Multivariate modified Fourier series and application to boundary value problems

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Abstract

In this paper we analyse the approximation-theoretic properties of modified Fourier series in Cartesian product domains with coefficients from full and hyperbolic cross index sets. We show that the number of expansion coefficients may be reduced significantly whilst retaining comparable error estimates. In doing so we extend the univariate results of Iserles, Nørsett and S. Olver. We then demonstrate that these series can be used in the spectral-Galerkin approximation of second order Neumann boundary value problems, which offers some advantages over standard Chebyshev or Legendre polynomial discretizations.

Introduction

Univariate modified Fourier series—eigenseries of the Laplace operator subject to homogeneous Neumann boundary conditions—were introduced in [14] as an adjustment of Fourier series. Combined with modern quadrature methods (as opposed to the Fast Fourier transform) to evaluate the coefficients, the benefit of using such series to approximate a non-periodic function f is a faster convergence rate (in particular the convergence is uniform and there is no Gibbs' phenomenon on the boundary). Moreover, the coefficients may be calculated adaptively in fewer operations without the restriction that the truncation parameter be a highly composite integer. In [13] these series and quadrature methods were generalized to Cartesian product domains.

In [11], alongside so-called polynomial subtraction (a familiar device for univariate Fourier series [15, 17]), the authors used a hyperbolic cross index set [2, 23], to accelerate convergence. Due to the method of calculating the coefficients, such a device can be readily exploited by modified Fourier series to produce approximations comprising a far reduced number of terms over approximations based on Fourier series or orthogonal polynomials. Thus in higher dimensions, modified Fourier series become an increasingly attractive option.

The aim of this paper is twofold. In Sections 1–3 we extend the work of [11, 13] and provide convergence analysis for modified Fourier series in various norms using various index sets. For reasons that we make clear, modified Fourier approximations are best analysed in so-called Sobolev spaces of *dominating mixed smoothness*, [20]. Using this framework, we prove uniform convergence of such series, and provide estimates for the convergence rate in the L^2, H^s, $s \geq 1$, and uniform norms. Furthermore, we demonstrate that using the hyperbolic cross index set does not unduly affect the convergence rate, aside from a logarithmic factor, provided additional (mixed) smoothness assumptions are imposed where necessary.

For univariate modified Fourier series it was observed in [14] and proved in [18] that the convergence rate was one power of N faster inside the interval than at the endpoints. We prove the same result for d-variate cubes using full and hyperbolic cross index sets. Finally, we demonstrate that the advantage of modified Fourier series over Fourier series can be expressed as the observation that the modified Fourier basis is dense not only in $L^2(\Omega)$, but also in the space $H^1_{\text{mix}}(\Omega)$ (the first Sobolev space of dominating mixed smoothness).

One significant use of Fourier series is the discretization of boundary value problems with periodic boundary conditions. This approach offers numerous benefits, including rapid convergence and low complexity. The application of Fourier series using hyperbolic cross index sets to the numerical solution of periodic boundary value problems has been studied in [16].

Because each modified Fourier basis functions satisfies homogeneous Neumann boundary conditions, modified Fourier expansions are best suited to discretizations of non-periodic boundary value problems with the same boundary conditions. In the second half of this paper we consider the application to linear, second order problems defined on d-variate cubes (see [1] for the case $d = 1$). Much like the Fourier spectral method, this technique possesses a number of beneficial properties, including reasonable conditioning and the availability of an optimal, diagonal preconditioner. Furthermore, due to the hyperbolic cross index set, the operational cost of this method grows only moderately with dimension. In d dimensions, we show that the so-called *modified Fourier–Galerkin* approximation comprises $\mathcal{O}\left(N(\log N)^{d-1}\right)$ coefficients which can be found in only $\mathcal{O}\left(N^2\right)$ operations using standard iterative techniques. In comparison, the efficient spectral-Galerkin methods of Shen, [9, 21, 22], based on Legendre and Chebyshev polynomials involve $\mathcal{O}\left(N^d\right)$ coefficients that can be found in at best $\mathcal{O}\left(N^{d+1}\right)$ operations.

The modified Fourier basis is best suited to Neumann boundary value problems. It can be applied to problems with other boundary conditions, however techniques for enforcing the boundary conditions are either increasingly complicated for $d \geq 2$ or lead to a loss of accuracy. For this reason, a better approach is to choose basis functions that satisfy the boundary conditions inherently. Given, for example, Robin boundary conditions, we use instead the basis of Laplace eigenfunctions subject to these boundary conditions. Such basis is very similar to the modified Fourier basis (the analysis of convergence is virtually identical), and the resulting Galerkin method possesses many similar features, including mild conditioning and low complexity. For this reason the modified Fourier–Galerkin method can be viewed as a particular example of a class of methods for second order boundary value problems, each with basis functions determined by the boundary conditions. In Section 5 we present some numerical examples of discretizations of Dirichlet and Robin boundary value problems in this manner.

For the task of function approximation, modified Fourier expansions converge faster than expansions based on, for example, Laplace–Dirichlet eigenfunctions (which do not converge uniformly unless the function being approximated also satisfies homogeneous Dirichlet boundary conditions). Hence they are the natural choice from such class of bases. However, for the purposes spectral discretizations (where the exact solution automatically satisfies the boundary conditions), each basis is immediately adapted to a particular problem.

The disadvantage of all such methods is that they converge only algebraically in terms of the truncation parameter. Standard orthogonal polynomial methods converge spectrally provided the solution is smooth. However, due to the much reduced complexity, for many test problems these methods give lower errors for moderate values of the truncation parameter. We present several such examples.

Notation: Throughout we shall write $(\cdot,\cdot)$ for the standard $L^2(\Omega)$ inner product on some domain Ω. We write $\|\cdot\|$ for the L^2 norm, $\|\cdot\|_q$ for the H^q norm, $q \geq 1$, and $\|\cdot\|_\infty$ for the uniform norm. N shall be a truncation parameter and I_N some finite index set. We denote by f a function in $L^2(\Omega)$ and $u \in H^q(\Omega)$ a function that satisfies certain derivative conditions on the boundary $\partial\Omega$. For a multi-index $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$, D^α will correspond to the derivative operator

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} ... \partial_{x_d}^{\alpha_d}} = \partial_{x_1}^{\alpha_1} ... \partial_{x_d}^{\alpha_d},$$

where $|\alpha| = \sum \alpha_i$. If $\alpha = (r, r, ..., r)$, $r \in \mathbb{N}$, we also write D^r, and, if $r = 1$, just D.

We define $[d]$ to be the set of ordered tuples of length at most d with entries in $\{1, ..., d\}$. For $t \in [d]$ we write $|t|$ for the length (number of elements) in t, so that $t = (t_1, ..., t_{|t|})$. If $j \in \{1, ..., d\}$ we write $j \in t$ if $j = t_l$ for some $l = 1, ..., |t|$. Given $t \in [d]$, we define $\bar{t} \in [d]$ as the tuple of length $d - |t|$ of elements not in t.

1 Modified Fourier series in $[-1,1]^d$

1.1 Definition and basic properties

The modified Fourier basis is the set of eigenfunctions of the Laplace operator subject to homogeneous Neumann boundary conditions. On the domain $\bar{\Omega}$, where $\Omega = (-1,1)^d$, these arise from Cartesian products of the univariate eigenfunctions

$$\phi_0^{[0]}(x) = \frac{1}{\sqrt{2}}, \quad \phi_n^{[0]}(x) = \cos n\pi x, \quad \phi_n^{[1]}(x) = \sin(n-\tfrac{1}{2})\pi x, \quad n = 1,2,..., \quad x \in [-1,1].$$

Given multi-indices $n = (n_1,...,n_d) \in \mathbb{N}^d$ and $i = (i_1,...,i_d) \in \{0,1\}^d$, the d-variate eigenfunctions are

$$\phi_n^{[i]}(x) = \prod_{j=1}^{d} \phi_{n_j}^{[i_j]}(x_j), \quad x = (x_1,...,x_d) \in [-1,1]^d, \tag{1.1}$$

with corresponding eigenvalues

$$\mu_n^{[i]} = \sum_{j=1}^{d} \mu_{n_j}^{[i_j]}, \quad \text{where} \quad \mu_0^{[0]} = 0, \quad \mu_n^{[0]} = n^2\pi^2, \quad \mu_n^{[1]} = (n-\tfrac{1}{2})^2\pi^2, \quad n = 1,2,...$$

For ease of notation we shall write $\phi_n^{[i]}$ and $\mu_n^{[i]}$ as above, with the understanding that $\phi_n^{[i]} = 0$ and $\mu_n^{[i]} = 0$ if $i_j = 1$ and $n_j = 0$ for some $j = 1,...,d$.

Concerning the density of such functions, we have the following:

Lemma 1. *The set $\{\phi_n^{[i]} : n \in \mathbb{N}^d,\ i \in \{0,1\}^d\}$ is an orthonormal basis of $L^2(-1,1)^d$.*

Proof. This is a standard result of spectral theory. □

For a function $f \in L^2(-1,1)^d$, truncation parameter N and finite index set $I_N \subset \mathbb{N}^d$, we define the truncated modified Fourier series of f as

$$\mathcal{F}_N[f](x) = \sum_{i\in\{0,1\}^d} \sum_{n\in I_N} \hat{f}_n^{[i]} \phi_n^{[i]}(x), \quad \text{where} \quad \hat{f}_n^{[i]} = \int_\Omega f(x)\phi_n^{[i]}(x)\, dx.$$

In [13, 14] quadrature routines are developed to evaluate these coefficients numerically. Using highly oscillatory methods, where applicable, and so-called exotic quadrature elsewhere, any M coefficients can be found in $\mathcal{O}(M)$ operations. We shall not discuss such routines here. Highly oscillatory methods are greatly advantageous for such approximations (they facilitate the use of hyperbolic cross index sets). However, there are a number of unresolved issues and open problems associated with their implementation, which we do not intend to address presently. We refer the reader to [13] and references therein for further detail. For the remainder of this paper we shall assume that the error in approximating the coefficients is insignificant in comparison to the error in approximating f by $\mathcal{F}_N[f]$.

If we define the finite dimensional space

$$\mathcal{S}_N = \text{span}\{\phi_n^{[i]} : n \in I_N,\ i \in \{0,1\}^d\},$$

then $\mathcal{F}_N : L^2(-1,1)^d \to \mathcal{S}_N$ is the orthogonal projection onto $\mathcal{S}_N$ with respect to the standard Euclidean inner product. We state, without proof, a version of Parseval's lemma for such series:

Lemma 2 (Parseval)**.** *Suppose that $f \in L^2(-1,1)^d$, $\cup_{N\geq 0} I_N = \mathbb{N}^d$ and $I_1 \subset I_2 \subset ... \subset \mathbb{N}^d$. Then $\mathcal{F}_N[f]$ is the best approximation to f from $\mathcal{S}_N$ in the L^2 norm, $\|f - \mathcal{F}_N[f]\| \to 0$ as $N \to \infty$ and*

$$\|f\|^2 = \sum_{i\in\{0,1\}^d} \sum_{n\in\mathbb{N}^d} |\hat{f}_n^{[i]}|^2. \tag{1.2}$$

Unlike its Fourier counterpart, the modified Fourier basis is not closed under differentiation. If we differentiate $\phi_n^{[i]}$ with respect to x_1, say, we obtain

$$\partial_{x_1}\phi_n^{[i]}(x) = (-1)^{1+i_1}(\mu_{n_1}^{[i_1]})^{\frac{1}{2}}\left[\psi_{n_1}^{[1-i_1]}(x_1)\prod_{j=2}^{d}\phi_{n_j}^{[i_j]}(x_j)\right],$$

where $\{\psi_n^{[i]} : i = 0,1,\ n = 1,2,...\}$ is the set of eigenfunctions of the univariate Laplace operator subject to homogeneous Dirichlet boundary conditions:

$$\psi_n^{[0]}(x) = \cos(n-\tfrac{1}{2})\pi x, \quad \psi_n^{[1]}(x) = \sin n\pi x, \quad n = 1,2,...$$

In particular the Laplace–Neumann and Laplace–Dirichlet operators share eigenvalues (aside from the 0 eigenvalue of the former). We conclude that $\partial_{x_1}\phi_n^{[i]}(x)$ is proportional to an eigenfunction of the Laplace operator on $[-1,1]^d$ which obeys homogeneous Dirichlet boundary conditions on the subset of the boundary $\Gamma_1^{\pm}$, where

$$\Gamma_j^{\pm} = \{x \in [-1,1]^d\ : x_j = \pm 1\}, \quad j = 1,...,d,$$

and homogeneous Neumann boundary conditions on $\Gamma\backslash(\Gamma_1^+ \cup \Gamma_1^-)$, where $\Gamma = \partial\Omega = \cup_j \Gamma_j^{\pm}$. Such eigenfunctions are orthogonal and dense in $L^2(-1,1)^d$. Repeating this argument for various j we obtain:

Lemma 3 (Duality)**.** *Suppose that $\alpha = (\alpha_1,...,\alpha_d) \in \mathbb{N}^d$. If we apply the operator D^α to the set of modified Fourier eigenfunctions we obtain, up to scalar multiples, the eigenfunctions of the Laplace operator that obey homogeneous Dirichlet boundary conditions on the faces $\Gamma_j^{\pm}$ where α_j is odd, and homogeneous Neumann boundary conditions elsewhere. Such eigenfunctions are orthonormal and dense in $L^2(-1,1)^d$.*

This duality is essential to proving many of the convergence properties of modified Fourier series. As mentioned, the modified Fourier basis is not only L^2 dense, but also dense in various other Sobolev norms. Using this lemma, we now show this for the H^1 norm:

Lemma 4. *Suppose that $f \in H^1(-1,1)^d$ and I_N is as in Parseval's Lemma. Then $\|f - \mathcal{F}_N[f]\|_1 \to 0$ as $N \to \infty$. Furthermore, we have*

$$\|f\|_1^2 = \sum_{i\in\{0,1\}^d}\sum_{n\in\mathbb{N}^d}(1+\mu_n^{[i]})|\hat{f}_n^{[i]}|^2, \tag{1.3}$$

and additionally, $\mathcal{F}_N[f]$ is the best approximation to f from $\mathcal{S}_N$ in the H^1 norm.

Proof. It suffices to prove that $\|\partial_{x_j}(f - \mathcal{F}_N[f])\| \to 0$ as $N \to \infty$ for each j. By symmetry, it is enough to consider the case $j = 1$. Now,

$$\partial_{x_1}\mathcal{F}_N[f](x) = \sum_{i\in\{0,1\}^d}\sum_{\substack{n\in I_N \\ n_1\neq 0}}\hat{f}_n^{[i]}(-1)^{1+i_1}(\mu_{n_1}^{[i_1]})^{\frac{1}{2}}\tilde{\phi}_n^{[i]}(x),$$

where

$$\tilde{\phi}_n^{[i]} = \psi_{n_1}^{[1-i_1]}(x_1)\prod_{j=2}^{d}\phi_{n_j}^{[i_j]}(x_j),$$

is an eigenfunction of the type introduced above. For $f \in H^1(-1,1)^d$ and $n_1 \neq 0$, we obtain, via Green's theorem,

$$\hat{f}_n^{[i]} = \int_\Omega f(x)\phi_n^{[i]}(x)\ dx = (-1)^{i_1}(\mu_{n_1}^{[i_1]})^{-\frac{1}{2}}\int_\Omega f(x)\partial_{x_1}\tilde{\phi}_n^{[i]}(x)\ dx = (-1)^{1+i_1}(\mu_{n_1}^{[i_1]})^{-\frac{1}{2}}\int_\Omega \partial_{x_1}f(x)\tilde{\phi}_n^{[i]}(x)\ dx.$$

Using the above relation, we see that $\partial_{x_1}\mathcal{F}_N[f](x)$ is precisely the orthogonal projection of $\partial_{x_1}f$ onto

$$\tilde{\mathcal{S}}_N = \mathrm{span}\{\tilde{\phi}_n^{[i]}\ : n \in I_N,\ n_1 \neq 0,\ i \in \{0,1\}^d\}.$$

By the Duality lemma, the set $\{\tilde{\phi}_n^{[i]} : n \in \mathbb{N}^d,\ i \in \{0,1\}^d\}$ is an orthonormal basis is of $L^2(-1,1)^d$. In particular, $\|\partial_{x_1}(f - \mathcal{F}_N[f])\| \to 0$ as $N \to \infty$ since $\partial_{x_1} f \in L^2(-1,1)^d$. Furthermore, using a version of Parseval's lemma for this basis we see that

$$\|\partial_{x_1} f\|^2 = \sum_{n\in\mathbb{N}^d} \sum_{i\in\{0,1\}^d} \mu_{n_1}^{[i_1]} |\hat{f}_n^{[i]}|^2.$$

Replacing 1 by $j = 2, ..., d$ in the above formula and summing each contribution gives (1.3). To conclude that $\mathcal{F}_N[f]$ is the best approximation in the H^1 norm, we merely notice that $\mathcal{F}_N : H^1(-1,1)^d \to \mathcal{S}_N$ is the orthogonal projection with respect to the H^1 inner product. □

Lemma 4 provides an equivalent characterization of the H^1 norm in terms of modified Fourier coefficients. Much the same is done for Fourier series and the periodic spaces $H^q(\mathbb{T}^d)$, for $q \geq 0$. However, we cannot do the same for modified Fourier series for $q \neq 0, 1$ unless we restrict to classes of functions where all the odd derivatives vanish on $\partial\Omega$—the analogue of periodicity for modified Fourier series. We shall not fully adopt this approach. Nonetheless, in the sequel it will be useful to consider the modified Fourier expansion of a function that satisfies a finite number of derivative conditions on the boundary. For this we have the following result:

Lemma 5. *Suppose that $u \in H^{2k+1}(-1,1)^d$ obeys homogeneous Neumann boundary conditions up to order k on $\partial\Omega$:*

$$\partial_{x_j}^{2r+1} u\big|_{\Gamma_j^{\pm}} = 0, \quad j = 1, ..., d, \quad r = 0, ..., k-1. \tag{1.4}$$

Then, for $r = 0, ..., 2k+1$, $\mathcal{F}_N[u]$ is the best approximation to u from $\mathcal{S}_N$ in the H^r norm, $\|u - \mathcal{F}_N[u]\|_r \to 0$ and we have the characterization:

$$\|u\|_r^2 = \sum_{i\in\{0,1\}^d} \sum_{n\in\mathbb{N}^d} \left[\sum_{|\alpha|\leq r} \prod_{j=1}^{d} (\mu_{n_j}^{[i_j]})^{\alpha_j} \right] |\hat{u}_n^{[i]}|^2. \tag{1.5}$$

Proof. This is very similar to Lemma 4. We may show (by repeated application of Green's theorem, noticing that the boundary integrals vanish due to (1.4)) that if u obeys the prescribed boundary conditions then $D^\alpha \mathcal{F}_N[u]$, $|\alpha| \leq 2k+1$, is precisely the orthogonal projection of $D^\alpha u$ onto the space spanned by eigenfunctions that satisfy homogeneous Dirichlet boundary conditions on faces $\Gamma_j^{\pm}$ when α_j is odd, and Neumann boundary conditions elsewhere. □

In the sequel we shall use a simple version of Bernstein's inequality, which now follows immediately:

Corollary 1 (Bernstein's Inequality)**.** *Suppose that $\phi \in \mathcal{S}_N$. Then, for $r \in \mathbb{N}$, we have*

$$\|\phi\|_r \leq \max_{n\in I_N} \left\{ (1 + \mu_n^{[0]})^{\frac{r}{2}} \right\} \|\phi\|. \tag{1.6}$$

Proof. For $i \in \{0,1\}^d$ and $n \in I_N$, $\mu_n^{[i]} \leq \mu_n^{[0]}$. Furthermore

$$(1 + \mu_n^{[i]})^r = \sum_{|\alpha|\leq r} c_{\alpha,r} \prod_{j=1}^{d} (\mu_{n_j}^{[i_j]})^{\alpha_j} \tag{1.7}$$

for some constants $c_{\alpha,r} \geq 1$. Hence, for $\phi \in \mathcal{S}_N$, we obtain

$$\|\phi\|_r^2 \leq \sum_{i\in\{0,1\}^d} \sum_{n\in I_N} (1 + \mu_n^{[i]})^r |\hat{\phi}_n^{[i]}|^2 \leq \max_{n\in I_N} \left\{ (1 + \mu_n^{[0]})^r \right\} \sum_{i\in\{0,1\}^d} \sum_{n\in I_N} |\hat{\phi}_n^{[i]}|^2,$$

and Parseval's lemma gives the result. □

An advantage of modified Fourier series is that there is no Gibbs' phenomenon on the boundary. Indeed the modified Fourier expansion of a sufficiently smooth function converges uniformly on $[-1,1]^d$. We shall now prove this. One reason for doing so is to be able to express the error as a convergent infinite series, which in turn will allow us to derive estimates for the pointwise and uniform rates of convergence. This shall require particular choices of the index set I_N, which we defer to the sequel. However, uniform convergence may be proved independently of the choice of index set. To do so we must consider Sobolev spaces of dominating mixed smoothness.

1.2 Sobolev spaces of dominating mixed smoothness

Sobolev spaces of dominating mixed smoothness are the standard setting whenever a hyperbolic cross index set is employed, [3, 20, 23]. In the particular case of modified Fourier series, even for full index sets, such spaces provide a suitable framework for analysis.

It turns out that the modified Fourier basis is not just dense in the space $H^1(-1,1)^d$, but also in the first Sobolev space of dominating mixed smoothness, which we denote $H^1_{\text{mix}}(-1,1)^d$. This fact ensures uniform convergence of $\mathcal{F}_N[f]$ to f which we shall prove in the next section.

Subsequently we shall also see that the corresponding norms are precisely those required to bound the modified Fourier coefficients $\hat{f}^{[i]}_n$ in inverse powers of $n_1 \ldots n_d$, which leads to quasi-optimal error estimates and justifies the use of a hyperbolic cross index set.

For $k \in \mathbb{N}$ we define the k^{th} Sobolev space of dominating mixed smoothness by

$$H^k_{\text{mix}}(-1,1)^d = \{f : D^\alpha f \in L^2(-1,1)^d,\ \forall\ \alpha : |\alpha|_\infty \leq k\}, \tag{1.8}$$

where $|\alpha|_\infty = \max\{\alpha_i\}$, with norm

$$\|f\|^2_{k,\text{mix}} = \sum_{|\alpha|_\infty \leq k} \|D^\alpha f\|^2. \tag{1.9}$$

This space is also commonly denoted by $S^{(k,\ldots,k)}_2 H(-1,1)^d$ in literature, [20, 23].

In an identical manner to Lemma 5, we may characterize the H^1_{mix} norm in terms of modified Fourier coefficients. We merely notice (recalling the proof of Lemma 4) that $D^\alpha \mathcal{F}_N[f]$ is the orthogonal projection onto some suitable finite dimensional space not just for $|\alpha| \leq 1$, but also for $|\alpha|_\infty \leq 1$. This yields:

Lemma 6. *Suppose that $f \in H^1_{\text{mix}}(-1,1)^d$. Then, $\mathcal{F}_N[f]$ is the best approximation to f from $\mathcal{S}_N$ in the H^1_{mix} norm, $\|f - \mathcal{F}_N[f]\|_{1,\text{mix}} \to 0$ and we have the characterization:*

$$\|f\|^2_{1,\text{mix}} = \sum_{i \in \{0,1\}^d} \sum_{n \in \mathbb{N}^d} \left[\sum_{|\alpha|_\infty \leq 1} \prod_{j=1}^d (\mu^{[i_j]}_{n_j})^{\alpha_j} \right] |\hat{f}^{[i]}_n|^2. \tag{1.10}$$

Furthermore, suppose that $u \in H^{2k+1}_{\text{mix}}(-1,1)^d$ satisfies the first k derivative conditions (1.4). Then, for $r = 0,1,\ldots,2k+1$, $\mathcal{F}_N[u]$ is the best approximation to u in the H^r_{mix} norm, $\|u - \mathcal{F}_N[u]\|_{r,\text{mix}} \to 0$ and

$$\|u\|^2_{r,\text{mix}} = \sum_{i \in \{0,1\}^d} \sum_{n \in \mathbb{N}^d} \left[\sum_{|\alpha|_\infty \leq r} \prod_{j=1}^d (\mu^{[i_j]}_{n_j})^{\alpha_j} \right] |\hat{u}^{[i]}_n|^2. \tag{1.11}$$

1.3 Uniform convergence

We commence with the following lemma:

Lemma 7. *Suppose that $f \in H^1_{\text{mix}}(-1,1)^d$. Then, $f \in C[-1,1]^d$ and there is a constant c independent of f such that*

$$\|f\|_\infty \leq c\|f\|_{1,\text{mix}}. \tag{1.12}$$

To prove this we need the following lemma:

Lemma 8. *Suppose that $f \in C^\infty[-1,1]^d$. Then*

$$f(x) = \sum_{t \in [d]} \int_{-1}^{x_{t_1}} \ldots \int_{-1}^{x_{t_{|t|}}} D_t f(x_t, -1)\ dx_{t_1} \ldots dx_{t_{|t|}} + f(-1,\ldots,-1), \quad x \in [-1,1]^d. \tag{1.13}$$

where $[d]$ is the set of ordered tuples of length at most d with entries in $\{1,\ldots,d\}$, $D_t = \partial_{x_{t_1}} \ldots \partial_{x_{t_{|t|}}}$ for $t = (t_1,\ldots,t_{|t|}) \in [d]$ and $(x_t,-1) \in \mathbb{R}^d$ has j^{th} entry x_j if $j \in t$ and -1 otherwise.

Proof. We use induction on d. For $d = 1$ we have $f(x) = \int_{-1}^{x} f'(x)\ dx + f(-1)$, so the result holds. Now assume that (1.13) holds for $d-1$. Then

$$
\begin{aligned}
f(x) &= \int_{-1}^{x_d} \partial_{x_d} f(x)\ dx_d + f(x_1, ..., x_{d-1}, -1) \\
&= \sum_{t\in[d-1]} \int_{-1}^{x_{t_1}} ... \int_{-1}^{x_{t_{|t|}}} \int_{-1}^{x_d} \partial_{x_d} D_t f((x_t, x_d), -1)\ dx_{t_1} ... dx_{t_{|t|}} dx_d \\
&\quad + \sum_{t\in[d-1]} \int_{-1}^{x_{t_1}} ... \int_{-1}^{x_{t_{|t|}}} D_t f(x_t, -1)\ dx_{t_1} ... dx_{t_{|t|}} + \int_{-1}^{x_d} \partial_{x_d} f(x_d, -1)\ dx_d + f(-1, ..., -1).
\end{aligned}
$$

Since the set $[d]$ is comprised of elements t, $(t, d) = (t_1, ..., t_{|t|}, d)$, (d), where $t \in [d-1]$, this expression reduces to (1.13). Hence the proof is complete. □

Proof of Lemma 7. We first prove the result for $f \in C^\infty[-1,1]^d$. We note that

$$
f(x_t, -1) = 2^{d-|t|} \int_{-1}^{1} ... \int_{-1}^{1} D_{\bar{t}} \left(f(x) \prod_{j\notin t} (x_j - 1) \right) dx_{\bar{t}_1} ... dx_{\bar{t}_{d-|t|}}, \quad \forall t \in [d],
$$

where $\bar{t} \in [d]$ is the tuple of length $d - |t|$ of elements not in t. Hence, using Lemma 8, we have

$$
\begin{aligned}
f(x) = \sum_{t\in[d]} \int_{-1}^{1} ... \int_{-1}^{1} \int_{-1}^{x_{t_1}} ... \int_{-1}^{x_{t_{|t|}}} D \left(f(x) \prod_{j\notin t} (x_j - 1) \right)\ dx_{t_1} ... dx_{t_{|t|}} dx_{\bar{t}_1} ... dx_{\bar{t}_{d-|t|}} \\
+ \int_{-1}^{1} ... \int_{-1}^{1} D \left(f(x) \prod_{j=1}^{d} (x_j - 1) \right) dx_1 ... dx_d.
\end{aligned}
$$

Each integrand involves terms of the form $D^\alpha f$ for some $|\alpha|_\infty \le 1$. Hence, using the Cauchy–Schwarz inequality and replacing suitable upper limits of integration by 1, we obtain (1.12) for $f \in C^\infty[-1,1]^d$.

We now proceed in the standard manner. If $f \in H^1_{\text{mix}}(-1,1)^d$ then f is the limit in $H^1_{\text{mix}}(-1,1)^d$ of a sequence of functions belonging to $C^\infty[-1,1]^d$. Since (1.12) holds for $f \in C^\infty[-1,1]^d$ this sequence converges uniformly on $[-1,1]^d$ to $\tilde{f} \in C[-1,1]^d$. Since $f = \tilde{f}$ a.e. the result follows. □

Theorem 2. *Suppose that $f \in H^1_{mix}(-1,1)^d$ and I_N satisfies the conditions of Parseval's lemma. Then, $\mathcal{F}_N[f]$ converges pointwise to f for all $x \in [-1,1]^d$. Moreover, the convergence is uniform.*

Proof. Replacing f by $f - \mathcal{F}_N[f]$ in (1.12) and applying Lemma 6 gives the result. □

Prior to considering various different choices of index set and the corresponding error estimates for modified Fourier series, we need to develop bounds for the modified Fourier coefficients:

1.4 Bounds for modified Fourier coefficients

We commence with the following lemma:

Lemma 9. *Suppose that $f \in H^1_{mix}(-1,1)^d$ and $n \in \mathbb{N}^d_+ = (\mathbb{N}\backslash\{0\})^d$. Then*

$$
\hat{f}^{[i]}_n = (-1)^{d+|i|} \left(\prod_{j=1}^{d} \mu^{[i_j]}_{n_j} \right)^{-\frac{1}{2}} \int_\Omega Df(x) \psi^{[1-i]}_n(x)\ dx, \tag{1.14}
$$

where $1 - i$ is the multi-index $(1 - i_1, ..., 1 - i_d)$.

Proof. This is obtained by repeated application of Green's theorem. □

To obtain robust bounds for the coefficients $\hat{f}_n^{[i]}$ we need to apply Green's theorem to the right hand side of (1.14). However, in this case the boundary integrals are non-vanishing, so we need some additional notation. Given a tuple $t \in [d]$ and $x = (x_1, ..., x_d) \in \mathbb{R}^d$ we define $x_t = (x_{t_1}, ..., x_{t_{|t|}}) \in \mathbb{R}^{|t|}$. Further, for $i \in \{0,1\}$ and $j = 1, ..., d$ we define the operator $\triangle_j^{[i]} : H^1_{\text{mix}}(-1,1)^d \to H^1_{\text{mix}}(-1,1)^{d-1}$ by

$$\triangle_j^{[i]}[g](x_1, ..., x_{j-1}, x_{j+1}, ..., x_d) = g(x_1, ..., x_{j-1}, 1, x_{j+1}, ..., x_d) + (-1)^{i+1} g(x_1, ..., x_{j-1}, -1, x_{j+1}, ..., x_d).$$

Given $t = (t_1, ..., t_{|t|}) \in [d]$ and $i \in \{0,1\}^{|t|}$ we define $\triangle_t^{[i]} : H^1_{\text{mix}}(-1,1)^d \to H^1_{\text{mix}}(-1,1)^{d-|t|}$ by

$$\triangle_t^{[i]}[g] = \triangle_{t_1}^{[i_1]}\left[\triangle_{t_2}^{[i_2]}\left[...\left[\triangle_{t_{|t|}}^{[i_{|t|}]}[g]\right]\right]\right].$$

Note that the operators $\triangle_{t_1}^{[i_{t_1}]}, ..., \triangle_{t_{|t|}}^{[i_{t_{|t|}}]}$ commute with each other and with partial differentiation in the independent variables. Further if g is a function of $x \in [-1,1]^d$ then $\triangle_t^{[i]}[g]$ is a function of $x_{\bar{t}}$.

Finally, given $t \in [d]$, $i \in \{0,1\}^d$ and $n \in \mathbb{N}^{d-|t|}$ we define $\Theta_{t,n}^{[i]} : H^1_{\text{mix}}(-1,1)^d \to \mathbb{R}$ by

$$\Theta_{t,n}^{[i]}[g] = \widehat{\triangle_t^{[i_t]}[D_{\bar{t}} g]}_n^{[i_{\bar{t}}]},$$

where $D_{\bar{t}} = \partial_{x_{\bar{t}_1}} ... \partial_{x_{\bar{t}_{|\bar{t}|}}}$. With this in hand we may now deduce the key result of this section:

Lemma 10. *Suppose that $g \in H^1_{mix}(-1,1)^d$ and $n \in \mathbb{N}_+^d$. Then*

$$\int_\Omega g(x)\psi_n^{[1-i]}(x)\ dx = (-1)^{|i|}\left(\prod_{j=1}^d \mu_{n_j}^{[i_j]}\right)^{-\frac{1}{2}}\left(\sum_{t\in[d]}(-1)^{|n_t|+|i_t|+|t|}\Theta_{t,n_{\bar{t}}}^{[i]}[g] + \widehat{Dg}_n^{[i]}\right).$$

Proof. It suffices to prove this result for $g \in C^\infty[-1,1]^d$. To cover the general case we use density, linearity and the bound

$$\left|\Theta_{t,n}^{[i]}[f]\right| \le c\|f\|_{1,\text{mix}}, \quad \forall f \in H^1_{\text{mix}}(-1,1)^d,$$

for some constant $c > 0$ (see Lemma 12). We proceed by induction on d. For $d = 1$ trivial integration by parts verifies the result. Indeed, we have

$$\int_{-1}^1 g(x)\psi_n^{[1-i]}(x)\ dx = (-1)^i\left(\mu_n^{[i]}\right)^{-\frac{1}{2}}\left((-1)^{n+i+1}\triangle_1^{[i]}[g] + \widehat{Dg}_n^{[i]}\right).$$

Suppose that the result is true for $d-1$. Then, for $g \in C^\infty[-1,1]^d$ we have

$$\int_\Omega g(x)\psi_n^{[1-i]}(x)\ dx = \int_{(-1,1)^{d-1}} h_{n_d}(x')\psi_{n'}^{[1-i']}(x')\ dx'$$

where $x' = (x_1, ..., x_{d-1})$, $i' = (i_1, ..., i_{d-1})$ and $n' = (n_1, ..., n_{d-1})$ are the first $d-1$ entries of x, i and n respectively, and

$$h_{n_d}(x') = \int_{-1}^1 g(x)\psi_{n_d}^{[1-i_d]}(x_d)\ dx_d.$$

By induction:

$$\int_\Omega g(x)\psi_n^{[1-i]}(x)\ dx = (-1)^{|i'|}\left(\prod_{j=1}^{d-1}\mu_{n'_j}^{[i'_j]}\right)^{-\frac{1}{2}}\left(\sum_{t'\in[d-1]}(-1)^{|n_{t'}|+|i_{t'}|+j}\Theta_{t',n'_{\bar{t}'}}^{[i']}[h_{n_d}] + \widehat{Dh_{n_d}}_{n'}^{[i']}\right). \tag{1.15}$$

Using the formula for $d = 1$ we may write

$$h_{n_d}(x') = (-1)^{i_d}\left(\mu_{n_d}^{[i_d]}\right)^{-\frac{1}{2}}\left((-1)^{n_d+i_d+1}\triangle_d^{[i_d]}[g](x') + \widehat{\partial_{x_d} g}_{n_d}^{[i_d]}\right).$$

Now we consider terms of (1.15) separately. We have

$$\Theta^{[i']}_{t',n'_{\bar{t}'}}[\triangle^{[i_d]}_d[D_{\bar{t}'}g]] = \Theta^{[i]}_{t,n_{\bar{t}}}[g],$$

where $t = (t', d) \in [d]$. Also

$$\Theta^{[i']}_{t',n'_{\bar{t}'}}\left[\widehat{\partial_{x_d} g}^{[i_d]}_{n_d}\right] = \Theta^{[i]}_{t,n_{\bar{t}}}[g],$$

where, in this case, $t = t' \in [d]$. For the second term of (1.15) we have

$$\begin{aligned}\widehat{Dh_{n_d}}^{[i']}_{n'} &= (-1)^{i_d}\left(\mu^{[i_d]}_{n_d}\right)^{-\frac{1}{2}}\left((-1)^{n_d+i_d+1}\widehat{\triangle^{[i_d]}_d[D_{\bar{d}}g]}^{[i']}_{n'} + \widehat{Dg}^{[i]}_n\right)\\ &= (-1)^{i_d}\left(\mu^{[i_d]}_{n_d}\right)^{-\frac{1}{2}}\left((-1)^{n_d+i_d+1}\Theta^{[i]}_{d,n}[g] + \widehat{Dg}^{[i]}_n\right),\end{aligned}$$

where $\bar{d} = (1, ..., d-1) \in [d]$. Combining these results, we obtain

$$\begin{aligned}\breve{g}^{[1-i]}_n = (-1)^{|i|}\left(\prod_{j=1}^{d}\mu^{[i_j]}_{n_j}\right)^{-\frac{1}{2}}&\left\{\sum_{t'\in[d-1]}(-1)^{|n_{t'}|+|i_{t'}|+|t'|}\left((-1)^{n_d+i_d+1}\Theta^{[i]}_{t,n_{\bar{t}}}[g] + \Theta^{[i]}_{t',n_{\bar{t}'}}[g]\right)\right.\\ &\left. + (-1)^{n_d+i_d+1}\Theta^{[i]}_{d,n}[g] + \widehat{Dg}^{[i]}_n\right\},\end{aligned}$$

where $t = (t', d)$. Now, the set $[d]$ contains precisely elements (t', d), t', and (d) where $t' \in [d-1]$. The first three terms in the above sum correspond to each of these. Furthermore

$$\begin{aligned}&\text{if}\quad t = (t', d),\quad \text{then}\quad |n_t| = |n_{t'}| + n_d,\quad |i_t| = |i_{t'}| + i_d,\quad |t| = |t'| + 1,\\ &\text{if}\quad t = t',\quad \text{then}\quad |n_t| = |n_{t'}|,\quad |i_t| = |i_{t'}|,\quad |t| = |t'|,\\ &\text{if}\quad t = (d),\quad \text{then}\quad |n_t| = n_d,\quad |i_t| = i_d,\quad |t| = 1.\end{aligned}$$

With this in hand, we obtain the result. □

Corollary 3. *Suppose that* $f \in H^2_{mix}(-1,1)^d$ *and* $n \in \mathbb{N}^d_+$. *Then*

$$\hat{f}^{[i]}_n = (-1)^d\left(\prod_{j=1}^{d}\mu^{[i_j]}_{n_j}\right)^{-1}\left(\sum_{t\in[d]}(-1)^{|n_t|+|i_t|+|t|}\Theta^{[i]}_{t,n_{\bar{t}}}[Df] + \widehat{D^2f}^{[i]}_n\right).$$

Iterating this formula immediately yields the following:

Theorem 4. *Suppose that* $f \in H^{2k}_{mix}(-1,1)^d$ *and* $n \in \mathbb{N}^d_+$. *Then*

$$\hat{f}^{[i]}_n = \sum_{r=0}^{k-1}\left((-1)^d\prod_{j=1}^{d}\mu^{[i_j]}_{n_j}\right)^{-(r+1)}\left(\sum_{t\in[d]}(-1)^{|n_t|+|i_t|+|t|}\Theta^{[i]}_{t,n_{\bar{t}}}[D^{2r+1}f]\right) + (-1)^{kd}\left(\prod_{j=1}^{d}\mu^{[i_j]}_{n_j}\right)^{-k}\widehat{D^{2k}f}^{[i]}_n.$$

If f *obeys the first* k *derivative conditions then the first* k *terms vanish.*

Proof. To obtain the formula we use Corollary 3. Now suppose that f obeys the prescribed derivative conditions. It follows that $D^{2r+1}f$, $r = 0, ..., k-1$, vanishes on the boundary. Since each of the first k terms involves at least one evaluation of $D^{2r+1}f$ on the boundary, we deduce the result. □

Thus far we have assumed that $n \in \mathbb{N}^d_+$. Now suppose that $n_{t_1} = ...n_{t_{|t|}} = 0$ for some $t \in [d]$. Then $\hat{f}^{[i]}_n = \widehat{f_t}^{[i_{\bar{t}}]}_{n_{\bar{t}}}$ where f_t is the function defined by

$$f_t(x_{\bar{t}}) = \int_{-1}^{1}...\int_{-1}^{1} f(x)\ dx_{t_1}...d_{x_{t_{|t|}}}.$$

Hence Theorem 4 may be used for f_t and in turn for $\hat{f}^{[i]}_n$ when $n \in \mathbb{N}^d$.

As examples of Theorem 4, consider the cases $d = 1, 2$. For $d = 1$ we have

$$\hat{f}^{[i]}_n = \sum_{r=0}^{k-1} \frac{(-1)^{n+i}}{(\mu^{[i]}_n)^{r+1}} \triangle^{[i]}[f^{(2r+1)}] + \frac{(-1)^k}{(\mu^{[i]}_n)^k} \widehat{f^{(2k)}}^{[i]}_n, \quad n \in \mathbb{N}, \quad i \in \{0, 1\},$$

and for $d = 2$ with $k = 1$ the corresponding result is

$$\hat{f}^{[i]}_n = \frac{1}{\mu^{[i_1]}_{n_1} \mu^{[i_2]}_{n_2}} \Bigg\{ (-1)^{n_1+n_2+i_1+i_2} \triangle^{[i]}[\partial_{x_1} \partial_{x_2} f] + (-1)^{n_1+i_1+1} \widehat{\triangle^{[i_1]}[\partial_{x_1} \partial^2_{x_2} f]}^{[i_2]}_{n_2}$$
$$+ (-1)^{n_2+i_2+1} \widehat{\triangle^{[i_2]}[\partial^2_{x_1} \partial_{x_2} f]}^{[i_1]}_{n_1} + \widehat{\partial^2_{x_1} \partial^2_{x_2} f}^{[i]}_n \Bigg\}, \quad n \in \mathbb{N}^2_+, \quad i \in \{0, 1\}^2,$$

$$\hat{f}^{[i_1,0]}_{n_1,0} = \frac{1}{\mu^{[i_1]}_{n_1}} \left\{ (-1)^{n_1+i_1} \widehat{\triangle^{[i_1]}[\partial_{x_1} f]}^{[0]}_0 - \widehat{\partial^2_{x_1} f}^{[i_1,0]}_{n_1,0} \right\}, \quad n_1 \in \mathbb{N}_+, \quad i_1 \in \{0, 1\},$$
$$\hat{f}^{[0,i_2]}_{0,n_2} = \frac{1}{\mu^{[i_2]}_{n_2}} \left\{ (-1)^{n_2+i_2} \widehat{\triangle^{[i_2]}[\partial_{x_2} f]}^{[0]}_0 - \widehat{\partial^2_{x_2} f}^{[0,i_2]}_{0,n_2} \right\}, \quad n_2 \in \mathbb{N}_+, \quad i_2 \in \{0, 1\}.$$

We now wish to derive bounds for the coefficients $\hat{f}^{[i]}_n$. To do so it is useful to consider the following Sobolev spaces:

$$G^k_{\text{mix}}(-1, 1)^d = \{ f : D^\alpha f \in L^1(-1, 1)^d, \ \forall\ \alpha : |\alpha|_\infty \leq k \}, \quad k \in \mathbb{N},$$

with norm

$$|||f|||_{k,\text{mix}} = \sum_{|\alpha|_\infty \leq k} \|D^\alpha f\|_{L^1(-1,1)^d}.$$

Regarding such spaces, we have the following result which we shall use in the sequel:

Lemma 11. *Suppose that $f \in H^k_{mix}(-1, 1)^d$, $k \in \mathbb{N}$. Then $f \in G^k_{mix}(-1, 1)^d$ and $|||f|||_{k,mix} \leq (2k+2)^{\frac{d}{2}} \|f\|_{k,mix}$.*

Proof. The first result follows immediately since $L^2(-1, 1)^d \subset L^1(-1, 1)^d$. For the second, we use the Cauchy–Schwarz inequality to obtain

$$|||f|||_{k,\text{mix}} \leq 2^{\frac{d}{2}} \sum_{|\alpha|_\infty \leq k} \|D^\alpha f\| \leq 2^{\frac{d}{2}} \left(\sum_{|\alpha|_\infty \leq k} 1 \right)^{\frac{1}{2}} \|f\|_{k,\text{mix}}.$$

Since there are $(k + 1)^d$ choices of $\alpha \in \mathbb{N}^d$ with $|\alpha|_\infty \leq k$ we obtain the result. □

To derive coefficient bounds we first need the following lemma:

Lemma 12. *Suppose that $f \in H^1_{mix}(-1, 1)^d$, $i \in \{0, 1\}^{|t|}$, $t \in [d]$ and $n \in \mathbb{N}^{d-|t|}$. Then*

$$\left|\Theta^{[i]}_{t,n}[f]\right| \leq |||f|||_{1,mix}$$

Proof. For $i \in \{0, 1\}$ and $j = 1, ..., d$ we have

$$\triangle^{[i]}_j[f] = \int_{-1}^{1} \partial_{x_j} \left(f(x) x^i_j \right) \, dx_j.$$

This implies that

$$\triangle^{[i]}_t[f] = \int_{-1}^{1} ... \int_{-1}^{1} D_t \left(f(x) \prod_{j \in t} x^{i_j}_j \right) \, dx_{t_1} ... dx_{t_{|t|}}, \quad t \in [d], \quad i \in \{0, 1\}^d.$$

Hence

$$\Theta_{t,n}^{[i]}[f] = \int_\Omega D\left(f(x)\prod_{j\in t} x_j^{i_j}\right)\prod_{j\notin t}\phi_{n_j}^{[i_j]}(x_j)\ dx.$$

We deduce that

$$\left|\Theta_{t,n}^{[i]}[f]\right| \le \int_\Omega \left|D\left(f(x)\prod_{j\in t} x_j^{i_j}\right)\right| dx \le |||f|||_{1,\text{mix}},$$

Note that the final inequality follows since the integral is a sum over derivatives $D^\alpha g$ with $|\alpha|_\infty \le 1$ each multiplied by $x_1^{\beta_1}...x_d^{\beta_d}$ for some suitable multi-index $|\beta|_\infty \le 1$. □

Using this lemma we deduce the following:

Theorem 5. *Suppose that $f \in H^{2k+2}_{mix}(-1,1)^d$ obeys the first k derivative conditions. Then*

$$\left|\hat{f}_n^{[i]}\right| \le 2^{\chi(n)}\left(\prod_{j:n_j>0}\mu_{n_j}^{[i_j]}\right)^{-(k+1)} |||f|||_{2k+2,mix}, \quad n\in\mathbb{N}^d,$$

where $\chi(n)$, the grade of n, is the number of non-zero entries.

Proof. Suppose first that $n \in \mathbb{N}_+^d$. Then, since f obeys the first k derivative conditions,

$$\begin{aligned}|\hat{f}_n^{[i]}| &\le \left(\prod_{j=1}^d \mu_{n_j}^{[i_j]}\right)^{-(k+1)}\left(\sum_{l=1}^d\sum_{|t|=l}\left|\Theta_{t,n}^{[i]}[D^{2k+1}f]\right| + \left|\widehat{D^{2k+2}f}_n^{[i]}\right|\right)\\ &\le \left(\prod_{j=1}^d \mu_{n_j}^{[i_j]}\right)^{-(k+1)}\left(1+\sum_{l=1}^d\sum_{|t|=l}1\right)|||f|||_{2k+2,\text{mix}} \le 2^d\left(\prod_{j=1}^d \mu_{n_j}^{[i_j]}\right)^{-(k+1)} |||f|||_{2k+2,\text{mix}},\end{aligned}$$

since there are $\left(\begin{array}{c} d\\ l\end{array}\right)$ choices of $t\in[d]$ with $|t|=l$. Now suppose that $n_j=0$ for $j\in t$,where $t\in[d]$. Then, using the previous result,

$$|\hat{f}_n^{[i]}| = |\hat{f_t}_n^{[i]}| \le 2^{\chi(n)}\left(\prod_{j:n_j>0}\mu_{n_j}^{[i_j]}\right)^{-(k+1)} |||f_t|||_{2k+2,\text{mix}}.$$

Moreover,

$$|||f_t|||_{2k+2,\text{mix}} = \sum_{\substack{|\alpha|_\infty\le 2k+2\\ \alpha\in\mathbb{N}^{\chi(n)}}}\int_{(-1,1)^{\chi(n)}}\left|D^\alpha f_t(x)\right| dx \le \sum_{\substack{|\alpha|_\infty\le 2k+2\\ \alpha\in\mathbb{N}^{\chi(n)}}}\int_{(-1,1)^d}\left|D^\alpha f(x)\right| dx \le |||f|||_{2k+2,\text{mix}}.$$

Hence we obtain the result. □

Using Lemma 11 we may also derive a bound for $\hat{f}_n^{[i]}$ in terms of $\|f\|_{2k+2,\text{mix}}$:

Corollary 6. *Suppose that $f\in H^{2k+2}_{mix}(-1,1)^d$ obeys the first k derivative conditions. Then*

$$\left|\hat{f}_n^{[i]}\right| \le 2^{\chi(n)+\frac{d}{2}}(2k+3)^{\frac{\chi(n)}{2}}\left(\prod_{j:n_j>0}\mu_{n_j}^{[i_j]}\right)^{-(k+1)}\|f\|_{2k+2,mix}, \quad n\in\mathbb{N}^d.$$

Proof. If $\chi(n) = d$ the result follows immediately from Theorem 5 and Lemma 11. Now suppose that $\chi(n) < d$. We then have

$$|\hat{f}_n^{[i]}| \leq 2^{\chi(n)} \left(\prod_{j:n_j>0} \mu_{n_j}^{[i_j]} \right)^{-(k+1)} |||f_t|||_{2k+2,\mathrm{mix}}.$$

Furthermore, $|||f_t|||_{2k+2,\mathrm{mix}} \leq (4k+6)^{\frac{\chi(n)}{2}} \|f_t\|_{2k+2,\mathrm{mix}}$ and

$$\|D^\alpha f_t\| \leq 2^{\frac{d}{2} - \frac{\chi(n)}{2}} \|D^\alpha f\|, \quad \alpha \in \mathbb{N}^{\chi(n)}.$$

Combining these observations we obtain $|||f_t|||_{2k+2,\mathrm{mix}} \leq 2^{\frac{d}{2}} (2k+3)^{\frac{\chi(n)}{2}} \|f\|_{2k+2,\mathrm{mix}}$, completing the proof. □

For the results of subsequent sections the following corollary is in fact more useful:

Corollary 7 (Coefficient bounds)**.** *Suppose that $f \in H^{2k+2}_{mix}(-1,1)^d$ obeys the first k derivative conditions. Then*

$$\left|\hat{f}_n^{[i]}\right| \leq 2^{\chi(n)+\frac{d}{2}} (2k+3)^{\frac{\chi(n)}{2}} (2^{|i|}\pi^{-\chi(n)})^{2(k+1)} (\bar{n}_1 ... \bar{n}_d)^{-2(k+1)} \|f\|_{2k+2,mix}, \quad n \in \mathbb{N}^d,$$

where $\bar{m} = \max\{m, 1\}$ for $m \in \mathbb{N}$.

Proof. For $n \in \mathbb{N}_+$ and $i \in \{0,1\}$ it is easily shown that $\mu_n^{[i]} \geq (2^{|i|}\pi^{-1})^{-2} n^2$. The result now follows immediately from Corollary 6. □

These are bounds precisely what is needed to provide quasi-optimal estimates for the error $f - \mathcal{F}_N[f]$ in various norms using various index sets, as we shall see in the next two sections.

2 Full index sets

The results of the previous sections do not make any assumptions regarding the index set I_N aside from the stipulations that I_N be finite, $\cup_N I_N = \mathbb{N}^d$ and $I_1 \subset I_2 \subset .... \subset \mathbb{N}^d$. The size of I_N determines the cost of constructing the approximation $\mathcal{F}_N[f]$: using numerical quadrature, the number of operations to evaluate the coefficients is $\mathcal{O}(|I_N|)$. Standard intuition leads to the *full index set*

$$I_N = \left\{ n \in \mathbb{N}^d : \max_{j=1,\dots,d} \{n_j\} \leq N \right\}, \tag{2.1}$$

which is just the hypercube of length $N+1$ in $\mathbb{N}^d$. Indeed, the prevalence of this index set in spectral discretizations is due to the fact that the method of choice for evaluating Fourier or Chebyshev coefficients, namely the FFT, computes all the coefficients in I_N in a non-adaptive way. However, $|I_N| = \mathcal{O}\left(N^d\right)$ and this figure grows exponentially with dimension. To alleviate this problem we employ a hyperbolic cross index set in the sequel. Nonetheless, for the purposes of comparison, in the remainder of this section we consider the approximation properties of modified Fourier series based on (2.1). In the univariate case, this has been thoroughly dealt with in [1], [14] and [18]. We now extend these results to the multivariate setting.

2.1 Pointwise and uniform convergence rate

Our first theorem generalizes the univariate result of S. Olver, [18], to d-variate cubes:

Theorem 8. *Suppose that $f \in H^3_{mix}(-1,1)^d$ and I_N is the full index set (2.1). Then the error $f(x) - \mathcal{F}_N[f](x)$ is $\mathcal{O}\left(N^{-2}\right)$ uniformly in any compact subset of $(-1,1)^d$.*

Proof. We can always rewrite a function f as a linear combination of terms of the form $f(\pm x_1, x_2, ..., x_d)$, $f(x_1, \pm x_2, x_3, ..., x_d)$, and so on. This allows us to express f as a sum of functions which only have non-zero modified Fourier coefficients for only one specific value of i. For example, if $d = 1$, we have

$$f(x) = \frac{1}{2}\left[f(x) + f(-x)\right] + \frac{1}{2}\left[f(x) - f(-x)\right],$$

where $\frac{1}{2}[f(x)+f(-x)]$ and $\frac{1}{2}[f(x)-f(-x)]$ have only non-zero modified Fourier coefficients for $i=0$ and $i=1$ respectively. Hence it suffices to consider a function f with non-zero modified Fourier coefficients for just one specific value of i. Since such cases are similar, we shall take $i=0$.

We use induction on d. The case $d=1$ has been dealt with in [18], so we shall not repeat it here. Suppose that the result is true for $d-1$. Then, to simplify matters we first reformulate f so that $\hat{f}_n$ (dropping the $[i]$ superscript) is nonzero only when every component of n is positive. This may be achieved as follows. We set $f_0(x)=f(x)$ and then, given $f_1,...,f_r$, we define f_{r+1} for $r\leq d-1$ by

$$f_{r+1}(x)=f_r(x)-2^{-(d-r)}\sum_{\substack{t\in[d]\\|t|=d-r}}\int_{\Omega_t}f_r(x)\ dx,$$

where Ω_t is the $(d-r)$-dimensional subset of Ω spanned by $x_{t_1},...,x_{t_{d-r}}$. The new function $\tilde{f}=f_d$ has nonzero modified Fourier coefficients only when each component of n is nonzero. To see this we first assert the following:

$$\int_{\Omega_t}f_{r+1}(x)\ dx=0,\quad \text{whenever}\quad t\in[d],\ |t|\geq d-r.$$

In particular

$$\int_{\Omega_t}\tilde{f}(x)\ dx=0,\quad t\in[d].$$

We prove this by induction. Since

$$f_1(x)=f(x)-2^{-d}\int_{\Omega}f(x)\ dx,$$

the result is true for $r=0$. Now suppose that the result holds for $r-1$. If we integrate f_{r+1} over any Ω_t with $|t|\geq d-r$ the first term cancels with the term in the sum corresponding to t. All other terms in the sum are zero since they are integrals of f_r over subsets of the boundary of dimension at least $d-r+1$. Thus we obtain the result.

From this, it is simple to show that $\tilde{f}$ satisfies the desired property. Note that $\tilde{f}$ may be expressed as $f+g$ where g is a linear combination of functions of $d-1$ variables. Thus, to prove the theorem it suffices to consider only $\tilde{f}$, which we now rewrite as f, and use induction on g to obtain the result. Finally, we note that $\tilde{f}$ and g have the same smoothness properties as f.

Dropping the $[i]$ superscript and using Corollary 3 gives

$$\begin{aligned}\hat{f}_n&=(-1)^d\left(\prod_{j=1}^{d}(n_j\pi)^{-2}\right)\left(\sum_{t\in[d]}(-1)^{|n_t|+|t|}\Theta_{t,n_{\bar{t}}}[Df]+\widehat{D^2f}_n\right)\\&=(-1)^d\left(\prod_{j=1}^{d}(n_j\pi)^{-2}\right)\left((-1)^{|n|}c_f+\sum_{|t|<d}(-1)^{|n_t|+|t|}\Theta_{t,n_{\bar{t}}}[Df]+\widehat{D^2f}_n\right),\end{aligned}\tag{2.2}$$

where is a constant depending on f only. Using Theorem 2 we may express the error as

$$f(x)-\mathcal{F}_N[f](x)=\sum_{t\in[d]}\sum_{\substack{n_j=1\\j\notin t}}^{N}\sum_{\substack{n_j>N\\j\in t}}\hat{f}_n\phi_n(x).\tag{2.3}$$

Suppose that we define the univariate functions

$$\Phi(N,x)=g(x)-\mathcal{F}_N[g](x)=\sum_{n>N}\frac{(-1)^n}{(n\pi)^2}\cos n\pi x,\quad \Psi(N,x)=\mathcal{F}_N[g](x)=\sum_{n=1}^{N}\frac{(-1)^n}{(n\pi)^2}\cos n\pi x,$$

where $g(x) = \frac{1}{4}x^2 - \frac{1}{12}$. Then the contribution of first term of (2.2) to (2.3) is

$$c_f \sum_{n \notin I_N} (-1)^{|n|} \prod_{j=1}^{d} (n_j \pi)^{-2} \cos n_j \pi x_j = c_f \sum_{t \in [d]} \prod_{j \notin t} \Psi(N, x_j) \prod_{j \in t} \Phi(N, x_j). \tag{2.4}$$

We have $\Phi(N,x) = \mathcal{O}\left(N^{-2}\right)$ uniformly for x in closed subsets of $(-1,1)$, [18]. Trivially $\Psi(N,x)$ is uniformly bounded in N and x. Hence (2.4) is $\mathcal{O}\left(N^{-2}\right)$ uniformly in any compact subset of $(-1,1)^d$.

We deal with the other terms in the expansion of $\hat{f}_n$ in a similar manner. Consider a term of the form

$$\left(\prod_{j=1}^{d} (n_j \pi)^{-2}\right) (-1)^{|n_s|+|s|} \Theta_{s,n_{\bar{s}}}[Df], \quad s \in [d],\ |s| < d.$$

The contribution of this term to the error is

$$\sum_{t \in [d]} \Big(\sum_{\substack{n_j=1 \\ j \notin t}}^{N} (n_j \pi)^{-2} \Big) \Big(\sum_{\substack{n_j > N \\ j \in t}} (n_j \pi)^{-2} \Big) (-1)^{|n_s|+|s|} \Theta_{s,n_{\bar{s}}}[Df]. \tag{2.5}$$

Using the same method as in the proof of Lemma 12 it can be shown that

$$\left|\Theta_{s,n_{\bar{s}}}[Df]\right| = \left|\triangle_s \widehat{[D^1 D_{\bar{s}} f]}_n\right| \leq c \|f\|_{3,\mathrm{mix}} \prod_{j \notin s} n_j^{-1}.$$

Suppose that x is in a compact subset of $(-1,1)^d$. We consider the terms of (2.5) separately for each $t \in [d]$. If t and s do not have any common entry then there must be an infinite sum of the form $\sum_{n>N} n^{-3} = \mathcal{O}\left(N^{-2}\right)$. On the other hand, if t and s have at least one common entry j then we may factor out a term of the form $\Phi(N, x_j)$, which is $\mathcal{O}\left(N^{-2}\right)$, giving the result.

To bound the contribution of the final term of (2.2) we merely note that $\widehat{D^2 f}_n = \mathcal{O}\left(n_1^{-1} \ldots n_d^{-1}\right)$. □

This theorem excludes subsets of the boundary from the error estimate. Dealing with such regions is much easier, we merely use a bound for the uniform error. For such a bound, we need lower smoothness:

Lemma 13. *Suppose that $f \in H^2_{mix}(-1,1)^d$ and I_N is the full index set (2.1). Then*

$$\|f - \mathcal{F}_N[f]\|_\infty \leq c_0 \|f\|_{2,mix} N^{-1} + \textit{lower order terms},$$

where $\zeta(\cdot)$ is the zeta function and

$$c_0 = \frac{d 5^d 2^{1+\frac{d}{2}}}{\pi^2} \sqrt{3} \left[1 + 2\sqrt{3}\pi^{-2}\zeta(2)\right]^{d-1}.$$

Proof. We have

$$\|f - \mathcal{F}_N[f]\|_\infty \leq \sum_{i \in \{0,1\}^d} \sum_{n \notin I_N} |\hat{f}_n^{[i]}| \leq \|f\|_{2,\mathrm{mix}} \sum_{i \in \{0,1\}^d} 2^{2|i|} \sum_{t \in [d]} \sum_{\substack{\bar{n}_j=1 \\ j \notin t}}^{N} \sum_{\substack{n_j > N \\ j \in t}} 2^{\chi(n)+\frac{d}{2}} 3^{\frac{\chi(n)}{2}} \pi^{-2\chi(n)} (\bar{n}_1 \ldots \bar{n}_d)^{-2}.$$

The largest contribution occurs when $|t| = 1$. Since there are d such possible $t \in [d]$ we obtain

$$\|f - \mathcal{F}_N[f]\|_\infty \leq d 5^d 2^{1+\frac{d}{2}} \sqrt{3} \pi^{-2} \|f\|_{2,\mathrm{mix}} \left(1 + 2\sqrt{3}\pi^{-2} \sum_{n=1}^{N} n^{-2}\right)^{d-1} \sum_{n>N} n^{-2} + \text{lower order terms}.$$

For $r > 1$, $\sum_{n=1}^{N} n^{-r-1} = \zeta(r+1) + \mathcal{O}\left(N^{-r}\right)$ and $\sum_{n>N} n^{-r-1} = r^{-1}N^{-r} + \mathcal{O}\left(N^{-r-1}\right)$, hence we obtain the result. □

For the purposes of spectral-Galerkin methods based on modified Fourier series, Theorem 8 is somewhat superfluous: the Galerkin approximation does not normally possess a faster convergence rate inside the interval. In that case, Lemma 13 is better suited.

When a function u satisfies the first k derivative conditions, these rates of convergence increase by a factor of N^{2k}. By identical arguments we obtain:

Theorem 9. *Suppose that $u \in H^{2k+2}_{mix}(-1,1)^d$ satisfies the first k derivative conditions and I_N is the full index set (2.1). Then*

$$\|u - \mathcal{F}_N[u]\|_\infty \le c_k \|u\|_{2k+2,mix} N^{-2k-1},$$

where

$$c_k = \frac{d(1+4^{k+1})^d 2^{1+\frac{d}{2}}}{(2k+1)\pi^{2(k+1)}}\sqrt{2k+3}\left[1+2\sqrt{2k+3}\pi^{-2(k+1)}\zeta(2k+2)\right]^{d-1}.$$

Moreover, if $u \in H^{2k+3}_{mix}(-1,1)^d$, then the error $u(x) - \mathcal{F}_N[u](x)$ is $\mathcal{O}\left(N^{-2k-2}\right)$ uniformly in any compact subset of $(-1,1)^d$.

2.2 Estimates in other norms

Concerning the error in the H^s norm, we have:

Lemma 14. *Suppose that $u \in H^{2k+1}(-1,1)^d$ satisfies the first k derivative conditions and I_N is the full index set (2.1). Then*

$$\|u - \mathcal{F}_N[u]\|_s \le c_{r,s} N^{s-r}\|u\|_r, \quad r = s, ..., 2k+1, \quad s = 0, ..., 2k+1, \tag{2.6}$$

for some positive constant $c_{r,s}$ independent of u and N.

Proof. For $n \notin I_N$, $\mu^{[i]}_n \ge (N\pi)^2$. Using this, Lemma 5 and (1.7) we have

$$\|u - \mathcal{F}_N[u]\|_s^2 \le \sum_{i\in\{0,1\}^d}\sum_{n\notin I_N}(1+\mu^{[i]}_n)^s|\hat{u}^{[i]}_n|^2 \le (N\pi)^{2(s-r)}\sum_{i\in\{0,1\}^d}\sum_{n\notin I_N}(1+\mu^{[i]}_n)^r|\hat{u}^{[i]}_n|^2$$

$$\le c_{r,s}N^{2(s-r)}\sum_{i\in\{0,1\}^d}\sum_{n\in\mathbb{N}^d}\sum_{|\alpha|\le r}\prod_{j=1}^d(\mu^{[i_j]}_{n_j})^{\alpha_j}|\hat{u}^{[i]}_n|^2 = c_{r,s}N^{2(s-r)}\|u-\mathcal{F}_N[u]\|_r^2,$$

as required. □

The conclusion of Lemma 14 may lead to the assertion that, for smooth u satisfying the first k odd derivative conditions, the H^1 error, for example, is $\mathcal{O}(N^{-2k})$. However, this is not the case: the H^1 error is $\mathcal{O}(N^{-2k-\frac{1}{2}})$, which we shall now prove. To show this, instead of using the above method of proof, we utilize the coefficient bounds of Section 1.4.

Lemma 15. *Suppose that $u \in H^{2k+2}_{mix}(-1,1)^d$ satisfies the first k derivative conditions and I_N is the full index set (2.1). Then*

$$\|u - \mathcal{F}_N[u]\|_s \le c_s N^{s-2k-\frac{3}{2}}\|u\|_{2k+2,mix} + \textit{lower order terms}, \quad s = 0, ..., 2k+1, \tag{2.7}$$

where

$$c_s = \frac{\sqrt{d}\left(1+16^{k+1}\right)^{\frac{d-1}{2}} 2^{1+\frac{d}{2}}}{\pi^{2(k+1)-s}(4k+3-2s)}\sqrt{2k+3}\left[1+4(2k+3)\pi^{-4(k+1)}\zeta(4k+4)\right]^{\frac{d-1}{2}}.$$

Proof. Using Lemma 5 we have

$$\|u - \mathcal{F}_N[u]\|_s^2 = \sum_{i\in\{0,1\}^d}\sum_{|\alpha|\le s}\sum_{t\in[d]}\sum_{\substack{\bar{n}_j=1\\ j\notin t}}^{N}\sum_{\substack{n_j>N\\ j\in t}}|\hat{u}^{[i]}_n|^2\prod_{j=1}^d(\mu^{[i_j]}_{n_j})^{\alpha_j}.$$

Since $\hat{u}_n^{[i]} = \mathcal{O}\left((\bar{n}_1...\bar{n}_d)^{-2k-2}\right)$ the largest contribution occurs when $|t| = 1$ and $\alpha_j = s$ if $j \in t$ and 0 otherwise. Hence, as in the proof of Lemma 13, we obtain

$$\|u - \mathcal{F}_N[u]\|_s^2 \leq d \sum_{i \in \{0,1\}^d} \sum_{\substack{\bar{n}_j=1 \\ j=1,...,d-1}}^{N} \sum_{n_d > N} |\hat{u}_n^{[i]}|^2 (\mu_{n_d}^{[i_d]})^s + \text{lower order terms}.$$

After an application of the Coefficient bounds corollary and using arguments similar to those given in Lemma 13 we obtain the result. □

3 Hyperbolic cross index sets

A hyperbolic cross index set is obtained by including only those terms in the expansion

$$\sum_{i \in \{0,1\}^d} \sum_{n \in \mathbb{N}^d} \hat{f}_n^{[i]} \phi_n^{[i]}(x),$$

whose absolute value in some norm is greater than some tolerance ϵ. To do so, we need some bound for the coefficients $\hat{f}_n^{[i]}$ and the functions $\phi_n^{[i]}$. Given a norm $|||\cdot|||$, we use the coefficient bounds of Section 1.4:

$$|||\hat{f}_n^{[i]} \phi_n^{[i]}||| \leq c\|f\|_{2,\text{mix}} |||\phi_n^{[i]}||| (\bar{n}_1...\bar{n}_d)^{-2}.$$

We shall consider the index set that originates from the L^2 and uniform norms. In this case $\|\phi_n^{[i]}\|_\infty = \|\phi_n^{[i]}\| = 1$, and $|||\hat{f}_n^{[i]} \phi_n^{[i]}||| \leq c\|f\|_{2,\text{mix}} (\bar{n}_1...\bar{n}_d)^{-2}$. The tolerance ϵ is precisely this upper bound with $n = (N, 1, ...1)$, in other words $\epsilon = c\|f\|_{2,\text{mix}} N^{-2}$. This yields the *hyperbolic cross*, [2, 23], index set

$$I_N = \{n \in \mathbb{N}^d : \bar{n}_1...\bar{n}_d \leq N\}. \tag{3.1}$$

We devote the remainder of this section to showing the benefit of this index set. There are two aspects to this: the reduced cost in forming the approximation—in other words, the reduced size of the hyperbolic cross index set—and the retention of the similar error estimates in comparison to approximations based on the full index set (2.1). We commence with the former:

Lemma 16. *Suppose that $\theta_d(t)$ is the number of terms $n = (n_1, ..., n_d) \in \mathbb{N}^d$ such that $\bar{n}_1...\bar{n}_d \leq t$. Then*

$$\theta_d(t) = \frac{t(\log t)^{d-1}}{(d-1)!} + \textit{lower order terms},$$

for large t.

For a proof of this in a more general setting we refer to [5]. A simple inductive argument appears in [11], which we now repeat here, since similar methods will be used in the sequel:

Proof. For $d = 1$, $\theta_1(t) = t$. Suppose that the result is true for $d - 1$. Then

$$\theta_d(t) = \sum_{n=1}^{t} \theta_{d-1}\left(\frac{t}{n}\right) \approx \frac{1}{(d-2)!} \sum_{n=1}^{t} \frac{t}{n} \left(\log\left(\frac{t}{n}\right)\right)^{d-2}$$
$$\approx \frac{1}{(d-2)!} \int_1^t \frac{t}{n} \left(\log\left(\frac{t}{n}\right)\right)^{d-2} dn = \frac{1}{(d-2)!} t \int_1^t x^{-1} (\log x)^{d-2}\, dx = \frac{t(\log t)^{d-1}}{(d-1)!},$$

where the last equality follows by the substitution $\frac{t}{n} = x$. □

Corollary 10. *The number of terms in the expansion $\mathcal{F}_N[f]$ based on the hyperbolic cross (3.1) is*

$$\frac{2^d}{(d-1)!} N(\log N)^{d-1} + \textit{lower order terms}. \tag{3.2}$$

Proof. For any n with strictly positive entries there are 2^d choices of $i \in \{0,1\}^d$. The total number of coefficients $\hat{f}_n^{[i]}$ where at least one entry of n is zero is of lower order than the leading term in (3.2). □

3.1 Convergence rate in various norms

We now assess the rate of convergence of the approximation $\mathcal{F}_N[u]$ based on the hyperbolic cross (3.1) in various norms:

Lemma 17. *Suppose that $u \in H^{2k+1}(-1,1)^d$ satisfies the first k derivative conditions (1.4) and I_N is the hyperbolic cross index set (3.1). Then, for some positive constant $c_{r,s}$ independent of u and N,*

$$\|u - \mathcal{F}_N[u]\|_s \leq c_{r,s} N^{\frac{s-r}{d}} \|u\|_r, \quad r = s, ..., 2k+1, \quad s = 0, ..., 2k+1. \tag{3.3}$$

If, additionally, $u \in H^{2k+1}_{mix}(-1,1)^d$, then

$$\|u - \mathcal{F}_N[u]\|_s \leq c_{r,s} N^{s-r} \|u\|_{r,mix}, \quad r = s, ..., 2k+1, \quad s = 0, ..., 2k+1. \tag{3.4}$$

Proof. Due to Lemma 5 and (1.7) we may write

$$\|u - \mathcal{F}_N[u]\|_s^2 \leq \sum_{i\in\{0,1\}^d} \sum_{n\notin I_N} |\hat{u}_n^{[i]}|^2 (1+\mu_n^{[i]})^s = \sum_{i\in\{0,1\}^d} \sum_{n\notin I_N} |\hat{u}_n^{[i]}|^2 (1+\mu_n^{[i]})^r (1+\mu_n^{[i]})^{s-r}.$$

By a standard inequality $1+\mu_n^{[i]} \geq c\,(\bar{n}_1 ... \bar{n}_d)^{\frac{2}{d}}$, and since $n \notin I_N$ we have $1+\mu_n^{[i]} \geq N^{\frac{2}{d}}$. Hence

$$\|u - \mathcal{F}_N[u]\|_s^2 \leq c_{r,s} N^{\frac{2(s-r)}{d}} \sum_{i\in\{0,1\}^d} \sum_{n\mathbb{N}^d} |\hat{u}_n^{[i]}|^2 (1+\mu_n^{[i]})^r \leq c_{r,s} N^{\frac{2(s-r)}{d}} \|u\|_r^2,$$

which gives (3.3). Next consider (3.4). We have

$$\|u - \mathcal{F}_N[u]\|_s^2 \leq \sum_{i\in\{0,1\}^d} \sum_{n\notin I_N} |\hat{u}_n^{[i]}|^2 (1+\mu_n^{[i]})^s = \sum_{i\in\{0,1\}^d} \sum_{n\notin I_N} |\hat{u}_n^{[i]}|^2 (1+\mu_n^{[i]})^s \frac{\sum_{|\alpha|_\infty \leq r} \prod_{j=1}^d (\mu_{n_j}^{[i_j]})^{\alpha_j}}{\sum_{|\alpha|_\infty \leq r} \prod_{j=1}^d (\mu_{n_j}^{[i_j]})^{\alpha_j}}.$$

For $r = s, ..., 2k+1$ and $n \notin I_N$ we have the inequality

$$\sum_{|\alpha|_\infty \leq r} \prod_{j=1}^d (\mu_{n_j}^{[i_j]})^{\alpha_j} \geq c_{r,s} (1+\mu_n^{[i]})^s \prod_{j:n_j>0} (\mu_{n_j}^{[i_j]})^{r-s} \geq c_{r,s} (1+\mu_n^{[i]})^s \prod_{j=1}^d (\bar{n}_j)^{2(r-s)} \geq c_{r,s} (1+\mu_n^{[i]})^s N^{2(r-s)}.$$

We therefore obtain

$$\|u - \mathcal{F}_N[u]\|_s^2 \leq c_{r,s} N^{2(s-r)} \sum_{i\in\{0,1\}^d} \sum_{n\notin I_N} |\hat{u}_n^{[i]}|^2 \sum_{|\alpha|_\infty \leq r} \prod_{j=1}^d (\mu_{n_j}^{[i_j]})^{\alpha_j} \leq c_{r,s} N^{2(s-r)} \|u\|_{r,\text{mix}}^2,$$

using Lemma 6. This gives the result. □

Typically our interest lies with functions of sufficient smoothness. For this, as before, we may use the coefficient bounds to derive error estimates:

Theorem 11. *Suppose that $u \in H^{2k+2}_{mix}(-1,1)^d$ obeys the first k derivative conditions and I_N is the hyperbolic cross index set (3.1). Then*

$$\|u - \mathcal{F}_N[u]\|_\infty \leq \frac{2^d c \|u\|_{2k+2,mix}}{(2k+1)(d-1)!} N^{-2k-1} (\log N)^{d-1} + \textit{lower order terms},$$

$$\|u - \mathcal{F}_N[u]\| \leq \frac{2^{\frac{d}{2}} c \|u\|_{2k+2,mix}}{\sqrt{(4k+3)(d-1)!}} N^{-2k-\frac{3}{2}} (\log N)^{\frac{d-1}{2}} + \textit{lower order terms},$$

$$\|u - \mathcal{F}_N[u]\|_s \leq c \sqrt{\frac{2d}{(4k+3)s^{d-1}} + \frac{d 2^d \omega_{4k+1,d}}{4k+3-2s}} \|u\|_{2k+2,mix} N^{s-2k-\frac{3}{2}} + \textit{lower order terms}, \quad s = 1, ..., 2k+1,$$

where c is the constant from the Coefficient bounds corollary with $\chi(n) = d$ and, for $r \geq 0$ and $d = 1, 2, ...,$ $\omega_{r,d} = (1+\zeta(r+2))^{d-1}$.

For this we need the following lemma:

Lemma 18. *Suppose that*

$$\gamma_{r,d}(t) = \sum_{\bar{n}_1 \ldots \bar{n}_d > t} (\bar{n}_1 \ldots \bar{n}_d)^{-r-1}, \quad r > 0, \quad d = 1, 2, \ldots.$$

Then

$$\gamma_{r,d}(t) = \frac{t^{-r}(\log t)^{d-1}}{r(d-1)!} + \textit{lower order terms} \tag{3.5}$$

for large t*. Furthermore, if*

$$\delta_{r,s,d}(t) = \sum_{\bar{n}_1 \ldots \bar{n}_d > t} (\bar{n}_1 \ldots \bar{n}_{d-1})^{-r-3} \bar{n}_d^{s-r-3},$$

for $r, s > 0$ *and* $s < 2 + r$*. Then*

$$\delta_{r,s,d}(t) = \left(\frac{1}{(r+2)s^{d-1}} + \frac{\omega_{r,d}}{r+2-s} \right) t^{s-r-2} + \textit{lower order terms.} \tag{3.6}$$

Proof. By induction. For $d = 1$ we have $\gamma_{r,1}(t) = \sum_{n>t} n^{-r-1} \sim \frac{t^{-r}}{r}$ for large t. Assume the result is true up to d. Then

$$\begin{aligned} \gamma_{r,d}(t) &= \gamma_{r,d-1}(t) + \sum_{n=1}^{t} n^{-r-1} \gamma_{r,d-1}\left(\frac{t}{n}\right) + \sum_{n>t} n^{-r-1} \gamma_{r,d-1}(1) \\ &= \sum_{n=1}^{t} n^{-r-1} \gamma_{r,d-1}\left(\frac{t}{n}\right) + \text{lower order terms} \\ &\approx \frac{t^{-r-1}}{r(d-2)!} \sum_{n=1}^{t} \frac{t}{n} \left(\log\left(\frac{t}{n}\right)\right)^{d-2} \approx \frac{t^{-r-1}}{r} \theta_d(t) = \frac{t^{-r}(\log t)^{d-1}}{r(d-1)!}, \end{aligned}$$

where θ_d is as in Lemma 16. Thus we obtain (3.5).

Next we consider

$$\begin{aligned} \delta_{r,s,d}(t) &= \gamma_{r+2,d-1}(t) + \sum_{n=1}^{t} n^{s-r-3} \gamma_{r+2,d-1}\left(\frac{t}{n}\right) + \gamma_{r+2,d-1}(1) \sum_{n>t} n^{s-r-3} \\ &= \sum_{n=1}^{t} n^{s-r-3} \gamma_{r+2,d-1}\left(\frac{t}{n}\right) + \frac{1}{s-r-2} \gamma_{r+2,d-1}(1) t^{s-r-2} + \text{lower order terms.} \end{aligned}$$

For the first term, we use (3.5) to obtain

$$\begin{aligned} \sum_{n=1}^{t} n^{s-r-3} \gamma_{r+2,d-1}\left(\frac{t}{n}\right) &= \frac{1}{(r+2)(d-2)!} \sum_{n=1}^{t} n^{s-r-3} \left(\frac{t}{n}\right)^{-r-2} \left(\log\left(\frac{t}{n}\right)\right)^{d-2} + \text{lower order terms} \\ &\approx \frac{1}{(r+2)(d-2)!} \int_1^t n^{s-1} \left(\log\left(\frac{t}{n}\right)\right)^{d-2} dn + \text{lower order terms} \\ &= \frac{t^{s-r-2}}{(r+2)(d-2)!} \int_1^t x^{-s-1} (\log x)^{d-2} \, dx = \frac{t^{s-r-2}}{(r+2)s^{d-1}} + \text{lower order terms.} \end{aligned}$$

Here the final equality follows from the observation that

$$\int_1^t x^{-s-1} (\log x)^d \, dx = \frac{d!}{s^{d+1}} + \text{lower order terms.}$$

For $\gamma_{r+2,d-1}(1)$ we have

$$\gamma_{r+2,d-1}(1) = \sum_{\bar{n}_1 \ldots \bar{n}_{d-1} \geq 1} (\bar{n}_1 \ldots \bar{n}_{d-1})^{-r-2} = \sum_{j=0}^{d-1} \binom{d-1}{j} \zeta(r+2)^{d-1-j} = \omega_{r,d}.$$

Hence we obtain the result. ∎

Proof of Theorem 11. This follows immediately from the Coefficient bounds corollary and the previous lemma. □

Theorem 11 indicates that the convergence rate of $\mathcal{F}_N[f]$ using the hyperbolic cross (3.1) is comparable to that of the approximation based on the full index set (2.1). Indeed, for the L^2 and uniform rates we only lose factors of $\mathcal{O}\left((\log N)^{d-1}\right)$ and $\mathcal{O}((\log N)^{\frac{d-1}{2}})$ respectively. The H^s rate, $s \geq 1$, remains the same.

As is common with hyperbolic cross approximations, additional (mixed) smoothness is required for the estimates of Lemma 17 in comparison to those of Lemma 14. If only H^r-regularity is imposed, the hyperbolic cross approximation will converge more slowly than its counterpart based on the full index set. However, for approximations based on either the full or hyperbolic cross index set the minimal regularity required to obtain an optimal convergence rate is the same (see Lemma 15 and Theorem 11 respectively).

It is also of interest to consider the affect of the hyperbolic cross on the pointwise rate of convergence. As we shall see in the next section, this also only deteriorates by a factor of $\mathcal{O}\left((\log N)^{d-1}\right)$. Moreover the smoothness assumption remains the same.

3.2 Pointwise convergence rate

To analyse the pointwise convergence rate of $\mathcal{F}_N[f](x)$ we consider a slight adjustment of the index set (3.1), namely the *step hyperbolic cross*. To simplify matters we shall assume that f only has nonzero modified Fourier coefficients when $i = 0$ and each component of n is positive. Suppose that $N = 2^r$, then we define

$$Q_r = \bigcup_{|k| \leq r} \rho(k), \quad \text{where} \quad \rho(k) = \{n \in \mathbb{N}^d : 2^{k_j} \leq n_j < 2^{k_j+1}, \quad j = 1, ..., d\}, \quad k \in \mathbb{N}^d. \tag{3.7}$$

We call Q_r the step hyperbolic cross of size r. Note that we have the inclusion $Q_r \subset I_N \subset Q_{r+d}$, (see, for example, [16]), where $I_N = \{n \in \mathbb{N}^d : n_1 ... n_d \leq N\}$ is the hyperbolic cross index set. Furthermore, we have the decomposition

$$\mathbb{N}^d \backslash Q_r = \bigcup_{k \notin J_r} \rho(k) \cup \bigcup_{k \in H_r} \rho(k),$$

where $J_r = \{k : 0 \leq k_j \leq r, \quad j = 1, ..., d\}$, and H_r is the finite set $H_r = \{k \in J_r : |k| > r\}$. Thus, when considering the pointwise error $f(x) - \mathcal{F}_N[f](x)$, based on the step hyperbolic cross we may decompose this into a part corresponding to the full index set (2.1) of size N and a part consisting of the coefficients n where $n \in \rho(k)$ for some $k \in H_r$. Since the rate of convergence of the approximation based on the full index set (2.1) has already been proved, it suffices to consider contribution of the latter term

$$\sum_{k \in H_r} \sum_{n \in \rho(k)} \hat{f}_n \phi_n(x). \tag{3.8}$$

Lemma 19. *Suppose that $f \in H^3_{mix}(-1,1)^d$. Then the term (3.8) is $\mathcal{O}\left(r^{d-1} 2^{-2r}\right)$ uniformly for x in compact subsets of $(-1,1)^d$.*

Proof. Consider first the sum

$$\sum_{n \in \rho(k)} \hat{f}_n \phi_n(x) = \sum_{n_1 = 2^{k_1}}^{2^{k_1+1}-1} \cdots \sum_{n_d = 2^{k_d}}^{2^{k_d+1}-1} \hat{f}_n \prod_{j=1}^{d} \phi_{n_j}(x_j).$$

Suppose that, for simplicity, that $\hat{f}_n = c_f (-1)^{|n|} \prod_{j=1}^{d} (n_j \pi)^{-2}$. Then

$$\sum_{n \in \rho(k)} \hat{f}_n \phi_n(x) = c_f \prod_{j=1}^{d} \tilde{\Phi}(k_j, x_j),$$

where $\tilde{\Phi}(k,x) = \sum_{n=2^k}^{2^{k+1}-1} (-1)^n (n\pi)^{-2} \cos n\pi x$. From univariate theory, for $x \in (-1,1)$, $\tilde{\Phi}(k,x) = \mathcal{O}\left(2^{-2k}\right)$. Thus we see that

$$\sum_{n \in \rho(k)} \hat{f}_n \phi_n(x) = \mathcal{O}\left(2^{-2|k|}\right),$$

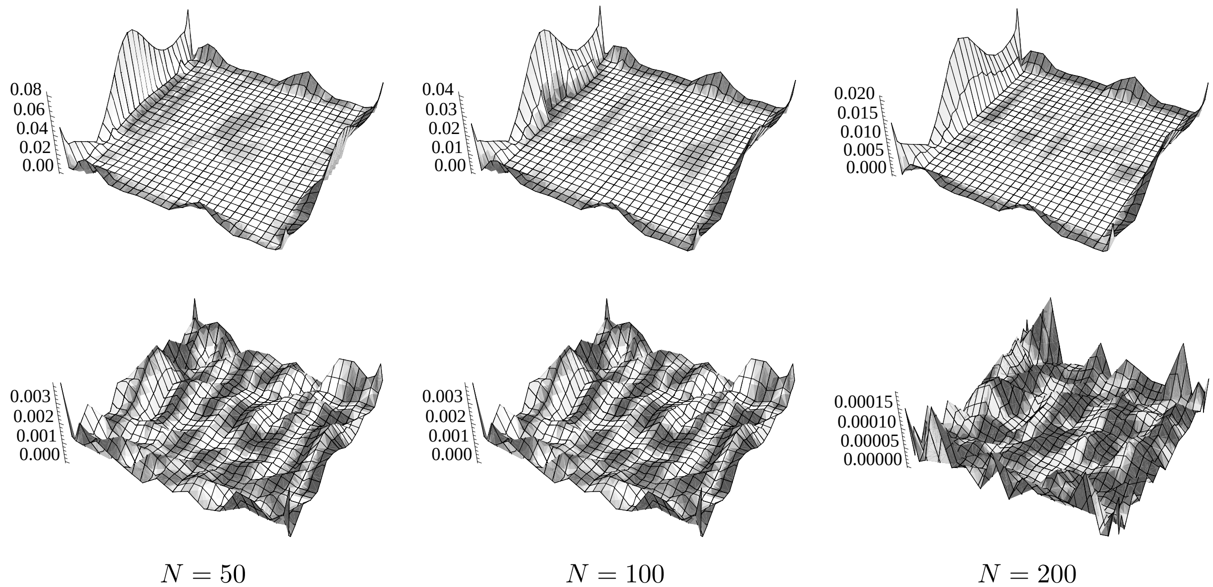


Figure 1: Absolute error $|f(x,y) - \mathcal{F}_N[f](x,y)|$ for $-1 \le x, y \le 1$ (top row), $-0.9 \le x, y \le 0.9$ (bottom row) and $N = 50, 100, 200$.

and, since $|H_r| < \infty$, we have

$$\sum_{k \in H_r} \sum_{n \in \rho(k)} \hat{f}_n \phi_n(x) = \mathcal{O}\left(\sum_{k \in H_r} 2^{-2|k|} \right).$$

Now, if $k' = (k_1, ..., k_{d-1})$ for $k = (k_1, ..., k_d)$, we have

$$\sum_{k \in H_r} 2^{-2|k|} = \sum_{\substack{0 \le k_j \le r \\ j=1,\dots,d \\ |k| > r}} 2^{-2|k|} = \sum_{\substack{0 \le k_j \le r \\ j=1,\dots,d-1}} 2^{-2|k'|} \sum_{k_d = r - |k'|}^{r} 2^{-2k_d} = 2 \sum_{\substack{0 \le k_j \le r \\ j=1,\dots,d-1}} 2^{-2|k'|} \left(2^{2|k'|-2r} - 2^{-2r} \right)$$

$$= 2^{-2r} \sum_{\substack{0 \le k_j \le r \\ j=1,\dots,d-1}} \left(1 - 2^{-2|\underline{k}'|} \right) = \mathcal{O}\left(r^{d-1} 2^{-2r} \right),$$

which gives the result. The case where $\hat{f}_n$ has more terms in its expansion can be dealt with similarly. $\square$

It is a trivial exercise to extend this to a function u that obeys a finite number of derivative conditions. We therefore obtain the following result:

Theorem 12. *Suppose that $u \in H^{2k+3}_{mix}(-1,1)^d$ obeys the first k derivative conditions. Then the error $u(x) - \mathcal{F}_N[u](x)$, where $\mathcal{F}_N[u]$ is the modified Fourier approximation of u based on the step hyperbolic cross (3.7), is $\mathcal{O}\left(N^{-2k-2} (\log N)^{d-1}\right)$ uniformly in compact subsets of $(-1,1)^d$.*

The inclusion $Q_r \subset I_N \subset Q_{r+d}$ indicates that this is also the case for the hyperbolic cross (3.1).

3.3 Numerical Results

In Figures 1 and 2 we give numerical results for the modified Fourier approximation $\mathcal{F}_N[f](x,y)$ using the hyperbolic cross (3.1) where

$$f(x,y) = e^{\sin 6x} \left(y + \tan^2(1 - y^2) \right).$$

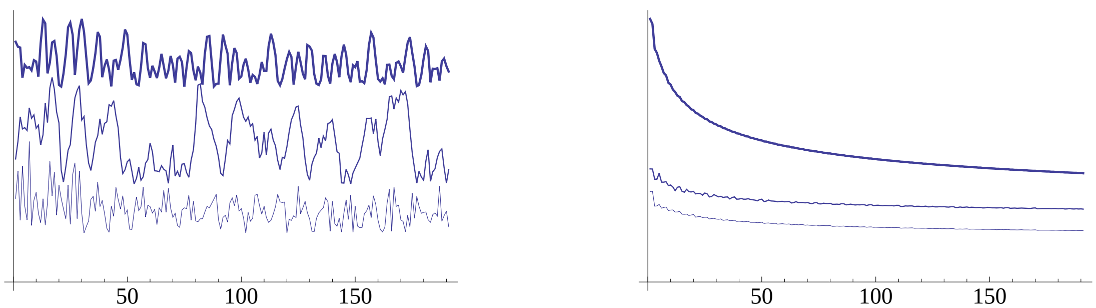


Figure 2: (left) scaled pointwise error $N^2(\log N)^{-1}|f(x,y) - \mathcal{F}_N[f](x,y)|$, $N = 10, ..., 200$, for $(x,y) = (\frac{3}{4}, \frac{2}{3})$ (thickest line),$(-\frac{7}{8}, -\frac{3}{4})$,$(-\frac{1}{10}, \frac{1}{8})$ (thinnest line). (right) scaled pointwise error $N(\log N)^{-1}|f(x,y) - \mathcal{F}_N[f](x,y)|$ for $(x,y) = (-1,0)$, $(-1,-1)$, $(1,1)$.

As Figure 1 demonstrates, the error inside the domain is much smaller than on the boundary, as predicted by Theorem 12. Observe further that when N doubles the uniform error roughly halves, whereas the error in $[-0.9,-0.9] \times [-0.9,-0.9]$ roughly quarters, again as predicted. Figure 2 verifies the results of Theorems 11 and 12.

4 The Modified Fourier–Galerkin method

In this section we consider the approximation of Neumann boundary value problems using the modified Fourier basis. As we will establish, the resulting method has a number of beneficial properties. In Section 5 we demonstrate how to discretize problems with other boundary conditions using closely related bases.

For the moment, we consider the linear boundary value problem

$$\mathcal{L}[u](x) = -\triangle u(x) + a \cdot \nabla u(x) + bu(x) = f(x), \quad x \in [-1,1]^d, \tag{4.1}$$

where $a = (a_1, ..., a_d)^\top \in \mathbb{R}^d$ and $b \in \mathbb{R}$ are constants (we consider the general case in the sequel) and f is some given function, with homogeneous Neumann boundary conditions

$$\partial_{x_j} u|_{\Gamma_j^{\pm}} = 0, \quad j = 1, ..., d. \tag{4.2}$$

Equivalently, in weak form, if $T : H^1(-1,1)^d \times H^1(-1,1)^d \to \mathbb{R}$ is the bilinear form

$$T(u,v) = (\nabla u, \nabla v) + (a \cdot \nabla u, v) + b(u,v), \quad \forall u, v \in H^1(-1,1)^d,$$

where $(\nabla u, \nabla v) = \int_\Omega \nabla u \cdot \nabla v$, then we may rewrite (4.1) as

$$\text{find} \quad u \in H^1(-1,1)^d : \quad T(u,v) = (f,v), \quad \forall v \in H^1(-1,1)^d.$$

We shall use the Lax–Milgram Theorem and Céa's Lemma (see, for example, [4, 19]) so it useful to know that the operator T is continuous and coercive provided

$$b - \frac{1}{4}\sum_{i=1}^{d} a_i^2 > 0, \tag{4.3}$$

in which case there are positive constants ω and γ such that

$$|T(u,v)| \leq \gamma \|u\|_1 \|v\|_1, \quad T(u,u) \geq \omega \|u\|_1^2, \quad \forall u, v \in H^1(-1,1)^d. \tag{4.4}$$

4.1 Galerkin's equations and iterative solution techniques

We consider the modified Fourier–Galerkin approximation of (4.1)–(4.2). Suppose that we write the approximation $u_N \in \mathcal{S}_N$ as

$$u_N(x) = \sum_{i \in \{0,1\}^d} \sum_{n \in I_N} a^{[i]}_n \phi^{[i]}_n(x), \quad x \in [-1,1]^d,$$

with coefficients $a^{[i]}_n$ that enforce Galerkin's equations $T(u_N,\phi) = (f,\phi)$, $\forall \phi \in \mathcal{S}_N$. Then we have:

Lemma 20. *The coefficients $a^{[i]}_n$ satisfy*

$$(b+\mu^{[i]}_n)a^{[i]}_n + \sum_{j=1}^{d} \sum_{\substack{m_j \in \mathbb{N},\\ (n,m_j) \in I_N}} a_j \delta^{[i_j]}_{n_j,m_j} a^{[(i,1-i_j)]}_{(n,m_j)} = \hat{f}^{[i]}_n, \quad i \in \{0,1\}^d, \quad n \in I_N, \tag{4.5}$$

where I_N is some appropriate index set

$$(n,m_j) = (n_1,...,n_{j-1},m_j,n_{j+1},...,n_d), \quad (i,1-i_j) = (i_1,...,i_{j-1},1-i_j,i_{j+1},...,i_d),$$

and

$$\delta^{[i]}_{n,m} = \int_{-1}^{1} \phi^{[i]}_n(x)(\phi^{[1-i]}_m)'(x)\; dx = 2(-1)^{n+m}\frac{\mu^{[1-i]}_m}{\mu^{[i]}_n - \mu^{[1-i]}_m}, \quad i=0,1, \quad n,m=0,...,N. \tag{4.6}$$

Proof. We set $\phi = \phi^{[i]}_n$, $i \in \{0,1\}^d$, $n \in I_N$ in Galerkin's equations. Due to the Laplace term, we obtain

$$T(u_N,\phi^{[i]}_n) = (b+\mu^{[i]}_n)a^{[i]}_n + \sum_{j=1}^{d} \sum_{l \in \{0,1\}^d} \sum_{m \in I_N} a_j (\partial_{x_j}\phi^{[l]}_m, \phi^{[i]}_n) a^{[l]}_m.$$

Now

$$(\partial_{x_j}\phi^{[l]}_m, \phi^{[i]}_n) = ((\phi^{[l_j]}_{m_j})', \phi^{[i_j]}_{n_j}) \prod_{k \neq j} (\phi^{[l_k]}_{m_k}, \phi^{[i_k]}_{n_k}) = \begin{cases} \delta^{[i_j]}_{n_j,m_j} & l=(i,1-i_j), \quad m_k = n_k, \quad k \neq j, \\ 0 & \text{otherwise}, \end{cases}$$

which gives the result. □

For spectral discretizations in Cartesian product domains Galerkin's equations are normally written in tensor product form. The advantage of this approach is that it facilitates the use of novel solution techniques such as the matrix diagonalization and Schur decomposition methods, [4]. Furthermore, the matrices involved, which in this case would correspond to univariate modified Fourier discretizations, are well understood and have a number of beneficial properties, [1].

We shall not pursue this approach. Due to the simple nature of the modified Fourier–Galerkin equations such techniques are unnecessary. Moreover, for approximations using the hyperbolic cross index set, Galerkin's equations do not naturally have a tensor product form.

Instead we consider standard iterative methods. Suppose that we write the discretization matrix as A_{G} and Galerkin's equations as $A_{\mathrm{G}}\hat{a} = \hat{f}$. In addition, we decompose $A_{\mathrm{G}} = M_{\mathrm{G}} + N_{\mathrm{G}}$, where M_{G} is the diagonal matrix corresponding to restriction of the operator $-\triangle + b\imath$ to $\mathcal{S}_N$, where $\imath$ is the identity operator.

For any spectral-Galerkin discretization it is easily established that the iterative scheme $M_{\mathrm{G}}\hat{a}^{k+1} = -N_{\mathrm{G}}\hat{a}^k + \hat{f}$, $k=0,1,2,...$, converges (provided (4.3) holds). Moreover the number of iterations required for convergence within some numerical tolerance is independent of the truncation parameter N. In the modified Fourier case the matrix M_{G} is diagonal, making this scheme practical. The overall cost is thus determined by the number of operations required to perform matrix-vector multiplications involving N_{G}. We have:

Lemma 21. *Suppose that $N \gg d$. Then the number of operations required to evaluate the matrix-vector multiplication $N_{\mathrm{G}}\hat{a}$ is*

$$d2^{d+1}N^{d+1} + \textit{lower order terms}, \tag{4.7}$$

in the case of the full index set (2.1) and

$$d2^{d+1}N^2\lceil (1+\zeta(2))^{d-1} \rceil + \textit{lower order terms}, \tag{4.8}$$

for the hyperbolic cross (3.1).

Proof. In view of Lemma 20 the number of operations is precisely

$$\sum_{i\in\{0,1\}^d}\sum_{n\in I_N}\sum_{j=1}^{d}\sum_{\substack{m_j\in\mathbb{N},\\(n,m_j)\in I_N}} 2.$$

If I_N is the full index set, we easily obtain (4.7). For the hyperbolic cross we have

$$\begin{aligned}\sum_{i\in\{0,1\}^d}\sum_{n\in I_N}\sum_{j=1}^{d}\sum_{\substack{m_j\in\mathbb{N},\\(n,m_j)\in I_N}} 2 &= d2^{d+1}\sum_{n\in I_N}\sum_{\substack{m_d\in\mathbb{N},\\(n,m_d)\in I_N}} 1 + \text{lower order terms}\\ &= d2^{d+1}\sum_{n\in I_N}\sum_{m=0}^{N(\bar{n}_1\ldots\bar{n}_{d-1})^{-1}} 1 = d2^{d+1}\sum_{n\in I_N}\frac{N}{\bar{n}_1\ldots\bar{n}_{d-1}} + \text{lower order terms}\\ &= d2^{d+1}N^2\sum_{\bar{n}_1,\ldots,\bar{n}_{d-1}=1}^{\infty}\frac{1}{(\bar{n}_1\ldots\bar{n}_{d-1})^2} + +\text{lower order terms.}\end{aligned}$$

Evaluating this final sum gives (4.8). □

The matrix M_{G} has one other significant use: it is an optimal preconditioner for A_{G} (we shall not prove this: the case $d = 1$ was demonstrated in [1] and the extension to $d \geq 2$ is simple). In the modified Fourier case, since M_{G} is diagonal, this preconditioner is practical. For this reason, Galerkin's equations can also be solved using preconditioned conjugate gradients. The cost of this approach is again determined by the number of operations required to evaluate matrix-vector multiplications involving N_{G}.

In view of Lemma 21 we conclude that Galerkin's equations can be solved in $\mathcal{O}\left(N^{d+1}\right)$ (full index set) or $\mathcal{O}\left(N^2\right)$ (hyperbolic cross index set) operations using either of the standard iterative methods outlined in this section.

Since the action of the matrix N_{G} corresponds to finding modified Fourier coefficients of derivatives of finite modified Fourier sums, a variant of the Fast Fourier Transform (FFT) can be employed in the full index set case. In this manner the figure of $\mathcal{O}\left(N^{d+1}\right)$ can easily be reduced to $\mathcal{O}\left(N^d \log N\right)$. For the hyperbolic cross index set, a variant of the sparse grid FFT could be employed, [7]. In this manner, the figure of $\mathcal{O}\left(N^2\right)$ could be reduced to $\mathcal{O}\left(N(\log N)^d\right)$. However, this technique is neither easy nor straightforward to implement.

4.2 Properties of the discretization matrix

The properties of A_{G}, in particular the (spectral) condition number and the existence of effective preconditioners, are of importance in spectral discretizations. In this section we demonstrate that the (spectral) condition number of the modified Fourier–Galerkin method is $\mathcal{O}\left(N^2\right)$. The results of this section are extensions of those found in [1].

Lemma 22. *Suppose that I_N is either the full or the hyperbolic cross index set. Then the spectral condition number of A_{G} is $\mathcal{O}\left(N^2\right)$ provided the operator $\mathcal{L}$ is coercive. Specifically, if λ is an eigenvalue of A_{G} then*

$$\omega \leq |\lambda| \leq \gamma(1 + N^2\pi^2 d), \quad \omega \leq |\lambda| \leq \gamma(1 + (d - 1 + N^2)\pi^2),$$

in the full and hyperbolic cross cases respectively.

Proof. For an eigenvalue λ with eigenfunction $u \in \mathcal{S}_N$ we have $\lambda(u, \phi) = (\mathcal{L}[u], \phi)$ for all $\phi \in \mathcal{S}_N$. In particular, $\omega\|u\|^2 \leq |\lambda|\|u\|^2$ and $|\lambda|\|u\|^2 \leq \gamma\|u\|_1^2$. Now, by Bernstein's Inequality (Corollary 1), $\|u\|_1^2 \leq \max_{n\in I_N}(1 + \mu_n^{[0]})\|u\|^2$. Moreover, for $n \in I_N$,

$$1 + \mu_n^{[0]} \leq 1 + N^2\pi^2 d, \quad 1 + \mu_n^{[0]} \leq 1 + (d - 1 + N^2)\pi^2, \tag{4.9}$$

where I_N is the full or hyperbolic cross index set respectively. □

We may also prove the same result for the L^2 condition number. To do so we need the following lemma:

Lemma 23. *Suppose that λ is an eigenvalue of $A_{\mathrm{G}}^{\top} A_{\mathrm{G}}$ with associated eigenfunction $u \in \mathcal{S}_N$. Then*

$$(\mathcal{F}_N[\mathcal{L}[u]], \mathcal{F}_N[\mathcal{L}[\phi]]) = \lambda(u, \phi), \quad \forall \phi \in \mathcal{S}_N. \tag{4.10}$$

Proof. This is a simple generalization of the proof given in [1], so is not presented here. □

Corollary 13. *Suppose that I_N is either the full or the hyperbolic cross index set. Then the L^2 condition number of A_{G}, $\kappa(A_{\mathrm{G}})$, is $\mathcal{O}\left(N^2\right)$ provided the operator $\mathcal{L}$ is coercive. Specifically, if $\gamma' > 0$ is such that $\|\mathcal{L}[u]\|^2 \leq \gamma' \|u\|_2^2$ for all $u \in H^2(-1,1)^d$, then we have the bounds*

$$\kappa(A_{\mathrm{G}}) \leq \omega^{-1} \gamma' (1 + N^2 \pi^2 d), \quad \kappa(A_{\mathrm{G}}) \leq \omega^{-1} \gamma' (1 + (d - 1 + N^2) \pi^2),$$

in the full and hyperbolic cross cases respectively.

Proof. Setting $\phi = u$ in (4.10) gives $\|\mathcal{F}_N[\mathcal{L}[u]]\|^2 = \lambda \|u\|^2$. Now

$$\|\mathcal{F}_N[\mathcal{L}[u]]\| = \sup_{g \in L^2(-1,1)^d} \frac{(\mathcal{F}_N[\mathcal{L}[u]], g)}{\|g\|} \geq \sup_{g \in \mathcal{S}_N} \frac{(\mathcal{L}[u], g)}{\|g\|}.$$

Suppose that we define $g \in \mathcal{S}_N$ by enforcing the condition $(\mathcal{L}[\phi], g) = (\phi, u)$ for all $\phi \in \mathcal{S}_N$. Note that the coefficients of g are the solution of a linear system involving $A_{\mathrm{G}}^{\top}$. Hence existence and uniqueness of g is guaranteed. Furthermore $(\mathcal{L}[u], g) = (u, u) = \|u\|^2$ and, since $\mathcal{L}$ is coercive, $\omega \|g\|_1 \leq \|u\|$. Hence

$$\lambda \|u\|^2 = \|\mathcal{F}_N[\mathcal{L}[u]]\|^2 \geq \left(\frac{(\mathcal{L}[u], g)}{\|g\|} \right)^2 = \frac{\|u\|^4}{\|g\|^2} \geq \omega^2 \|u\|^2.$$

To derive an upper bound for λ, we note that

$$\lambda \|u\|^2 = \|\mathcal{F}_N[\mathcal{L}[u]]\|^2 \leq \|\mathcal{L}[u]\|^2 \leq \gamma' \|u\|_2^2 \leq \max_{n \in I_N} (1 + \mu_n^{[0]})^2 \|u\|^2,$$

by Bernstein's Inequality. The result now follows immediately from (4.9). □

Note that the lower bounds for the minimal eigenvalues of A_{G} and $A_{\mathrm{G}}^{\top} A_{\mathrm{G}}$ are independent of the Galerkin discretization used. The upper bounds, however, rely on Bernstein-type estimates which vary from method to method.

4.3 Convergence rate and numerical results

The results of Sections 1–3 allow us to immediately provide estimates for the convergence rate in the H^1 norm. From Céa's lemma we have

$$\|u - u_N\|_1 \leq \frac{\gamma}{\omega} \inf_{\phi \in \mathcal{S}_N} \|u - \phi\|_1.$$

For modified Fourier series, in light of Lemma 4, this infimum is precisely $\|u - \mathcal{F}_N[u]\|_1$. Hence:

Theorem 14. *Suppose that u_N is the modified Fourier–Galerkin approximation based on the full index set (2.1). Then*

$$\|u - u_N\|_1 \leq \gamma \omega^{-1} c_{r,1} N^{1-r} \|u\|_r, \quad r = 1, 2, 3, \quad \|u - u_N\|_1 \leq \gamma \omega^{-1} c_1 N^{-\frac{5}{2}} \|u\|_{4,\mathrm{mix}},$$

where $c_{r,1}$ and c_1 are the constants from Lemmas 14 and 15 respectively. If u_N is the approximation based on the hyperbolic cross (3.1) then

$$\|u - u_N\|_1 \leq \gamma \omega^{-1} c_{r,1} N^{\frac{1-r}{d}} \|u\|_r, \quad \|u - u_N\|_1 \leq \gamma \omega^{-1} c_{r,1} N^{1-r} \|u\|_{r,\mathrm{mix}}, \quad r = 1, 2, 3,$$

and $\|u - u_N\|_1 \leq \gamma \omega^{-1} c_1 N^{-\frac{5}{2}} \|u\|_{4,\mathrm{mix}}$, where $c_{r,1}$ and c_1 are the constants from Lemma 17 and Theorem 11 respectively.

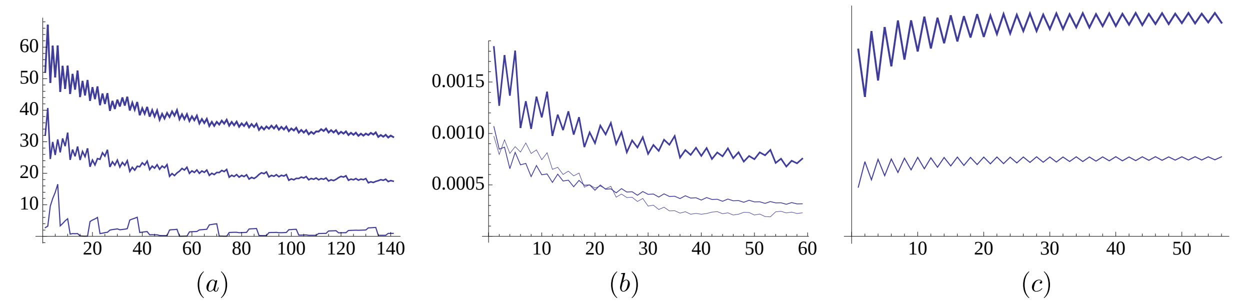


Figure 3: (a) scaled pointwise error $N^3(\log N)^{-1}|u(x,y)-u_N(x,y)|$ for the problem (P1), where $(x,y)=(1,\frac{7}{10})$ (thickest line), $(\frac{1}{2},\frac{3}{4})$, $(\frac{5}{8},\frac{3}{4})$ (thinnest line). (b) scaled pointwise error $N^3(\log N)^{-2}|u(x,y,z)-u_N(x,y,z)|$ for (P2), where $(x,y,z)=(1,1,1)$ (thickest line), $(\frac{3}{4},\frac{3}{4},\frac{3}{4})$, $(\frac{11}{20},\frac{11}{20},\frac{11}{20})$ (thinnest line). (c) Scaled H^1 error $N^{\frac{5}{2}}\|u-u_N\|_1$ for (P1) (thicker line) and (P2) (thinner line).

As in Section 3, when $u\notin H^4_{\mathrm{mix}}(-1,1)^d$ the method based on the hyperbolic cross requires additional smoothness to attain the same convergence rate as its counterpart based on the full index set. However, provided the solution $u\in H^4_{\mathrm{mix}}(-1,1)^d$ both the full and hyperbolic cross index sets offer the same convergence rate. Since the latter involves far reduced complexity, we shall focus on it in the remainder of this paper.

In Figure 3 we give numerical results for the modified Fourier–Galerkin method based on the hyperbolic cross index set (3.1) applied to the following problems:

$$\text{(P1)}\quad \begin{aligned}&d=2, a_1=-1, a_2=2, b=4,\\ &u(x,y)=\left(\cosh 3x-\tfrac{3}{2}(x^2-1)\sinh 3\right)\left(\sin 2(y+1)+\sin^2 2(1+y)^2-2y+4\right),\end{aligned}$$

$$\text{(P2)}\quad \begin{aligned}&d=3, a_1=-1, a_2=2, a_3=1, b=5,\\ &u(x,y,z)=\frac{1}{8}\left(3+e^{\frac{1}{8}(1+x)}-\tfrac{1}{32}(x+1)\left(3-x+e^{\frac{1}{4}}(x+1)\right)\right)\\ &\qquad\times\left(\sin\tfrac{1}{2}(y+1)-\tfrac{1}{8}(y+1)\left(3+\cos 1+(\cos 1-1)y\right)\right)(z-2)(z+1)^2.\end{aligned}$$

Figure 3(c) confirms Theorem 14 for these examples. Figures 3(a),(b) indicate that the uniform error for this method is $\mathcal{O}\left(N^{-3}(\log N)^{d-1}\right)$, precisely the same as for function approximation using modified Fourier series (note that, unless $a=0$, $u_N\neq\mathcal{F}_N[u]$, so the results of Section 3 do not apply directly). However, unlike the latter, the modified Fourier–Galerkin method does not offer faster convergence inside the domain.

Concerning the rate of uniform convergence, we have:

Theorem 15. *Suppose that $u\in L^\infty[-1,1]^d$ and that u_N is the modified Fourier–Galerkin approximation based on the hyperbolic cross index set (3.1). Then, for some positive constant c independent of u and N,*

$$\|u-u_N\|_\infty\leq cN^{\frac{1}{2}-\frac{1}{d}}(\log N)^{\frac{d-1}{2}}\|u-u_N\|_1+\|u-\mathcal{F}_N[u]\|_\infty.$$

Proof. Theorem 17 gives $\|v-\mathcal{F}_N[v]\|_1\leq cN^{-\frac{1}{d}}\|v\|_2$ for any $v\in H^2(-1,1)^d$ satisfying the first derivative condition. By a standard duality argument (see, for example, [10, p.190]), we obtain

$$\|u-u_N\|\leq cN^{-\frac{1}{d}}\|u-u_N\|_1.$$

Writing $e_N=\mathcal{F}_N[u]-u_N$ this result yields $\|e_N\|\leq cN^{-\frac{1}{d}}\|u-u_N\|_1$. Further, for any $\phi\in\mathcal{S}_N$ we have

$$\begin{aligned}\|\phi\|_\infty&\leq\sum_{i\in\{0,1\}^d}\sum_{n\in I_N}|\hat{\phi}^{[i]}_n|\leq\left(\sum_{i\in\{0,1\}^d}\sum_{n\in I_N}1\right)^{\frac{1}{2}}\left(\sum_{i\in\{0,1\}^d}\sum_{n\in I_N}|\hat{\phi}^{[i]}_n|^2\right)^{\frac{1}{2}}\\&\leq c|I_N|^{\frac{1}{2}}\|\phi\|\leq cN^{\frac{1}{2}}(\log N)^{\frac{d-1}{2}}\|\phi\|.\end{aligned}$$

Since $e_N \in \mathcal{S}_N$ we obtain $\|e_N\|_\infty \leq cN^{\frac{1}{2}-\frac{1}{d}}(\log N)^{\frac{d-1}{2}}\|u-u_N\|_1$. A simple application of the triangle inequality

$$\|u-u_N\|_\infty \leq \|e_N\|_\infty + \|u-\mathcal{F}_N[u]\|_\infty,$$

now yields the result. □

Corollary 16. *Suppose that u_N is the modified Fourier–Galerkin approximation based on the hyperbolic cross index set (3.1). Then, for some positive constant c independent of u and N,*

$$\|u-u_N\|_\infty \leq cN^{\frac{3}{2}-\frac{1}{d}-r}(\log N)^{\frac{d-1}{2}}\|u\|_{r,mix}, \quad r=1,2,3, \quad \|u-u_N\|_\infty \leq cN^{-2-\frac{1}{d}}(\log N)^{\frac{d-1}{2}}\|u\|_{4,mix}.$$

In light Figure 3 this result is non-optimal for $d \geq 2$.

4.4 Numerical comparison

Standard methods based on Chebyshev or Legendre polynomials yield spectral convergence whenever the solution is smooth. However, due to its lower complexity, for certain examples the modified Fourier method based on the hyperbolic cross index set (3.1) offers a lower error for moderate N. We consider three such examples, each with parameters $d=3$, $b=2$, $a=0$ and exact solutions

$$u(x,y,z) = \sinh\left(8x^3(2x^2-2)^2\right)\left(\sinh 2y - (2\cosh 2)\,y\right)\left(3z^5-5z^3\right), \tag{4.11}$$

$$u(x,y,z) = \frac{1}{4}e^{8x^4-16x^2+1}(\cosh 2y - y^2\sinh 2)(z^4-2z^2) \tag{4.12}$$

$$u(x,y,z) = \sin(2x(2x^2-2)^2)(\sin y - y\cos 1)(z^5-5z) \tag{4.13}$$

respectively.

In Figures 4–6 we plot the error against numbers of terms for this method and the Legendre–Galerkin approximation, [9, 21, 22], (the Chebyshev–Galerkin approximation gives similar results). As is evident the modified Fourier method offers a smaller error until the number of terms is moderately large. In particular, at least 3375 terms are required before the Legendre approximations to (4.11)–(4.13), which involve $\mathcal{O}\left(N^3\right)$ terms in comparison to $\mathcal{O}\left(N(\log N)^2\right)$, become superior.

Note that these plots do not take into account the operational cost of finding the coefficients of both methods. As we know from the previous discussion, finding the modified Fourier coefficients involves $\mathcal{O}\left(N^2\right)$ operations, whereas for the Legendre method, even if the coefficients of f are known exactly, this value is $\mathcal{O}\left(N^4\right)$, [22]. Thus the modified Fourier method is likely to perform even better if we were to take this into account.

These examples all have smooth solutions. Unlike the univariate case—where the so-called *shift theorem* guarantees smoothness of the solution provided the coefficients and inhomogeneous terms are smooth—the solution to (4.1) is only guaranteed $H^2(-1,1)^d$ regularity, [8], even if $f \in C^\infty[-1,1]^d$. In this situation, the modified Fourier method is competitive in terms of both complexity and convergence rate.

For $d > 3$, in both the smooth and non-smooth cases this effect becomes more pronounced. As mentioned, due to its $\mathcal{O}\left(N^{d+1}\right)$ complexity the Legendre method becomes impractical for such higher dimensional problems.

4.5 Variable coefficient problems

The extension of this method to variable coefficient problems, where $a_j : (-1,1)^d \to \mathbb{R}$ and $b : (-1,1)^d \to \mathbb{R}$ are (sufficiently smooth) functions, can be achieved in a simple manner. Concerning the approximation error, Theorems 14 and 15 are easily generalized. Further, estimates for the condition number remain valid, and an optimal, diagonal preconditioner can be easily obtained.

Much like in the Fourier case, the matrix A_{G} has entries that involve modified Fourier (and Laplace–Dirichlet) coefficients of the functions a_j and b. As with the inhomogeneous term f, these may be calculated by numerical quadrature. Efficient solution of Galerkin's equations can be achieved (as in the constant coefficient case) either by splitting the matrix A_{G} suitably or by using conjugate gradients.

This matrix is typically dense, thus direct evaluation of matrix-vector products involving A_{G} requires $\mathcal{O}\left(N^2(\log N)^{2(d-1)}\right)$ operations. However, since the action of N_{G} corresponds to finding modified Fourier

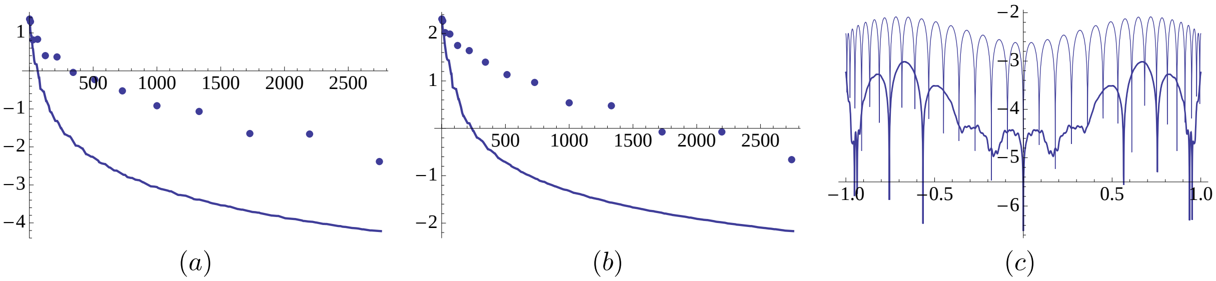


Figure 4: Comparison of the modified Fourier (line) and Legendre–Galerkin (dots) methods applied to (4.1) with parameters $d = 3$, $b = 2$, $a = 0$ and exact solution (4.11). (a) log L^2 error $\log_{10} \|u - u_N\|$ against number of terms, (b) log H^1 error $\log_{10} \|u - u_N\|_1$ against number of terms, (c) log pointwise error $\log_{10} |u(x, 1, 1) - u_N(x, 1, 1)|$ for $-1 \leq x \leq 1$ where N is chosen so that the number of terms for each method is approximately 2750.

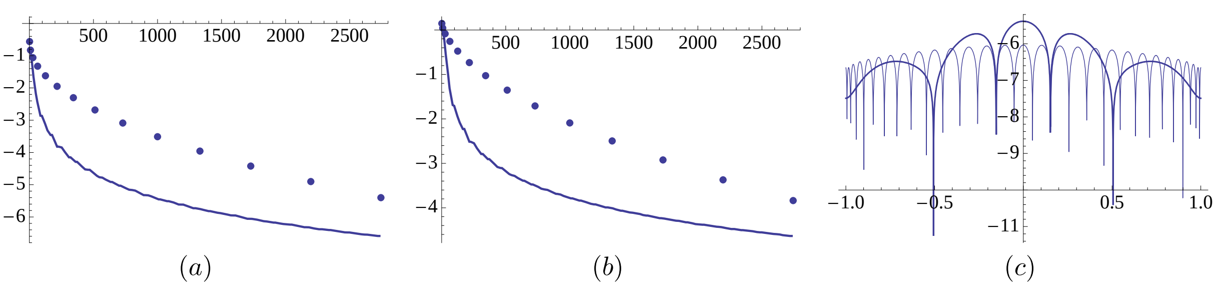


Figure 5: Comparison of the modified Fourier and Legendre–Galerkin methods applied to (4.1) with exact solution (4.12). (a) log L^2 error $\log_{10} \|u - u_N\|$ against number of terms, (b) log H^1 error $\log_{10} \|u - u_N\|_1$, (c) log pointwise error $\log_{10} |u(x, 1, 1) - u_N(x, 1, 1)|$ using approximately 2750 terms.

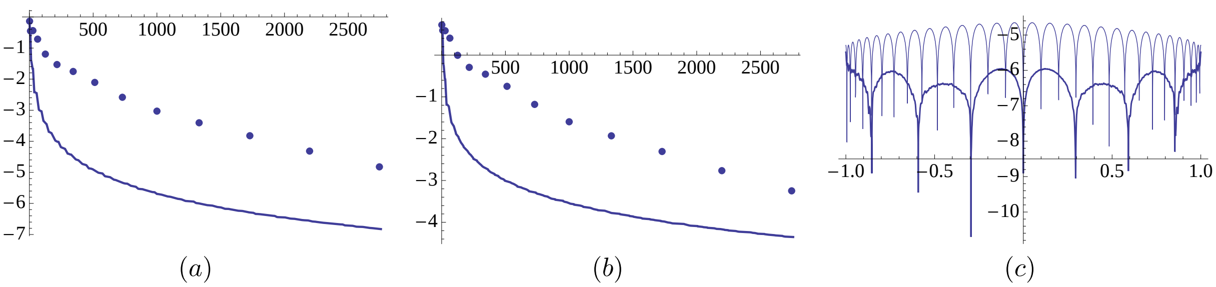
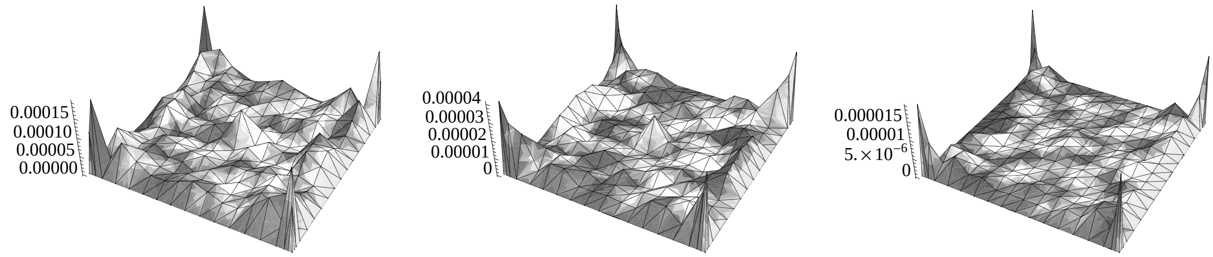


Figure 6: Comparison of the modified Fourier and Legendre–Galerkin methods applied to (4.1) with exact solution (4.13). (a) log L^2 error $\log_{10} \|u - u_N\|$ against number of terms, (b) log H^1 error $\log_{10} \|u - u_N\|_1$, (c) log pointwise error $\log_{10} |u(x, 1, 1) - u_N(x, 1, 1)|$ using approximately 2750 terms.



$N = 50$ $N = 100$ $N = 150$

Figure 7: Absolute error $|u(x, y) - u_N(x, y)|$, $-1 \le x, y \le 1$, for the Laplace–Dirichlet Galerkin approximation to (5.1) for $N = 50, 100, 150$.

coefficients of products and derivatives of finite modified Fourier sums, this figure can be reduced to $\mathcal{O}\left(N(\log N)^d\right)$ as in the constant coefficient case.

5 Discretization of Dirichlet and Robin boundary value problems

As mentioned in the Introduction, the modified Fourier basis is suited for spectral discretizations of homogeneous Neumann boundary value problems. Analogously, for a homogeneous Dirichlet boundary value problem (for example), we discretize using Laplace–Dirichlet eigenfunctions $\psi_n^{[i]}$ (see Section 1.1). The resulting method exhibits many similar properties to the modified Fourier–Galerkin method (unsurprisingly, given the duality enjoyed by the two bases). In particular, the equations may be solved in $\mathcal{O}\left(N^2\right)$ operations, and there is an optimal, diagonal preconditioner.

In Figure 7 we plot the error for the Galerkin approximation based on Laplace–Dirichlet eigenfunctions applied to the boundary value problem (4.1) subject to homogeneous Dirichlet boundary conditions with parameters $a_1 = 1$, $a_2 = -1$, $b = 3$ and exact solution

$$u(x, y) = (x^2 - 1)^2 (y^2 - 1). \tag{5.1}$$

Observe that doubling N reduces the error by roughly a factor of 4. This indicates an $\mathcal{O}\left(N^{-2}\right)$ uniform error. However, inside the domain—unlike the modified Fourier–Galerkin approximation—the error is much smaller. Numerical results indicate that an analogue of Theorem 3.2 holds for Laplace–Dirichlet Galerkin approximations.

For Robin boundary conditions

$$\partial_{x_j} u + \theta u\big|_{\Gamma_j^\pm} = 0, \quad j = 1, ..., d,$$

where $\theta \in \mathbb{R}$, we may also use a similar approach. The Laplace eigenfunctions subject to such boundary conditions are Cartesian products of the univariate eigenfunctions given explicitly by

$$\phi_0^{[0]}(x) = (\theta^{-1} \sinh(2\theta))^{-\frac{1}{2}} e^{-\theta x}, \quad \phi_n^{[0]}(x) = (n^2\pi^2 + \theta^2)^{-\frac{1}{2}} \left(n\pi \cos n\pi x - \theta \sin n\pi x\right), \quad n \in \mathbb{N}_+,$$
$$\phi_n^{[1]}(x) = ((n - \tfrac{1}{2})^2\pi^2 + \theta^2)^{-\frac{1}{2}} \left((n - \tfrac{1}{2})\pi \sin(n - \tfrac{1}{2})\pi x + \theta \cos(n - \tfrac{1}{2})\pi x\right), \quad n \in \mathbb{N}_+.$$

As in the Dirichlet case, the resulting method is shares many features with the modified Fourier method.

In Figure 8 we plot the error for the Galerkin approximation based on these eigenfunctions applied to the boundary value problem with parameters $a_1 = 2$, $a_2 = 3$, $b = 5$ subject to homogeneous Robin boundary conditions with $\theta = 3$ and exact solution

$$u(x, y) = \frac{1}{26}\left(16 e^{\frac{1-x}{2}} - 8e(x + 1) + (3e - 1)(1 + x)^2\right)(y + 1)^2 (2y - 3), \tag{5.2}$$

These results indicate an $\mathcal{O}\left(N^{-3}\right)$ uniform error, as was observed for the modified Fourier method. Moreover, unlike the Dirichlet approximation, the rate of convergence away from the boundary is not of higher order.

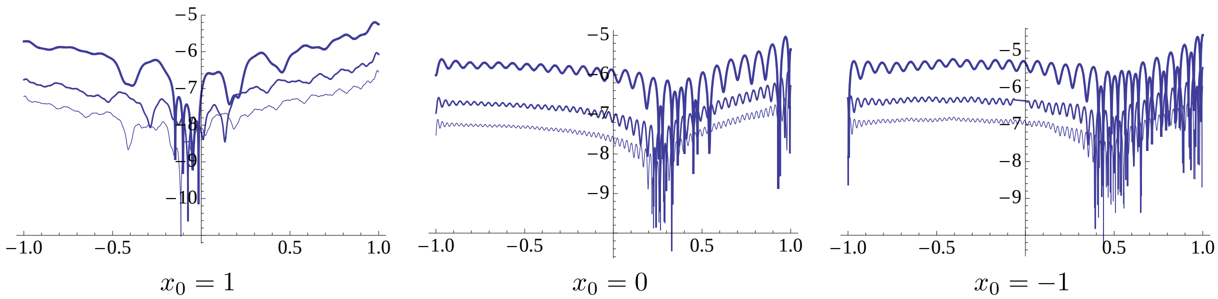


Figure 8: Log pointwise error $\log_{10}|u(x_0, y) - u_N(x_0, y)|$, $-1 \leq y \leq 1$, for the Laplace–Robin Galerkin approximation to (5.2) with $N = 50$ (thickest line), $N = 100$ and $N = 150$ (thinnest line).

Conclusions

We have developed the approximation-theoretic properties of modified Fourier series in Cartesian product domains using full and hyperbolic cross index sets. In particular we have proved uniform convergence and extended the results of [18] concerning the rate of pointwise convergence.

In the second half of this paper we have applied such series in the spectral-Galerkin approximation of Neumann boundary value problems. We have shown that the resulting approximation consists of $\mathcal{O}\left(N(\log N)^{d-1}\right)$ terms which may be found in $\mathcal{O}\left(N^2\right)$ operations using generic iterative methods. Despite offering only algebraic convergence, we have shown that these methods are more effective than standard high order polynomial approaches for moderate values of the truncation parameter and for certain problems. Moreover, the discretization matrix has a condition number of $\mathcal{O}\left(N^2\right)$ and there is an optimal, diagonal preconditioner. Finally, we have demonstrated how very similar methods can be developed for problems with Dirichlet or Robin boundary conditions.

There are a number of potential areas for further investigation. First, the Laplace–Neumann eigenfunctions are known explicitly on the equilateral triangle, hence such series can be developed on this domain. Existing spectral algorithms for triangular domains are complicated to implement, so the modified Fourier approach may offer benefits in this respect.

In [1] a method is developed to accelerate the convergence rate of the modified Fourier–Galerkin method. Unfortunately this approach is not easily applicable in higher dimensions, except in the simple case of the Helmholtz equation. Nonetheless, there are other devices for convergence acceleration which may be suitable, one of which is Eckhoff's method, [6]. Incorporating this technique into spectral discretizations is an area for future investigation.

As demonstrated in Section 5, closely related techniques can be developed for Dirichlet and Robin boundary value problems. Unfortunately more complicated boundary conditions cannot be tackled so easily. For example, so-called co-normal boundary conditions are outside the scope of this approach, since the relevant Laplace eigenfunctions cannot be expressed as simple Cartesian products. However, certain higher, even-order differential operators also have simple eigenfunctions, [12]. These may have application in the spectral discretization of higher-order boundary value problems.

Finally, throughout this paper we have assumed that modified Fourier coefficients are calculated to sufficiently high accuracy using numerical quadrature as outlined in [13]. This area alone warrants further investigation. In particular, robust error estimates are largely lacking and the quadrature routines become more complicated as d increases. For this reason efficient implementation of the modified Fourier–Galerkin method requires further research. Nonetheless, this approach allows us to immediately exploit hyperbolic cross index sets, leading to the favourable properties outlined in this paper.

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