On modified asymptotic series involving confluent hypergeometric functions

Alfredo Deaño
DAMTP, Centre for Mathematical Sciences,
University of Cambridge,
Wilberforce Road, CB3 0WA, UK.

Nico M. Temme
CWI, P.O. Box 94079, 1090 GB Amsterdam, The Netherlands.
e-mail: ad495@cam.ac.uk, nicot@cwi.nl

July 31, 2008

Abstract

A modification of standard Poincaré asymptotic expansions for functions defined by means of Laplace transforms is analyzed. This modification is based on an alternative power series expansion of the integrand, and the convergence properties are seen to be superior to those of the original asymptotic series. The resulting modified asymptotic expansion involves confluent hypergeometric functions $U(a, c, z)$, which can be computed by means of continued fractions. Numerical examples are included, such as the incomplete Gamma function $\Gamma(a, z)$ and the modified Bessel function $K_\nu(z)$ for large values of z. The same procedure can be applied to uniform asymptotic expansions when extra parameters become large as well.

1 Introduction

Many special functions admit integral representations in term of Laplace or Fourier transforms:

$$F(z) = \int_0^\infty e^{-zt} f(t) dt, \tag{1}$$

where $\Re z > 0$ and $f(t)$ may depend on one or several extra parameters. In some cases, this formulation is obtained after some suitable transformations of a contour integral in the complex plane, for example through the classical saddle point method. For instance, the modified Bessel function $K_\nu(z)$ of order ν can be written as:

$$K_\nu(z) = \frac{\sqrt{\pi}(2z)^\nu e^{-z}}{\Gamma(\nu+\frac{1}{2})} \int_0^\infty e^{-2zt}[t(1+t)]^{\nu-\frac{1}{2}} dt, \tag{2}$$

and this expression is valid for $\Re(\nu) > -\frac{1}{2}$ and $\Re(z) > 0$.

When z is real the integrand is exponentially decaying, and for the purposes of numerical evaluation a quadrature rule such as Gauss-Laguerre should be quite effective for approximating the value of (1). However, if z is complex (particularly if $\Im z$ is large and/or $\Re z$ is small), then the integrand will become oscillatory. In this case, an asymptotic expansion in terms of z can be interesting from a numerical point of view.

The standard procedure for deriving an asymptotic expansion for large values of z from the integral (1) is either successive integration by parts (provided that the function $f(t)$ is regular enough) or, more generally, an application of Watson's lemma [6], [12]. When applying Watson's lemma we suppose that we have a convergent or asymptotic expansion

$$f(t) \sim \sum_{n=0}^{\infty} a_n t^{\lambda_n - 1}, \qquad t \to 0^+, \tag{3}$$

where $\lambda_{n+1} > \lambda_n$ for all non-negative n and $\lambda_0 > 0$, and also that the integral (1) exists for large enough values of z. Then integration term by term yields

$$F(z) \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\lambda_n)}{z^{\lambda_n}}, \qquad z \to \infty. \tag{4}$$

This Poincaré asymptotic expansion can be very useful for large values of z, but its main disadvantage is that in general it will be divergent for fixed values of z. This situation is due to the fact that the expansion (3) will typically have a finite radius of convergence (limited by the closest singularity of $f(t)$ to the origin). Thus, integration from 0 to ∞ in the variable t in (1) will yield a divergent expression in (4).

In order to circumvent the problem of the divergence of the asymptotic series, several possibilities have been presented in the literature. One of them is the use of Hadamard expansions (see [7] and subsequent papers in the series). Taking into account the location of the singularities of $f(t)$, the interval $[0,\infty)$ in (1) is decomposed into a union of finite intervals, and then Watson's lemma is applied in each of them to yield a convergent expansion.

The terms in the series are no longer Gamma functions, as in (4), but incomplete Gamma functions, by virtue of the formula:

$$P(a,z) = \frac{1}{\Gamma(a)} \int_0^z t^{a-1} e^{-t} dt, \qquad \Re a > 0. \tag{5}$$

The outcome is a convergent approximation consisting of a infinite number of contributions:

$$F(z) = \sum_{n=0}^{\infty} e^{-\Omega_n z} S_n(z), \tag{6}$$

where the parameters Ω_n depend on the partition of the original interval, and each $S_n(z)$ is a series involving incomplete Gamma functions as smoothing factors. For details and numerical examples we refer the reader to [7].

Another modification of the standard asymptotic series gives the so called factorial series, which have been proposed as approximations of Bessel functions, see [4]. Convergence in a half plane $\Re z \geq \varepsilon > 0$ can be obtained through this procedure, and the coefficients can be computed by means of a recurrence relation, although they are not given explicitly in general. Other methods are based on interpolation of the integrand together with successive integration in a similar way as Watson's lemma, see [5, Sec. 2.4.4] for a general review.

A different possibility, discussed in [11] and [5], is a modification of the power series expansion (3), followed by integration term by term. This gives an expansion analogous to (4), but including confluent hypergeometric functions instead of inverse powers of z as asymptotic sequence. The main advantages of this approach with respect to other methods are two: firstly, the domain of convergence of the modified expansion is larger than the one of the original series, and in fact in rather general cases the modified approximation can be shown to be valid for values of z without the original restriction $\Re z > 0$ given by the Laplace integral (1). Secondly, the coefficients of the modified series can be given in closed form in some quite general cases.

The purpose of this paper is to analyze some features of this modification, namely the convergence properties of the modified asymptotic series and some techniques that can be used to compute the confluent hypergeometric functions involved in the approximation. This is studied in Section 2. As a general example, modified expansions for confluent hypergeometric functions are considered in Section 3, and as particular cases expansions for the incomplete Gamma function $\Gamma(a,z)$ and the modified Bessel function $K_\nu(z)$ are studied. In Section 4 we investigate a similar modification applied to uniform asymptotic expansions, and we present the function $K_\nu(\nu z)$ for large values of ν as an example.

2 Modified asymptotic series

Consider the Laplace integral

$$F(z) = \int_0^\infty e^{-zt} t^{\alpha-1} h(t) dt, \tag{7}$$

where $\alpha > 0$, $\Re z > 0$ and $h(t)$ is analytic in a domain containing the positive real t axis.

The usual method to obtain an asymptotic expansion of this integral for large values of z is to expand $h(t)$ as a power series in t and integrate term by term, but unless $h(t)$ is entire in the complex plane, the resulting expression will be divergent, as a consequence of integrating in t from $(0, \infty)$ regardless of the (finite) singularities of $h(t)$.

In this section we propose an alternative expansion for $h(t)$, which is based on a different power series and in general exhibits better behaviour. The modified asymptotic series does not contain inverse powers of z, but confluent hypergeometric U functions, that can be efficiently computed by means of continued fraction representations.

2.1 Construction

First we consider the basic aspects of the construction of the modified asymptotic series.

Proposition 2.1 *Let $h(t)$ be analytic in a certain domain $D \subset \mathbb{C}$, which contains the origin. If we consider the two following expansions:*

$$h(t) = \sum_{j=0}^{\infty} a_j t^j, \qquad h(t) = \sum_{k=0}^{\infty} b_k \left(\frac{t}{1+t} \right)^k, \tag{8}$$

which converge inside D, then it is true that $b_0 = a_0$, and for k=1,2,...

$$b_k = \sum_{j=1}^{k} a_j \frac{(j)_{k-j}}{(k-j)!}. \tag{9}$$

We use the standard Pochhammer symbol

$$(a)_0 = 1, \quad (a)_m = \frac{\Gamma(a+m)}{\Gamma(a)}, \quad m \geq 1. \tag{10}$$

Proof 2.2 *The equality* $a_0 = b_0$ *is clear by comparing powers of* t *of order* 0 *in both expansions. By using the change of variable* $s = t/(1+t)$ *it follows that for* $k \ge 1$

$$b_k = \frac{1}{2\pi i}\int_{C_s} \frac{h(s/(1-s))}{s^{k+1}}\,ds, \tag{11}$$

where C_s *is a small circle around the origin inside* D. *Returning to the* t *variable we have*

$$b_k = \frac{1}{2\pi i}\int_{C_t} \frac{h(t)(1+t)^{k-1}}{t^{k+1}}\,dt, \tag{12}$$

where C_t *is a contour around the origin, which again can be taken as a small circle. By using the first expansion in* (8), *it follows that*

$$b_k = \sum_{j=0}^{\infty} a_j B_{j,k}, \quad B_{j,k} = \frac{1}{2\pi i}\int_{C_t} \frac{t^j(1+t)^{k-1}}{t^{k+1}}\,dt. \tag{13}$$

We see that $B_{j,k} = 0$ *if* $j = 0$ *and if* $j > k$. *Also,*

$$B_{j,k} = \binom{k-1}{k-j} = \frac{(j)_{k-j}}{(k-j)!}, \quad j = 1, 2, \ldots, k, \tag{14}$$

which proves the result.

We observe that the relation for b_k also holds for formal series, so analytic functions are not really needed in the proof. Alternatively, the result can be proved by manipulating the series in (8) directly.

Remark 2.1 This modification can be seen as a particular case of a more general transformation of series, as exposed in [8]. We can write

$$h(t) = \frac{1}{1-\lambda t}\sum_{k=0}^{\infty} \hat{b}_k \left(\frac{\lambda t}{1-\lambda t}\right)^k, \tag{15}$$

where λ is chosen in an optimal way, taking into account the singularities of the function $h(t)$. The coefficients $\hat{b}_k$ can be written in a similar form as b_k in (9).

In the examples of Section 3, the optimal value $\lambda = -1/2$ seems to give minor improvements on the convergence of the series with respect to $\lambda = -1$. However, as follows from Scraton's paper, in more general cases (such as the uniform expansions that we consider in Section 4) the optimal value of λ depends on t. This is not convenient when we use the expansion (15) in integral transforms, because the identification of the terms in the asymptotic series as known special functions is not possible anymore. For this reason, in this paper we will only consider the case $\lambda = -1$.

2.2 Asymptotic properties

In this section we will analyze the integrals that result when integrating term by term the modified power series that we have constructed. For integer $K > 0$ consider the partial sum:

$$h_K(z) = \sum_{k=0}^{K} b_k \left(\frac{t}{1+t} \right)^k , \tag{16}$$

then

$$F_K(z) = \sum_{k=0}^{K} b_k \int_0^\infty e^{-zt} t^{\alpha+k-1} (1+t)^{-k} dt. \tag{17}$$

These integrals can be written as confluent hypergeometric functions, by virtue of the integral representation [1, Eq. 13.2.5]:

$$U(a,c,z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt} t^{a-1} (1+t)^{c-a-1} dt, \tag{18}$$

valid for $\Re a > 0$, $\Re z > 0$. Identifying parameters we obtain:

$$F_K(z) = \sum_{k=0}^{K} b_k \Gamma(\alpha+k) U(\alpha+k, \alpha+1, z), \tag{19}$$

for $\Re z > 0$. We note that using the identity [1, Eq. 13.1.29]

$$U(a,c,z) = z^{1-c} U(a+1-c, 2-c, z), \tag{20}$$

we can write (19) in the form:

$$F_K(z) = z^{-\alpha} \sum_{k=0}^{K} b_k \Gamma(\alpha+k) U(k, 1-\alpha, z). \tag{21}$$

We can show that for large z this series presents nice asymptotic properties. This follows from the next proposition.

Proposition 2.3 *For fixed $\alpha > 0$, the functions $\phi_k(z) := U(k, 1-\alpha, z)$, $k = 0, 1, \ldots$, form an asymptotic sequence when $z \to \infty$ in $|\arg z| < 3\pi/2$.*

Proof 2.4 *Following the definition of asymptotic sequence given by Olver [6, Pg. 25], we need to show that, for all $k \geq 0$, $\phi_{k+1}(z) = o(\phi_k(z))$ when*

$z \to \infty$, $|\arg z| < 3\pi/2$. This can be obtained from the asymptotic estimate [1, 13.5.2]:

$$U(a,c,z) \sim z^{-a}, \qquad z \to \infty, \qquad |\arg z| < \tfrac{3}{2}\pi, \tag{22}$$

valid for fixed values of a and c. Therefore:

$$\frac{\phi_{k+1}(z)}{\phi_k(z)} \sim \frac{1}{z}, \qquad z \to \infty, \tag{23}$$

for $k \geq 0$, which proves the result.

Remark 2.2 We point out that this result is not uniform with respect to the parameter α. We will see later on that large values of this parameter can cause important numerical problems.

2.3 Convergence

Up to this point, the construction of the modified asymptotic series has been formal. In this section we investigate the convergence properties of the approximation.

As it is well known, the radius of convergence of the first series in (8), say R, is determined by the singularities of the function $h(t)$ (in the complex plane), in the sense that if the singularity of $h(t)$ that is closest to the origin is t_0, then $R = |t_0|$. If we use the change of variable

$$s = \frac{t}{1+t}, \tag{24}$$

then the singularity will be moved from t_0 to $s_0 = t_0/(1+t_0)$. Let us denote $\rho = |s_0|$. We have the following result:

Proposition 2.5 *Let t_0 be the singularity of $h(t)$ which is closest to the origin. With the change of variables (24), it is true that:*

- *If $\rho \geq 1$ then the second series in (8) converges for $t > 0$.*
- *If $\rho < 1$ then the second series in (8) converges for $0 < t < t_+$, where $t_+ \geq |t_0|$.*

Proof 2.6 *The domain of convergence of the series is given by $|s| < \rho$, that is $|t| < \rho|1+t|$.*

If $\rho > 1$, this domain is the exterior of the circle

$$D = \left\{ t = x + iy : \left(x - \frac{\rho^2}{1-\rho^2} \right)^2 + y^2 = \frac{\rho^2}{(1-\rho^2)^2} \right\},$$

which includes the real axis $t > 0$.

If $\rho = 1$, then $|t| < |1+t|$ holds for $t > -1/2$.

If $0 < \rho < 1$ then the domain of convergence is the interior of D. This includes the part of the real axis $0 < t < t_+$, where

$$t_+ = \frac{\rho^2}{1-\rho^2} + \frac{\rho}{1-\rho^2} = \frac{\rho}{1-\rho}.$$

Now, since $0 < \rho < 1$, it follows that $|1+t_0| > |t_0|$, and then

$$\frac{\rho}{1-\rho} = \frac{|t_0|}{|1+t_0| - |t_0|} \geq |t_0|.$$

The following corollary will be useful when dealing with Laplace transforms:

Corollary 2.7 *If $\rho \geq 1$ then the sequence*

$$F_K(z) = \sum_{k=0}^{K} b_k \int_0^\infty e^{-zt} t^{\alpha-1} \left(\frac{t}{1+t} \right)^k dt$$

is convergent for $|\arg z| < \frac{\pi}{2}$, and its limit is

$$F(z) = \lim_{K\to\infty} F_K(z) = \int_0^\infty e^{-zt} t^{\alpha-1} h(t) dt$$

Proof 2.8 *The result follows directly from the convergence of the power series for $h(t)$, uniformly on compact intervals of $(0, \infty)$, when $\rho \geq 1$.*

In most of the cases that we will consider, the first part of the proposition can be applied, and the modified series will be convergent.

Remark 2.3 *It is important to observe that the convergence of the expansions* (19) *and* (21) *can also be established when we have information on the coefficients b_k. From [9, Pg. 81] we have the following estimation for the terms in the sum* (21)*:*

$$\Gamma(\alpha + k) U(k, 1-\alpha, z) \sim 2(kz)^{\frac{\alpha}{2}} \; e^{\frac{z}{2}} \; K_{-\alpha}(2\sqrt{kz}), \quad k \to \infty \tag{25}$$

inside the sector $-\pi < \arg z < \pi$. *For the modified Bessel function we have the asymptotic relation (see [1, Eq. 9.7.2])*

$$K_\mu(z) \sim \sqrt{\frac{\pi}{2z}}\, e^{-z}, \quad z \to \infty \tag{26}$$

inside the sector $-\frac{3}{2}\pi < \arg z < \frac{3}{2}\pi$. *So*

$$\Gamma(\alpha+k)U(k, 1-\alpha, z) \sim \sqrt{\pi}(kz)^{\frac{2\alpha-1}{4}}\, e^{\frac{z}{2}-2\sqrt{kz}}, \quad k \to \infty. \tag{27}$$

Combining the information on b_k *with the large* k *behavior of the Kummer functions gives the convergence properties of the expansions. We also note that this analysis can be used to obtain an analytic continuation of* $F(z)$ *for values of* $\arg z$ *different from the ones imposed by the Laplace integral representation* (1), *that is* $|\arg z| < \frac{\pi}{2}$.

2.4 Numerical aspects

As can be seen in formulas (19) and (21), the modified asymptotic series involves confluent hypergeometric functions as the asymptotic sequence. In this section we will analyze possible strategies for the numerical computation of these functions.

It is known that the functions $f_k(z) := U(a+k, c, z)$ satisfy a three term recurrence relation of the form:

$$f_{k+1}(z) + \beta_k f_k(z) + \alpha_k f_{k-1}(z) = 0, \tag{28}$$

where α_k and β_k are rational functions in the parameters a and c and the variable z. In principle this enables us to generate the sequence of $f_k(z)$ needed for the modified asymptotic series with two initial values, $f_0(z)$ and $f_1(z)$.

However, as noted in [11], see also [2] and [5], the function $f_k(z)$ is the minimal solution of the recursion for increasing k, and hence the computation in the forward direction (increasing k) is numerically ill conditioned. Instead, the backward direction or equivalently the associated continued fraction should be used.

The recursion for increasing k reads ([1, Eq. 13.4.15]):

$$y_{k+1}(z) + \frac{c-2a-2k-z}{(a+k)(a+k-c)} y_k(z) + \frac{1}{(a+k)(a+k-c)} y_{k-1}(z) = 0, \tag{29}$$

for $k = 1, 2, \ldots$, with initial values y_0 and $y_1(z)$. A second solution is given by:

$$g_k(z) = \frac{1}{\Gamma(a+k+1-c)} \, {}_1F_1(a+k, c, z), \tag{30}$$

in terms of the confluent hypergeometric function of the first kind or Kummer function. This is a dominant solution for increasing k.

From the recursion (28) we can construct the associated continued fraction:

$$\frac{f_k}{f_{k-1}} = \frac{-\alpha_k}{\beta_k+} \, \frac{-\alpha_{k+1}}{\beta_{k+1}+} \, \frac{-\alpha_{k+2}}{\beta_{k+2}+} \, \ldots, \tag{31}$$

where for $k \geq 0$ we have:

$$\alpha_k = 1, \quad \alpha_{k+j} = (a+k+j-1)(a+k+j-1+c), \quad j = 1, 2, 3, \ldots, \tag{32}$$

$$\beta_{k+j} = c - 2a - 2k - 2j - z, \qquad j = 0, 1, 2, \ldots \tag{33}$$

Since the continued fraction will give the value of a ratio f_k/f_{k-1}, then it is convenient to compute the series of the form (21):

$$F_K = \sum_{k=0}^{K} d_k f_k \tag{34}$$

in the following way (provided that $f_k \neq 0$):

$$F_K = d_0 f_0 \left(1 + \frac{d_1}{d_0}\frac{f_1}{f_0}\left(1 + \frac{d_2}{d_1}\frac{f_2}{f_1}\left(\ldots + \left(1 + \frac{d_K}{d_{K-1}}\frac{f_K}{f_{K-1}}\right)\right)\right)\right). \tag{35}$$

The advantage of this formulation is that it may prevent overflow or underflow if f_k and f_{k-1} are very large or very small but the ratio is of moderate size, and it exploits the structure that the coefficients d_k have in most of the cases.

An algorithm for the evaluation of this series could be:

- Choose an integer K, which may be estimated from the terms of the series (for more details see the discussion in [11]).
- Compute the continued fraction for the ratio $r_K := f_K/f_{K-1}$, using e.g. the modified Lentz-Thompson method [5, Ch. 6].
- The ratios r_k can be easily updated once we have r_K, since:

$$r_k = \frac{-\alpha_k}{\beta_k + r_{k+1}}, \qquad j = K, K-1, \ldots, 1. \tag{36}$$

We observe that the coefficients d_k are easily obtained once we know b_k, since $d_k = b_k\Gamma(\alpha+k)$ for $k \geq 0$. Moreover, $d_0 = b_0\Gamma(\alpha)$ and $f_0 = 1$, so in this setting there is no need for the actual computation of the confluent hypergeometric functions.

We also point out a potential problem here. The convergence of the continued fraction (31) to the ratio of U functions is ensured by Pincherle's theorem [5], but for large values of the parameter c and small values of k the convergence can be poor. This phenomenon has been analyzed in [3] and [5] for several recursions for Gauss and Kummer functions, and it is related to the fact that the minimal or dominant character of the solutions can be temporarily reversed for small values of k, resulting in an anomalous behaviour of both the recursion and the associated continued fraction.

This problem will be present here when the parameter α is large, see equations (21) and (41). In these cases one possible solution is to consider uniform asymptotic expansions. This type of expansion is (necessarily) more complex than the one presented before, but nevertheless it lends itself to a similar transformation. For an example we refer to Section 4.1.

3 Examples

3.1 The confluent hypergeometric function $U(a,c,z)$

As an important example, we can derive the modified asymptotic expansion for the confluent hypergeometric U function itself. Starting from the Laplace integral (18), if we expand:

$$h(t) = (1+t)^{c-a-1} = \sum_{j=0}^{\infty} \binom{c-a-1}{j} t^j, \tag{37}$$

then a standard application of Watson's lemma gives the known asymptotic expansion:

$$U(a,c,z) \sim z^{-a} \sum_{j=0}^{\infty} \frac{(a)_j(a+1-c)_j}{j!}(-z)^{-j}, \tag{38}$$

which is valid for $|\arg z| < 3\pi/2$, see [1, Eq. 13.5.2]. The modification of this asymptotic expansion, along the lines explained before, gives an expression of the form (21), with $\alpha = a$:

$$U(a,c,z) = \sum_{k=0}^{\infty} (a)_k b_k U(a+k, a+1, z), \tag{39}$$

In this case the coefficients b_k can indeed be written in compact form, namely:

$$b_k = (-1)^k \binom{a+1-c}{k} = \frac{(c-a-1)_k}{k!}, \tag{40}$$

where we have used the Pochhammer symbols given in (10). Therefore, using (20) we can write (39) in the form:

$$U(a,c,z) = z^{-a} \sum_{k=0}^{\infty} \frac{(a)_k (c-a-1)_k}{k!} \, U(k, 1-a, z), \tag{41}$$

which may be seen as a modification of the expansion (38).

Regarding convergence of this expansion, we have a straightforward application of the results of the preceding section.

Proposition 3.1 *The series* (39) *is convergent for bounded values of a and for z in $|\arg z| < \pi$.*

Proof 3.2 *Since the function $h(t) = (1+t)^{c-a-1}$ has a singularity at $t_0 = -1$, then $\lambda = \infty$ and convergence follows from Corollary 2.7 for $|\arg z| < \frac{\pi}{2}$. This domain can be extended to all $z \neq 0$, inside the sector $|\arg z| < \pi$, using Remark 2.3.*

We note that the convergence can be also established by means of (27), together with the fact that $b_k \sim k^{c-a-2}$ when $k \to \infty$, which follows directly from (40). Naturally, for complex z, one would expect the convergence to get slower when z is close to the negative imaginary axis, since in this case the decay of the exponential term $e^{-2\sqrt{kz}}$ is much less pronounced.

As particular cases of the confluent hypergeometric function $U(a,c,z)$ we have several special functions of importance. In the next sections we address some examples.

3.2 The incomplete Gamma function $\Gamma(a,z)$

We consider the incomplete Gamma function:

$$\Gamma(a,z) = \int_z^{\infty} e^{-t} t^{a-1} dt = z^a e^{-z} \int_0^{\infty} e^{-zt} (1+t)^{a-1} dt, \tag{42}$$

where we assume that $|\arg z| < \pi$. The relation with the confluent U-function is:

$$\Gamma(a,z) = z^a e^{-z} U(1, a+1, z) = e^{-z} U(1-a, 1-a, z), \tag{43}$$

see for instance [10, Pg. 186]. Hence, the standard asymptotic expansion for large z follows directly from (38):

$$\Gamma(a,z) \sim e^{-z} z^{a-1} \sum_{j=0}^{\infty} (1-a)_j (-z)^{-j}, \tag{44}$$

when $|\arg z| < 3\pi/2$. Alternatively, one can expand the function $h(t) = (1+t)^{a-1}$ in powers of t and apply Watson's lemma. The divergence of this expansion for fixed values of z is shown in Figure 3.1.

The modified asymptotic series can be obtained from (39):

$$\Gamma(a,z) \sim z^a e^{-z} \sum_{k=0}^{\infty} (a-1)_k U(1+k,2,z), \tag{45}$$

and the convergence of this expansion for $|\arg z| < \pi$ follows from the more general case in Proposition 3.1.

It is important to note that the parameter a does not appear in the U functions. However, large values of a will slow down the convergence of the series (45). This can be seen by considering the estimations (25) and (26), which yield:

$$(a-1)_k U(1+k,2,z) \sim \frac{\sqrt{\pi} z^{-3/4} e^{z/2}}{\Gamma(a+1)}\, k^{a-7/4} e^{-2\sqrt{kz}}, \qquad k \to \infty. \tag{46}$$

The effect of a on the convergence of the modified asymptotic series is illustrated in Figure 3.1. Indeed, for large values of a we observe that the approximation is quite poor.

3.3 The modified Bessel function $K_\nu(z)$

The modified Bessel function $K_\nu(z)$, also called MacDonald function, can be written as:

$$K_\nu(z) = \sqrt{\pi}(2z)^\nu e^{-z} U\left(\nu + \tfrac{1}{2}, 2\nu+1, 2z\right), \tag{47}$$

see for instance [10, Eq. 9.45]. The corresponding asymptotic approximation follows directly from (41), with parameters $a = \nu + \frac{1}{2}$, $c = 2\nu + 1$. Namely,

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z} \sum_{k=0}^{\infty} \frac{(\nu-\frac{1}{2})_k (\nu+\frac{1}{2})_k}{k!}\, U\left(k, \frac{1}{2} - \nu, 2z\right), \tag{48}$$

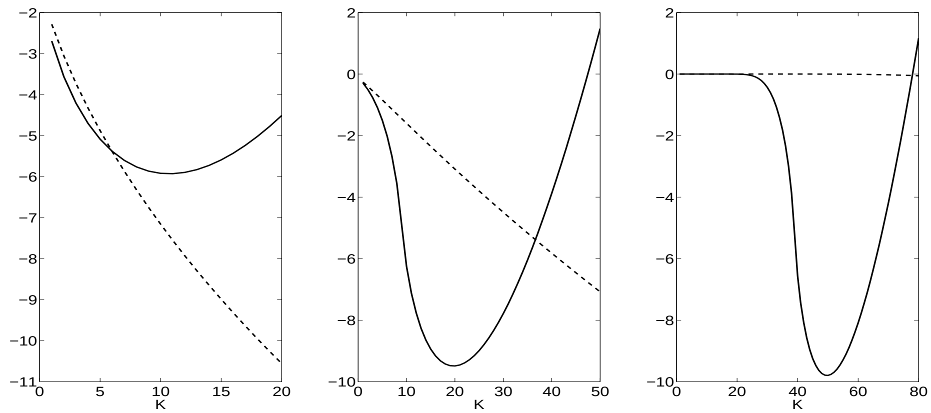

Figure 1: Relative error (in $\log_{10}$ scale) in the computation of the incomplete Gamma, using the standard (solid line) and modified series (dashed line) with K terms and $z = 10.23$. Left, $a = 1.5$, center $a = 10.5$ and right $a = 40.5$.

which is convergent for $\nu > -1/2$ and $|\arg z| < \pi$, again as a consequence of Proposition 3.1.

This expansion can also be obtained from the integral representation:

$$K_\nu(z) = \frac{\sqrt{\pi}(2z)^\nu e^{-z}}{\Gamma(\nu + \frac{1}{2})} \int_0^\infty e^{-2zt}[t(1+t)]^{\nu - \frac{1}{2}} dt, \tag{49}$$

which is valid for $\Re(\nu) > -\frac{1}{2}$ and $\Re(z) > 0$, and in turn can be easily obtained from the standard formula [1, Eq. 9.6.23]:

$$K_\nu(z) = \frac{\sqrt{\pi}(\frac{z}{2})^\nu}{\Gamma(\nu + \frac{1}{2})} \int_1^\infty e^{-zt}(t^2 - 1)^{\nu - \frac{1}{2}} dt. \tag{50}$$

In Figure 3.2 we illustrate the computation of the modified series for the function $K_\nu(z)$ in Matlab for three different values of z, and we plot the error with respect to the direct evaluation of the Bessel function using the Matlab internal subroutine. Similarly to what happened with the incomplete Gamma function, we note that large values of ν give worse results.

3.4 Other examples

These techniques can be applied to several other examples within the family of confluent hypergeometric functions. For example, by using the following

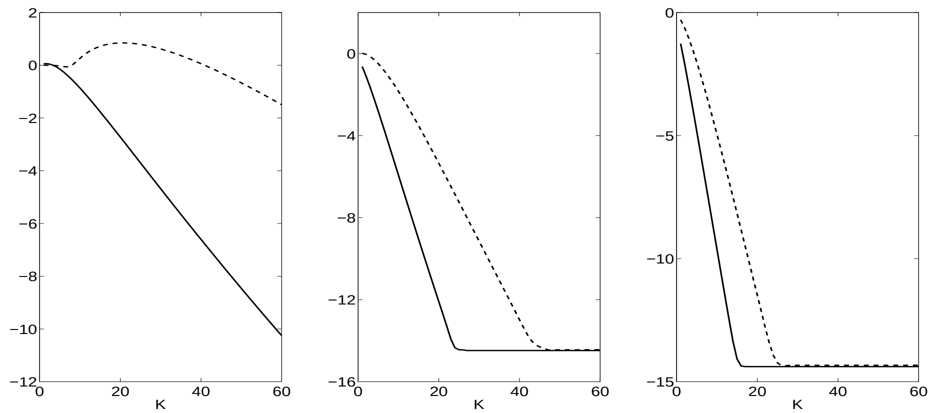

Figure 2: Relative error (in $\log_{10}$ scale) in the computation of the Bessel function $K_\nu(z)$, using the series involving Kummer U functions. Left, $z = 10+11.1i$, center $z = 50.1+42.5i$ and right $z = 100.1+120.5i$. Here $\nu = 10.1$ (solid line) and $\nu = 20.1$ (dashed line).

identities [1, Eq. 9.6.4]:

$$H_\nu^{(1)}(z) = \frac{2}{\pi i}\, e^{-\frac{\nu\pi i}{2}} K_\nu\left(z e^{-\frac{\pi i}{2}}\right), \qquad -\frac{\pi}{2} < \arg z \leq \pi, \tag{51}$$

$$H_\nu^{(2)}(z) = -\frac{2}{\pi i}\, e^{\frac{\nu\pi i}{2}} K_\nu\left(z e^{\frac{\pi i}{2}}\right), \qquad -\pi < \arg z \leq \frac{\pi}{2}, \tag{52}$$

it is possible to derive the modified asymptotic expansions for large z corresponding to the Hankel functions (and hence to the standard Bessel functions $J_\nu(z)$ and $Y_\nu(z)$):

$$H_\nu^{(1)}(z) = \sqrt{\frac{2}{\pi z}}\, e^{iz - \frac{\nu\pi i}{2} - \frac{\pi i}{4}} \sum_{k=0}^{\infty} \frac{(\nu-1/2)_k(\nu+1/2)_k}{k!}\, U\left(k, \tfrac{1}{2}-\nu, -2iz\right), \tag{53}$$

which is valid for $-\frac{\pi}{2} < \arg z < \pi$, and

$$H_\nu^{(2)}(z) = \sqrt{\frac{2}{\pi z}}\, e^{-iz + \frac{\nu\pi i}{2} + \frac{\pi i}{4}} \sum_{k=0}^{\infty} \frac{(\nu-1/2)_k(\nu+1/2)_k}{k!}\, U\left(k, \tfrac{1}{2}-\nu, 2iz\right), \tag{54}$$

for $-\pi < \arg z \ < \frac{\pi}{2}$.

Other examples are furnished by the Weber parabolic cylinder functions, see [1, Ch.19]. Using [10, Eq. 7.21]

$$U(a,z) = 2^{-3/4-a/2} e^{-z^2/4} z \; U\left(\tfrac{3}{4} + \tfrac{1}{2}a, \tfrac{3}{2}, \tfrac{1}{2}z^2\right), \tag{55}$$

we get the modified expansion:

$$U(a,z) = z^{-1/2-a} e^{-z^2/4} \sum_{k=0}^{\infty} b_k U\left(k, \tfrac{1}{4} - \tfrac{1}{2}a, \tfrac{1}{2}z^2\right), \tag{56}$$

where

$$b_k = \frac{(\frac{3}{4} + \frac{1}{2}a)_k (-\frac{1}{4} - \frac{1}{2}a)_k}{k!}. \tag{57}$$

Once more, large values of a will slow down the convergence of this modified asymptotic series.

4 Modified uniform asymptotic expansions

As can be seen from the previous examples, one problem with the modified asymptotic expansions is that, though being convergent in many cases, they are not uniform with respect to other parameters, such as a for the incomplete Gamma function and ν for the modified Bessel function. Large values of these parameters with respect to z will slow down the numerical convergence.

A way to overcome this difficulty is to use an asymptotic expansion for large values of the parameters that remains uniformly valid with respect to z, and then apply a modification similar to the one that we used before. As an illustrative example, we investigate again the modified Bessel function.

4.1 A modified uniform asymptotic expansion for $K_\nu(\nu z)$

4.1.1 Construction

An asymptotic expansion for large values of ν which is uniform with respect to z can be found in [1, Eq. 9.7.8]:

$$K_\nu(\nu z) \sim \sqrt{\frac{\pi}{2\nu}} \frac{e^{-\nu\eta}}{(1+z^2)^{1/4}} \left(1 + \sum_{k=1}^{\infty} (-1)^k \frac{u_k(t)}{\nu^k}\right), \tag{58}$$

which holds when $\nu \to \infty$, uniformly with respect to z such that $|\arg z| < \pi/2$. Here,

$$t = \frac{1}{\sqrt{1+z^2}}, \qquad \eta = \sqrt{1+z^2} + \log \frac{z}{1+\sqrt{1+z^2}}. \tag{59}$$

The first coefficients $u_k(t)$ are [1, Eq. 9.3.9]:

$$u_0(t) = 1, \qquad u_1(t) = \frac{3t - 5t^3}{24}, \qquad u_2(t) = \frac{81t^2 - 462t^4 + 385t^6}{1152}, \tag{60}$$

and other coefficients can be obtained by applying the formula

$$u_{k+1}(t) = \tfrac{1}{2}t^2(1-t^2)u_k'(t) + \tfrac{1}{8}\int_0^t (1-5s^2)u_k(s)ds, \qquad k = 0,1,2,\ldots \tag{61}$$

This expansion can be obtained in the following way: consider the integral representation [1, Eq. 9.6.24]

$$K_\nu(\nu z) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-\nu\phi(v)}\,dv, \quad \phi(v) = z\cosh v - v. \tag{62}$$

When z is real, the function $\phi(v)$ has a real saddle point located at $v_0 = \operatorname{arcsinh}(1/z)$. We apply the following transformation:

$$\phi(v) - \phi(v_0) = \tfrac{1}{2}\phi''(v_0)w^2, \quad \operatorname{sign}(w) = \operatorname{sign}(v - v_0), \tag{63}$$

where $\phi''(v_0) = \sqrt{1+z^2} = 1/t$ and with t as before. This gives

$$K_\nu(\nu z) = \tfrac{1}{2}\, e^{-\nu\eta}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\nu\phi''(u_0)w^2}\,\frac{dv}{dw}\,dw, \tag{64}$$

where η is given in (59). If we expand $dv/dw = \sum_{k=0}^{\infty} c_k w^k$ and integrate term by term we obtain (58), with

$$u_k(t) = (-1)^k(2t)^k\left(\tfrac{1}{2}\right)_k c_{2k}, \quad k = 0,1,\ldots\,. \tag{65}$$

An alternative expansion can be obtained as follows. Write

$$K_\nu(\nu z) = \tfrac{1}{2}\, e^{-\nu\eta}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\nu\phi''(u_0)w^2}\, f(w)\,dw, \tag{66}$$

where $f(w)$ is the even part of du/dw (considered as a function of w). That is,

$$f(w) = \sum_{k=0}^{\infty} a_k w^{2k}, \tag{67}$$

where $a_k = c_{2k}$, and the c_{2k} can be computed from the functions $u_k(t)$ using (65). To obtain an alternative expansion we write

$$f(w) = \sum_{k=0}^{\infty} b_k\left(\frac{w^2}{1+w^2}\right)^k. \tag{68}$$

The relation between a_k and b_k is given by (9), and this gives:

$$K_\nu(\nu z) = \tfrac{1}{2} e^{-\nu\eta} \sum_{k=0}^{\infty} b_k \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\nu\phi''(u_0)w^2} w^{2k}}{(1+w^2)^k}\, dw. \tag{69}$$

These integrals can be expressed in terms of the Kummer $U-$function. Indeed, using (18):

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\nu\phi''(u_0)w^2} w^{2k}}{(1+w^2)^k} = \Gamma\left(k+\tfrac{1}{2}\right) U\left(k+\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{1}{2}\nu\sqrt{1+z^2}\right). \tag{70}$$

Therefore, the expansion can be written as follows:

$$K_\nu(\nu z) = \tfrac{1}{2} e^{-\nu\eta} \sum_{k=0}^{\infty} b_k\ \Gamma\left(k+\tfrac{1}{2}\right) U\left(k+\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{1}{2}\nu\sqrt{1+z^2}\right). \tag{71}$$

The coefficients b_k can be expressed in terms of $u_k(t)$, using (65), (9) and [1, Eq. 9.3.9]. The first few are:

$$\begin{aligned}
b_0 &= a_0 = 1,\\
b_1 &= a_1 = \tfrac{5}{24}t^2 - \tfrac{1}{8},\\
b_2 &= a_1 + a_2 = \tfrac{385}{3456}t^4 + \tfrac{43}{576}t^2 - \tfrac{13}{128},\\
b_3 &= a_1 + 2a_2 + a_3 = \tfrac{17017}{248832}t^6 + \tfrac{13783}{138240}t^4 + \tfrac{89}{230400}t^2 - \tfrac{85}{1024},\\
b_4 &= a_1 + 3a_2 + 3a_3 + a_4\\
&= \tfrac{1062347}{23887872}t^8 + \tfrac{979693}{9953280}t^6 + \tfrac{83633}{1720320}t^4 - \tfrac{159049}{4300800}t^2 - \tfrac{2237}{32768},
\end{aligned} \tag{72}$$

where we recall that $t = 1/\sqrt{1+z^2}$. Although the expressions become rather cumbersome, we note that, with the aid of symbolic computation in mathematical software like Maple or Mathematica, it is not difficult to generate and store a sequence of $u_k(t)$ using (61), which can then be used to compute the coefficients b_k.

In Figure 4.1 we give an example of this expansion, taking the first few terms. We consider the same values of the variable as before (though now we scale to evaluate at νz) and plot the relative error with respect to Matlab internal routine for the Bessel K function, for increasing values of ν.

We observe that, as expected, large values of the parameter ν improve the results. In fact, we have (see (25))

$$U\left(\tfrac{1}{2}+k, \tfrac{3}{2}, \xi\right) \sim \frac{1}{\Gamma(k)}\ 2(k\xi)^{-\frac{1}{4}} e^{\frac{1}{2}\xi} K_{\frac{1}{2}}(2\sqrt{k\xi}), \qquad k\to\infty, \tag{73}$$

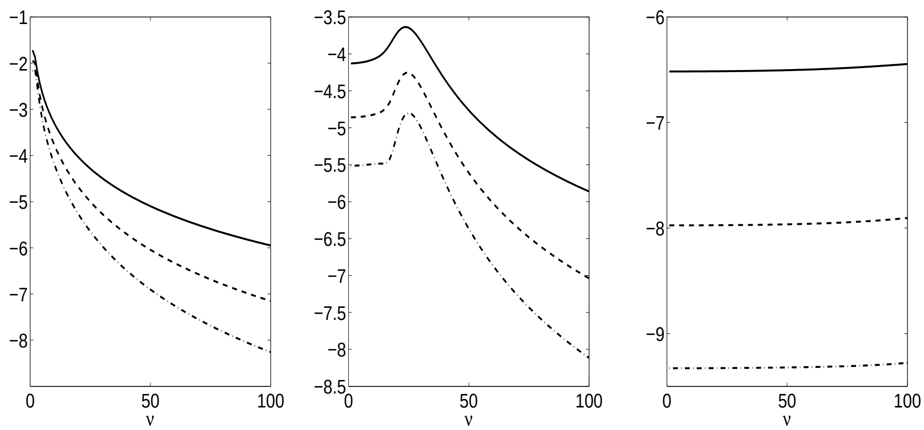

Figure 3: Relative error (in $\log_{10}$ scale) in the computation of the Bessel function $K_\nu(\nu z)$, using 3 terms (solid line), 4 terms (dashed line) and 5 terms (dashed-dotted line) of the series involving Kummer $U-$functions. Left, $\nu z = 1 + i$, center $\nu z = 10.1 + 20.5i$ and right $\nu z = 100.1 + 120.5i$.

uniformly with respect to ξ in $|\arg \xi| < \pi$, where

$$\xi = \tfrac{1}{2}\nu\sqrt{1+z^2}. \tag{74}$$

For the modified Bessel function $K_{\frac{1}{2}}(2\sqrt{k\xi})$ we have the exact relation

$$K_{\frac{1}{2}}(2\sqrt{k\xi}) = \tfrac{1}{2}\sqrt{\pi}(k\xi)^{-\frac{1}{4}}e^{-2\sqrt{k\xi}}, \tag{75}$$

and hence

$$U\left(\tfrac{1}{2}+k, \tfrac{3}{2}, \xi\right) \sim \frac{1}{\Gamma(k)}\sqrt{\frac{\pi}{k\xi}}\, e^{\frac{1}{2}\xi - 2\sqrt{k\xi}}, \qquad k \to \infty, \tag{76}$$

uniformly with respect to ξ in $|\arg \xi| < \pi$.

4.1.2 Convergence properties

The domain of convergence of the standard and modified asymptotic expansions can be analyzed by considering the singularities of the respective integrands in the complex plane, as shown in Section 2. For simplicity, in this section we will restrict ourselves to real values of $z > 0$.

We note that the change of variables (63) introduces singularities of the function dv/dw in the complex w-plane that we can use to analyze the convergence of the series that results from Watson's lemma applied to the integral (64). Indeed, the (complex) solutions of (63) are:

$$v_k(z) = (-1)^k \text{arcsinh}\frac{1}{z} + k\pi i, \qquad k = 0, \pm 1, \pm 2, \dots, \tag{77}$$

the case $k = 0$ corresponding to the saddle point which is real when z is real. The next relevant saddle points are $w_{\pm 1}$, which will give the closest singularities of dv/dw to the origin in the w variable. A direct manipulation using (63) yields:

$$w_{\pm 1}^2 = -\frac{4\eta \pm 2\pi i}{\sqrt{1+z^2}}, \tag{78}$$

where again η is given in (59). Hence, the radius of convergence of the series obtained by application of Watson's lemma to (64) is $|w_{\pm 1}|$.

In Figure 4.2 we show the location of these two saddle points in the complex w-plane, for $z = 0.1, 0.2, \dots, 20$:

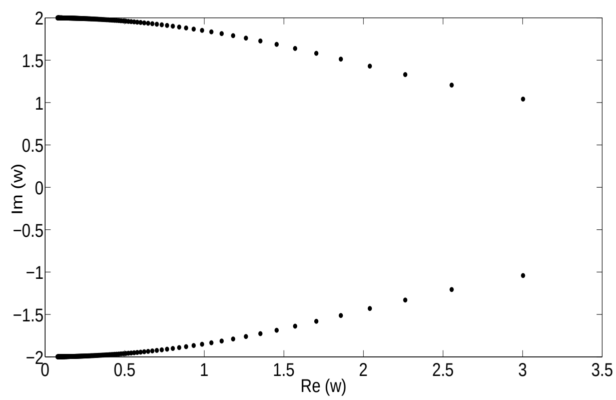

Figure 4: Saddle points w_1 (negative imaginary part) and w_{-1} (positive imaginary part), for $z = 0.1, 0.2, \dots, 20$ (from right to left in the graphic).

It is clear from (59) and (78) that when $z \to 0^+$ then $w_{\pm 1}^2 \to +\infty \mp 2\pi i$, and when when $z \to +\infty$ then $w_{\pm 1}^2 \to -4$.

We expand in the form:

$$f(w) = \sum_{k=0}^{\infty} b_k \left(\frac{w^2}{1+w^2}\right)^k = \sum_{k=0}^{\infty} b_k s^{2k}, \tag{79}$$

where

$$s = \frac{w}{\sqrt{1+w^2}}. \tag{80}$$

The singularities of the new variable s can be computed from the ones in w obtained from (77), and they will determine the domain of convergence of the modified asymptotic series. More precisely, we can prove the following result:

Proposition 4.1 *If $z > 1$ then $|s_{\pm 1}| > 1$.*

Proof 4.2 *From* (80) *we obtain:*

$$|s_1|^2 = \left| \frac{w_1^2}{1 + w_1^2} \right| \tag{81}$$

If we write $w_1(z) = re^{i\theta}$, then the condition $|s_1|^2 > 1$ is seen to be equivalent to $\Re w_1^2(z) < -1/2$. From (78) *it follows that*

$$\Re w_1^2(z) = -\frac{4\eta}{\sqrt{1+z^2}} = -4 - \frac{1}{\sqrt{1+z^2}} \log \frac{z}{1+\sqrt{1+z^2}}. \tag{82}$$

As a function of z, $\Re w_1^2(z)$ is decreasing for $z > 0$, and $\Re w_1^2(z) < -1/2$, which proves the result. The same reasoning can be applied to s_{-1}.

As a corollary we have:

Corollary 4.3 *The series* (79) *is convergent for all real w if $z > 1$.*

Proof 4.4 *This is a consequence of Proposition 4.1 and Corollary 2.7.*

It is clear that in these results the value $z = 1$ is set for clarity and can be refined to be the solution of (82) equal to $-1/2$. Numerical computation gives approximately $z^* = 0.753$. For $z < z^*$ we do not have convergence of the modified expansion, and the series (71) should be understood in an asymptotic sense.

Other singular points in the $w-$plane of the mapping in (63) occur when $\phi(v) = \phi(v_0)$ at points different from the point $v = v_0$ inside the strip $-\pi < \Im v < \pi$. It is not difficult to verify that this cannot happen when $z > 0$.

Figure 4.3 illustrates the location of the points $s_{\pm 1}$ in the complex plane for different values of z.

We recall that we can replace the expansion in (68) by a more efficient modified expansion of the form (15), where we take into account the singularities of $f(w)$. However, as follows from [8] and from the singular points of $f(w)$, the value of λ for an optimal choice gives an expansion in which λ depends on w. When we take such an optimal λ a transformation of the uniform expansion (58) into an expansion in terms of the Kummer $U-$functions is not possible anymore.

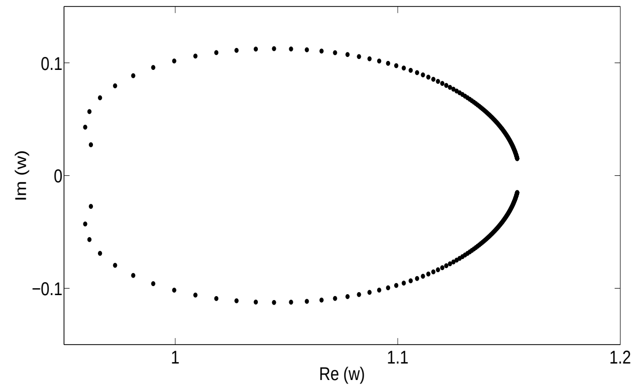

Figure 5: Saddle points s_1 (negative imaginary part) and s_{-1} (positive imaginary part), for $z = 0.1, 0.2, \ldots, 20$ (from left to right in the graphic).

4.2 A modified uniform asymptotic expansion for the $U-$function

As a final example, we give a few details for a uniform asymptotic expansion of the Kummer $U-$function that generalizes the expansion for $K_\nu(\nu z)$ given in (58).

We write (18) in the form

$$U(\nu + \tfrac{1}{2}, 2\nu + 1 + b, 2\nu z) = \frac{1}{\Gamma(\nu + \frac{1}{2})} \int_0^\infty e^{-\nu\phi(t)} \frac{(1+t)^b}{\sqrt{t(1+t)}}\, dt, \tag{83}$$

where

$$\phi(t) = 2zt - \ln t(1+t). \tag{84}$$

It is clear that for $b = 0$ this $U-$function can be written in terms of the modified Bessel function $K_\nu(\nu z)$, see formula (47). When $z > 0$ there is a positive saddle point:

$$t_0 = \frac{1 - z + \sqrt{1+z^2}}{2z}. \tag{85}$$

We have

$$\phi(t_0) = 1 - z + \ln(2z) + \eta, \quad \phi''(t_0) = \frac{4z^2\sqrt{1+z^2}}{1+\sqrt{1+z^2}}, \tag{86}$$

where η is given in (59). We apply the transformation:

$$\phi(t) - \phi(t_0) = \tfrac{1}{2}\phi''(t_0) w^2, \quad \operatorname{sign}(w) = \operatorname{sign}(t - t_0), \tag{87}$$

and obtain

$$U(\nu+\tfrac{1}{2}, 2\nu+1+b, 2\nu z) = \frac{e^{-\nu+\nu z-\nu\eta}}{\Gamma(\nu+\frac{1}{2})(2z)^{\nu}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\nu\phi''(t_0)w^2} f(w)\, dw, \quad (88)$$

where

$$f(w) = \frac{(1+t)^b}{\sqrt{t(1+t)}} \frac{dt}{dw}. \quad (89)$$

Expanding now $f(w) = \sum_{k=0}^{\infty} f_k w^k$ we obtain the asymptotic expansion:

$$U(\nu+\tfrac{1}{2}, 2\nu+1+b, 2\nu z) \sim \frac{\sqrt{\pi/\nu}(1+t_0)^b e^{-\nu+\nu z-\nu\eta}}{\Gamma(\nu+\frac{1}{2})(2z)^{\nu}(1+z^2)^{1/4}} \left(1+\sum_{k=1}^{\infty} \frac{U_k(b,z)}{\nu^k}\right), \quad (90)$$

where

$$U_k(b,z) = \frac{f_{2k}}{f_0} \frac{2^k(\frac{1}{2})_k}{(\phi''(t_0))^k}. \quad (91)$$

We have

$$U_1(b,z) = \tfrac{1}{24}(-1-3b^2z+6b^2+(-3-3b^2z+3bz)t+(3bz-6b)t^2+5t^3), \quad (92)$$

where again $t = 1/\sqrt{1+z^2}$ (as in (59)). In the case $b=0$ we obtain the expansion in (58) when we use the estimation

$$\frac{\sqrt{2\pi}e^{-\nu}\nu^{\nu}}{\Gamma(\nu+\frac{1}{2})} \sim 1+\sum_{k=1}^{\infty} \frac{\gamma_k}{\nu^k}, \qquad \nu\to\infty, \quad (93)$$

together with the asymptotic identity

$$\left(1+\sum_{k=1}^{\infty}\frac{\gamma_k}{\nu^k}\right)\left(1+\sum_{k=1}^{\infty}\frac{U_k(0,z)}{\nu^k}\right) \sim 1+\sum_{k=1}^{\infty}(-1)^k\frac{u_k(t)}{\nu^k}. \quad (94)$$

5 Acknowledgements

The first author acknowledges useful discussions with A. Iserles and B. Adcock (DAMTP, University of Cambridge), and financial support from the Program of postdoctoral research grants (*Programa de becas postdoctorales*) of the Spanish Ministry of Education and Science (*Ministerio de Educación y Ciencia*). The authors acknowledge financial support from the Spanish Ministry of Education and Science (*Ministerio de Educación y Ciencia*), under project MTM2006–09050.

References

[1] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. U.S. Government Printing Office, Washington, 1964.

[2] A. Deaño, J. Segura, and N.M. Temme. Identifying minimal and dominant solutions for Kummer recursions. *Math. Comp.*, 77:2277–2293, 2008.

[3] A. Deaño and J. Segura. Transitory minimal solutions of hypergeometric recursions and pseudoconvergence of associated continued fractions. *Math. Comp*, 76(258):879–901, 2007.

[4] T. M. Dunster and D. A. Lutz. Convergent factorial series expansions for Bessel functions. *SIAM J. Math. Anal*, 22:1156–1172, 1991.

[5] A. Gil, J. Segura, and N. M. Temme. *Numerical methods for special functions*. SIAM, Philadelphia, PA, 2007.

[6] F. W. J. Olver. *Asymptotics and special functions*. AKP Classics. A K Peters Ltd., Wellesley, MA, 1997. Reprint of the 1974 original [Academic Press, New York].

[7] R. B. Paris. On the use of Hadamard expansions in hyperasymptotic evaluation of Laplace-type integrals. I: real variable. *J. Comp. Appl. Math*, 167:293–319, 2004.

[8] R. E. Scraton. A note on the summation of divergent power series. *Proc. Cambridge Philos. Soc*, 66:109–114, 1969.

[9] L. J. Slater. *Confluent hypergeometric functions*. Cambridge University Press, New York, 1960.

[10] N. M. Temme. *Special functions*. A Wiley-Interscience Publication. John Wiley & Sons Inc., New York, 1996. An introduction to the classical functions of mathematical physics.

[11] N.M. Temme. Numerical evaluation of functions arising from transformations of formal series. *J. Math. Anal. Appl*, 51:678–694, 1975.

[12] R. Wong. *Asymptotic approximations of integrals*, volume 34 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001. Corrected reprint of the 1989 original.