) - \frac{1}{2}\mathcal{B}(t,0,0)\right].
\end{aligned} \tag{4.12}$$

Unfortunately, the explicit solution of the functional equation (4.12) is unknown. Thus, the present state of knowledge is as follows: the formula (4.7) is true and the coefficients $\beta_{r,m,k}$ can be obtained by the recursion (4.9), alternatively by the (easier) recursions (4.10) and (4.11). Yet, the explicit form of the $\beta_{r,m,k}$s is unknown.

4.3 Assembling the Neumann expansion

Because of the symmetry inherent in (4.8), we deduce from (4.7) that

$$\mathcal{K}_\omega^r[\mathrm{e}^{\mathrm{i}\omega(1-y)}] = \sum_{m=0}^{r} \frac{1}{(\mathrm{i}\omega)^m} \sum_{k=0}^{m} (-1)^{m-k} \frac{\beta_{r,m,k}}{(r-m)!} [(2k+1) - (-1)^k y]^{r-m} \mathrm{e}^{\mathrm{i}\omega[(2k+1)-(-1)^k y]}, \tag{4.13}$$

with the same coefficients $\beta_{r,m,k}$ as before.

We compute

$$\begin{aligned}
&\sum_{r=0}^{\infty}\mu^r\mathcal{K}_\omega^r[\mathrm{e}^{\mathrm{i}\omega(1+y)}]\\
=\ &\sum_{r=0}^{\infty}\mu^r\sum_{m=0}^{r}\frac{1}{(\mathrm{i}\omega)^m}\sum_{k=0}^{m}\frac{(-1)^{m-k}}{(r-m)!}\beta_{r,m,k}[(2k+1)+(-1)^k y]^{r-m}\mathrm{e}^{\mathrm{i}\omega[(2k+1)+(-1)^k y]}\\
=\ &\sum_{m=0}^{\infty}\frac{\mu^m}{(\mathrm{i}\omega)^m}\sum_{k=0}^{m}(-1)^{m-k}\left\{\sum_{r=0}^{\infty}\frac{\mu^r}{r!}\beta_{r+m,m,k}[(2k+1)+(-1)^k y]^r\right\}\mathrm{e}^{\mathrm{i}\omega[(2k+1)+(-1)^k y]}\\
=\ &\sum_{m=0}^{\infty}\frac{\mu^m}{(\mathrm{i}\omega)^m}\sum_{k=0}^{m}(-1)^{m-k}U_{m,k}(\mu[(2k+1)+(-1)^k y])\mathrm{e}^{\mathrm{i}\omega[(2k+1)+(-1)^k y]},
\end{aligned}$$

where

$$U_{m,k}(t)=\sum_{r=0}^{\infty}\frac{\beta_{r+m,m,k}}{r!}t^r.$$

However,

$$\begin{aligned}
U_{m,k}(t) &= \beta_{m,m,k}+\sum_{r=1}^{\infty}\frac{1}{r!}(\beta_{r+m-1,m,k}+\beta_{r+m-1,m,k+1})t^r\\
&= \beta_{m,m,k}+\sum_{r=0}^{\infty}\frac{1}{(r+1)!}\beta_{r+m,m,k}t^{r+1}+\sum_{r=0}^{\infty}\frac{1}{(r+1)!}\beta_{r+m,m,k+1}t^{r+1}\\
&= \beta_{m,m,k}+\int_0^t U_{m,k}(\tau)\,\mathrm{d}\tau+\int_0^t U_{m,k+1}(\tau)\,\mathrm{d}\tau.
\end{aligned}$$

Differentiating, we obtain the ODE

$$U'_{m,k}(t)=U_{m,k}(t)+U_{m,k+1}(t),$$

with the solution

$$U_{m,k}(t)=\mathrm{e}^t U_{m,k}(0)+\int_0^t \mathrm{e}^{t-\tau}U_{m,k+1}(\tau)\,\mathrm{d}\tau.$$

For $k=m$ we have $\beta_{r,m,m}=1/2^m$, therefore

$$U_{m,m}(t)=\frac{1}{2^m}\mathrm{e}^t,$$

hence $U_{m,m-1}(0)=\beta_{m,m,m-1}=m/2^{m-1}$ implies that

$$U_{m,m-1}(t)=\frac{t+2m}{2^m}\mathrm{e}^t.$$

Likewise, $\beta_{m,m,m-2}=(m-1)(2m+1)/2^m$ yields

$$U_{m,m-2}(t)=\frac{1}{2^m}\left[\frac{1}{2}t^2+2mt+(m-1)(2m+1)\right]\mathrm{e}^t,$$

while $\beta_{m,m,m-3}=\frac{2}{3}(m-2)m(2m+1)/2^m$ results in

$$U_{m,m-3}(t)=\frac{1}{2^m}[\frac{1}{6}t^3+mt^2+(m-1)(2m+1)t+\frac{2}{3}(m-2)m(2m+1)]\mathrm{e}^t.$$

In general, trivial induction confirms that

$$U_{m,m-s}(t)=\frac{1}{2^m}p_{m,s}(t)\mathrm{e}^t,$$

where $p_{m,0}(t) \equiv 1$ and

$$p_{m,s+1}(t) = p_{m,s+1}(0) + \int_0^t p_{m,s}(\tau)\,\mathrm{d}\tau \in \mathbb{P}_{s+1}[t].$$

Of course, $q_{m,s} = p_{m,s}(0) = 2^m \beta_{m,m,m-s}$.

Using the recurrence, we obtain

$$q_{m+1,s} = q_{m,s} + 2q_{m,s-1} + q_{m,s-2}, \tag{4.14}$$

and this results in

$$\begin{aligned} q_{m,0} &\equiv 1, \\ q_{m,1} &= 2m, \\ q_{m,2} &= (m-1)(2m+1), \\ q_{m,3} &= \frac{2}{3}(m-2)m(2m+1), \\ q_{m,4} &= \frac{1}{6}(m-3)m(2m-1)(2m+1), \\ q_{m,5} &= \frac{1}{15}(m-4)(m-1)m(2m-1)(2m+1). \end{aligned}$$

Proposition 4.3 *It is true that*

$$q_{m,s} = \frac{m-s+1}{m+2}\binom{2m+1}{s}, \qquad s = 0,\ldots,m, \quad s \in \mathbb{Z}_+. \tag{4.15}$$

Proof Follows at once by induction from (4.14), because $\beta_{m,0} = 1$. □

Proposition 4.4 *Each $p_{m,s}$ is a degree-s polynomial, given by the recurrence formula*

$$\begin{aligned} p_{m,0}(t) &\equiv 1, \\ p_{m,s}(t) &= \frac{m+1-s}{m+1}\binom{2m+2}{s} + \int_0^t p_{m,s-1}(\tau)\,\mathrm{d}\tau, \qquad s = 1,2,\ldots,m, \end{aligned}$$

therefore

$$p_{m,s}(t) = \sum_{k=0}^{s} \frac{m+1+k-s}{m+1}\binom{2m+2}{s-k}\frac{t^k}{k!}. \tag{4.16}$$

Proof The assertion (4.16) can be easily proved by induction. □

We thus deduce that

$$\begin{aligned} &\sum_{r=0}^{\infty} \mu^r \mathcal{K}_\omega^r[\mathrm{e}^{\mathrm{i}\omega(1\pm y)}] \\ \sim\ &\sum_{m=0}^{\infty} \frac{\mu^m}{(2\mathrm{i}\omega)^m} \sum_{k=0}^{m} (-1)^{m-k} p_{m,m-k}(\mu[(2k+1) \pm (-1)^k y])\mathrm{e}^{(\mathrm{i}\omega+\mu)[(2k+1)\pm(-1)^k y]}. \end{aligned}$$

Now, let us go back to (4.5). We have

$$\begin{aligned} f(y,\omega) &= \sum_{r=0}^{\infty} \mu^r \mathcal{K}_\omega^r[a] \\ &\sim a + \mu\sum_{r=0}^{\infty}\mu^r\mathcal{M}_r[a] + \mu\sum_{m=0}^{\infty}\frac{1}{(\mathrm{i}\omega)^{2m}}\sum_{l=1}^{\infty}\frac{(-2\mu)^l}{(\mathrm{i}\omega)^l}\binom{m+l-1}{m}\sum_{r=0}^{\infty}\mu^r\mathcal{M}_r[a^{(2m)}] \\ &\quad + \sum_{m=0}^{\infty}\frac{1}{(\mathrm{i}\omega)^{2m}}a^{(2m)}(y)\sum_{r=1}^{\infty}\frac{(-2\mu)^r}{(\mathrm{i}\omega)^r}\binom{m+r-1}{m}. \end{aligned}$$

although it is easy to prove that the last term equals

$$-2\mu\sum_{m=0}^{\infty}\frac{1}{(\mathrm{i}\omega)^{m+1}}\frac{a^{(2m)}(y)}{(\mathrm{i}\omega+\mu)^m},$$

we prefer to leave it as an unsummed series.

To render everything in easier notation, let

$$c_k(y)=(2k+1)+(-1)^k y.$$

Therefore,

$$\begin{aligned}V(y) &= \sum_{r=0}^{\infty}\mu^r\mathcal{K}_\omega^r[\mathrm{e}^{\mathrm{i}\omega(1+y)}]\\ &\sim \sum_{m=0}^{\infty}\frac{\mu^m}{(2\mathrm{i}\omega)^m}\sum_{k=0}^{m}(-1)^{m-k}p_{m,m-k}(\mu c_k(y))\mathrm{e}^{(\mathrm{i}\omega+\mu)c_k(y)}\end{aligned}$$

and, using (4.4).

$$\begin{aligned}&\sum_{r=0}^{\infty}\mu^r\mathcal{M}_r[b]\\ \sim\ & V(y)\sum_{n=0}^{\infty}\frac{1}{(\mathrm{i}\omega)^{n+1}}b^{(n)}(-1)+V(-y)\sum_{n=0}^{\infty}\frac{1}{(\mathrm{i}\omega)^{n+1}}(-1)^n b^{(n)}(1).\end{aligned}$$

We have at present all the ingredients to construct $f_r(y,\omega)$ such that

$$f_r(y,\omega)\sim f(y,\omega)+O(\omega^{-r-1}).$$

The steps are as follows:

1. Use (4.16) to produce $p_{m,s}$ for $0\le s\le m\le r$.
2. Form
$$V_r(y)=\sum_{m=0}^{r}\frac{\mu^m}{(2\mathrm{i}\omega)^m}\sum_{k=0}^{m}(-1)^{m-k}p_{m,m-k}(\mu c_k(y))\mathrm{e}^{(\mathrm{i}\omega+\mu)c_k(y)}.$$
3. Form
$$V_r(y)\sum_{n=0}^{r-1}\frac{1}{(\mathrm{i}\omega)^{n+1}}a^{(n)}(-1)-V_r(-y)\sum_{n=0}^{r-1}\frac{1}{(-\mathrm{i}\omega)^{n+1}}a^{(n)}(1)$$
and truncate ω^{-j} terms for $j\ge r+1$. This yields $W_{0,r}(y)$.
4. For every $m=1,\ldots,\lfloor r/2\rfloor$ form
$$V_r(y)\sum_{n=0}^{r-1}\frac{1}{(\mathrm{i}\omega)^{n+1}}a^{(n+2m)}(-1)-V_r(-y)\sum_{n=0}^{r-1}\frac{1}{(-\mathrm{i}\omega)^{n+1}}a^{(n+2m)}(1)$$
and truncate ω^{-j} terms for $j\ge r+1$. This results in $W_{m,r}(y)$.
5. Truncate ω^{-j} terms for $j\ge r+1$ in
$$\sum_{m=0}^{\lfloor r/2\rfloor}\sum_{l=1}^{r-2m}\frac{(-2\mu)^l}{(\mathrm{i}\omega)^{2m+l}}\binom{m+l-1}{m}W_{m,r}(y)$$
to obtain $Y_r(y)$.

6. Form
$$X_r(y) = \sum_{m=0}^{\lfloor r/2\rfloor}\sum_{l=1}^{r-2m} \frac{1}{(\mathrm{i}\omega)^{2m+l}}(-2\mu)^l \binom{m+l-1}{m} a^{(2m)}(y).$$

7. Add all the ingredients, whereby
$$f_r(y,\omega) = a(y) + \mu W_{0,r}(y) + \mu Y_r(y) + X_r(y).$$

To recap, we have proved in this section

Theorem 4.2 *Assume that $a \in C^\infty(I)$. If $\mu^{-1} \notin \sigma(\mathcal{F}_\omega)$, then the (unique) solution of the second-kind Fredholm integral equation (4.1) has, for $\omega \gg 1$, the expansion*

$$\begin{aligned} f(x;\omega) \quad\sim\quad & a(x) + \mu\sum_{r=0}^{\infty}\mu^r\mathcal{M}_r[a](x) \\ & + \mu\sum_{m=0}^{\infty}\frac{1}{(\mathrm{i}\omega)^{2m}}\sum_{l=1}^{\infty}\frac{(-2\mu)^l}{(\mathrm{i}\omega)^l}\binom{m+l-1}{m}\sum_{r=0}^{\infty}\mu^r\mathcal{M}_r[a^{(2m)}](x) \\ & + \sum_{m=0}^{\infty}\frac{1}{(\mathrm{i}\omega)^{2m}}a^{(2m)}(x)\sum_{r=1}^{\infty}\frac{(-2\mu)^r}{(\mathrm{i}\omega)^r}\binom{m+r-1}{m}. \end{aligned} \tag{4.17}$$

We have also expanded $f(x,\omega)$ in Neumann series in an explicit form.

5 From asymptotics to numerics

So far, this paper was mostly about asymptotics, except that we have already revealed our hand in the introduction: in a highly oscillatory setting asymptotics often provides the right avenue for effective numerics.

There are two numerical challenges commonly associated with Fredholm operators of the second kind. Firstly, the calculation of the spectrum of $\mathcal{F}_\omega$ and, secondly, the solution of the equation (4.1). Insofar as the kernel $K_\omega(x,y) = \mathrm{e}^{\mathrm{i}\omega|x-y|}$ is concerned, the spectral problem has been solved completely (up to the solution of the zeros of the transcendental equations (2.13)) in Section 2. We defer the discussion of more general spectral problems to the next section.

Insofar as the solution of the Fredholm equation of the second kind (4.1) is concerned, we have implicitly introduced in this paper (again, in the case $K_0(x,y) \equiv 1$) two methods. Firstly, let λ_m be the eigenvalues of $\mathcal{F}_\omega$ and $\tilde{\phi}_m$ the corresponding eigenfunctions, normalised so that $\|\tilde{\phi}_m\|_{L^2[-1,1]} = 1$, $m = 1,2,\ldots$. Note that these can be obtained from (2.5) and (2.9), once we have solved the transcendental equations (2.13). Note further that the countable set $\{\tilde{\phi}_m\}_{m\geq 1}$ is dense in $L^2[-1,1]$ [18]. Therefore we can expand $a \in L^2[-1,1]$ in the eigenfunctions,

$$a(x) = \sum_{m=1}^{\infty}\hat{a}_m\tilde{\phi}_m,$$

where

$$\hat{a}_m = \int_{-1}^{1} a(y)\tilde{\phi}_m(y)\,\mathrm{d}y, \qquad m \in \mathbb{N}. \tag{5.1}$$

Substituting $f(y) = \sum_{m=1}^{\infty}\hat{f}_m\tilde{\phi}_m$ into (4.1), bearing in mind that $\mathcal{F}_\omega[\tilde{\phi}_m] = \lambda_m\tilde{\phi}_m$ and that $\mu\lambda_m \neq 1$, $m \in \mathbb{N}$, we easily derive

$$\hat{f}_m = \frac{\hat{a}_m}{1-\mu\lambda_m}, \qquad m \in \mathbb{N}. \tag{5.2}$$

The implementation of this spectral algorithm proceeds in three steps:

1. Compute the least M solutions of the transcendental equations (2.13). Note that the proof of Theorem 3.1 provides excellent initial approximations for λ_m when m is large(ish): in our experience, it is possible to obtain the λ_ms to machine precision in a very small number of Newton–Raphson iterations. Once the solutions ν_m are available, we compute λ_m and $\tilde{\phi}_m$ from (2.5) and (2.9) respectively.

2. We compute $\hat{a}_m$, $m = 1, \ldots, M$. Note that

$$\tilde{\phi}_m(x) = \alpha_m \mathrm{e}^{\mathrm{i}\nu_m x} + \beta_m \mathrm{e}^{-\mathrm{i}\nu_m x}, \qquad m \in \mathbb{N},$$

   for some constants α_m and β_m. Therefore, (5.1) reduces (for large values of m) to the calculation of highly oscillatory integrals with Fourier oscillators, a task that can be accomplished very rapidly by the methods of [10, 12, 16]. Overall, the cost of this calculation scales like $O(M)$.

3. Once the expansion coefficients of a are available, we evaluate $\hat{f}_m$ using (5.2) and form the solution f as their linear combination.

An alternative to this spectral method is to use directly the Neumann expansion (4.17): note that all its constituents can be calculated using the material of Section 4. In the present setting this approach is inferior to the spectral method, yet it is worth mentioning for the following reason. Neumann expansions are typically considered a method of last resort, with exceedingly poor convergence [1, 6]. However, *high oscillation makes Neumann expansions converge faster!* The reason is that the amplitude of the iterated integrals $\mathcal{F}_\omega^r$ is decreasing with r because of high oscillation. It is clear from (4.17) that the convergence of the Neumann expansion is governed by the size of the operators $\mathcal{M}_r[a]$. We can now use (4.6–7) to demonstrate that the $\mathcal{M}_r[a]$s are small for large r and that they are becoming smaller when ω increases. The more rapid the oscillation, the faster the convergence of Neumann series!

6 Outlook: More general complex-symmetric integral operators

6.1 General oscillators $g(x, y)$ with stationary points

The results on the asymptotic spectral properties of Section 3 are curently being extended to highly oscillatory Fredholm integral operators (1.1–2) with more general oscillators like the ones described in (1.5) and (1.6). (Compare also [5].)

6.2 Fredholm integral operators with weakly singular kernels

Do the qualitative spectral properties of Section 3 remain valid if the Fredholm integral operator $\mathcal{F}_\omega$ is replaced by the weakly singular operator

$$(\mathcal{F}_{\omega,\alpha} f)(x) := \int_{-1}^{1} K_{\omega,\alpha}(x, y) f(y) \, \mathrm{d}y, \quad x \in I, \tag{6.1}$$

whose kernel now has the form

$$K_{\omega,\alpha}(x, y) := K_0(x, y)|x - y|^{-\alpha} \mathrm{e}^{\mathrm{i}\omega|x-y|}, \quad \alpha \in (0, 1) \text{ ?}$$

Preliminary results indicate that weak singularity plays role fairly similar to stationary points in classical asymptotic analysis of highly oscillatory integrals.

6.3 The finite section method

We already have an extensive body of results of the computation of fairly general highly oscillatory spectra using the finite section method with appropriate conditioning. This includes the Li–Fox oscillator (1.5) and other Fredholm operators of relevance in applications. It will feature in future papers.

6.4 The Neumann method for highly oscillatory Fredholm operators

The results of Section 4 are unlikely to be replicated in an equally comprehensive manner for more general Fredholm operators. Having said so, their main thrust, namely that the Neumann method is highly efficient in the presence of oscillation, is an observation of great generality and, we believe, importance. Here the challenge is in the computation of multivariate highly oscillatory integrals. Although there exist significant numerical theory and powerful algorithms for multivariate highly oscillatory integrals with various kernels [11, 13, 17], the challenge is to identify the salient feature of such 'Neumann integrals', in particular the structure of their stationary points. This, as our experience indicates, is a nontrivial issue. In particular, Abel-type kernels $K_\omega(x,y) = K_0(x,y)\mathrm{e}^{\mathrm{i}\omega g(x-y)}$ are likely to have a continuum of stationary points along the line $x = y$ and this calls for further developments in the numerical theory of multivariate highly oscillatory integrals.

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References

[1] K.E. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge University Press, Cambridge, 1997.

[2] M. Berry, Fractal modes of unstable lasers with polygonal and circular mirrors, *Optics Comm.* **200** (2001), 321–330.

[3] M. Berry, Mode degeneracies and the Petermann excess-noise factor for unstable lasers, *J. Modern Optics* **50** (2003), 63–81.

[4] M. Berry, C. Strom and W. van Saarloos, Theory of unstable laser modes: edge waves and fractality, *Optics Comm.* **197** (2001), 393–402.

[5] H. Brunner, A. Iserles and S.P. Nørsett, Modified Fourier expansions and spectral problems for highly oscillatory Fredholm operators, in preparation (see also: `www.newton.cam.ac.uk/programmes/HOP/hopw05`).

[6] J.A. Cochran, *Analysis of Linear Integral Equations*, McGraw-Hill, New York, 1972.

[7] J.A. Cochran and E.W. Hinds, Eigensystems associated with the complex-symmetric kernels of laser theory, *SIAM J. Appl. Math.* **26** (1974), 776–786.

[8] A.G. Fox and T. Li, Resonant modes in a maser interferometer, *Bell Systems Tech. J.* **40** (1961), 453–488.

[9] H. Hochstadt, On the eigenvalues of a class of integral equations arising in laser theory, *SIAM Review* **8** (1966), 62–65.

[10] D. Huybrechs and S. Vandewalle, On the evaluation of highly oscillatory integrals by analytic continuation, *SIAM J. Num. Anal.* **44** (2006), 1026–1048.

[11] D. Huybrechs and S. Vandewalle, The construction of cubature rules for multivariate highly oscillatory integrals, *Maths Comp.* **76** (2007), 1955–1980.

[12] A. Iserles and S. P. Nørsett, Efficient quadrature of highly oscillatory integrals using derivatives, *Proc. Royal Soc. A* **461** (2004), 755–772.

[13] A. Iserles and S.P. Nørsett, Quadrature methods for multivariate highly oscillatory integrals using derivatives, *Maths Comp.* **75** (2006), 1233–1258.

[14] H.J. Landau, On Szegő's eigenvalue distribution theorem and non-Hermitian kernels, *J. Analyse Math.* **28** (1975), 335–357.

[15] H.J. Landau, The notion of approximate eigenvalues applied to an integral equation of laser theory, *Quart. Appl. Math.* **35** (1977/78), 165–172.

[16] S. Olver, Moment-free numerical integration of highly oscillatory functions, *IMA J. Num. Anal.* **26** (2006), 213–227.

[17] S. Olver, On the quadrature of multivariate highly oscillatory integrals over non-polytope domains, *Numer. Math.* **103** (2006), 643–665.

[18] W. Rudin, *Functional Analysis* (Second Edition), McGraw-Hill, New York, 1991.

[19] F. Ursell, Integral equations with a rapidly oscillating kernel, *J. London Math. Soc.* **44** (1969), 449–459.