Asymptotic expansion and quadrature of composite highly oscillatory integrals

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Abstract

We consider in this paper asymptotic and numerical aspects of highly oscillatory integrals of the form $\int_a^b f(x)g(\sin[\omega\theta(x)])\,\mathrm{d}x$, where $\omega \gg 1$. Such integrals occur in the simulation of electronic circuits, but they are also of an independent mathematical interest.

The integral is expanded in asymptotic series in inverse powers of ω. This expansion clarifies the behaviour for large ω and also provides powerful means to design effective computational algorithms. In particular, we introduce and analyse Filon-type methods for this integral.

1 Introduction

The theory of highly oscillatory integrals is a mature subject and our theoretical understanding of the behaviour of integrals of the form

$$\int_a^b f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\,\mathrm{d}x \tag{1.1}$$

and their multivariate counterparts is fairly complete (Olver 1974, Stein 1993, Wong 2001). In the last few years this has been complemented by comprehensive understanding of the numerical quadrature of such integrals by a range of methods: Filon-type (Iserles & Nørsett 2004, Iserles & Nørsett 2005), Levin-type (Levin 1996, Olver 2006) and numerical steepest descent (Huybrechs & Vandewalle 2006). It is however in the nature of mathematical research that, no sooner than we declare a theory 'complete' , a new application comes to light, provides a counter-example and challenges our understanding of the subject.

A major stumbling block in the simulation of electronic circuits is the resolution of high frequency modulated signals. An important example is the modulation of a *diode rectifier circuit*, which is modelled by a nonlinear ordinary differential equation of the form

$$Cv' = -\frac{v}{R} + I_0[\mathrm{e}^{b(t)-v} - 1], \quad t \geq 0, \qquad v(0) = v_0, \tag{1.2}$$

where C is the capacitance, R the resistance and I_0 the diode inverse bias current, $b(t)$ is the input signal and the unknown v is the voltage (Dautbegovic, Condon & Brennan 2005). *Analogue modulation* of (1.2) is associated with the input signal $b(t) = \kappa sin(\omega t)$, where $\omega \gg 1$. Effective numerical solution of (1.2) with this value of b, using waveform relaxation, requires efficient computation of integrals of the form

$$\int_a^b f(x)\mathrm{e}^{\kappa \sin(\omega x)}\,\mathrm{d}x \tag{1.3}$$

(Condon, Deaño, Iserles, Maczynski & Xu 2008*b*). The integral (1.3) does not fit into the classical pattern (1.1) of highly oscillatory integrals or of its generalizations, whereby $\exp[\mathrm{i}\omega g(x)]$ is replaced by a suitable fundamental solution of a linear ordinary differential equation (Olver 2008) and, indeed, as will transpire shortly, its behaviour is altogether different. In particular, while (for $-\infty < a < b < \infty$) the integral (1.1) behaves like $\mathcal{O}\big(\omega^{-1/(r+1)}\big)$ for $\omega \to \infty$, where r is the highest degree of a stationary point ($r = 0$ if $g' \neq 0$ in the closed interval), the integral (1.3) is $\mathcal{O}(1)$ for $\omega \to \infty$.

It is possible to analyse (and, indeed, compute) the integral (1.3) by exploiting a serendipitous identity,

$$\mathrm{e}^{\kappa \sin\theta} = I_0(\kappa) + 2\sum_{m=0}^{\infty}(-1)^m I_m(\kappa)\sin[(2m+1)\theta] + 2\sum_{m=1}^{\infty}(-1)^m I_{2m}(\kappa)\cos(2m\theta),$$

where I_m is the mth modified Bessel function (Abramowitz & Stegun 1964, p. 376), and this has been already done in (Condon, Deaño & Iserles 2008*a*). The purpose of the present paper is more ambitious, namely to consider integrals of the form

$$\boldsymbol{I}[f,g] = \int_a^b f(x)g(\sin(\omega x))\,\mathrm{d}x, \qquad \omega \gg 1, \tag{1.4}$$

where $f \in \mathrm{C}^\infty[a,b]$, g is analytic in the disc $|z| < r$ for some $r > 1$ and $\omega \gg 1$. In that case we no longer enjoy the benefits of serendipity and need to work out the asymptotic behaviour of (1.4) from basic premises.

An interesting generalisation of (1.4) is to the integral

$$\boldsymbol{I}[f,g,\theta] = \int_a^b f(x)g(\sin[\omega\theta(x)])\,\mathrm{d}x, \tag{1.5}$$

where $\theta \in \mathrm{C}^\infty[a,b]$.[1] Provided that $\theta' \neq 0$ in $[a,b]$, our analysis can be easily extended to this setting. Matters are more difficult when θ' is allowed to vanish, a situation similar to the

[1]Here and elsewhere the $\mathrm{C}^\infty[a,b]$ assumption can be relaxed and replaced by $\mathrm{C}^\nu[a,b]$ for some $\nu \geq 1$, except that in that case only finite number of leading terms in our expansions are valid.

presence of stationary points in classical theory. In that case we indicate how to obtain an expansion using an approach similar to (Iserles & Nørsett 2005).

In Section 2 we expand (1.4) into asymptotic series in inverse powers of ω, while Section 3 is devoted to efficient numerical quadrature of this integral by means of a variant of a Filon-type method.

We note in passing that our approach is applicable also to the integral

$$\int_a^b f(x)g(\cos(\omega x))\,\mathrm{d}x, \qquad \omega \gg 1.$$

Since an identical methodology applies and the results are similar, we do not dwell further on this.

2 The asymptotic expansion of the integral

2.1 The basic expansion

Expanding the analytic function g into Taylor series, we have

$$\boldsymbol{I}[f,g] = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} \boldsymbol{S}_n[f], \tag{2.1}$$

where

$$\boldsymbol{S}_n[f] = \int_a^b f(x)\sin^n(\omega x)\,\mathrm{d}x, \qquad n \in \mathbb{Z}_+.$$

Thus, our first goal is to expand each functional $\boldsymbol{S}_n[f]$ asymptotically in ω.

Noting that

$$\begin{aligned}\boldsymbol{S}_n[f] &= \frac{1}{(2\mathrm{i})^n}\int_a^b f(x)(\mathrm{e}^{\mathrm{i}\omega x} - \mathrm{e}^{-\mathrm{i}\omega x})^n\,\mathrm{d}x \\ &= \frac{1}{(2\mathrm{i})^n}\sum_{m=0}^{n}(-1)^{n-m}\binom{n}{m}\int_a^b f(x)\mathrm{e}^{\mathrm{i}\omega(2m-n)x}\,\mathrm{d}x,\end{aligned}$$

we need to distinguish between even and odd values of n. This leads to calculations which, although lengthy, are quite elementary.

For even n we have

$$\begin{aligned}\boldsymbol{S}_{2n}[f] = \frac{(-1)^n}{4^n}&\left\{(-1)^n\binom{2n}{n}\int_a^b f(x)\,\mathrm{d}x\right. \\ &\left.+\sum_{m=0}^{n-1}(-1)^m\binom{2n}{m}\int_a^b f(x)[\mathrm{e}^{2\mathrm{i}\omega(n-m)x} + \mathrm{e}^{-2\mathrm{i}\omega(n-m)x}]\,\mathrm{d}x\right\} \\ \sim \frac{1}{4^n}&\binom{2n}{n}\int_a^b f(x)\,\mathrm{d}x - \frac{(-1)^n}{4^n}\sum_{m=0}^{n-1}(-1)^m\binom{2n}{m}\sum_{k=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{k+1}}\end{aligned}$$

$$
\begin{aligned}
&\times \Bigg\{ f^{(k)}(b) \left[\frac{\mathrm{e}^{2\mathrm{i}\omega(n-m)b}}{(2n-2m)^{k+1}} + \frac{\mathrm{e}^{-2\mathrm{i}\omega(n-m)b}}{(2m-2n)^{k+1}} \right] \\
&\qquad\qquad - f^{(k)}(a) \left[\frac{\mathrm{e}^{2\mathrm{i}\omega(n-m)a}}{(2n-2m)^{k+1}} + \frac{\mathrm{e}^{-2\mathrm{i}\omega(n-m)a}}{(2m-2n)^{k+1}} \right] \Bigg\} \\
&= \frac{1}{4^n}\binom{2n}{n}\int_a^b f(x)\,\mathrm{d}x - \frac{(-1)^n}{4^n}\sum_{m=0}^{n-1}(-1)^m\binom{2n}{m}\sum_{k=0}^{\infty}\frac{1}{[-2\mathrm{i}\omega(n-m)]^{k+1}} \\
&\times \Big\{ f^{(k)}(b)[\mathrm{e}^{2\mathrm{i}\omega(n-m)b} - (-1)^k \mathrm{e}^{-2\mathrm{i}\omega(n-m)b}] \\
&\qquad\qquad - f^{(k)}(a)[\mathrm{e}^{2\mathrm{i}\omega(n-m)a} - (-1)^k \mathrm{e}^{-2\mathrm{i}\omega(n-m)a}] \Big\} \\
&= \frac{1}{4^n}\binom{2n}{n}\int_a^b f(x)\,\mathrm{d}x + \frac{2(-1)^n}{4^n}\sum_{m=0}^{n-1}(-1)^m\binom{2n}{m} \\
&\times \Bigg\{ \sum_{k=0}^{\infty}\frac{(-1)^k}{[2\omega(n-m)]^{2k+1}}[f^{(2k)}(b)\sin 2\omega(n-m)b - f^{(2k)}(a)\sin 2\omega(n-m)a] \\
&+ \sum_{k=0}^{\infty}\frac{(-1)^k}{[2\omega(n-m)]^{2k+2}}[f^{(2k+1)}(b)\cos 2\omega(n-m)b \\
&\qquad\qquad\qquad - f^{(2k+1)}(a)\cos 2\omega(n-m)a] \Bigg\} \\
&= \frac{1}{4^n}\binom{2n}{n}\int_a^b f(x)\,\mathrm{d}x \\
&+ \frac{2(-1)^n}{4^n}\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+1}}f^{(2k)}(b)\sum_{m=0}^{n-1}(-1)^m\binom{2n}{m}\frac{\sin 2\omega(n-m)b}{[2(n-m)]^{2k+1}} \\
&- \frac{2(-1)^n}{4^n}\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+1}}f^{(2k)}(a)\sum_{m=0}^{n-1}(-1)^m\binom{2n}{m}\frac{\sin 2\omega(n-m)a}{[2(n-m)]^{2k+1}} \\
&+ \frac{2(-1)^n}{4^n}\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+2}}f^{(2k+1)}(b)\sum_{m=0}^{n-1}(-1)^m\binom{2n}{m}\frac{\cos 2\omega(n-m)b}{[2(n-m)]^{2k+2}} \\
&- \frac{2(-1)^n}{4^n}\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+2}}f^{(2k+1)}(a)\sum_{m=0}^{n-1}(-1)^m\binom{2n}{m}\frac{\cos 2\omega(n-m)a}{[2(n-m)]^{2k+2}} \\
&= \frac{1}{4^n}\binom{2n}{n}\int_a^b f(x)\,\mathrm{d}x + \frac{2}{4^n}\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+1}}f^{(2k)}(b)\sum_{m=1}^{n}(-1)^m\binom{2n}{n-m}\frac{\sin 2\omega mb}{(2m)^{2k+1}} \\
&- \frac{2}{4^n}\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+1}}f^{(2k)}(a)\sum_{m=1}^{n}(-1)^m\binom{2n}{n-m}\frac{\sin 2\omega ma}{(2m)^{2k+1}} \\
&+ \frac{2}{4^n}\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+2}}f^{(2k+1)}(b)\sum_{m=1}^{n}(-1)^m\binom{2n}{n-m}\frac{\cos 2\omega mb}{(2m)^{2k+2}} \\
&- \frac{2}{4^n}\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+2}}f^{(2k+1)}(a)\sum_{m=1}^{n}(-1)^m\binom{2n}{n-m}\frac{\cos 2\omega ma}{(2m)^{2k+2}}.
\end{aligned}
$$

Thus, we have expanded $\boldsymbol{S}_{2n}[f]$ in inverse powers of ω. Similarly to the familiar expansions of integrals (1.1) in the absence of stationary points, the coefficients of the expansion depend on f and its derivatives at the endpoints. However, an important difference is that, while the integral in (1.1) tends to zero for $\omega \to \infty$, we have

$$\boldsymbol{S}_{2n}[f] \sim \frac{1}{4^n}\binom{2n}{n}\int_a^b f(x)\,\mathrm{d}x, \qquad \omega \to \infty.$$

Note that it is sufficient that $f \in \mathrm{C}[a,b]$ for this to be true.

The expansion for odd index, is similar,

$$\begin{aligned}
\boldsymbol{S}_{2n+1}[f] &= -\frac{1}{(2\mathrm{i})^{2n+1}}\sum_{m=0}^{2n+1}(-1)^m\binom{2n+1}{m}\int_a^b f(x)\mathrm{e}^{\mathrm{i}\omega(2m-2n-1)x}\,\mathrm{d}x \\
&= -\frac{(-1)^n\mathrm{i}}{2\cdot 4^n}\sum_{m=0}^{n}(-1)^m\binom{2n+1}{m}\int_a^b f(x)[\mathrm{e}^{\mathrm{i}\omega(2n-2m+1)x}-\mathrm{e}^{-\mathrm{i}\omega(2n-2m+1)x}]\,\mathrm{d}x \\
&\sim \frac{(-1)^n\mathrm{i}}{2\cdot 4^n}\sum_{m=0}^{n}(-1)^m\binom{2n+1}{m}\sum_{k=0}^{\infty}\frac{1}{(-\mathrm{i}\omega)^{k+1}}\left\{f^{(k)}(b)\left[\frac{\mathrm{e}^{\mathrm{i}\omega(2n-2m+1)b}}{(2n-2m+1)^{k+1}}\right.\right. \\
&\quad \left. + (-1)^k\frac{\mathrm{e}^{-\mathrm{i}\omega(2n-2m+1)b}}{(2n-2m+1)^{k+1}}\right] - f^{(k)}(a)\left[\frac{\mathrm{e}^{\mathrm{i}\omega(2n-2m+1)a}}{(2n-2m+1)^{k+1}}\right. \\
&\quad \left.\left. + (-1)^k\frac{\mathrm{e}^{-\mathrm{i}\omega(2n-2m+1)a}}{(2n-2m+1)^{k+1}}\right]\right\} \\
&= \frac{(-1)^n}{4^n}\sum_{m=0}^{n}(-1)^m\binom{2n+1}{m}\left\{-\sum_{k=0}^{\infty}\frac{(-1)^k}{[\omega(2n-2m+1)]^{2k+1}}\right. \\
&\quad \times [f^{(2k)}(b)\cos\omega(2n-2m+1)b - f^{(2k)}(a)\cos\omega(2n-2m+1)a] \\
&\quad + \sum_{k=0}^{\infty}\frac{(-1)^k}{[\omega(2n-2m+1)]^{2k+2}}[f^{(2k+1)}(b)\sin\omega(2n-2m+1)b \\
&\qquad\qquad \left. - f^{(2k+1)}(a)\sin\omega(2n-2m+1)a]\right\} \\
&= \frac{(-1)^n}{4^n}\left[-\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+1}}f^{(2k)}(b)\sum_{m=0}^{n}(-1)^m\binom{2n+1}{m}\frac{\cos\omega(2n-2m+1)b}{(2n-2m+1)^{2k+1}}\right. \\
&\quad + \sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+1}}f^{(2k)}(a)\sum_{m=0}^{n}(-1)^m\binom{2n+1}{m}\frac{\cos\omega(2n-2m+1)a}{(2n-2m+1)^{2k+1}} \\
&\quad + \sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+2}}f^{(2k+1)}(b)\sum_{m=0}^{n}(-1)^m\binom{2n+1}{m}\frac{\sin\omega(2n-2m+1)b}{(2n-2m+1)^{2k+2}} \\
&\quad \left. - \sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+2}}f^{(2k+1)}(a)\sum_{m=0}^{n}(-1)^m\binom{2n+1}{m}\frac{\sin\omega(2n-2m+1)a}{(2n-2m+1)^{2k+2}}\right] \\
&= -\frac{1}{4^n}\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+1}}f^{(2k)}(b)\sum_{m=0}^{n}(-1)^m\binom{2n+1}{n-m}\frac{\cos\omega(2m+1)b}{(2m+1)^{2k+1}}
\end{aligned}$$

$$+\frac{1}{4^n}\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+1}}f^{(2k)}(a)\sum_{m=0}^{n}(-1)^m\binom{2n+1}{n-m}\frac{\cos\omega(2m+1)a}{(2m+1)^{2k+1}}$$
$$+\frac{1}{4^n}\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+2}}f^{(2k+1)}(b)\sum_{m=0}^{n}(-1)^m\binom{2n+1}{n-m}\frac{\sin\omega(2m+1)b}{(2m+1)^{2k+2}}$$
$$-\frac{1}{4^n}\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+2}}f^{(2k+1)}(a)\sum_{m=0}^{n}(-1)^m\binom{2n+1}{n-m}\frac{\sin\omega(2m+1)a}{(2m+1)^{2k+2}}.$$

We substitute asymptotic expansions of $\boldsymbol{S}_n[f]$ for even and odd values of n into (2.1). The outcome is an asymptotic expansion of the integral $\boldsymbol{I}[f,g]$ in inverse powers of large parameter ω,

$$\begin{aligned}
I[f] \sim\ & 2\sum_{n=0}^{\infty}\frac{g^{(2n)}(0)}{2^{2n}}\Bigg[\frac{1}{2}\frac{1}{(n!)^2}\int_a^b f(x)\,\mathrm{d}x \\
&+\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+1}}f^{(2k)}(b)\sum_{m=1}^{n}\frac{(-1)^m}{(n-m)!(n+m)!}\frac{\sin 2\omega mb}{(2m)^{2k+1}} \\
&-\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+1}}f^{(2k)}(a)\sum_{m=1}^{n}\frac{(-1)^m}{(n-m)!(n+m)!}\frac{\sin 2\omega ma}{(2m)^{2k+1}} \\
&+\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+2}}f^{(2k+1)}(b)\sum_{m=1}^{n}\frac{(-1)^m}{(n-m)!(n+m)!}\frac{\cos 2\omega mb}{(2m)^{2k+2}} \\
&-\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+2}}f^{(2k+1)}(a)\sum_{m=1}^{n}\frac{(-1)^m}{(n-m)!(n+m)!}\frac{\cos 2\omega ma}{(2m)^{2k+2}}\Bigg] \\
&+2\sum_{n=0}^{\infty}\frac{g^{(2n+1)}(0)}{2^{2n+1}}\Bigg[-\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+1}}f^{(2k)}(b)\sum_{m=0}^{n}\frac{(-1)^m}{(n-m)!(n+m+1)!}\frac{\cos\omega(2m+1)b}{(2m+1)^{2k+1}} \\
&+\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+1}}f^{(2k)}(a)\sum_{m=0}^{n}\frac{(-1)^m}{(n-m)!(n+m+1)!}\frac{\cos\omega(2m+1)a}{(2m+1)^{2k+1}} \\
&+\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+2}}f^{(2k+1)}(b)\sum_{m=0}^{n}\frac{(-1)^m}{(n-m)!(n+m+1)!}\frac{\sin\omega(2m+1)b}{(2m+1)^{2k+2}} \\
&-\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+2}}f^{(2k+1)}(a)\sum_{m=0}^{n}\frac{(-1)^m}{(n-m)!(n+m+1)!}\frac{\sin\omega(2m+1)a}{(2m+1)^{2k+2}}\Bigg] \\
=\ & \sum_{n=0}^{\infty}\frac{g^{(2n)}(0)}{(n!)^2}\frac{1}{2^{2n}}\int_a^b f(x)\,\mathrm{d}x \\
&+2\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+1}}f^{(2k)}(b)\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m)^{2k+1}}\frac{\sin 2\omega mb}{2^{2m}}\sum_{n=0}^{\infty}\frac{g^{(2m+2n)}(0)}{n!(2m+n)!}\frac{1}{4^n} \\
&-2\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+1}}f^{(2k)}(a)\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m)^{2k+1}}\frac{\sin 2\omega ma}{2^{2m}}\sum_{n=0}^{\infty}\frac{g^{(2m+2n)}(0)}{n!(2m+n)!}\frac{1}{4^n}
\end{aligned}$$

$$
\begin{aligned}
&-2\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+1}}f^{(2k)}(b)\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^{2k+1}}\frac{\cos\omega(2m+1)b}{2^{2m+1}}\sum_{n=0}^{\infty}\frac{g^{(2m+2n+1)}(0)}{n!(2m+1+n)!}\frac{1}{4^n}\\
&+2\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+1}}f^{(2k)}(a)\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^{2k+1}}\frac{\cos\omega(2m+1)a}{2^{2m+1}}\sum_{n=0}^{\infty}\frac{g^{(2m+2n+1)}(0)}{n!(2m+1+n)!}\frac{1}{4^n}\\
&+2\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+2}}f^{(2k+1)}(b)\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m)^{2k+2}}\frac{\cos 2\omega mb}{2^{2m}}\sum_{n=0}^{\infty}\frac{g^{(2m+2n)}(0)}{n!(2m+n)!}\frac{1}{4^n}\\
&-2\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+2}}f^{(2k+1)}(a)\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m)^{2k+2}}\frac{\cos 2\omega ma}{2^{2m}}\sum_{n=0}^{\infty}\frac{g^{(2m+2n)}(0)}{n!(2m+n)!}\frac{1}{4^n}\\
&+2\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+2}}f^{(2k+1)}(b)\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^{2k+2}}\frac{\sin\omega(2m+1)b}{2^{2m+1}}\sum_{n=0}^{\infty}\frac{g^{(2m+2n+1)}(0)}{n!(2m+1+n)!}\frac{1}{4^n}\\
&-2\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+2}}f^{(2k+1)}(a)\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^{2k+2}}\frac{\sin\omega(2m+1)a}{2^{2m+1}}\sum_{n=0}^{\infty}\frac{g^{(2m+2n+1)}(0)}{n!(2m+1+n)!}\frac{1}{4^n}.
\end{aligned}
$$

Let

$$
\rho_m=\frac{1}{2^{m-1}}\sum_{n=0}^{\infty}\frac{g^{(m+2n)}(0)}{n!(m+n)!}\frac{1}{4^n},\qquad m\in\mathbb{Z}_+, \tag{2.2}
$$

and set for all $k\in\mathbb{Z}_+$

$$
\begin{aligned}
U_k(t)&=\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m)^{2k+1}}\rho_{2m}\sin(2\omega mt)-\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^{2k+1}}\rho_{2m+1}\cos[\omega(2m+1)t],\\
V_k(t)&=\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m)^{2k+2}}\rho_{2m}\cos(2\omega mt)+\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^{2k+2}}\rho_{2m+1}\sin[\omega(2m+1)t].
\end{aligned}
$$

Then

$$
\begin{aligned}
\boldsymbol{I}[f,g]\sim\tfrac{1}{2}\rho_0\int_a^b f(x)\,\mathrm{d}x&+\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+1}}[f^{(2k)}(b)U_k(b)-f^{(2k)}(a)U_k(a)] \qquad (2.3)\\
&+\sum_{k=0}^{\infty}\frac{(-1)^k}{\omega^{2k+2}}[f^{(2k+1)}(b)V_k(b)-f^{(2k+1)}(a)V_k(a)].
\end{aligned}
$$

Note that the function g enters the asymptotic expansion solely through the sequence $\boldsymbol{\rho}=\{\rho_m\}_{m\in\mathbb{Z}_+}$, while f contributes both through a nonoscillatory integral and derivatives at the endpoints.

2.2 The sequence $\boldsymbol{\rho}$

In the present subsection we consider the sequence $\boldsymbol{\rho}=\{\rho_m\}_{m\in\mathbb{Z}_+}$. Our main concern is with the asymptotic behaviour of the sequence. Examining the functions U_k and V_k, it is clear that geometric decay, $\rho_m\approx c\tilde{\rho}^m$ for some $\tilde{\rho}\in(0,1)$, is sufficient for their existence and boundedness for all $k\in\mathbb{Z}_+$, a *sine qua non* for the applicability of the expansion (2.3)

We commence with three detailed examples. Firstly we let $g(x) = \mathrm{e}^{\kappa x}$, $\kappa \in \mathbb{R} \setminus \{0\}$, therefore consider the integral (1.3). This yields

$$\rho_m = 2^{1-m}\kappa^m \sum_{n=0}^{\infty} \frac{1}{n!(m+n)!}\left(\frac{\kappa}{2}\right)^{2n} = 2\frac{(\kappa/2)^m}{m!}\,{}_0F_1\left[\begin{array}{c} -; \\ m+1; \end{array} \frac{\kappa^2}{2}\right] = 2I_m(\kappa),$$

where we have used the hypergeometric representation of modified Bessel functions (Rainville 1960, p. 116). After brief manipulation, we recover the serendipitous asymptotic expansion from (Condon et al. 2008*a*). Note that standard asymptotic expansions of modified Bessel functions for large index demonstrate at once the extraordinarily rapid decay of the sequence $\boldsymbol{\rho}$, because $I_m(\kappa) = \mathcal{O}(1/m!)$.

We next consider the case $g(x) = \sin \kappa x$, where $\kappa \in \mathbb{R} \setminus \{0\}$. Similar algebra conforms that

$$\rho_{2m} = 0, \quad \rho_{2m+1} = 2(-1)^m J_{2m+1}(\kappa),$$

where J_m is the Bessel function. Again, the sequence $\boldsymbol{\rho}$ decays at a faster-than-exponential speed.

A more interesting example, of which we will make use in the sequel, is $g(x) = (1 - \kappa x)$, where $|\kappa| < 1$. Therefore $g^{(n)}(0) = n!\kappa^n$ and

$$\rho_m = \frac{1}{2^{m-1}} \sum_{n=0}^{\infty} \frac{(m+2n)!\kappa^{m+2n}}{n!(m+n)!4^n} = 2\left(\frac{\kappa}{2}\right)^m {}_2F_1\left[\begin{array}{c} \frac{m+1}{2}, \frac{m+2}{2}; \\ m+1; \end{array} \frac{\kappa^2}{4}\right].$$

Let

$$\chi(z) = {}_2F_1\left[\begin{array}{c} \frac{m+1}{2}, \frac{m+2}{2}; \\ m+1; \end{array} z\right], \qquad |z| < 1.$$

We use the identity

$$\frac{1}{(1+x)^{2a}}\,{}_2F_1\left[\begin{array}{c} a, b; \\ 2b; \end{array} \frac{4x}{(1+x)^2}\right] = {}_2F_1\left[\begin{array}{c} a, a-b+\frac{1}{2}; \\ b+\frac{1}{2}; \end{array} x^2\right]$$

(Rainville 1960, p. 65) with $a = m/2 + 1$, $b = (m+1)/2$ and $x = (2 - t - 2\sqrt{1-t})/t$. After long, yet elementary algebra, and bearing in mind that for our values of a and b it is true that

$${}_2F_1\left[\begin{array}{c} a, a-b+\frac{1}{2}; \\ b+\frac{1}{2}; \end{array} x^2\right] = {}_2F_1\left[\begin{array}{c} \frac{m+2}{2}, 1; \\ \frac{m+2}{2}; \end{array} x^2\right] = \frac{1}{1-x^2},$$

we have

$$\chi(z) = \frac{1}{\sqrt{1-z}}\left[\frac{2(1-\sqrt{1-z})}{z}\right]^m = \frac{1}{\sqrt{1-z}}\left(\frac{2}{1+\sqrt{1-z}}\right)^m.$$

In particular, recalling that $|\kappa| < 1$, we deduce

$$\rho_m = \frac{1}{\sqrt{1-\kappa^2/4}}\left(\frac{\kappa}{1+\sqrt{1-\kappa^2/4}}\right)^m, \qquad m \in \mathbb{Z}_+,$$

an exponential decay.

Note that the above identity from (Rainville 1960) is true only when $|x| < 1$ and $4|x| < |1+x|^2$. This is not a problem, because we can extend our explicit form of χ elsewhere by analytic continuation. The only subtle point about it is that we might need to change the sign of the square root once we cross the branch cut $\mathrm{Re}\, z = 0$.

Theorem 1 *Given any analytic function g with radius of convergence $r > 1$, it is true that $\lim_{m\to\infty} \rho_m = 0$. Moreover, if $r < \infty$ then $\rho_m = o(r^{-2m})$, while if g is entire, i.e. $r = +\infty$, then $\boldsymbol{\rho}$ decays faster than a reciprocal of a polynomial.*

Proof Applying the Cauchy test to the Taylor expansion of g about the origin,

$$\frac{1}{r} = \left|\frac{g^{(n)}(0)}{n!}\right|^{1/n} = \frac{\mathrm{e}|g^{(n)}(0)|^{1/n}}{n} + o(1), \qquad n \to \infty,$$

where we have used the Stirling formula to approximate the factorial. Next, we apply the Cauchy test to the definition of ρ_m: convergence is equivalent to

$$1 > \limsup_{n\to\infty} \left|\frac{g^{(m+2n)}(0)}{n!(m+n)!4^n}\right|^{1/n} = \limsup_{n\to\infty} \frac{\left[\left|g^{(m+2n)}(0)\right|^{1/(m+2n)}\right]^{2+m/n}}{4(n!)^{1/n}[(m+n)!]^{1/n}}.$$

Using again the Stirling formula and exploiting the fact that

$$\limsup_{n\to\infty} \frac{\left|g^{(m+2n)}(0)\right|^{1/(m+2n)}}{m+2n} \le \limsup_{n\to\infty} \frac{\left|g^{(n)}(0)\right|^{1/n}}{n} = \frac{1}{\mathrm{e}r},$$

we deduce that the limsup on the right is $r^{-2} < 1$, hence convergence.

The assertion on speed of convergence follows at once from the Cauchy criterion. □

Noting the importance of Theorem 1 in justifying the validity of the asymptotic expansion (2.3), we also observe an important potential shortcoming of the definition (2.2). Numerical computation of high derivatives of a function g, its smoothness notwithstanding, is a notoriously ill-conditioned problem. We will return to this issue in the sequel, but at present we generalize from our examples and provide an explicit formula for a very important subset of functions g.

Recall that a *generalized hypergeometric function* is

$${}_pF_q\left[\begin{array}{l}\alpha_1, \alpha_2, \ldots, \alpha_p;\\ \beta_1, \beta_2, \ldots, \beta_q;\end{array} x\right] = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\prod_{i=1}^p (\alpha_i)_n}{\prod_{i=1}^q (\beta_i)_n} x^n,$$

where the *Pochhammer symbol* $(z)_n$ is defined recursively, $(z)_0 = 1$ and $(z)_n = (z)_{n-1}(z + n - 1)$, $n \in \mathbb{N}$ (Rainville 1960). The parameters α_i and β_i are arbitrary complex numbers, except that the β_is can be neither zero nor negative integers. Not just our three examples but many other important functions in applied mathematics and theoretical physics can be written using hypergeometric functions.

Let us assume that $a = -1$, $b = 1$ and

$$g(x) = {}_pF_q\left[\begin{array}{l}\alpha_1, \alpha_2, \ldots, \alpha_p;\\ \beta_1, \beta_2, \ldots, \beta_q;\end{array} \kappa x\right].$$

To assure ourselves of analyticity in $[-1, 1]$, we require $q + 1 \ge p$ and that if $q + 1 = p$ then $|\kappa| < 1$. Since

$$g^{(n)}(0) = \frac{\prod_{i=1}^p (\alpha_i)_n}{\prod_{i=1}^q (\beta_i)_n} \kappa^n,$$

substitution in (2.2) yields

$$\rho_m = \frac{1}{2^{m-1}} \sum_{n=0}^{\infty} \frac{1}{n!(m+n)!} \frac{\prod_{i=1}^{p} (\alpha_i)_{m+2n}}{\prod_{i=1}^{q} (\beta_i)_{m+2n}} \kappa^{m+2n}.$$

But

$$(\gamma)_{m+2n} = (\gamma)_m (\gamma+m)_{2n} = 4^n (\gamma)_m \left(\tfrac{\gamma+m}{2}\right)_n \left(\tfrac{\gamma+m+1}{2}\right)_n, \qquad \gamma \in \mathbb{C}, \quad m,n \in \mathbb{Z}_+,$$

therefore

$$\begin{aligned}
\rho_m &= 2 \frac{\prod_{i=1}^{p} (\alpha_i)_m}{m! \prod_{i=1}^{q} (\beta_i)_m} \left(\frac{\kappa}{2}\right)^m \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \left(\frac{\alpha_i+m}{2}\right)_n \left(\frac{\alpha_i+m+1}{2}\right)_n}{n!(m+1)_n \prod_{i=1}^{q} \left(\frac{\beta_i+m}{2}\right)_n \left(\frac{\beta_i+m+1}{2}\right)_n} \frac{\kappa^{2n}}{4^{(1+q-p)n}} \\
&= 2 \frac{\prod_{i=1}^{p} (\alpha_i)_m}{m! \prod_{i=1}^{q} (\beta_j)_m} \left(\frac{\kappa}{2}\right)^m {}_{2p}F_{2q+1} \left[\begin{matrix} \tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_{2p}; \\ m+1, \tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_{2q}; \end{matrix} \frac{\kappa^2}{4^{1+q-p}} \right],
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\alpha}_i &= \frac{\alpha_i + m}{2}, & \tilde{\alpha}_{p+i} &= \frac{\alpha_i + m + 1}{2}, & i &= 1, 2, \ldots, p; \\
\tilde{\beta}_i &= \frac{\beta_i + m}{2}, & \tilde{\beta}_{q+i} &= \frac{\beta_i + m + 1}{2}, & i &= 1, 2, \ldots, q.
\end{aligned}$$

Therefore ρ_m itself can be expressed as a generalized hypergeometric function. Note that $q+1 \geq p$ implies that $(2q+1)+1 \geq 2p$ and ρ_m is analytic in $[-1,1]$ as a function of κ (or, for that matter, of $\kappa/4^{1+q-p}$). This is consistent with Theorem 1.

The importance of this hypergeometric representation of $\boldsymbol{\rho}$ is that most modern software, e.g. MAPLE. MATHEMATICA and MATLAB, is well equipped to calculate generalized hypergeometric functions. This avoids the need to calculate high derivatives and provides an effective means to compute $\boldsymbol{\rho}$, hence the asymptotic expansion (2.3), when g is a generalized hypergeometric function.

2.3 A generalised oscillator

Our results can be generalized with ease to the integral $\boldsymbol{I}[f,g,\theta]$ defined in (1.5), provided that $\theta' \neq 0$ in $[a,b]$. Letting $t = \theta(x)$, a trivial change of variables results in

$$\boldsymbol{I}[f,g,\theta] = \int_{\theta(a)}^{\theta(b)} \frac{f(\theta^{-1}(t))}{\theta'(\theta^{-1}(t))} g(\sin \omega t)\, \mathrm{d}t = \boldsymbol{I}[\tilde{f}, g], \tag{2.4}$$

where

$$\tilde{f}(x) = \frac{f(\theta^{-1}(x))}{\theta'(\theta^{-1}(x))}.$$

Therefore, since

$$\int_{\theta(a)}^{\theta(b)} \tilde{f}(t)\, \mathrm{d}t = \int_a^b f(x)\, \mathrm{d}x,$$

we deduce from (2.3) that

$$\begin{aligned}\boldsymbol{I}[f,g,\theta] \sim{}& \tfrac{1}{2}\rho_0 \int_a^b f(x)\,\mathrm{d}x \\ &+ \sum_{k=0}^{\infty} \frac{(-1)^k}{\omega^{2k+1}} [\tilde{f}^{(2k)}(\theta(b))U_k(\theta(b)) - \tilde{f}^{(2k)}(\theta(a))U_k(\theta(a))] \\ &+ \sum_{k=0}^{\infty} \frac{(-1)^k}{\omega^{2k+2}} [\tilde{f}^{(2k+1)}(\theta(b))V_k(\theta(b)) - \tilde{f}^{(2k+1)}(\theta(a))V_k(\theta(a))].\end{aligned} \tag{2.5}$$

Matters are more complicated in the presence of stationary points, where θ' vanishes. We may assume without loss of generality that $\theta'(b) = 0$ and $\theta'(x) \neq 0$ for $x \in [a,b)$, since any integral (1.5) with $r \geq 1$ stationary points can be written as a sum of $r+1$ integrals with a stationary point at an endpoint and the latter can be assumed at the larger endpoint through possible linear transformation of variable.

The change of variables (2.4) remains valid but the expansion (2.5) is unusable, because $\tilde{f}$ and its derivatives have singularity at b. Yet, we can extract from (2.4) an important morsel of information. Thus, suppose that $\theta^{(i)}(b) = 0$, $i = 1,2,\ldots,q$, and $\theta^{(q+1)}(b) \neq 0$. We assume in addition, without loss of generality, that $\theta(b) = 0$. It is trivial to deduce that $\theta'(\theta^{-1}(t)) \approx c(b-t)^{q/(q+1)}$ as $t \to b$ for some $c \in \mathbb{C} \setminus \{0\}$ and it follows at once that

$$\boldsymbol{I}[f,g,\theta] \sim \mathcal{O}\Big(\omega^{-1/(q+1)}\Big), \qquad \omega \to \infty,$$

a result similar to the classical *van der Corput lemma* for integrals (1.1) (Stein 1993).

To obtain an asymptotic expansion, we take a leaf off (Iserles & Nørsett 2005). Assume for simplicity that $q = 1$, therefore $\theta'(b) = 0$ and $\theta''(b) \neq 0$. We add and subtract $f(b)$,

$$\begin{aligned}\boldsymbol{I}[f,g,\theta] &= f(b)\int_a^b g(\sin[\omega\theta(x)])\,\mathrm{d}x + \int_a^b [f(x)-f(b)]g(\sin[\omega\theta(x)])\,\mathrm{d}x \\ &= f(b)\int_a^b g(\sin[\omega\theta(x)])\,\mathrm{d}x + \int_{\theta(a)}^{\theta(b)} \frac{f(\theta^{-1}(t)) - f(b)}{\theta'(\theta^{-1}(t))} g(\sin\omega t)\,\mathrm{d}t.\end{aligned}$$

The function

$$\breve{f}(x) = \frac{f(\theta^{-1}(x)) - f(b)}{\theta'(\theta^{-1}(x))}$$

has a removable singularity at $\theta(b)$ and is $\mathrm{C}^\infty[\theta(a),\theta(b)]$. Therefore we can again use (2.3) to expand

$$\begin{aligned}\boldsymbol{I}[f,g,\theta] \sim{}& f(b)\int_a^b g(\sin[\omega\theta(x)])\,\mathrm{d}x + \tfrac{1}{2}\rho_0 \int_a^b [f(x)-f(b)]\,\mathrm{d}x \\ &+ \sum_{k=0}^{\infty} \frac{(-1)^k}{\omega^{2k+1}} [\breve{f}^{(2k)}(\theta(b))U_k(\theta(b)) - \breve{f}^{(2k)}(\theta(a))U_k(\theta(a))] \\ &+ \sum_{k=0}^{\infty} \frac{(-1)^k}{\omega^{2k+2}} [\breve{f}^{(2k+1)}(\theta(b))V_k(\theta(b)) - \breve{f}^{(2k+1)}(\theta(a))V_k(\theta(a))].\end{aligned} \tag{2.6}$$

Note two important differences between (2.5) and (2.6). Firstly, in the latter equation there is an additional integral which, in general, is unknown – except that, by our analysis, we know that it is $\sim \mathcal{O}(\omega^{-1/2})$. Secondly, while each $\tilde{f}^{(k)}(\theta(b))$ can be written as a linear combination of $f^{(i)}(b)$ for $i = 0, 1, \ldots, k$, evaluating $\breve{f}^{(k)}(\theta(b))$ requires $f^{(i)}(b)$ for $i = 0, 1, \ldots, 2k$, roughly twice the data, because the endpoint is a removable singularity.

3 The quadrature of $\boldsymbol{I}[f, g]$

3.1 Asymptotic method

An obvious approach toward the calculation of the integral $\boldsymbol{I}[f, g]$ is to truncate the asymptotic expansion (2.3). In the terminology of (Iserles & Nørsett 2005), the outcome is the *asymptotic method*

$$\begin{aligned} \boldsymbol{A}_s[f, g] = \tfrac{1}{2}\rho_0 \int_a^b f(x)\,\mathrm{d}x + \sum_{k=0}^{s-1} \frac{(-1)^k}{\omega^{2k+1}} [f^{(2k)}(b)U_k(b) - f^{(2k)}(a)U_k(a)] \\ + \sum_{k=0}^{s-1} \frac{(-1)^k}{\omega^{2k+2}} [f^{(2k+1)}(b)V_k(b) - f^{(2k+1)}(a)V_k(a)], \end{aligned} \tag{3.1}$$

where $s \in \mathbb{N}$ is given. Comparing (2.3) with (3.1) demonstrates at once that

$$\boldsymbol{A}_s[f, g] = \boldsymbol{I}[f, g] + \mathcal{O}(\omega^{-2s-1}), \qquad \omega \to \infty. \tag{3.2}$$

Therefore, similarly to the numerical theory for classical highly oscillatory integrals (1.1), the more are we willing to invest in computing derivatives of f at the endpoints, the better the accuracy for large ω.

Figure 3.1 displays the scaled error committed by the asymptotic method (3.1), as applied to $f(x) = \mathrm{e}^x$, $g(x) = (2 - x)^{-1}$ in the interval $[-1, 1]$. For each $\omega \in [0, 100]$ we have computed the absolute error $|\boldsymbol{A}_s[f] - I[f]|$ and scaled it by ω^{2s+1}. The asymptotic formula (3.2) indicates that $\omega^{2s+1}|\boldsymbol{A}_s[f] - I[f]|$ should be bounded away from infinity for $\omega \gg 1$ and this is confirmed by the figure. As a matter of fact, the onset of asymptotic behaviour is almost immediate!

An implicit assumption underlying (3.2) is that all the quantities in (3.1) are available in an explicit form. In reality, we require three levels of approximation;

1. For general f we need to replace the integral by quadrature;

2. The sequence $\boldsymbol{\rho}$ might need to be approximated; and

3. The functions U_k and V_k are given as infinite sums and need be truncated in practical computation.

We address these three approximations in detail

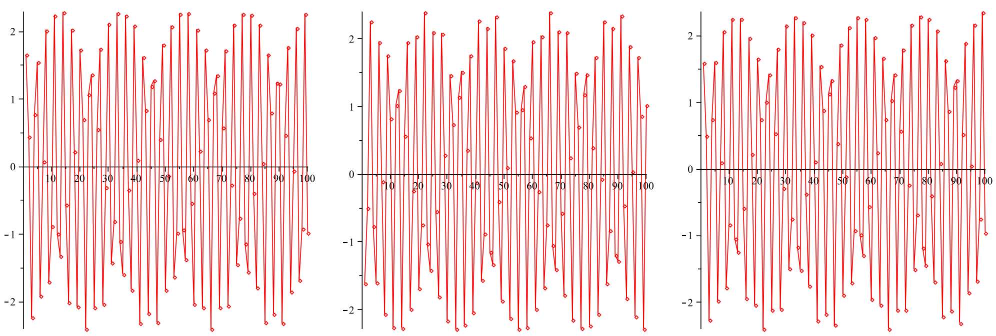

Figure 3.1: Scaled errors $\omega^{2s+1}|\boldsymbol{A}_s[f]-I[f]|$ for $f(x)=\mathrm{e}^x$, $g(x)=(2-x)^{-1}$, $[a,b]=[-1,2]$ and $s=1,2,3$.

3.1.1 Computing the non-oscillatory integral

Often the integral of f is known. Otherwise we propose to compute it using an approach from (Iserles & Nørsett 2008). In order to implement (3.1), we need to compute $f^{(k)}(a)$ and $f^{(k)}(b)$ for $k=0,\ldots,2s-1$. The idea is to reuse these values in a *Birkhoff quadrature* of the form

$$\int_a^b f(x)\,\mathrm{d}x \approx \sum_{k=0}^{2s-1}[w_k^a f^{(k)}(a)+w_k^b f^{(k)}(b)], \tag{3.3}$$

where the weights w_k^a and w_k^b are chosen to maximize classical quadrature order. It is easy to verify that (3.3) can be made exact for all polynomials f of order $4s-1$ and to compute the weights for any s and interval $[a,b]$. For example, for $[a,b]=[-1,1]$ we have $w_k^b=(-1)^k w_k^a$, $k=0,1,\ldots,2s-1$, and

$$s=1:\quad \boldsymbol{w}^a=\begin{bmatrix}1\\ \frac{1}{3}\end{bmatrix},\qquad s=2:\quad \boldsymbol{w}^a=\begin{bmatrix}1\\ \frac{3}{7}\\ \frac{2}{21}\\ \frac{1}{105}\end{bmatrix},\qquad s=3:\quad \boldsymbol{w}^a=\begin{bmatrix}1\\ \frac{5}{11}\\ \frac{4}{33}\\ \frac{2}{99}\\ \frac{1}{495}\\ \frac{1}{10395}\end{bmatrix}.$$

One way of improving upon the quadrature (3.3) is to allow the computation of intermediate points,

$$\int_a^b f(x)\,\mathrm{d}x \approx \sum_{k=0}^{2s-1}[w_k^a f^{(k)}(a)+w_k^b f^{(k)}(b)]+\sum_{l=1}^{\nu} w_k^i f(c_k), \tag{3.4}$$

where $c_k\in(a,b)$ are given internal nodes. It is easy to verify that for $[a,b]=[-1,1]$ the optimal choice of internal nodes is the zeros of the Jacobi polynomial $P_\nu^{(2s,2s)}$: the outcome is exact for all polynomials f of degree $4s-1+2\nu$. This can be easily extended to arbitrary finite intervals by linear translation.

Table 1: Errors of different Birkhoff quadrature rules

$f(x)$	(3.3), $s = 1$	(3.3), $s = 2$	(3.3), $s = 3$	(3.4), $s = \nu = 3$
e^x	4.77_{-02}	2.11_{-05}	1.47_{-09}	1.11_{-19}
$(2-x)^{-1}$	6.18_{-02}	5.14_{-03}	4.76_{-04}	8.85_{-09}
$\cos x$	4.14_{-02}	1.93_{-05}	1.38_{-09}	1.07_{-19}

Addition of internal nodes improves drastically the accuracy, as illustrated by Table 1. We present there the absolute error of different quadrature schemes (3.3) and (3.4) for three functions f. The dramatic improvement upon the addition of internal points is self evident.

3.1.2 The approximation of ρ_m

The second step in need of approximation is the computation of the sequence $\boldsymbol{\rho}$ defined in (2.2). We have already seen in Subsection 2.2 that the sequence can be explicitly computed for some functions g and that, once g is a generalized hypergeometric function, each ρ_m can be expressed in terms of generalized hypergeometric functions. Such functions can be computed efficiently, often to machine accuracy, with most leading software packages.

Given general function g which cannot be reduced to a generalized hypergeometric form, an alternative is to interpolate it by a polynomial φ, say, at points $\vartheta_1 < \vartheta_2 < \cdots < \vartheta_\nu$ in $[-1,1]$ – in other words

$$g(x) \approx \varphi(x) = \sum_{j=1}^{\nu} \ell_j(x) g(\vartheta_j),$$

where the ℓ_js are cardinal polynomials of Lagrange's interpolation at $\vartheta_1, \vartheta_2, \ldots, \vartheta_\nu$. Other things being equal, the natural choice of ϑ_ks is as *Chebyshev points*, since this renders the error $\|g-\varphi\|_{\mathrm{L}_\infty[-1,1]}$ small. We then approximate

$$\begin{aligned}\rho_m \approx \tilde{\rho}_m &= \frac{1}{2^{m-1}} \sum_{n=0}^{\infty} \frac{\varphi^{(m+2n)}(0)}{m!(m+n)!4^n} \\ &= \frac{1}{2^{m-1}} \sum_{j=1}^{\nu} g(c_j) \sum_{n=0}^{\lfloor(\nu-m-1)/2\rfloor} \frac{\ell_j^{(m+2n)}(0)}{n!(m+n)!4^n}.\end{aligned} \tag{3.5}$$

Note that the computation of each $\tilde{\rho}_m$ is a finite sum. However, $\tilde{\rho}_m = 0$ for $m \geq \nu$. This, as well as the need to attain good accuracy, means that we need a sufficiently large value of ν.

An alternative to Lagrangian interpolation is rational approximation. Mindful of the need to ensure analyticity, we consider a function of the form

$$\psi(x) = \sum_{j=1}^{\nu} \frac{\alpha_j}{x-\beta_j}$$

where $\beta_1, \beta_2, \ldots, \beta_\nu \notin [-1,1]$. For example, we can fix the poles β_j along a perimeter of an ellipse surrounding the interval $[-1,1]$ and determine $a_1, a_2, \ldots, a_\nu$ by imposing interpolation conditions $\psi(\vartheta_j) = g(\vartheta_j)$ for some $\vartheta_1, \vartheta_2, \ldots, \vartheta_\nu \in [a,b]$.

Since

$$\psi^{(m)}(x) = (-1)^m m! \sum_{j=1}^{\nu} \frac{\alpha_j}{(x-\beta_j)^{m+1}}, \qquad \text{we have} \qquad \psi^{(m)}(0) = -m! \sum_{j=1}^{\nu} \frac{\alpha_j}{\beta_j^{m+1}}$$

and we approximate

$$\rho_m \approx \bar{\rho}_m = \frac{1}{2^{m-1}} \sum_{n=0}^{\infty} \frac{\psi^{(m+2n)}(0)}{n!(m+n)!4^n} = -\frac{1}{2^{m-1}} \sum_{j=1}^{\nu} \frac{\alpha_j}{\beta_j} \sum_{n=0}^{\infty} \frac{(m+2n)!}{n!(m+n)!} \frac{1}{(2\beta_j)^{m+2n}}.$$

Bearing in mind that $(m+n)! = m!(m+1)_n$ and $(m+2n)! = 4^n m!((m+1)/2)_m((m+2)/2)_n$, we deduce that

$$\bar{\rho}_m = -\frac{1}{2^{m-1}} \sum_{j=1}^{\nu} \frac{\alpha_j}{\beta_j^{m+1}} \chi(\beta_j^{-2}) = -\sum_{j=1}^{\nu} \frac{\alpha_j}{\sqrt{\beta_j^2 - 1}} \frac{1}{\left(\beta_j + \sqrt{\beta_j^2 - 1}\right)^m}, \tag{3.6}$$

where the function χ has been defined and discussed in Subsection 2.2. Note that we must be careful in the choice of the sign of $\sqrt{\beta_j^2 - 1}$: since the branch cut is along the line $\mathrm{Re}\, z = 0$, its sign must be the same as the sign of $\mathrm{Re}\, \beta_j$.

Note that $\beta_j \notin [-1, 1]$ implies $\left|\beta_j + \sqrt{\beta_j^2 - 1}\right| > 1$, therefore the sequence $\{\bar{\rho}\}_{m \in \mathbb{Z}_+}$ decays geometrically. Similarly to (3.5), the computation of each $\bar{\rho}_m$ consists of a finite summation and is probably better conditioned since it does not require the computation of derivatives.

We did not analyse in detail the approximations (3.5) and (3.6) but preliminary computational results indicate the superiority of using rational interpolation.

3.1.3 Computing U_k and V_k

Because of the rapid decay of $\boldsymbol{\rho}$, the four infinite sums required for the computation of $\{U_m(t), V_k(t)\}$ converge rapidly. Therefore we can afford to truncate the sums, retaining relatively small number of terms – the largest k, the fewer terms we require.

3.2 A Filon-type method

Let $a = c_1 < c_2 < \cdots < c_\nu = b$ by arbitrary *quadrature nodes* and $\mu_1, \mu_2, \ldots, \mu_\nu \in \mathbb{N}$ corresponding *multiplicities*. Similarly to Filon-type methods for integrals (1.1) in (Iserles & Nørsett 2005), we interpolate the function f by a polynomial ϕ of degree $N = \sum_{k=1}^{\nu} \mu_k - 1$,

$$\phi^{(i)}(c_k) = f^{(i)}(c_k), \qquad i = 0, 1, \ldots, \mu_k - 1, \quad k = 1, 2, \ldots, \nu, \tag{3.7}$$

and set

$$\boldsymbol{F}_r[f, g] := \boldsymbol{I}[\phi, g], \tag{3.8}$$

where $r = \min\{\mu_1, \mu_\nu\}$.

An important feature of the *Filon-type method* (3.8) is that $\boldsymbol{I}[\varphi, g]$ can be evaluated explicitly, since the asymptotic expansion (2.3) terminates for polynomials,

$$\boldsymbol{F}[\phi, g] = \tfrac{1}{2}\rho_0 \int_a^b \phi(x)\,\mathrm{d}x + \sum_{k=0}^{\lfloor N/2 \rfloor} \frac{(-1)^k}{\omega^{2k+1}} [\phi^{(2k)}(b)U_k(b) - \phi^{(2k)}(a)U_k(a)] \qquad (3.9)$$
$$+ \sum_{k=0}^{\lfloor (N-1)/2 \rfloor} \frac{(-1)^k}{\omega^{2k+2}} [\phi^{(2k+1)}(b)V_k(b) - \phi^{(2k+1)}(a)V_k(a)].$$

Theorem 2 *The error of the Filon-type method (3.9) is*

$$\boldsymbol{F}_r[f, g] - \boldsymbol{I}[f, g] \sim \boldsymbol{E}[\phi, f] + \mathcal{O}\left(\omega^{-r-1}\right), \qquad \omega \to \infty, \qquad (3.10)$$

where

$$\boldsymbol{E}[\phi, f] = \left| \int_a^b [\phi(x) - f(x)]\,\mathrm{d}x \right|$$

is the error of the underlying Birkhoff quadrature.

Proof Since $\boldsymbol{F}_r[f, g] - \boldsymbol{I}[f, g] = \boldsymbol{I}[\phi - f, g]$, the theorem follows at once by substituting $\phi - f$ in place of f in the asymptotic expansion (2.3). □

The two above error components, one originating in non-oscillatory quadrature and the other asymptotic, are similar to what we have observed for the asymptotic method and they have been already discussed for the special case $g(x) = \mathrm{e}^{\kappa x}$ in (Condon et al. 2008*a*).

In case $\int_a^b f(x)\,\mathrm{d}x$ is known, we replace (3.8) with

$$\tilde{\boldsymbol{F}}[\phi, g] = \tfrac{1}{2}\rho_0 \int_a^b f(x)\,\mathrm{d}x + \sum_{k=0}^{\lfloor N/2 \rfloor} \frac{(-1)^k}{\omega^{2k+1}} [\phi^{(2k)}(b)U_k(b) - \phi^{(2k)}(a)U_k(a)] \qquad (3.11)$$
$$+ \sum_{k=0}^{\lfloor (N-1)/2 \rfloor} \frac{(-1)^k}{\omega^{2k+2}} [\phi^{(2k+1)}(b)V_k(b) - \phi^{(2k+1)}(a)V_k(a)].$$

It is now trivial to observe that, in place of (3.10), we have

$$\tilde{\boldsymbol{F}}_r[f, g] - \boldsymbol{I}[f, g] \sim \mathcal{O}\left(\omega^{-r-1}\right), \qquad \omega \to \infty,$$

without any derogatory influence of Birkhoff quadrature error.

In Fig. 3.2 we display scaled error for six Filon-type approximations (3.11): note that in our example the non-oscillatory integral is trivial and we computed it explicitly but, in fairness, so it was in Fig. 3.1 hence we compare alike with alike.

Our first observation is that, even when the asymptotic and Filon-type methods employ exactly the same information (i.e., when $\nu = 2$, $\mu_1 = \mu_2$ and there are no internal nodes), the Filon-type method is more precise. This is clear when comparing Figs 3.1a with 3.2a and 3.1b with 3.2f respectively. This is consistent with a comparison of asymptotic and Filon-type methods for integrals (1.1) and can break down when interpolation of f by ϕ is of poor quality (the Runge example), cf. (Olver 2008).

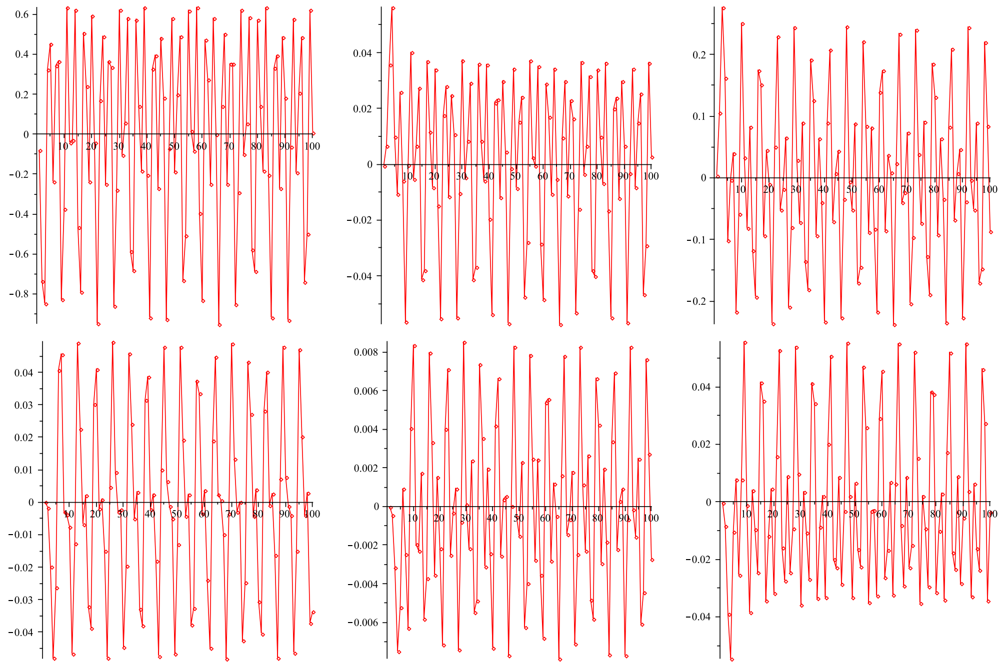

Figure 3.2: Scaled errors $\omega^{r+1}|\tilde{\boldsymbol{F}}_s[f,g] - \boldsymbol{I}[f,g]|$ for $f(x) = \mathrm{e}^x$, $g(x) = (2-x)^{-1}$, $[a,b] = [-1,2]$ and six different Filon-type methods (3.11): (a) $r = 2$, $\nu = 2$; (b) $r = 2$, $\nu = 4$; $c_2 = 0$, $c_3 = 1$; (c) $r = 3$, $\nu = 2$; (d) $r = 3$, $\nu = 3$, $c_2 = 1$; (e) $r = 3$, $\nu = 4$, $c_2 = 0$, $c_3 = 1$; (f) $r = 4$, $\nu = 2$. In all these methods $\mu_1 = \mu_\nu = r$, otherwise $\mu_k = 1$.

Our second observation is again in line with the theory of Filon-type methods for 'classical' integrals (1.1). The addition of internal nodes leaves the asymptotic error $\mathcal{O}\big(\omega^{-r-1}\big)$ intact. Yet, at a small extra cost, it significantly lowers the amplitude of the error. This is apparent when comparing Figs 2.2a and 2.2b or Figs 2.2c, 2.2d and 2.2e. The intuitive reason is that addition of internal points makes the interpolation error smaller, and this is reflected in the error of the Filon-type method.

3.3 Computing the generalised oscillator

As long as $\theta' \neq 0$, the scope of both asymptotic and Filon-type methods generalizes at once to the integral $\boldsymbol{I}[f,g,\theta]$. In place of the asymptotic method (3.1) we truncate (2.5), while expressing each $\tilde{f}^{(k)}(x)$ for $x \in \{a,b\}$ as a linear combination of $f^{(i)}(x)$, $i = 0,1,\ldots,k$, with coefficients that depend upon θ.

A generalization of the Filon-type method to this setting is even simpler. We again choose nodes $a = c_1 < c_2 < \cdots < c_\nu = b$ and corresponding weights $\mu_1, \mu_2, \ldots, \mu_\nu \geq 1$. The function $\tilde{f}$ is interpolated by

$$\tilde{\phi}(t) = \sum_{l=0}^{N} \phi_l [\theta^{-1}(t)]^l.$$

Specifically, we require

$$\tilde{\phi}^{(i)}(\theta(c_k)) = \tilde{f}^{(i)}(\theta(c_k)), \qquad i = 0, \ldots, \mu_k - 1, \quad k = 1, 2, \ldots, \nu. \tag{3.12}$$

However, since each $\tilde{f}^{(i)}(\theta(x))$ is a linear combination of $f(x), f'(x), \ldots, f^{(i)}(x)$, it follows at once that (3.12) is equivalent to (3.7) where, again, ϕ is an nth-degree algebraic polynomial. Therefore, in the absence of stationary points, a Filon-type method for (1.5) is identical to that for (1.4).

The computation of the generalised oscillator (1.5) in the presence of stationary points is made considerably more complicated by the presence of the term

$$\zeta(\omega) = \int_a^b g(\sin[\omega\theta(x)])\,\mathrm{d}x.$$

In principle, we can expand ζ similarly to (2.1),

$$\zeta(\omega) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} \int_a^b \sin^n[\omega\theta(x)]\,\mathrm{d}x,$$

and proceed like in Subsection 2.1, but fairly rapidly we are faced by fairly unpleasant expressions. Although all this can be accomplished, with great deal of effort, for simple functions θ, e.g. $\theta(x) = x^q$ for $q \in \mathbb{N}$, we see no point of embarking on such long calculation without further motivation of a specific application.

Once ζ can be computed, we generalize our method along similar lines to the Filon-type method in the presence stationary points in (Iserles & Nørsett 2005). Thus, in addition to multiplicity-r interpolation at the endpoint, we interpolate to multiplicity $(m+1)r$ at any stationary point c of degree $m \geq 1$ (that is, $\theta^{(i)}(c) = 0$, $i = 1, 2, \ldots, m$, $\theta^{(m+1)}(c) \neq 0$), and perhaps at additional points.

4 Discussion

The main purpose of this paper has been to explore asymptotic and numerical features of a new model for composite highly oscillatory integrals. This model is of relevance in the simulation of electronic circuits, but we believe that its potential importance ranges wider and that it is of an independent mathematical interest.

We have singled Filon-type methods as the method of choice for our integrals. It might well have been possible to extend Levin-type methods to this setting. However, the remaining member of the triad of modern methods for highly oscillatory quadrature, the method of numerical steepest descent of Huybrechs & Vandewalle (2006), is probably of no use in the current situation. The idea underlying this approach is to integrate in the complex plane, along trajectories where the integral decays exponentially. This can be done very effectively with integrals of the form (1.1) but not, say, (1.3), since there is no trajectory $z(x + \mathrm{i}y)$ linking a, ∞ and b in the complex plane along which $\sin[\omega z(x + \mathrm{i}y)]$ decays exponentially along the segments linking a and b to ∞.

References

Abramowitz, M. & Stegun, I. A., eds (1964), *Handbook of Mathematical Functions*, National Bureau of Standards, Washington, DC.

Condon, M., Deaño, A. & Iserles, A. (2008*a*), On highly oscillatory problems arising in electronic engineering, Technical Report NA2008/10, DAMTP, University of Cambridge.

Condon, M., Deaño, A., Iserles, A., Maczynski, K. & Xu, T. (2008*b*), On numerical methods for highly oscillatory problems in circuit simulation, Technical Report NA2008/12, DAMTP, University of Cambridge.

Dautbegovic, E., Condon, M. & Brennan, C. (2005), 'An efficient nonlinear circuit simulation technique', *IEEE Trans. Microwave Theory & Techniques* **53**, 548–555.

Huybrechs, D. & Vandewalle, S. (2006), 'On the evaluation of highly oscillatory integrals by analytic continuation', *SIAM J. Num. Anal.* **44**, 1026–1048.

Iserles, A. & Nørsett, S. P. (2004), 'On quadrature methods for highly oscillatory integrals and their implementation', *BIT* **44**, 755–772.

Iserles, A. & Nørsett, S. P. (2005), 'Efficient quadrature of highly oscillatory integrals using derivatives', *Proc. Royal Soc. A* **461**, 1383–1399.

Iserles, A. & Nørsett, S. P. (2008), 'From high oscillation to rapid approximation I: Modified Fourier expansions', *IMA J. Num. Anal.* **28**, 862–887.

Levin, D. (1996), 'Fast integration of rapidly oscillatory functions', *J. Comput. Appl. Maths* **67**, 95–101.

Olver, F. W. J. (1974), *Asymptotics and Special Functions*, Academic Press, New York.

Olver, S. (2006), 'Moment-free numerical integration of highly oscillatory functions', *IMA J. Num. Anal.* **26**, 213–227.

Olver, S. (2008), Numerical Approximation of Highly Oscillatory Integrals, PhD thesis, DAMTP, University of Cambridge.

Rainville, E. D. (1960), *Special Functions*, Macmillan, New York.

Stein, E. M. (1993), *Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, NJ.

Wong, R. (2001), *Asymptotic Approximations of Integrals*, SIAM, Philadelphia.